Computed Tomography

1. Introduction

Computerized Axial Transverse (CAT) scanning is an imaging technique which uses X-rays that was introduced in the early 1970's. Today, it is more commonly referred to as Computed Tomography or CT. It is primarily used in the medical field but is also used for non-medical imaging purposes.

Our discussion will be very basic, relying on a 2-dimensional representation of what is happening. We will also leave out the more advanced considerations as well as later generational types of scanning.

2. Background

The attenuation of X-rays is essentially a probabilistic process. The basic idea is that as a photon of a given energy enters a block of material, it either passes through unobstructed or interacts with an intervening electron. The chance of this happening depends on the energy of the photon, the density and type of the material and the thickness of the block. Considering a beam consisting of intensity I_0 (this is a function of the number of photons) of a fixed energy, the intensity of the beam which emerges from the block is given by:

$$I = I_0 e^{-\mu l}$$

where μ is the **linear attenuation coefficient** of the material in question and l is the thickness of the block. It is usually convenient to measure $A = \frac{I}{I_0}$, this being the attenuation of the beam. Note that A is just the probability that any given photon emerges from the block (without interacting with an electron).

3. Method

Our basic scanner will consist of an X-ray emitter which emits a tight beam of a fixed energy (it is monochromatic) and a corresponding detector. An object is divided into a number of regular blocks called voxels. A given voxel is assumed to consist of only one type of material. Then an X-ray beam of known intensity is passed through the object and then into a detector. The boxes which the beam passes through and the amount of attenuation of the beam are recorded. See Figure 1.

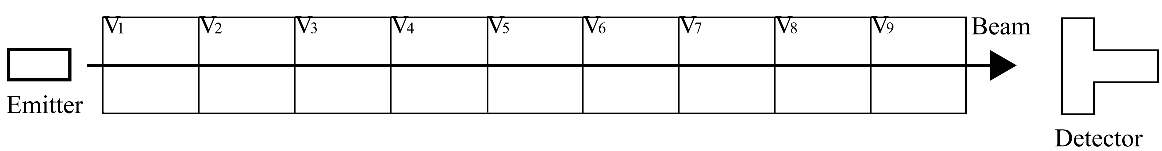

Figure 1

Suppose that the beam passes through voxels $v_1, ..., v_n$ and that the length of the beam path through v_i is l_i, then the amount of attenuation of the beam is given by the product of the attenuations in each individual voxel:

$$A = e^{-\mu_1 l_1} \cdots e^{-\mu_n l_n} = e^{-(\mu_1 l_1 + \cdots + \mu_n l_n)}$$

where μ_i is the linear attenuation coefficient of the material which comprises v_i. It is these μ_i which we wish to know as different types of material have different linear attenuation coefficients. Some typical μ for a beam energy of 60keV are:

Material	μ
Healthy Tissue	0.2
Tumorous Tissue	0.3
Bone	0.4
Lead	50

Taking the natural log of both sides of this formula gives us:

$$\ln(A) = -(\mu_1 l_1 + \cdot + \mu_n l_n) \tag{1}$$

For convenience in these notes we will ignore the thickness of the individual voxels and assume that $l_i = 1$; however, in practice, we must consider l_i. With this assumption, our formula reduces to:

$$\ln(A) = -(\mu_1 + \cdots + \mu_n) \tag{2}$$

or

$$-\ln(A) = \mu_1 + \cdots + \mu_n \tag{3}$$

which is just a linear equation involving n variables.

In order to find the individual linear attenuation coefficients, we must make enough scans. In particular, the resulting linear system must have a unique solution and will, more than likely, be over-determined. In practice such a system will be inconsistent necessitating the use of least squares, but in these notes the setup will be considered ideal and the resulting linear systems will be consistent.

Note also that as we adjust our emitter and detector, new voxels will also be scanned. So, although an individual scan may pass through n voxels, the actual linear system will have many more unknowns. For example, if we divide our object into a 10×10 grid of square voxels then each individual horizontal scan will have 10 unknowns but the overall linear system will have 100 unknowns.

4. An Example

Suppose that we have an object which we wish to image using a 2×2 grid. We will need a minimum of 4 scans as there are 4 unknowns. See Figure 2.

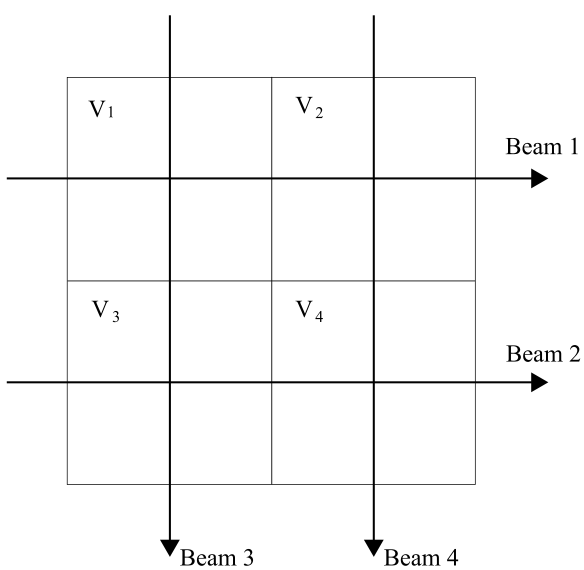

Figure 2

We record the following attenuation values A_i for the 4 indicated beams:

$$\ln(A_1) = -0.79, \ln(A_2) = -3.26, \ln(A_3) = -2.4 \text{ and } \ln(A_4) = -1.65$$

This gives us the following 4×4 linear system:

$$\begin{aligned} \mu_1 + \mu_2 &= 0.79 \\ \mu_3 + \mu_4 &= 3.26 \\ \mu_1 + \mu_3 &= 2.4 \\ \mu_2 + \mu_4 &= 1.65 \end{aligned}$$

and corresponding augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0.79 \\ 0 & 0 & 1 & 1 & 3.26 \\ 1 & 0 & 1 & 0 & 2.4 \\ 0 & 1 & 0 & 1 & 1.65 \end{array}\right)$$

Solving this system using MATLAB leads to a solution which has a free variable and thus has an infinite number of solutions. In order to find a unique solution, we will add one more scan: See Figure 3.

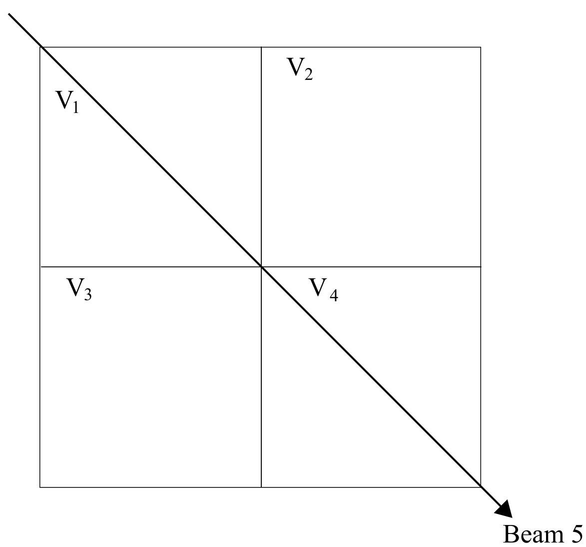

FIGURE 3

Note that this beam passes though v_1 and v_4 only. For this beam we find $\ln(A_5) = -1.28$. So now we have the following 5×4 linear system:

$$\begin{aligned} \mu_1 + \mu_2 &= 0.79 \\ \mu_3 + \mu_4 &= 3.26 \\ \mu_1 + \mu_3 &= 2.4 \\ \mu_2 + \mu_4 &= 1.65 \\ \mu_1 + \mu_4 &= 1.28 \end{aligned}$$

and corresponding augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0.79 \\ 0 & 0 & 1 & 1 & 3.26 \\ 1 & 0 & 1 & 0 & 2.4 \\ 0 & 1 & 0 & 1 & 1.65 \\ 1 & 0 & 0 & 1 & 1.28 \end{array}\right)$$

Solving again using MATLAB we see that

$$\mu_1 = 0.21, \mu_2 = 0.58, \mu_3 = 2.19 \text{ and } \mu_4 = 1.07$$