Equivariant intersection cohomology of semi-stable points

By

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In this paper we show that the equivariant intersection cohomology of the semi-stable points on a *complex projective variety* for the action of a complex reductive group may be determined from the equivariant intersection cohomology of the semi-stable points for the action of a maximal torus. This extends the work of Brion (see [Br]) who considered the smooth case using equivariant cohomology. The equivariant intersection cohomology we employ is due to Brylinski and the second author - see [Bryl-2] or [J-1]. As an application, we find a surprising relation between the intersection cohomology of Chow hypersurfaces - see §3. The second author would like to thank the Institut Fourier and the first author for an enjoyable visit.

§1. (1.0) We will first recall some basic terminology from [Br] p.126. Let G denote a complex connected reductive group, with T a chosen maximal torus and $W = N_G(T)/T$ its Weyl group. S will denote the symmetric algebra on the group of characters of T tensored with $\mathbb{Q}$ and S^W will denote the invariants for the action of W on S. We make S into a graded algebra by assigning the degree 2 to the characters of T. $\mathcal{H}$ will denote the graded W sub-module of S formed of the elements that are W-harmonic, ie. killed by every differential operator with constant coefficients, invariant by W and with no constant term. Then $\mathcal{H}$ is isomorphic to the regular representation of W and the multiplication of S induces an isomorphism of $S^W \otimes \mathcal{H}$ with S as modules over W and over S^W. Let ϵ denote the signature of W. For every W-module M, one denotes by M^a the set of *anti-invariant elements* of M, i.e. eigen-vectors of W on M with weight ϵ. The sub-space $\mathcal{H}^a$ of $\mathcal{H}$ has dimension 1 over $\mathbb{Q}$; one chooses an element $D \neq 0$ that serves as a generator of $\mathcal{H}^a$ and one sets $N = deg(D)$.

(1.1) Let X denote a complex projective variety, possibly singular, provided with the action of a complex reductive group G and provided with a G-linearized ample line bundle L. Recall (see [M-F-K] p.36) a point $x \,\epsilon\, X$ is *semi-stable* (relative to L and G) if there exists an integer $n > 0$ and a section $s \,\epsilon\, H^0(X; L^n)^G$ such that $s(x) \neq 0$; we will let X^{ss} denote the set of all such semi-stable points. (Recall also that such a point is *stable* if in addition the isotropy subgroup G_x is finite and the orbit $G.x$ is closed in X^{ss}.) Since L is also T-linearized, one may let X^{ss}_T denote the set of T-semi-stable points. Now one has the relation (i.e. the Hilbert-Mumford criterion):

(1.1.*) $X^{ss} = \bigcap_{g \,\epsilon\, G} g.X^{ss}_T$.

[1]Supported in part by a seed grant from the office of research, Ohio State University, Columbus, Ohio, 43210.

The main result of our note will be the following theorem.

(1.2)**Theorem**. (See [Br] p.128 for the smooth case.) With the previous notations, there exists a natural isomorphism:

$$IH^*_G(X^{ss};\mathbb{Q}) \cong IH^{*+N}_T(X^{ss}_T;\mathbb{Q})^a$$

where intersection cohomology is the one with the middle perversity.

Let $\alpha : X^{ss} \to X^{ss}_T$ denote the obvious *open immersion*. The theorem will follow from the following results.

(1.3.1) The map

$$\alpha^* : IH^*_T(X^{ss}_T)^a \to IH^*_T(X^{ss})^a$$

is an isomorphism.

(1.3.2) Moreover one also obtains an isomorphism:

$$\mathcal{H} \otimes IH^*_G(X^{ss};\mathbb{Q}) \xrightarrow[\cong]{u} IH^*_T(X^{ss};\mathbb{Q})$$

On taking the anti-invariant part of (1.3.2) one obtains the isomorphism:

$$\mathcal{H}^a \otimes IH^*_G(X^{ss};\mathbb{Q}) \xrightarrow[\cong]{u^a} IH^{*+N}_T(X^{ss};\mathbb{Q})^a$$

Now the map : $x \to u^a(x)^{-1}/D$ defines an isomorphism: $IH^*_T(X^{ss};\mathbb{Q})^a \xrightarrow{\cong} IH^{*-N}_G(X^{ss};\mathbb{Q})$ which will prove the theorem. Recall the intersection cohomology of the geometric quotient $X//G$ is isomorphic to the equivariant intersection cohomology of the G-semi-stable points if every G-semi-stable point is also stable. Therefore, if every semi-stable point is also stable, (1.2) provides the isomorphism

$$IH^*(X//G;\mathbb{Q}) \cong IH^{*+N}_T(X^{ss}_T;\mathbb{Q})^a.$$

Since the stratification of X with respect to the action of T is simpler than that with respect to the action G, the above isomorphism is often useful in computing $IH^*(X//G;\mathbb{Q})$. We will demonstrate this by considering some examples in the third section of the paper.

§2. We will provide a complete proof of Theorem (1.2) in this section. Throughout the rest of the paper we will freely make use of the results as well as the terminology discussed in the appendix.

(2.1.0) We will assume the basic situation of (1.1); however in (2.1.1) through (2.3) we will let X be also *quasi-projective*. Now one obtains the fibration

$$G/T \to B(G/T, G, X) \xrightarrow{\pi.} EG\underset{G}{\times}X$$

(2.1.1)**Lemma**. Under the above assumptions one obtains the isomorphism:

$$H^*_G(X;\, R\pi._*(\underline{\mathbb{Q}})) \cong H^*_G(X;\,\mathbb{Q}) \otimes H^*(G/T;\,\mathbb{Q}).$$

Proof. For this one may also consider the fibration:

$$B/T \cong U \to G/T \to G/B$$

where B is a Borel subgroup containing T, and where U is the unipotent radical of U. Since U is isomorphic to an affine space, one readily obtains an isomorphism $H^*(G/T;\mathbb{Q}) \cong H^*(G/B;\mathbb{Q})$. One may apply the Kunneth formula to each of the projections

$$\pi_n : B(G/T, G, X)_n = G/T \times G^n \times X \to (EG\underset{G}{\times}X)_n = G^n \times X$$

(and similarly to $\bar{\pi}_n : B(G/B, G, X)_n \to B(Spec\ \mathbb{C}, G, X)_n = (EG\underset{G}{\times}X)_n)$ to obtain the identifications

$$R^m\pi_{n*}(\underline{\mathbb{Q}}) = H^m(G/T;\mathbb{Q}),\ R^m\bar{\pi}_{n*}(\underline{\mathbb{Q}}) = H^m(G/B;\mathbb{Q}).$$

It follows that the natural map of the Leray spectral sequences for π and $\bar{\pi}$:

$$E_2^{s,t} = H^s_G(X;\, R^t\pi_*(\underline{\mathbb{Q}})) \Rightarrow H^{s+t}_G(X; R\pi_*(\underline{\mathbb{Q}})) \text{ and}$$

$$E_2^{s,t} = H^s_G(X;\, R^t\bar{\pi}_*(\underline{\mathbb{Q}})) \Rightarrow H^{s+t}_G(X; R\bar{\pi}_*(\underline{\mathbb{Q}}))$$

is an isomorphism from the E_2-level onwards and hence at the abutments. In view of the above observations, it suffices to prove (2.1.1) with $\pi.$ replaced by $\bar{\pi}.$ Since G/B is projective and smooth, the corresponding Leray spectral sequence degenerates at the E_2-level by an application of Deligne's degeneration condition (see (A.5)) and identifies

(2.1.1.*) $H^*_G(X;\, R\bar{\pi}_*(\underline{\mathbb{Q}})) \cong \mathbb{H}^*_G(X;\, \underset{n}{\oplus} R^n\bar{\pi}_*(\underline{\mathbb{Q}})[-n]).$

(For this observe that if $\mathcal{L}$ is a G-linearized ample line bundle on G/B, it has an equivariant Chern-class c_1 in $H^*(B(G/B, G, X);\underline{\mathbb{Q}})$. Iterated cup-product with the c_1 defines a map:

$$R^{n-k}\bar{\pi}._*(\underline{\mathbb{Q}}) \to R^{n+k}\bar{\pi}._*(\underline{\mathbb{Q}})$$

By Hard-Lefschetz this is an isomorphism. Now apply (A.5).)

Now one has the pull-back square

$$\begin{array}{ccc} B(G/B, G, Spec\ \mathbb{C}) & \xleftarrow{\alpha'} & B(G/B, G, X) \\ \bar{\pi}.\downarrow & & \downarrow\bar{\pi}. \\ B(Spec\ \mathbb{C}, G, Spec\ \mathbb{C}) & \xleftarrow{\alpha} & B(Spec\ \mathbb{C}, G, X) \end{array}$$

Since G/B is projective, it follows that each of the maps $\tilde{\pi}_n$ and $\bar{\pi}_n$ is proper. Now proper-base change provides the isomorphism: $R^n\bar{\pi}_{.*}(\underline{\mathbb{Q}}) \cong \alpha^*(R^n\tilde{\pi}_{.*}(\underline{\mathbb{Q}}))$ for each n. Since G is connected, $BG. = B(Spec\ \mathbb{C}, G, Spec\ \mathbb{C})$ is simply connected and hence the locally constant sheaves $R^n\tilde{\pi}_{.*}(\underline{\mathbb{Q}})$ on $BG.$ are *constant*; hence so are $R^n\bar{\pi}_{.*}(\underline{\mathbb{Q}})$. Now (2.1.1.*) and the above observation complete the proof of the lemma.

(2.2)**Proposition**. Assume the above hypotheses. Let $K = \{K_n|n\}$ denote a bounded complex of sheaves of $\mathbb{Q}$-vector spaces on $EG\underset{G}{\times}X$. (See (A.1.5).) Now one obtains the natural isomorphism:

$$\mathbb{H}^*_T(X;\, \pi.^*(K)) \cong \mathbb{H}^*_G(X;\, R\pi._*\pi.^*(K)) \cong H^*(G/T;\mathbb{Q}) \otimes \mathbb{H}^*_G(X;\, K).$$

Proof. For each $n \geq 0$, let $\pi_n : B(G/T, G, X)_n \to (EG\underset{G}{\times}X)_n$ denote the obvious map induced by π. Now one obtains the identification:

$$R\pi_{n*}\pi^*_n(K_n) \simeq R\pi_{n*}(\underline{\mathbb{Q}}) \otimes K_n, \text{ natural in } n \text{ and } K$$

This follows from the projection formula. Since the above identification is natural in n, it follows that one has the identification

$$\{R\pi_{n*}\pi^*_n(K_n)|n\} \simeq \{R\pi_{n*}(\underline{\mathbb{Q}}) \otimes K_n|n\}$$

natural in K. i.e. $R\pi._*\pi.^*(K) \simeq R\pi_*(\underline{\mathbb{Q}}) \otimes K$. Now one takes the equivariant hypercohomology with respect to G to obtain the isomorphism:

$$(2.2.1)\ \mathbb{H}^*_G(X;\, R\pi._*\pi.^*(K)) \cong \mathbb{H}^*_G(X;\, R\pi._*(\underline{\mathbb{Q}}) \otimes K)$$

$$\cong \mathbb{H}^*_G(X;\, R\pi._*(\underline{\mathbb{Q}})) \underset{\mathbb{H}^*_G(X;\mathbb{Q})}{\overset{L}{\otimes}} \mathbb{H}^*_G(X;\, K)$$

where we applied the Kunneth formula (see (A.3)) to obtain the last isomorphism. Now (2.1) shows one may identify the last term with

$$H^*(G/T;\, \mathbb{Q}) \otimes \mathbb{H}^*_G(X;\, K)$$

One may identify the left-hand side of (2.2.1) with $\mathbb{H}^*(B(G/T, G, X);\, \pi^*(K)) \cong \mathbb{H}^*_T(X;\, K)$ where the last isomorphism follows from (A.2.1). This completes the proof of the proposition.

(2.3) **Examples**. (i) Take $K = \underline{\mathbb{Q}}$, the obvious constant sheaf. (2.2) now becomes (2.1).

(ii). Take $K = IC^G_p(\underline{\mathbb{Q}})$ = the equivariant intersection cohomology complex as in (A.4.0) where p is an arbitrary perversity. This shows that

$$IH^*_{T,p}(X;\, \mathbb{Q}) \cong H^*(G/T;\mathbb{Q}) \otimes IH^*_{G,p}(X;\, \mathbb{Q}).$$

(iii) Let $i : Y \to X$ denote the immersion of a locally closed G-stable subvariety of X. We will let i denote the corresponding induced maps $EG\underset{G}{\times}Y \to EG\underset{G}{\times}X$, $B(G/T, G, Y) \to B(G/T, G, X)$ as well. Let $K = Ri_*Ri^!(IC_p^G(\underline{\mathbb{Q}}))$. Now (2.2) provides the isomorphism:

$$IH^*_{T,Y,p}(X;\mathbb{Q}) \cong H^*(G/T;\mathbb{Q}) \otimes IH^*_{G,Y,p}(X;\mathbb{Q})$$

where $IH^*_{G,Y,p}(X;\mathbb{Q}) = \mathbb{H}^*_G(X; Ri_*Ri^!(IC_p^G(\underline{\mathbb{Q}}))$ while

$$IH^*_{T,Y,p}(X;\mathbb{Q}) \cong \mathbb{H}^*(B(G/T,G,X); Ri_*Ri^!(IC_p^{(B(G/T,G,X))}(\underline{\mathbb{Q}})))$$

with $IC_p^{(B(G/T,G,X))}(\underline{\mathbb{Q}})$ being the complex defined in (A.4.1). (We skip the verification of the above isomorphism.)

(iv). Take $K =$ the constant sheaf $\underline{\mathbb{Q}}$ and $X = \text{Spec } \mathbb{C}$ in (2.2). Now we obtain the isomorphism

$$H^*(BT;\mathbb{Q}) \cong H^*(G/T;\mathbb{Q}) \otimes H^*(BG;\mathbb{Q}) \cong H^*(G/T;\mathbb{Q}) \otimes (H^*(BT;\mathbb{Q}))^W$$

One may now identify $\mathcal{H}$ (as in (1.0)) with $H^*(G/T;\mathbb{Q})$. It follows that one may identify the class $D \in \mathcal{H}$ with a class in $H^*(BT;\mathbb{Q})$.

(2.4) Once again assume that X is a *projective* complex variety as in (1.1). We let $\pi : \tilde{X} \to X$ denote a G-equivariant resolution of singularities provided by Hironaka - see [Hir]. Now one has the following (see [Kir] p. 157):

(i) if L is a very ample G-linearized line bundle on X, then $L_d = \pi^*(L^{\otimes d}) \otimes \mathcal{O}(-E)$, ($E$ being the exceptional divisor), is a very ample G-linearized line bundle on $\tilde{X}$ for d large,

(iii) let $\{\Sigma_\gamma|\gamma\}$ denote the stratification for $\tilde{X}$ as defined in [Kir] p.158; if $\{\Sigma_\beta|\beta\}$ is the corresponding stratification of X, then the strict-transform $\pi^{-1}(\Sigma_\beta)$ of Σ_β is a union of strata of $\tilde{X}$.

Recall that the open stratum on X and $\tilde{X}$ is always the set of semi-stable points for the G-action. If Z is any G-stable subvariety of X, Z_T^{ss} will denote $Z \cap X_T^{ss}$ and $\pi^{-1}(Z)$ will denote the strict transform of Z by π.

(2.5). **Lemma**. Let $\tilde{U}$ and $\tilde{V}$ denote the unions of strata in the above stratification of $\tilde{X}$ so that $\pi^{-1}(X^{ss}) \subseteq \tilde{U} \subseteq \tilde{V}$ and so that each is open in $\tilde{X}$. Assume moreover that $\tilde{\Sigma} = \tilde{V} - \tilde{U}$ is a *smooth* variety. Then the image of the class

$D \in H^*(BT;\mathbb{Q})$ in $H^*_T(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss});\mathbb{Q})$ is zero.

Proof. Let $\tilde{\Sigma}_c$ denote the disjoint union of the lowest-dimensional strata in $\tilde{\Sigma}$. Now one has the long-exact sequence:

$$... \to H_T^*(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}), (\tilde{\Sigma} - \tilde{\Sigma}_c) \cap \pi^{-1}(X_T^{ss}); \mathbb{Q}) \to H_T^*(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}); \mathbb{Q})$$

$$\to H_T^*((\tilde{\Sigma} - \tilde{\Sigma}_c) \cap \pi^{-1}(X_T^{ss}); \mathbb{Q}) \to ...$$

and the Thom-Gysin isomorphism:

$$H_T^*(\tilde{\Sigma}_c \cap \pi^{-1}(X_T^{ss}); \mathbb{Q}) \cong H_T^*(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}), (\tilde{\Sigma} - \tilde{\Sigma}_c) \cap \pi^{-1}(X_T^{ss}); \mathbb{Q})$$

which is also W-equivariant. (Observe that the assumption $\tilde{\Sigma}$ is smooth is needed to obtain such an isomorphism.) By ascending induction on the number of strata in $\tilde{V} - \tilde{U}$ (and the above long-exact sequence in T-equivariant cohomology) we may now assume $\tilde{V} - \tilde{U} = \tilde{\Sigma}$ is a single stratum for the G-action on $\tilde{X}$. Since $\pi^{-1}(X^{ss}) \subseteq \tilde{U}$, one may further assume that $\pi(\tilde{\Sigma})$ is contained in a stratum Σ of X so that $\Sigma \subseteq X - X^{ss}$.

It is proven in [Br] Lemme 2, that, under the above assumptions, the image of the class D in $H_T^*(\Sigma_T^{ss}; \mathbb{Q})$ is *zero*. Now $\pi^* : H_T^*(\Sigma_T^{ss}; \mathbb{Q}) \to H_T^*(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}); \mathbb{Q})$ maps the above class to zero as well. This proves the lemma.

(2.6) **Proposition**. Assume the situation in (2.4). Let V, U denote unions of G-strata in X so that $X^{ss} \subseteq U \subseteq V$ and both are open. Assume further that $\Sigma = V - U$ is *projective* and that the strict transform $\pi^{-1}(V - U)$ is also *smooth*. Under the above hypotheses

$$H_T^*(\pi^{-1}(V_T^{ss}), \pi^{-1}(U_T^{ss}); \mathbb{Q})^a = 0$$

Proof. We will let $\tilde{U} = \pi^{-1}(U)$, $\tilde{V} = \pi^{-1}(V)$ and $\tilde{\Sigma} = \pi^{-1}(\Sigma)$ in (2.5). Since $\tilde{\Sigma}$ is smooth and G-stable one has the Thom-Gysin isomorphism which is *W-equivariant*:

(2.6.1) $H_T^*(\tilde{V}, \tilde{U}; \mathbb{Q}) \cong H_T^*(\tilde{\Sigma}; \mathbb{Q})$ and $H_T^*(\tilde{V} \cap \pi^{-1}(X_T^{ss}), \tilde{U} \cap \pi^{-1}(X_T^{ss}); \mathbb{Q})) \cong H_T^*(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}); \mathbb{Q})$

Recall $\tilde{\Sigma}$ is stable by G; now (2.2) and (A.2.4) provide the isomorphisms:

(2.6.2) $H_T^*(\tilde{\Sigma}; \mathbb{Q}) \cong \mathcal{H} \otimes H_G^*(\tilde{\Sigma}; \underline{\mathbb{Q}}) \cong \mathcal{H} \otimes (H_T^*(\tilde{\Sigma}; \mathbb{Q}))^W$

Next observe that $\tilde{\Sigma}$ is obtained by equivariant blow-ups from Σ and that it is *projective and smooth.* It follows that $\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss})$ is a union of strata for the T-action on $\tilde{\Sigma}$ according to [Kir] p.157 and the above stratification is *T-equivariantly perfect.* Now one may use an ascending induction on the number of such strata in $\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss})$ to prove the restriction

(2.6.3) $H_T^n(\tilde{\Sigma}; \mathbb{Q}) \to H_T^n(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}); \mathbb{Q})$

is *surjective* for each n. On taking anti-invariant elements, (2.6.1) and (2.6.2) show that every class x in $(H_T^*(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}); \mathbb{Q}))^a$ may be written as $x = \bar{D}.y$, for some $y \in H_T^*(\tilde{\Sigma} \cap \pi^{-1}(X_T^{ss}); \mathbb{Q})$ where $\bar{D}$ is the image of D. However $\bar{D}$ is zero, as observed in (2.5), which proves that x itself is zero. This proves the proposition.

Remark. While proving the next result we will need to use the stratification for the action of the reductive group G along with a similar stratification defined for the T-action. These will be referred to as the G-stratification and T-stratification, respectively.

(2.7) **Remark**. Assume the situation of (2.4) and let U, V denote unions of G-strata in X so that $U \subseteq V$ and both are open and non-empty. For any given U and V, by additional G-equivariant blow-ups one may always assume that $\pi^{-1}(V-U)$ is also smooth.

(2.8)**Proposition**. Let U denote a G-stable open subvariety of a *projective* variety X as in (2.4) so that it is a union of G-strata from the corresponding stratification. Assume that $X^{ss} \subseteq U$. Then :

$$IH^i_T(X^{ss}_T, U^{ss}_T; \mathbb{Q})^a = 0$$

Proof. Throughout the proof we will adopt the following notation: We will let $\pi : \tilde{X} \to X$ denote a G-equivariant resolution of singularities so that $\pi^{-1}(X-U)$ is also *smooth.* For each G-stable subvariety Y of X, $\tilde{Y}$ will denote the strict transform $\pi^{-1}(Y)$.

Next the decomposition theorem in equivariant intersection cohomology (see [J-2] section 4 for e.g.) applied to π shows that

$$(2.8.1)\quad R\pi_*(\underline{\mathbb{Q}}) = IC^T(X) \oplus \sum_S IC^T(\mathcal{L}_S)[d_S]$$

where the sum ranges over a finite collection of T-stable subvarieties S of X, $\mathcal{L}_S$ is a T-equivariant locally constant sheaf of $\mathbb{Q}$-vector spaces on S and d_S is a certain shift. Though this decomposition is far from natural in general, it has the following property (see [Kir] p.157). Let U be a G-stable open subvariety of X and let $\pi_0 : \tilde{U} = \pi^{-1}(U) \to U$ denote the induced G-equivariant resolution of singularities. Now the decomposition in (2.8.1) restricts to give a decomposition of $R\pi_{0*}(\underline{\mathbb{Q}})$. Moreover the square

$$(2.8.2)\quad \begin{array}{ccc} IH^*_T(X;\mathbb{Q}) & \longleftarrow & H^*_T(\tilde{X};\mathbb{Q}) \\ \downarrow & & \downarrow \\ IH^*_T(U;\mathbb{Q}) & \longleftarrow & H^*_T(\tilde{U};\mathbb{Q}) \end{array}$$

commutes, where the horizontal maps are *split epimorphisms.* (One may similiarly use (2.8.1) to obtain a commutative diagram:

$$
\text{(2.8.2')}\qquad
\begin{array}{ccc}
IH^*_T(X;\,\mathbb{Q}) & \longrightarrow & H^*_T(\tilde{X};\,\mathbb{Q})\\
\downarrow & & \downarrow\\
IH^*_T(U;\,\mathbb{Q}) & \longrightarrow & H^*_T(\tilde{U};\,\mathbb{Q})
\end{array}
$$

where the horizontal maps are *split monomorphisms.*)

One may use this observation to obtain the commutative diagram:

(2.8.3)

$$
\begin{array}{ccccccc}
IH^*_T(X;\mathbb{Q}) & & & \xrightarrow{\quad\gamma\quad} & & & IH^*_T(U;\mathbb{Q})\\
 & \nwarrow & & & & \nearrow & \\
\Big\downarrow{\scriptstyle\phi} & & H^*_T(\tilde{X};\mathbb{Q}) & \xrightarrow{\ \beta\ } & H^*_T(\tilde{U};\mathbb{Q}) & & \Big\downarrow{\scriptstyle\phi'}\\
 & & \downarrow{\scriptstyle\tilde{\phi}} & & \downarrow{\scriptstyle\tilde{\phi}'} & & \\
 & & H^*_T(\pi^{-1}(X^{ss}_T);\mathbb{Q}) & \xrightarrow{\ \beta'\ } & H^*_T(\pi^{-1}(U^{ss}_T);\mathbb{Q}) & & \\
 & \swarrow & & & & \searrow & \\
IH^*_T(X^{ss}_T;\mathbb{Q}) & & & \xrightarrow{\quad\gamma'\quad} & & & IH^*_T(U^{ss}_T;\mathbb{Q})
\end{array}
$$

Recall X and hence $\tilde{X}$ are projective. Now consider the stratification on $\tilde{X}$ for the T-action; the strict-transform of strata on X is a union of strata in $\tilde{X}$ and the above stratification on $\tilde{X}$ is equivariantly perfect for T-equivariant cohomology. It follows the map $\tilde{\phi}$ in the central square is an *epimorphism.*

Taking $V = X$ in (2.6), one observes that $\Sigma = V - U$ is closed in X and hence projective; by the hypotheses $\pi^{-1}(V - U)$ is smooth. Hence the hyptheses of (2.6) are satisfied. Therefore (2.6) shows that the map

$$\beta'^a : H^*_T(\pi^{-1}(X^{ss}_T);\, \mathbb{Q})^a \rightarrow H^*_T(\pi^{-1}(U^{ss}_T);\, \mathbb{Q})^a$$

is an *isomorphism.* The commutativity of the central square now shows that the map $\tilde{\phi}'^a$ is also an *epimorphism.* These observations along with (2.8.2) applied to different choices of U show that the maps ϕ and ϕ'^a are both *epimorphisms* as well. (Here, as well as elsewhere, we use the observation that taking the anti-invariant parts is an exact-functor.)

Now observe the identification (see (2.2) and (2.3) (ii)):

$IH^*_T(X;\,\mathbb{Q}) \cong H^*(G/T;\,\mathbb{Q}) \otimes IH^*_G(X;\,\mathbb{Q})$ and

$IH^*_T(U;\,\mathbb{Q}) \cong H^*(G/T;\,\mathbb{Q}) \otimes IH^*_G(U;\,\mathbb{Q})$.

Since the G-stratification on $\tilde{X}$ is equivariantly perfect for equivariant intersection cohomology with respect to G, it follows that the map γ is an *epimorphism.* Since ϕ and ϕ'^a are already *epimorphisms,* it follows that so is the map γ'^a.

Finally one has the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(\gamma'^a) & \longrightarrow & IH^*_T(X^{ss}_T\ ; \mathbb{Q})^a & \xrightarrow{\gamma'^a} & IH^*_T(U^{ss}_T; \mathbb{Q})^a & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ker(\beta'^a) & \longrightarrow & H^*_T(\pi^{-1}(X^{ss}_T); \mathbb{Q})^a & \xrightarrow{\beta'^a} & H^*_T(\pi^{-1}(U^{ss}_T); \mathbb{Q})^a & \longrightarrow & 0 \end{array}$$

where the last two vertical maps are split mono-morphisms by (2.8.2'). It follows $ker(\gamma'^a)$ *injects* into $ker(\beta'^a)$. However

$ker(\gamma'^a) = IH^*_T(X^{ss}_T, U^{ss}_T; \mathbb{Q})^a$ and $ker(\beta'^a) = H^*_T(\pi^{-1}(X^{ss}_T), \pi^{-1}(U^{ss}_T); \mathbb{Q})^a$

by the surjectivity of the maps β'^a and γ'^a. Since $H^*_T(\pi^{-1}(X^{ss}_T), \pi^{-1}(U^{ss}_T); \mathbb{Q})^a = 0$ as shown in (2.6), the proposition follows.

(2.9)**Corollary**. Assume the situation of (2.8). Now

$(IH^n_T(X^{ss}_T); \mathbb{Q}))^a \cong (IH^n_T(U^{ss}_T; \mathbb{Q}))^a$ for all n.

Proof. Observe that the long exact sequence

$$... \to IH^n_T(X^{ss}_T, U^{ss}_T\ ; \mathbb{Q}) \to IH^n_T(X^{ss}_T; \mathbb{Q}) \to IH^n_T(U^{ss}_T; \mathbb{Q}) \to ...$$

remains exact after taking the anti-invariant parts. Therefore (2.8) proves (2.9).

Conclusion. Clearly (2.9) proves (1.3.1) by taking $U = X^{ss}$, since $X^{ss} = X^{ss} \cap X^{ss}_T$. This completes the proof of the theorem (1.2).

§3. **Examples**.

(3.1) To any irreducible subvariety Z of codimension k in $\mathbb{P}^n$, one associates classically a Chow hypersurface H_Z in $G_{k-1,n}$, the Grassmanian of $(k-1)$-planes in $\mathbb{P}^n$ (see e.g. [A-N] pp.40-43.) Recall that H_Z is the set of all $(k-1)$-planes that meet Z. A variant of this construction yields a hypersurface H'_Z in $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (k times): the closure of the set of k-tuples in $\mathbb{P}^n$ which generate a $k-1$-plane meeting Z. Observe that H'_Z is stable under the natural action of the symmetric group S_k on $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (k times) by permutations. Therefore S_k acts on the intersection cohomology $IH^*(H'_Z; \mathbb{Q})$. From our main theorem, we will derive the following suprising result.

(3.1.1)**Proposition** There is a canonical isomorphism:

$IH^*(H_Z; \mathbb{Q}) \cong IH^{*+k(k-1)}(H'_Z; \mathbb{Q})^a$

Proof. Consider the linear action of $G = SL_k$ on $V = Hom_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{n+1})$. We identify V with $(\mathbb{C}^{n+1})^k$; the maximal torus T of diagonal matrices in SL_k acts on V by

$$(t_1, ..., t_k).(x_1, ..., x_k) = (t_1.x_1, ..., t_k.x_k).$$

It follows easily that

$$\mathbb{P}(V)^s_T = \mathbb{P}(V)^{ss}_T = \{[x_1, ..., x_k] \in \mathbb{P}(V) | x_i \neq 0 \text{ for all } i \}$$

and that

$$\mathbb{P}(V)^s_T/T \cong \mathbb{P}^n \times ... \times \mathbb{P}^n \ (k \text{ times})$$

Moreover, by the Hilbert-Mumford criterion, we have:

$$\mathbb{P}(V)^s = \mathbb{P}(V)^{ss} = \{[x_1, ..., x_k] | x_1, ..., x_k \text{ are linearly independent in } \mathbb{C}^{n+1} \}$$

and hence $\mathbb{P}(V)^s/G = G_{k-1,n}$ (the Grassmanian of $(k-1)$-planes in $\mathbb{P}^n$).

Now define a subvariety $X \subseteq \mathbb{P}(V)$ as the closure of the image in $\mathbb{P}(V)$ of the set of k-tuples $(x_1, ..., x_k) \in (\mathbb{C}^{n+1})^k$ such that $[x_1], ..., [x_k]$ generate a $(k-1)$-plane in $\mathbb{P}^n$ which meets Z. Then X is stable by the action of SL_k and it is clear that

$$X^{ss}_T/T = H'_Z \text{ and } X^s/G = H_Z.$$

By Theorem (1.2), we have a canonical isomorphism:

$$IH^*_G(X^{ss}; \mathbb{Q}) \cong IH^{*+k(k-1)}_T(X^{ss}_T; \mathbb{Q})^a$$

(Observe that $N = k(k-1)$ for SL_k.) Moreover, any semi-stable point (either for G or for T) is stable and hence:

$$IH^*_G(X^{ss}; \mathbb{Q}) \cong IH^*(X^s/G; \mathbb{Q})$$

To see this, consider the Leray spectral sequence $E_2^{p,q} = \mathbb{H}^p(X^s/G; R^q\pi_*(IC^G(\mathbb{Q}))) \Rightarrow IH^*_G(X^s; \mathbb{Q})$, where $\pi : EG\underset{G}{\times}X^s \to X^s/G$. Here X^s/G is viewed as a simplicial scheme in the obvious manner. Using the terminology of (A.2.0) one may identify the fiber over $\bar{x}$ of this simplicial map with the simplicial scheme $B(G/G(x), G, Spec\ \mathbb{C})$ where $G(x)$ is the stabiliser of x, $\bar{x} =$ the orbit of x. Now (A.2.1) provides the identification:

$$\mathbb{H}^*(B(G/G(x), G, Spec\ \mathbb{C}); IC^G(\mathbb{Q})) \cong \mathbb{H}^*(BG(x); i^*(IC^G(\mathbb{Q})))$$

where $i : BG(x) \to B(G/G(x), G, Spec\ \mathbb{C})$ is the map in (A.2.0) with $H = G(x)$ and $X = Spec\ \mathbb{C}$. Since $G(x)$ is trivial for each $x \in X^s$ (in this example), one readily obtains the following identification of the E_2-terms for the spectral sequence above:

$E_2^{p,q} = 0$ if $q > 0$ and $E_2^{p,0} \cong IH^p(X^s/G; \mathbb{Q})$.

By a similar argument, one obtains the isomorphism:

$IH_T^*(X_T^s; \mathbb{Q}) \cong IH^*(X_T^s/T; \mathbb{Q})$.

This completes the proof of the proposition.

Remark. If the irreducible variety $Z \subseteq \mathbb{P}^n$ is not a linear subspace, and if its codimension k is at least 2, then H_Z and H'_Z are singular in codimension 1. This justifies the use of intersection cohomology in the above example. To see this, consider the variety

$V_Z = \{(x, L) \in Z \times G_{k-1,n} | x \in L\}$ and the first projection $p : V_Z \to H_Z$.

Then it is easily checked that p is birational, but that the fibers $p^{-1}(L)$ are not connected for all L in some divisor in H_Z. By Zariski's main theorem, H_Z is singular in codimension 1. The same holds for H'_Z, by a similar argument.

(3.2) Consider three positive integers r, k and n such that $r < k$. Let $Y_r \subseteq \mathbb{P}^n \times ... \times \mathbb{P}^n$ (k times) be the set of k-tuples which generate a linear subspace of dimension at most r in $\mathbb{P}^n$. Then Y_r is a normal, (in general) singular variety, with an obvious action of the symmetric group S_k. We claim that the representation of S_k on $IH^*(Y_r; \mathbb{Q})$ does not contain the sign representation, i.e.

$IH^*(Y_r; \mathbb{Q})^a = 0$

Namely, with the notation of the first example, define $X \subseteq \mathbb{P}((\mathbb{C}^{n+1})^k)$ to be the set of all $[x_1, ..., x_k]$ such that the linear span of $x_1, ..., x_k$ has dimension at most $r + 1$. Then X is closed and stable under the action of SL_k. Moreover, we have:

$X_T^s/T = Y_r$ and $X^{ss} = \emptyset$ (since $r < k$).

Now the conclusion follows as in the first example.

Appendix. In this appendix we review the basics of equivariant intersection cohomology and the equivariant derived category.

(A.1.0). We will assume the basic situation of (1.1). Accordingly X denotes a complex quasi-projective variety provided with the action of a complex linear algebraic group G.

(A.1.1) In this situation one first forms the simplicial spaces $EG\underset{G}{\times}X$ in the usual manner (See [Fr]p. 4 for eg.) Observe that $(EG\underset{G}{\times}X)_n = G^n \times X$ with the usual structure maps. Each of the face maps $d_i : (EG\underset{G}{\times}X)_n \to (EG\underset{G}{\times}X)_{n-1}$ is induced by the group-action $\mu : G \times X \to X$ and the projection $\pi_2 : G \times X \to X$ and hence is smooth.

(A.1.2) Assume in addition to the hypotheses of (A.1.0) that Y is another variety acted on by the group G. Now one may form the double-bar construction $B(X, G, Y)$ which is a simplicial space given in degree n by $B(X, G, Y)_n = X \times G^n \times Y$ with the structure maps defined similarly. (See [Fr] p. 4 for eg.).

(A.1.3) Now assume $X.$ is a simplicial scheme (i.e. a simplicial object in the category of schemes), for e.g one of the simplicial schemes obtained as above. One puts a *Grothendieck topology* on $X.$ by defining the objects to be maps $u : U \to X_n$, where u is the inclusion of an open set in X_n for some n. Given two such open sets $u : U \to X_n$ and $v : V \to X_m$ for some n, M a map $\alpha : u \to v$ is given by a map $\alpha : U \to V$ that lies over a structure map $\alpha' : X_n \to X_m$ of the simplicial scheme $X.$. *If X_n is a complex algebraic variety for each n, we will always consider only the transcendental topology on X_n.* The corresponding topology on $X.$ will be denoted by $Top(X.)$.

(A.1.4). We will only consider sheaves of Q-vector spaces in this paper. A sheaf F of Q-vector spaces on a simplicial scheme $X.$ consists of a collection $\{F_n|n\}$, where each F_n is sheaf of Q-vector spaces on X_n, provided with a collection of maps $\phi(\alpha) : \alpha^*(F_n) \to F_m$ associated to each structure map $\alpha : X_m \to X_n$ of the simplicial space $X.$ satisfying certain obvious compatibility conditions as in ([Fr] p.14, for eg.). The category of such sheaves will be denoted by $Sh(X.)$. We will let $D_b(X.; Q)$ denote the derived category all bounded complexes of sheaves of Q-vector spaces. A sheaf F as above on $X.$ has *descent property* provided the maps $\phi(\alpha) : \alpha^*(F_n) \to F_m$, as above associated to each structure map α, is an isomorphism at each stalk. A complex $K^\cdot$ in $Sh(X.)$ or in $Sh(X.)$ *has descent* if all its cohomology sheaves $\{H^i(K_n)|n\}$ have descent. In the special case where $X.$ is the simplicial space $EG\underset{G}{\times}X$ as in (A.1.1) and $F = \{F_n|n\}$ is a sheaf on $EG\underset{G}{\times}X$ with descent, we will say F *is a G-equivariant sheaf on $EG\underset{G}{\times}X$* and that F_0 *is a G-equivariant sheaf on X*. In other words F is a lift to the equivariant derived category of the sheaf F_0 on X. (Conversely any sheaf K on X is equivariant if there is a sheaf

$F = \{F_n|n\}$ on $EG\underset{G}{\times}X$ with descent so that $K = F_0$.)

(A.1.5) In the above situation, the category $D_b^{des}(X.; Q)$ will denote the full subcategory of the derived category $D_b(X.; Q)$ of complexes of sheaves whose cohomology sheaves have descent. $D_b^{c,des}(X.; Q)$ will denote the corresponding full subcategory of complexes with constructible cohomology. If $X. = EG\underset{G}{\times}X$ = the simplicial space associated to the action of a group G on a space X (as in (A.1.0)), we will denote the category $D_b^{des}(X.; Q)$ ($D^{c,des}(X.; Q)$) by $D_b^G(X; Q)$ ($D_b^{c,G}(X; Q)$, respectively).

(A.1.6). Next assume that $p : Y. \to X.$ is a map of simplicial schemes. Now the category of sheaves of Q-vector spaces on $Y.$ has enough injectives as observed in [Fr] p.15. The derived functor $Rp_*(K) = \{Rp_{n*}(K_n)|n\}$, if $K = \{K_n|n\} \epsilon D^{c,G}(Y.; Q)$. One defines equivariant hypercohomology by:

$$\mathbb{H}_G^*(X; K) = \mathbb{H}^*(EG\underset{G}{\times}X; K)$$

(A.2.0) Assume the situation of (A.1.2). Clearly there exists a natural map $i : EH\underset{H}{\times}X \to B(G/H, G, X).$ of simplicial schemes which is a closed immersion in each degree. Moreover there exists an obvious projection map $\pi : B(G/H, G, X). \to EG\underset{G}{\times}X = B(Spec\ \mathbb{C}, G, X).$ induced by the projection $G/H \to Spec\ \mathbb{C}$. Let $K \epsilon D_b^{c,G}(X; Q)$. Clearly $\pi^*(K)$ is a complex of presheaves on the simplicial space $B(G/H, G, X).$ and $i^*(\pi^*(K))$ is a complex of presheaves on $EH\underset{H}{\times}X$. Now i induces an isomorphism:

(A.2.1) $\mathbb{H}_H^*(X; i^*(\pi^*K)) \xrightarrow{\simeq} \mathbb{H}^*(B(G/H, G, X); \pi^*(K))$

Remark. Under the above assumptions, observe that the complex $i^*\pi^*(K)$ is merely the restriction of K to $EH\underset{H}{\times}X$. If G is a complex reductive group and $H = N(T)$ = the normalizer of a maximal torus, then one knows $G/N(T)$ is acyclic rationally. It follows one obtains the isomorphism:

(A.2.2) $H_{N(T)}^*(X; K) \cong H_G^*(X; K)$.

Next assume that H is normal and has finite index in G. Now one obtains the isomorphism:

(A.2.3) $(H_H^*(X; K))^{G/H} \cong H_G^*(X; K)$

If $G = N(T)$ and $H = T$, combining (A.2.2) with (A.2.3) one obtains the isomorphism:

(A.2.4) $(H_T^*(X; K))^W \cong H_G^*(X; K)$.

(A.3). **Kunneth formula**. (See [J-3] appendix.)

$$\mathbb{H}_G^*(X; M \otimes N) \cong \mathbb{H}_G^*(X; M) \underset{H_G^*(X;\mathbb{Q})}{\overset{L}{\otimes}} \mathbb{H}_G^*(X; N),$$

whenever $M, N \epsilon D_b^{G,c}(X; Q)$.

(A.4.0) Next we recall the construction of equivariant intersection cohomology from [J-1]. (See [Bryl-1] or [Bryl-2] for a somewhat different approach.) Let X be a *quasi-projective* complex variety provided with the action of a complex linear algebraic group G. Let $\{U_i | 1 \leq i \leq n+1\}$ denote a G-invariant filtration of X by open subschemes so that each $U_i - U_{i-1}$ is smooth. Let H denote a closed subgroup of G. Now one forms the simplicial spaces $EH \underset{H}{\times} U_i$ as in (A.1.1); here the space in degree m is $H^m \times U_i$. It is clear one obtains a filtration of the simplicial space $EH \underset{H}{\times} X$ in this manner; we will denote the map $EH \underset{H}{\times} U_i \rightarrow EH \underset{H}{\times} U_{i+1}$ by j_i. Starting with an H-equivariant locally constant sheaf $\mathcal{L}$ of Q-vector spaces on U_i, one obtains the complex

$$IC_p^H(\mathcal{L}) = \sigma_{\leq p(n)} Rj_{n*} \circ ... \sigma_{\leq p(0)} Rj_{0*}(\mathcal{L})$$

called *the equivariant intersection cohomology complex relative to* H with the perversity $= p$.

(A.4.1) One may now consider the filtration $B(G/H, G, U_i) \subseteq B(G/H, G, U_{i+1})$ of $B(G/H, G, X)$. If j_i also denotes the above open immersion, one may define a complex

$$IC_p^{B(G/H,G,X)}(\mathcal{L}) = \sigma_{\leq p(n)} Rj_{n*} \circ ... \sigma_{\leq p(0)} Rj_{0*}(\mathcal{L})$$

where $\mathcal{L}$ denotes a G-equivariant locally constant sheaf on X. One may readily verify that if $i : EH \underset{H}{\times} X \rightarrow B(G/H, G, X)$ is the map as in (A.2.0), then $IC_p^H(\mathcal{L}) \simeq i^*(IC_p^{B(G/H,G,X)}(\mathcal{L}))$. Now (A.2.1) provides the identification:

$$\mathbb{H}^*(B(G/H, G, X);\, IC_p^{B(G/H,G,X)}(\mathcal{L})) \cong IH^*_{H,p}(X;\, \mathcal{L}).$$

(A.5) *The degeneration criterion of Deligne* (See [De] or [J-1] appendix).)

Let $p : Y. \rightarrow X.$ denote a map of simplicial schemes and let $F. \,\epsilon\, D_b^{c,G}(Y; Q)$. Let $u \,\epsilon\, H^2(Y.;\, \mathbb{Q})$ denote a class. Then cup-product with u defines maps

$$R^i p_*(u) : R^i p_*(F) \rightarrow R^{i+2} p_*(F), \text{ for each } i.$$

Now the Leray spectral sequence for the map p degenerates if there exists an integer n so that the maps:

$$(R^{n-i} p_*(u)))^i : R^{n-i} p_*(F) \rightarrow R^{n+i} p_*(F)$$

are isomorphisms for each $i \geq 0$.

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