Review 1 Solutions

(1a.) $A = \left(\begin{array}{ccccc|c} 3 & -2 & 4 & 3 & -1 & 17 \\ 2 & 0 & 4 & 1 & 2 & 8 \\ -2 & 2 & -2 & -1 & 2 & -10 \\ -2 & -2 & -6 & 2 & -6 & 0 \end{array}\right)$ **(1b.)** $\text{rref}(A) = \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

(1c.) It is consistent. From (1b.) we have the corresponding linear system: $\begin{aligned} x_1 + 2x_2 + x_5 &= 3 \\ x_2 + x_3 + x_5 &= -1 \\ x_4 &= 2 \end{aligned}$ Note that x_3 and x_5 are free variables. This gives the general solution $\mathbf{x} = \begin{pmatrix} 3 - 2x_3 - x_5 \\ -1 - x_3 - x_5 \\ x_3 \\ 2 \\ x_5 \end{pmatrix}$

(2.) A reduces to $\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 5 & -11 & -5 \\ 0 & 0 & \alpha + 17 & \beta + 7 \end{array}\right)$

- **(a.)** There is no solution if $\alpha = -17$ and $\beta \neq -7$.
- **(b.)** There is a unique solution if $\alpha \neq -17$.
- **(c.)** There are infinitely many solutions if $\alpha = -17$ and $\beta = -7$.

(3a.) $\mathcal{E} = D_2(-3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **(3b.)** $\mathcal{E} = E_{32}(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

(4a.) Note that the choices for $\mathcal{E}_i$ are not unique. Once such choice is:

$$\mathcal{E}_1 = P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathcal{E}_2 = E_{21}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathcal{E}_3 = D_3(1/7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}$$

(4b.) As $\mathcal{E}_3\mathcal{E}_2\mathcal{E}_1 A = B$,

$$\det(\mathcal{E}_3)\det(\mathcal{E}_2)\det(\mathcal{E}_1)\det(A) = \det(B)$$

Thus $-4 = \det(B) = (1/7)(1)(-1)\det(A)$ which gives $\det(A) = 28$.

(5.) Row reduce A with the following steps (not unique):

$$A \rightarrow_{E_{12}(-2)} \rightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -3 \\ -2 & 4 & 1 \end{pmatrix} \rightarrow_{E_{13}(2)} \rightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -3 \\ 0 & 10 & 11 \end{pmatrix} \rightarrow_{E_{23}(5)} \rightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \end{pmatrix} = B$$

Therefore

$$A = E_{12}(-2)^{-1}E_{13}(2)^{-1}E_{23}(5)^{-1}B = E_{12}(2)E_{13}(-2)E_{23}(5)B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \end{pmatrix}$$

(6a.) $A^{-1} = \begin{pmatrix} -0.0213 & -0.1064 & 0.3830 \\ 0.2340 & 0.1702 & -0.2128 \\ 0.0638 & 0.3191 & -0.1489 \end{pmatrix}$ and $\det(A) = 47$

(6b.) $A^{-1} = \begin{pmatrix} 0.3421 & -0.6579 & 0.0263 & -0.1579 \\ 0.0526 & 0.0526 & 0.1579 & 0.0526 \\ 0.0263 & 0.0263 & 0.0789 & 0.5263 \\ 0.2368 & 0.2368 & -0.2895 & -0.2632 \end{pmatrix}$ and $\det(A) = 38$

(7a.) False - generally not true.

(7b.) False - generally not true.

(7c.) False - a unique solution means there are no free variables, hence there is at most one solution to $A\mathbf{x} = \mathbf{b}_2$.

(7d.) False - can be inconsistent.

(7e.) False - the $n \times n$ zero matrix is not invertible.

(7f.) True - $A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$

(8a.)

$$\begin{aligned} AA^{-1} &= I \\ \Rightarrow \det(AA^{-1}) &= \det(I) = 1 \\ \Rightarrow \det(A)\det(A^{-1}) &= 1 \\ \Rightarrow \det(A^{-1}) &= \frac{1}{\det(A)} \end{aligned}$$

(8b.) Let $\mathbf{b}$ be given, then $A(A^{-1}\mathbf{b}) = I\mathbf{b}$ so that $A^{-1}\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Thus a solution exists. Now we need to show that it is unique: Suppose that $A\mathbf{y} = \mathbf{b}$, then $A^{-1}A\mathbf{y} = A^{-1}\mathbf{b} \Rightarrow \mathbf{y} = A^{-1}\mathbf{b}$. Thus $A^{-1}\mathbf{b}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$.

(8c.) Since A is invertible, it can be row reduced to the identity matrix I. In terms of elementary matrices, this means that there are elementary matrices $\mathcal{E}_1, \ldots, \mathcal{E}_k$ such that $\mathcal{E}_k \cdots \mathcal{E}_1 A = I$. Therefore $A = (\mathcal{E}_1 \cdots \mathcal{E}_k)^{-1} = \mathcal{E}_k^{-1} \cdots \mathcal{E}_1^{-1}$. Note that we could adjust our notation to match the question as written.