AUTUMN 2011, MATH 580
NOTES ON THE PROOF OF THEOREM 5.14

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1. Introduction

As you may recall, the proof of Theorem 5.14 that I gave in class is incomplete. There is one statement that was left unproved. My attempt to proving this statement led me to think that the proof of Theorem 5.14 given in the textbook misses some details. The purpose of this note is to present a more detailed proof of Theorem 5.14.

2. Proof of Theorem 5.14

Let $G \subset O(2)$ be a finite subgroup. Either G is contained in $SO(2)$ or it's not. If G is contained in $SO(2)$ then we are in case (a) and the proof in this case is given in class and in the textbook. So we'll omit this part.

Now suppose G is not contained in $SO(2)$. In this case we have

Claim. Define $H := G \cap SO(2)$. Then there is a reflection $R \in O(2)$ such that

$$G = H \cup H \circ R.$$

Proof. Let $r_L \in O(2)$ be reflection on $\mathbb{R}^2$ with respect to a line L that passes through the origin. Then as explain in class, we have $O(2) = SO(2) \cup SO(2) \circ r_L$. Hence

$$G = G \cap O(2) = G \cap (SO(2) \cup SO(2) \circ r_L) = (G \cap SO(2)) \cup (G \cap SO(2) \circ r_L).$$

Since $H = G \cap SO(2)$ by definition, it suffices to show that

$$G \cap SO(2) \circ r_L = H \circ R$$

for *some* reflection R. If $G \cap SO(2) \circ r_L = \emptyset$ then $G \subset SO(2)$ and the Claim is clearly true. So we may assume that $G \cap SO(2) \circ r_L \neq \emptyset$. Let $a \in G \cap SO(2) \circ r_L$. Consider the following map

$$\phi : G \cap SO(2) \circ r_L \to O(2), \quad \phi(g) := g \circ a.$$

First we identify the image $\phi(G \cap SO(2) \circ r_L)$. Give $g \in G \cap SO(2) \circ r_L$, since both g and a are elements in the group G, we have

$$g \circ a \in G.$$

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Since both g and a are elements in $SO(2) \circ r_L$, there exists rotations $\rho, \rho_0 \in SO(2)$ such that

$$g = \rho \circ r_L, \quad a = \rho_0 \circ r_L.$$

Hence

$$g \circ a = \rho \circ r_L \circ \rho_0 \circ r_L = \rho \circ \rho_0^{-1} \in SO(2),$$

where in the second equality we used Chapter 2 Exercise 10 (b). The discussion above shows that $g \circ a \in G \cap SO(2) = H$, hence $\phi(G \cap SO(2) \circ r_L) = H$. Now note that ϕ is invertible: the inverse is given by $\phi^{-1}(g) := g \circ a^{-1}$. Hence

$$G \cap SO(2) \circ r_L = \phi^{-1}(\phi(G \cap SO(2) \circ r_L)) = \phi^{-1}(H) = H \circ a^{-1}.$$

Set $R := a^{-1}$, we have $G = H \cup H \circ R$. It remains to show that $R = a^{-1}$ is a reflection. To see this we compute

$$R^2 = a^{-1} \circ a^{-1} = r_L \circ \rho_0^{-1} \circ r_L \circ \rho_0^{-1} = \rho_0 \circ \rho_0^{-1} = Id,$$

where in the third equality we again used Chapter 2 Exercise 10 (b). Since R is an invertible linear transformation with $R^2 = Id$, the two eigenvalues of R must be either 1 or -1. The eigenvalues cannot both be 1, since that means that $R = Id$, which is not the case. The eigenvalues cannot both be -1, since that means that $R = -Id$, which is a rotation. So the two eigenvalues must be $1, -1$. The eigenspace with eigenvalue 1 is fixed by R and the eigenspace with eigenvalue -1 is flipped. It follows that R is the reflection with respect to the eigenspace with eigenvalue 1.

With Claim now proved, we may proceed in the way described in class. If $H = \{Id\}$ is the trivial group, then we are in case (b). If $H =< \rho >$ is a nontrivial cyclic group generated by a rotation ρ, then we are in case (c). This concludes the proof of Theorem 5.14.

3. Discussion

The main difference between the proof given here and the one in the textbook is the following. In the proof in the textbook one starts with $O(2) = SO(2) \cup SO(2) \circ r_L$ and deduce from this the equality $G = H \cup H \circ r_L$. I am not sure how to deduce this equality. In the proof given here, we obtain $G = H \cup H \circ R$ for *some* reflection not necessarily r_L. This is enough to prove Theorem 5.14.

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