

## 1.5 Bayes' Rule

### 1. INTRODUCTION

Suppose that we have two coins. One is fair and has an equal chance of landing on either side when flipped, the other is biased in that it has a  $\frac{2}{3}$  chance of landing on heads. Suppose that a coin is chosen at random and flipped with the result being a head. What is the probability that the coin is fair?

Consider this question by letting  $A$  be the event that a head occurs when flipped,  $B_1$  be the event that the fair coin is picked and  $B_2$  be the event that the biased coin is flipped. We are then trying to find  $P(B_1|A)$ , that is we are trying to find the probability of the event  $B_1$  happening prior to the occurrence of  $A$ . Notice that we do know that  $P(B_1) = P(B_2) = \frac{1}{2}$ ,  $P(A|B_1) = \frac{1}{2}$  and  $P(A|B_2) = \frac{2}{3}$ . We have enough previous results to solve this, but we will first derive a very useful formula.

### 2. DEFINITIONS

Suppose that  $B_1, \dots, B_n$  is a partition of  $\Omega$ . Recall that

$$P(A|B_i) = \frac{P(A \cap B_i)}{P(B_i)} \text{ and } P(B_i|A) = \frac{P(A \cap B_i)}{P(A)}$$

and

$$P(A) = P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n)$$

By combining these two observations, we get a formula which is known as **Bayes' Rule**:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n)}$$

**Example 1:** Consider the first example concerning fair and biased coins. We know that  $P(B_1) = P(B_2) = \frac{1}{2}$ ,  $P(A|B_1) = \frac{1}{2}$  and  $P(A|B_2) = \frac{2}{3}$ , so using Bayes' Rule to find  $P(B_1|A)$  we get:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} = \frac{3}{7}$$

**Example 2:** Suppose there are three chests each having two drawers. The first chest has a gold coin in each drawer, the second has a gold coin in one drawer and a silver coin in the other, and the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer opened. If the drawer contains a gold coin, what is the probability that the first chest was chosen?

Let  $A$  be the event that a gold coin is in the opened drawer and let  $B_i$  be the event that chest  $i$  was chosen. We are interested in finding  $P(B_1|A)$  and we will use Bayes' Rule to find this probability. First note that  $P(B_i) = \frac{1}{3}$  for all  $i$ . Secondly, note that  $P(A|B_1) = 1$ ,  $P(A|B_2) = \frac{1}{2}$  and  $P(A|B_3) = 0$ . Therefore, by Bayes' Rule:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)} = \frac{2}{3}$$