Systems of Linear Equations and Matrices

In these notes, we define a linear system and their associated matrices. We also indicate the algebra which can be preformed on these objects.

1. Definitions and Notation

A **linear equation in n variables** is an equation of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

and a **system of m linear equations in n variables** is a collection of linear equations in the same n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We will often refer to a system of linear equations as a **linear system** or as a **system**. A **solution** to a system of linear equations in n variables is an n-tuple $(x_1, x_2, \ldots, x_n)$ which satisfies **all** of the equations in the system. We say that a system of linear equations is **consistent** if it has at least one solution; otherwise we say that it is **inconsistent**. A system of linear equations may have more than one solution (we will see later that it must have infinitely many solutions in this case) and the collection of all solutions of a linear system is called its **solution set**.

Consider the following two linear systems:

$$\begin{aligned} 5x_1 + 3x_2 &= 8 \\ 7x_1 - 2x_2 &= 5 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 &= 1 \\ x_2 &= 1 \end{aligned}$$

Notice that they have exactly the same solution set, namely $\{(1, 1)\}$. We say that these two systems are equivalent. More generally, two systems of linear equations (in the same variables) are said to be **equivalent** if they have the same solution set.

Also in the systems above, it is easier to see what the solution is for the second system than the first system. Thus our method in solving a linear system will be to transform it into an equivalent system where the solution is as easy as possible to find.

There are three basic types of algebraic operations which we may apply to a system to obtain an equivalent system. The overriding idea is the standard notion of "whatever you do to one side of an equation, you must do to the other side". The operations are:

(1.) We may interchange any two equations.

(2.) We may multiply any equation by a **non-zero** number.

(3.) We may add a multiple of one equation to any other equation.

2. Matrices

We now introduce the concept of a matrix as it relates to linear systems. Suppose that

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

is a linear system. Then notice that the there is both a vertical and horizontal index relating coefficients to a variable and an equation. We define the **coefficient matrix** to be the matrix:

$$\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}$$

Given a linear system, there is a unique coefficient matrix associated with it. On the other hand, there are infinitely many linear systems with the same coefficient matrix. To account for this, we define the **augmented matrix** of the system above to be:

$$\left(\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right)$$

Here there is a one-to-one correspondence between systems of linear equations and augmented matrices (up to the choice of variables). This correspondence extends to the algebraic steps mentioned above, in which case they are called row operations.

DEFINITION 2.1. *An* **Elementary Row Operation** *on a matrix is one of the following row operations:*

(1.) *Interchanging any two rows.*

(2.) *Multiplying any row by a non-zero number.*

(3.) *Adding a multiple of one row to another row.*

The process of transforming a matrix into a different matrix is called **row reduction**. It is our goal then to row reduce a given augmented matrix to an augmented matrix in which the corresponding linear system has an "easy" solution.

3. More on Augmented Matrices

We defined the augmented matrix of a linear system in the previous section. Here we will extend the definition. First consider our original definition: Given the linear system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_{11} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_{12} \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_{1m} \end{aligned}$$

Then the corresponding augmented matrix is:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} \end{array}\right)$$

Note how the constants in the linear system are indexed by two indices. Suppose that:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_{12} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_{22} \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_{m2} \end{aligned}$$

is another linear system where the coefficient side of the equations are exactly the same as the last system, only the constants (ie. b_{ij}) are different. We can form an augmented matrix that has the all the information from both systems:

$$\left(\begin{array}{cccc|cc} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & b_{m2} \end{array}\right)$$

In general, if we have a collection of k linear systems in which only the constants are different, we can form the augmented matrix of this collection:

$$\left(\begin{array}{cccc|ccc} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & \cdots & b_{1k} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & \cdots & b_{2k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & \cdots & b_{m2} \end{array}\right)$$