Introduction to Vector Spaces

This section introduces the concept of vector space. In reality, linear algebra is the study of vector spaces and the functions of vector spaces (linear transformations). They form the fundamental objects which we will be studying throughout the remaining course. Once we define a vector space, we will go on to study the properties of vector spaces. Their importance lies in the fact that many mathematical questions can be rephrased as a question about vector spaces. Thus each fact that we can prove about vector spaces gives us corresponding information about many different mathematical questions.

1. Definition and a Theorem

In this first section, we will give the precise definition of a vector space.

Definition 1.1. *A* **vector space over** $\mathbb{F}$ *consists of a set* V *(of vectors) and a set* $\mathbb{F}$ *(of scalars) along with the operations* $+$ *(called vector addition) and* $\cdot$ *(called scalar multiplications) such that the following are true:*

(a.) *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* V*, then* $\mathbf{x}+\mathbf{y}$ *is a vector in* V*.*

(b.) *If* $\mathbf{x}$ *is a vector in* V *and* α *is a scalar in* $\mathbb{F}$ *then* $\alpha\cdot\mathbf{x}$ *is a vector in* V*.*

Moreover, $+$ *and* $\cdot$ *have the following properties:*

(1.) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ *for all* $\mathbf{x},\mathbf{y}$ *in* V*.*

(2.) $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ *for all* $\mathbf{x},\mathbf{y},\mathbf{z}$ *in* V*.*

(3.) *There is a vector* $\mathbf{0}$ *such that* $\mathbf{0}+\mathbf{x}=\mathbf{x}$ *for all* $\mathbf{x}$ *in* V*.*

(4.) *For each* $\mathbf{x}$ *in* V *there is a vector* $-\mathbf{x}$ *such that* $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.

(5.) $(\alpha+\beta)\cdot\mathbf{x}=(\alpha\cdot\mathbf{x})+(\beta\cdot\mathbf{x})$ *for all* α,β *in* $\mathbb{F}$ *and* $\mathbf{x}$ *in* V*.*

(6.) $\alpha\cdot(\mathbf{x}+\mathbf{y})=(\alpha\cdot\mathbf{x})+(\alpha\cdot\mathbf{y})$ *for all* α *in* $\mathbb{F}$ *and* $\mathbf{x},\mathbf{y}$ *in* V*.*

(7.) $(\alpha\beta)\cdot\mathbf{x}=\alpha\cdot(\beta\cdot\mathbf{x})$ *for all* α,β *in* $\mathbb{F}$ *and* $\mathbf{x}$ *in* V*.*

(8.) $1\cdot\mathbf{x}=\mathbf{x}$ *for all* $\mathbf{x}$ *in* V*.*

In the above definition, (a.) is called being **closed under vector addition** and (b.) is called being **closed under scalar multiplication**. As of now, the only way to show that an object is a vector space is to individually verify all of the criteria above. In subsequent notes, we will show how to determine if a subset of a vector space is also a vector space using less work.

Essentially, the properties that a vector space must have allow us to use our usual algebraic techniques. So we can solve vector equations as we solve any other algebraic equation.

Usually the set of scalars in known, so we just refer to the **vector space** V and omit the reference to the scalars. Most of the vector spaces we will study are **real vector spaces**. These are vector spaces in which the scalars are real numbers, (ie. $\mathbb{F}=\mathbb{R}$). In this case, we will simply refer to the (real) vector space V. Also, for notational convenience, we will omit the $\cdot$ for scalar multiplication and write $\alpha\mathbf{x}$ for $\alpha\cdot\mathbf{x}$. Finally, we will generally write vectors in bold typeface and scalars in normal typeface, so that $\mathbf{x}$ denotes a vector and α denotes a scalar.

Theorem 1.2. *Let V be a vector space. The the following are true:*

(1.) $0 \cdot \mathbf{x} = \mathbf{0}$.

(2.) *If* $\mathbf{x} + \mathbf{y} = \mathbf{0}$ *then* $\mathbf{y} = -\mathbf{x}$.

(3.) $-1 \cdot \mathbf{x} = -\mathbf{x}$

The vector $-\mathbf{x}$ is called the **additive inverse** of $\mathbf{x}$ and (2.) states that this inverse is unique.

2. EXAMPLES

2.1. $\mathbb{R}^n$.

2.2. $\mathbb{R}^{m\times n}$.

2.3. $\mathbb{P}_n$.

2.4. $C[a, b]$.

2.5. $D(a, b)$.

2.6. $C^n[a, b]$.