Math 4507 Review

Euclidean Spaces: $\mathbb{R}^3$

standard coordinates: $(\hat{x}, \hat{y}, \hat{z})$.

Dot product: $(\hat{a}_1, \hat{b}_1, \hat{c}_1) \cdot (\hat{a}_2, \hat{b}_2, \hat{c}_2) := \hat{a}_1\hat{a}_2 + \hat{b}_1\hat{b}_2 + \hat{c}_1\hat{c}_2$.

$$= (\hat{a}_1, \hat{b}_1, \hat{c}_1) \begin{pmatrix} \hat{a}_2 \\ \hat{b}_2 \\ \hat{c}_2 \end{pmatrix}$$

Distance: $d(\hat{X}_1, \hat{X}_2) := \sqrt{(\hat{x}_1 - \hat{x}_2)^2 + (\hat{y}_1 - \hat{y}_2)^2 + (\hat{z}_1 - \hat{z}_2)^2}$

where $\hat{X}_1 = (\hat{x}_1, \hat{y}_1, \hat{z}_1)$

$\hat{X}_2 = (\hat{x}_2, \hat{y}_2, \hat{z}_2)$

length of vectors: $|\hat{V}_1| := \sqrt{\hat{V}_1 \cdot \hat{V}_1}$.

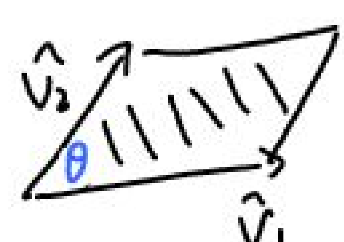
<u>Angle</u>: Consider

. $\boxed{\text{Law of Cosine}}$ says

$$|\hat{V}_2 - \hat{V}_1|^2 = |\hat{V}_1|^2 + |\hat{V}_2|^2 - 2|\hat{V}_1||\hat{V}_2| \cos\theta.$$

As a corollary: $\cos\theta = \dfrac{\hat{V}_1 \cdot \hat{V}_2}{|\hat{V}_1||\hat{V}_2|}$

Area of parallelograms:



, then Area $= |\hat{V}_1||\hat{V}_2| \sin\theta$

Also, Area $= |\hat{V}_1 \times \hat{V}_2|$, Where $\hat{V}_1 \times \hat{V}_2$ is the <u>cross-product</u>

defined by: $\hat{V}_1 = (\hat{a}_1, \hat{b}_1, \hat{c}_1)$, $\hat{V}_2 = (\hat{a}_2, \hat{b}_2, \hat{c}_2)$

$$\hat{V}_1 \times \hat{V}_2 = \begin{vmatrix} i & j & k \\ \hat{a}_1 & \hat{b}_1 & \hat{c}_1 \\ \hat{a}_2 & \hat{b}_2 & \hat{c}_2 \end{vmatrix} = \begin{vmatrix} \hat{b}_1 & \hat{c}_1 \\ \hat{b}_2 & \hat{c}_2 \end{vmatrix} i - \begin{vmatrix} \hat{a}_1 & \hat{c}_1 \\ \hat{a}_2 & \hat{c}_2 \end{vmatrix} j + \begin{vmatrix} \hat{a}_1 & \hat{b}_1 \\ \hat{a}_2 & \hat{b}_2 \end{vmatrix} k$$

Curves in $\mathbb{R}^3$: $\hat{X} : [a,b] \longrightarrow \mathbb{R}^3$

$t \longmapsto (\hat{x}(t), \hat{y}(t), \hat{z}(t))$

tangent vectors: $\dfrac{d\hat{X}}{dt} = \left(\dfrac{d\hat{x}}{dt}, \dfrac{d\hat{y}}{dt}, \dfrac{d\hat{z}}{dt} \right)$.

length: $$\int_a^b \sqrt{\frac{d\hat{X}}{dt} \cdot \frac{d\hat{X}}{dt}}\, dt$$

<u>New coordinates:</u>

$R > 0$

$$\begin{cases} x = \hat{x} \\ y = \hat{y} \\ z = \hat{z}/R \end{cases}$$

effect: the north pole of the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$ coordinate is $(0,0,R)$, which becomes $(0,0,1)$ in (x,y,z) coordinates.

eq'n of R-sphere in $\hat{x}$-$\hat{y}$-$\hat{z}$ coordinate: $\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2$

in x,y,z-coordinates: $\dfrac{1}{R^2}(x^2+y^2) + z^2 = 1$. or $\boxed{K(x^2+y^2) + z^2 = 1}$

$\boxed{K := \dfrac{1}{R^2}}$

<u>3 geometries</u>:

$K > 0$: spherical

$K = 0$: planar

$K < 0$: hyperbolic.

<u>Important</u>: Need to express dot product in x,y,z-coordinates

<u>Ans</u>: K-dot product $V_1 \cdot_K V_2 := V_1 \begin{pmatrix} 1 & & \\ & 1 & \\ & & K^{-1} \end{pmatrix} V_2^t$

<u>Point</u>: $V_1 \cdot_K V_2 = \hat{V}_1 \cdot \hat{V}_2$.

<u>Consequence</u>: quantities expressed in terms of dot product can now be expressed in terms of K-dot product.

Rigid motion: $\hat{M} = 3\times 3$ matrix. The map $\hat{X} \mapsto \hat{X}\hat{M}$ is a rigid motion if distances are preserved, i.e.

$$d(\hat{X}_1\hat{M}, \hat{X}_2\hat{M}) = d(\hat{X}_1, \hat{X}_2) \quad \forall\ \hat{X}_1, \hat{X}_2.$$

i.e. $\hat{M}\hat{M}^t = I$ (def'n: $\hat{M}$ is an orthogonal matrix).

Examples: Rotations about coordinate axis.

In xyz coordinates: the map $X \mapsto XM$ is a rigid motion (i.e. preserves distances) iff

$$M \begin{pmatrix} 1 & & \\ & 1 & \\ & & K^{-1} \end{pmatrix} M^t = \begin{pmatrix} 1 & & \\ & 1 & \\ & & K^{-1} \end{pmatrix}.$$

(def'n K-orthogonal matrix)

Neutral Geometry A 2-dim'l geometry that satisfies the first 4 of Euclid's axioms:

axiom 1: Given $P \neq Q$, $\exists!$ line thru P & Q

axiom 2: Given $\overline{AB}$, $\overline{CD}$, $\exists$ $\overline{AE}$ s.t. $B \in \overline{AE}$ and $|\overline{BE}| = |\overline{CD}|$

axiom 3 Given P, and $r>0$, $\exists!$ circle centered at P and w/ radius r.

axiom 4 All right angles are congruent.

Def'n 2 lines are parallel if they don't intersect.

- Criteria for congruence of triangles (SSS, SAS, ASA) hold in neutral geometry.
- There are several interesting results about angles proved in class. Perhaps the most interesting is

Thm: In neutral geometry, the sum of interior angles of a triangle is $\leq 180°$.

Euclidean geometry is a neutral geometry w/ one more axiom:

Axiom 5 (Parallel postulate) Given line L and point $p \notin L$, $\exists!$ line M s.t. $p \in M$ and $L \parallel M$.

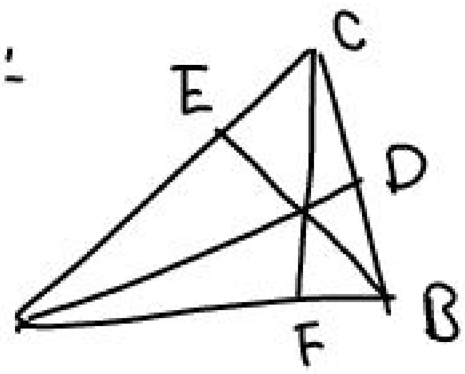
This implies many things. In particular:

Thm In Euclidean geometry, the sum of interior angles of a triangle is $180°$.

- We had the notion of dilation, which leads to the notion of similarity. Criterion for similarity of triangles: SSS, AAA.

Notable Theorems in Euclidean Geometry

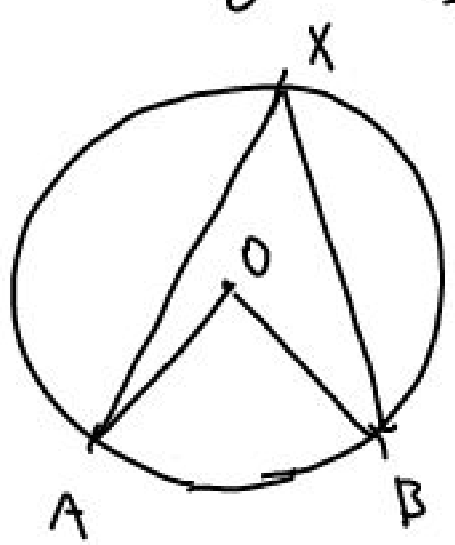
Ceva's Thm:



If $\frac{|\overline{AF}|}{|\overline{FB}|}\frac{|\overline{BD}|}{|\overline{DC}|}\frac{|\overline{CE}|}{|\overline{EA}|} = 1$, then $\overline{AD}, \overline{BE}, \overline{CF}$ meet at a common pt.

The converse is also true.

Inscribed angles: Thm The measure of an inscribed angle is half the measure of the corresponding central angle



Cross-ratios: $t_1, t_2, t_3, t_4 \in \mathbb{R} \cup \{\infty\}$ Their cross-ratio is defined to be
$$(t_1 : t_2 : t_3 : t_4) := \frac{t_1 - t_2}{t_3 - t_2} \Big/ \frac{t_1 - t_4}{t_3 - t_4}$$

- Cross-ratio is preserved under fractional linear transform:

$$t_i \mapsto \frac{a t_i + b}{c t_i + d}.$$

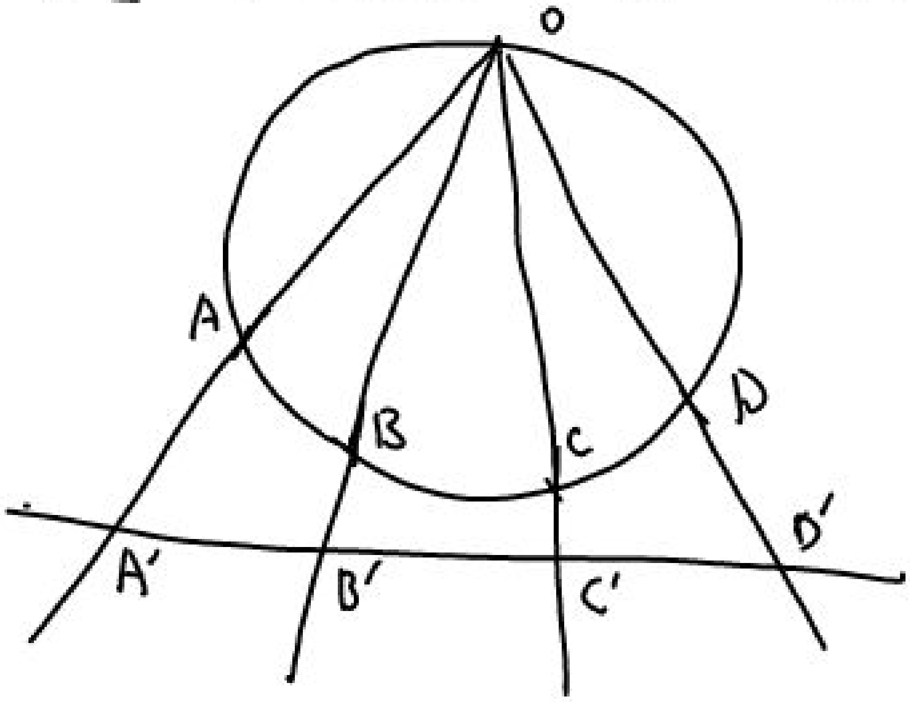
- A, B, C, D = four ordered pts on a circle, their cross ratio is

$$(A : B : C : D) := \frac{|\overline{AB}|}{|\overline{CB}|} \Big/ \frac{|\overline{AD}|}{|\overline{CD}|}.$$

- A', B', C', D' = four ordered pts on a line. their cross ratio is

$$(A' : B' : C' : D') := \frac{|\overline{A'B'}|}{|\overline{C'B'}|} \Big/ \frac{|\overline{A'D'}|}{|\overline{C'D'}|}.$$

<u>Thm</u>: Cross ratio is invariant under stereographic projection, i.e.



Then $(A : B : C : D) = (A' : B' : C' : D')$

<u>Thm</u> (Ptolemy) If four ordered pts A, B, C, D lie on a circle, then $|\overline{AC}||\overline{BD}| = |\overline{AD}||\overline{BC}| + |\overline{AB}||\overline{CD}|$.

<u>K-geometry</u>: model : $K(x^2+y^2)+z^2 = 1$

$K > 0$: spherical	$K < 0$: hyperbolic
$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$	same

useful K-rigid motions

$$\begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} \cosh\varphi & 0 & |K|^{1/2}\sinh\varphi \\ 0 & 1 & 0 \\ |K|^{-1/2}\sinh\varphi & 0 & \cosh\varphi \end{pmatrix}$$

Useful K-rigid motions II

- As discussed, a given (point, tangent vector) can be moved to another given (point, tangent vector) by a suitable K-rigid motion. (K-geometry is homogeneous.)

<u>Coordinates</u>

	Central projection	Stereographic projection
Coordinates	$x_c = X/Z$ $y_c = Y/Z$	$x_s = 2X/(Z+1)$ $y_s = 2Y/(Z+1)$
Construction	$r(x_c, y_c, 1) = (X, Y, Z)$	$\rho(x_s, y_s, 1-(-1)) = (X, Y, Z-(-1))$
inverse	$X = r x_c,\ Y = r y_c,\ Z = r$ $r^2 = \dfrac{1}{K(x_c^2+y_c^2)+1}$	$X = \rho x_s,\ Y = \rho y_s,\ Z = 2\rho - 1$ $\rho = \dfrac{1}{\frac{K}{4}(x_s^2+y_s^2)+1}$
dot product	$V_1^c \cdot_c V_2^c := V_1^c P_c (V_2^c)^t$ $P_c = \begin{pmatrix} r^2(1-r^2Kx_c^2) & -r^4Kx_cy_c \\ -r^4Kx_cy_c & r^2(1-r^2Ky_c^2) \end{pmatrix}$	$V_1^s \cdot_s V_2^s := V_1^s P_s (V_2^s)^t$ $P_s = \begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix}$
Area formula	$\int_{G_c} r^3\, dx_c\, dy_c$	$\int_{G_s} \rho^4\, dx_s\, dy_s$

Image of $ax+by+z=0$	$ax_c+by_c+1=0$	$\left(x_s-\frac{2a}{K}\right)^2+\left(y_s-\frac{2b}{K}\right)^2=\frac{4(K+a^2+b^2)}{K^2}$
Image of $ax+by=0$	$ax_c+by_c=0$	$ax_s+by_s=0$

Various measurements

- $\text{Volume (Pyramid)} = \frac{1}{3}\underbrace{B}_{\text{area of base}}\overbrace{h}^{\text{height}}$

(Recall:
- Cavalieri's principle
- Magnification principle)

- V = volume of R-sphere
  S = surface area of R-sphere, then $V=\frac{1}{3}R\cdot S$, $\boxed{S=4\pi R^2}$

- $\text{Area (Collar)} = 2\pi\cdot\frac{r_t+r_b}{2}\cdot S$: S = slant height
  r_t = radius of top circle
  r_b = radius of bottom circle

- $\text{Area}\left(\alpha\text{-lune on } R\text{-sphere}\right) = 2\alpha R^2$

- $\text{Area}\left(\text{triangle on } R\text{-sphere}\right) = R^2\left(\left(\text{Sum of interior angles}\right)-\pi\right)$

  Recall $R^2 = 1/K$

Lines in K-geometry

- formal def'n
- Thm: lines are given by intersecting K-geometry surface w/ planes passing thru the origin

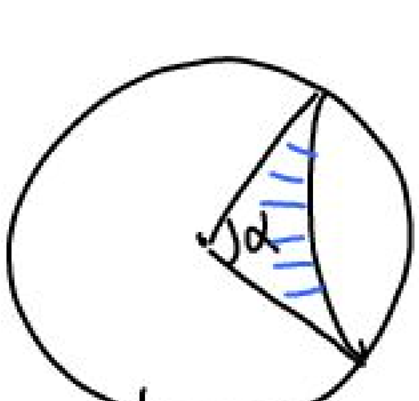
- Important : understand how the Thm is proved.

Hyperbolic geometry

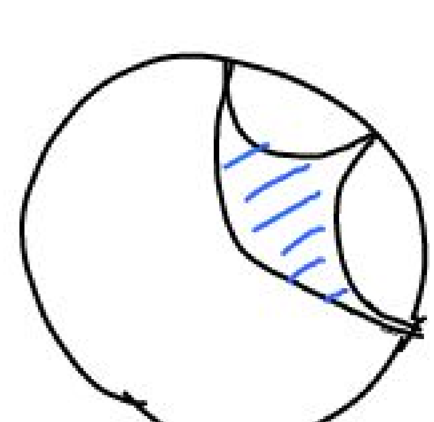
- K-dot product is nonnegative on tangent vectors
- The K-distance between $(|K|^{-1/2}\sinh\varepsilon, 0, \cosh\varepsilon)$ and $(0,0,1)$ is $|K|^{-1/2}\varepsilon$.

	Klein model	Poincaré model
model	$x_c^2+y_c^2<\frac{1}{\lvert K\rvert}$	$x_s^2+y_s^2<\frac{4}{\lvert K\rvert}$
lines	chords	Circular arcs perpendicular to the edge of universe
lines thru the north pole	diameters	diameters
K-distances	can be computed by cross-ratios	?

Various areas in Poincare model



$$\text{Area} = \frac{\pi-\alpha}{|K|}$$



$$\text{Area} = \frac{\pi}{|K|}$$

$$\text{Area}\left(\begin{array}{c}\text{hyperbolic}\\ \text{triangle}\end{array}\right) = \frac{\left(\pi - \left(\begin{array}{c}\text{sum of interior}\\ \text{angles}\end{array}\right)\right)}{|K|}.$$