Math 151
Spring 2004

Solutions to Exam 2 Review

(1a.)

$$
\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\
&= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\
&= \frac{-1}{2x\sqrt{x}} = -\frac{1}{2}x^{-3/2}
\end{aligned}
$$

(1b.)

$$
\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{[(x+h)^3+(x+h)]-[x^3+x]}{h} \\
&= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3+x+h-x^3-x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3+h}{h} \\
&= \lim_{h\to 0} 3x^2+3xh+h^2+1 = 3x^2+1
\end{aligned}
$$

(2a.)

$$
\begin{aligned}
f'(x) &= 3\sin^2(2x+1)D_x(2x+1) \text{ (chain rule)} \\
&= 3\sin^2(2x+1)[D_x(2x)+D_x(1)] \text{ (sum of der. = der. of sum)} \\
&= 3\sin^2(2x+1)[2D_x(x)+D_x(1)] \text{ (der. of const. times function = const. times der. of function)} \\
&= 3\sin^2(2x+1)[2+0] \text{ (power rule)} \\
&= 6\sin^2(2x+1)
\end{aligned}
$$

$$f'(x) = \frac{(\sec x)D_x(x^2+1)-(x^2+1)D_x(\sec x)}{\sec^2 x} \text{ (quotient rule)}$$

$$= \frac{(\sec x)[D_x(x^2)+D_x(1)]-(x^2+1)D_x(\sec x)}{\sec^2 x} \text{ (sum of der. = der. of sum)}$$

(2b.)

$$= \frac{(\sec x)(2x+0)-(x^2+1)D_x(\sec x)}{\sec^2 x} \text{ (power rule)}$$

$$= \frac{(\sec x)(2x+0)-(x^2+1)\sec x\tan x}{\sec^2 x} \text{ (trig. functions)}$$

$$= \frac{2x(\sec x)-(x^2+1)\sec x\tan x}{\sec^2 x}$$

Note: you may write $f(x) = (x^2+1)\cos x$ and then use the product rule.

(3a.) $f(x) = (4-x^2)^{-3/2}$ so $f'(x) = \frac{-3}{2}(4-x^2)^{-5/2}(-2x)$

(3b.) $f'(x) = \frac{-5}{(2x-1)^2}$ and $f''(x) = \frac{20}{(2x-1)^3}$

(3c.) $\frac{dr}{dt} = \sec^2[\sin(t^2)]\cos(t^2)(2t)$

(3d.) $D_x(y) = \cos(\cos 5x)(-\sin 5x)(5) + \frac{7}{3}x^{4/3}$

(3e.) $\frac{dy}{dx} = (\sec x^3)(\tan x^3)(3x^2)$ and $\frac{d^2y}{dx^2} = (\sec x^3)[(\tan x^3)(6x)+(3x^2)(\sec^2 x^3)(3x^2)]+(\tan x^3)(3x^2)(\sec x^3)(\tan x^3$

(3f.) $g'(x) = \frac{(x^2+1)(1)-(x)(2x)}{(x^2+1)^2}$

(4a.) $v(t) = s'(t) = \frac{\pi}{4}(-\sin\frac{\pi t}{4}) \Rightarrow v(1) = \frac{-\sqrt{2}\pi}{8}$

$a(t) = v'(t) = s''(t) = \frac{\pi^2}{16}(-\cos\frac{\pi t}{4}) \Rightarrow a(1) = \frac{-\sqrt{2}\pi^2}{32}$

(4b.) $v(t) = s'(t) = 10(t^2+1)^9(2t) \Rightarrow v(1) = 10240$

$a(t) = v'(t) = s''(t) = 20[(t^2+1)^9 + 9(t^2+1)^8(2t)] \Rightarrow a(1) = 102400$

(5a.) $2y\frac{dy}{dx} = \frac{(x^2+y^2)-x(2x+2y\frac{dy}{dx})}{(x^2+y^2)^2} \Rightarrow \frac{dy}{dx} = \frac{x^2+y^2-2x^2}{2y(x^2+y^2)^2+2xy}$

(5b.) $2xy\frac{dy}{dx}+y^2 = \sec(xy)\tan(xy)(x\frac{dy}{dx}+y)+\sec^2(xy^2)(2xy\frac{dy}{dx}+y^2) \Rightarrow \frac{dy}{dx} = \frac{\sec(xy)\tan(xy)(y)+\sec^2(xy^2)(y^2)-y^2}{2xy-\sec(xy)\tan(xy)(x)-sec^2(xy^2)(2xy)}$

(6a.) $3x^2 + 3(x\frac{dy}{dx} + y) - 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-3x^2-3y}{3x-3y^2} \Rightarrow \frac{dy}{dx}(0,-1) = -1$

So the equation of the tangent line is $y + 1 = -1(x - 0)$.

(6b.) $\frac{dy}{dx} = \sec^2(xy)(x\frac{dy}{dx} + y) \Rightarrow \frac{dy}{dx} = \frac{\sec^2(xy)(y)}{1-\sec^2(xy)(x)} \Rightarrow \frac{dy}{dx}(\frac{\pi}{4}, 1) = \frac{2}{1-\frac{\pi}{2}}$

So the equation of the tangent line is $y - 1 = \frac{2}{1-\frac{\pi}{2}}(x - \frac{\pi}{4})$.

(7.) See Fig 1. From the diagram, we have:

$$h^2 + b^2 = (10)^2$$

Differentiating both sides with respect to t gives:

$$2h\frac{dh}{dt} + 2b\frac{db}{dt} = 0$$
$$h\frac{dh}{dt} + b\frac{db}{dt} = 0$$
$$\frac{dh}{dt} = \frac{-b}{h}\frac{db}{dt}$$

From the problem, we have: $\frac{db}{dt} = 3$ and $h = 6$. Using $h^2 + b^2 = (10)^2$, we see that $b = 8$. We substitute these values to get:

$$\frac{dh}{dt} = \frac{-24}{6} = -4$$

So the ladder is moving away from the wall at $-4ft/sec$

(8.) See Fig 2. From the hint:

$$A = \frac{l_1 l_2 \sin\theta}{2}$$

Differentiating both sides with respect to t gives:

$$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{l_1 l_2 \sin\theta}{2}\right) = \frac{1}{2}\left[l_1\left(l_2\frac{d(\sin\theta)}{dt} + \sin\theta\frac{dl_2}{dt}\right) + l_2\sin\theta\frac{dl_1}{dt}\right]$$

$$\frac{dA}{dt} = \frac{1}{2}\left[l_1\left(l_2\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dl_2}{dt}\right) + l_2\sin\theta\frac{dl_1}{dt}\right]$$

From the problem, we have: $\frac{dl_1}{dt} = 3$, $\frac{dl_2}{dt} = -2$, $\frac{d\theta}{dt} = 1, l_1 = 4, l_2 = 5$ and $\theta = \pi/4$. We substitute these values to get:

$$\frac{dA}{dt} = 3\sqrt{2}$$

So the area is increasing at a rate of $3\sqrt{2}\text{ft}^2/\text{sec}$.