

## Systems of Linear Equations and Matrices

In these notes, we define a linear system and their associated matrices. We also indicate the algebra which can be performed on these objects.

### 1. DEFINITIONS AND NOTATION

A **linear equation in  $n$  variables** is an equation of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

and a **system of  $m$  linear equations in  $n$  variables** is a collection of linear equations in the same  $n$  variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A **solution** to a system of linear equations in  $n$  variables is an vector  $[s_1, s_2, \dots, s_n]$  such that the components satisfy **all** of the equations in the system when we set  $x_i = s_i$ . We say that a system of linear equations is **consistent** if it has at least one solution; otherwise we say that it is **inconsistent**. A system of linear equations may have more than one solution (we will see later that it must have infinitely many solutions in this case) and the collection of all solutions of a linear system is called its **solution set**.

Consider the following two linear systems:

$$\begin{array}{ccc} x_1 + x_2 = 2 & & x_1 = 1 \\ & \text{and} & \\ x_1 - x_2 = 0 & & x_2 = 1 \end{array}$$

Notice that they have exactly the same solution set, namely  $\{[1, 1]\}$ . We say that these two systems are equivalent. More generally, two systems of linear equations (in the same variables) are said to be **equivalent** if they have the same solution set.

Also in the systems above, it is easier to see what the solution is for the second system than the first system. Thus our method in solving a linear system will be to transform it into an equivalent system where the solution is as easy as possible to find.

There are three basic types of algebraic operations which we may apply to a system to obtain an equivalent system. The overriding idea is the standard notion of “whatever you do to one side of an equation, you must do to the other side”. The operations are:

- (1.) We may interchange any two equations.
- (2.) We may multiply any equation by a **non-zero** number.
- (3.) We may add a multiple of one equation to any other equation.

## 2. MATRICES

We now introduce the concept of a matrix as it relates to linear systems. Suppose that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is a linear system. Then notice that there is both a vertical and horizontal index relating coefficients to a variable and an equation. We define the **coefficient matrix** to be the matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Given a linear system, there is a unique coefficient matrix associated with it. On the other hand, there are infinitely many linear systems with the same coefficient matrix. To account for this, we define the **augmented matrix** of the system above. Let

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

then the augmented matrix of this system is:

$$[A|\mathbf{b}] = \left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

Here there is a one-to-one correspondence between systems of linear equations and augmented matrices (up to the choice of variables). This correspondence extends to the algebraic steps mentioned above, in which case they are called row operations.

**Definition 2.1.** An **Elementary Row Operation** on a matrix is one of the following row operations:

- (1.) *Interchanging any two rows.*
- (2.) *Multiplying any row by a non-zero number.*
- (3.) *Adding a multiple of one row to another row.*

The process of transforming a matrix into a different matrix is called **row reduction**. It is our goal then to row reduce a given augmented matrix to an augmented matrix in which the corresponding linear system has an “easy” solution.

### 3. VECTOR FORM OF A SYSTEM

Suppose that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is a linear system. In the previous section we saw that there we could form the augmented matrix

$$[A|\mathbf{b}] = \left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

that corresponds to this system. Moreover this correspondence is one-to-one and is respected by the algebra of elementary operations. There is another equivalent way to express this linear system, that is by using a vector equation. We define the vectors  $\mathbf{a}_i$  by setting them equal to the  $i$ -th column of  $A$ :

$$\mathbf{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$$

Then we form the following vector equation:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

The solutions to this vector equation are identical to the solutions to the original linear system. In particular we have three ways in which to express a linear system, first the standard linear system, secondly as an augmented matrix and thirdly as a vector equation. These three constructions are equivalent to each other. In particular, solutions obtained for one are solutions to the other two.