The intersection cohomology and the derived category of algebraic stacks

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Abstract

The present paper provides an extension of the theory of perverse sheaves to algebraic stacks and therefore to moduli problems, $\mathbb{Q}$-varieties, algebraic spaces etc. We also include a detailed study of the intersection cohomology of algebraic stacks and their associated moduli spaces. Smooth group scheme actions on schemes and the associated derived categories turn up as special cases of the more general results on algebraic stacks in the sense of Artin.

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The main goals of the present paper are as follows:

(i) generalize much of the basic theory of perverse sheaves as in ($[B-B-D]$) to algebraic stacks; *as a result the main results on perverse sheaves (for eg. the decomposition theorems for direct images of perverse sheaves by a proper map) are shown to hold in much more generality and apply in much wider contexts, for eg. moduli problems, Q-varieties, algebraic spaces etc.* (See (3.4.7) for a brief discussion of the problems involved in obtaining such a generalization.)

(ii) define and study the intersection cohomology of algebraic stacks (and their associated moduli spaces) in arbitrary characteristics. Recall that the only previous study of the intersection cohomology of moduli spaces is by Kirwan (see [Kir-1], [Kir-2]); however her study is from an entirely different point of view and is only valid for complex varieties.

(iii). using the observation (see [Ar] p.180) that algebraic stacks may be viewed as groupoid objects in the category of algebraic spaces, we are able to include the study of smooth group-scheme actions and the associated 'equivariant' derived category as a special case of our general study of algebraic stacks. This provides an alternate construction of the equivariant derived category along with all the relevant machinery; the equivariant derived category becomes the natural home of the equivariant intersection cohomology complexes introduced in $[J-2]$ and has further applications in $[J-3]$. (See also $[J-4]$.)

We begin section 1 by reviewing the basic theory of algebraic spaces and stacks. In section 2 we establish several basic results on the category of sheaves on algebraic stacks. This is continued in section 3 where we study the derived categories (and various t-structures) associated to algebraic stacks. Section 4 provides a detailed study of perverse sheaves on algebraic stacks. We recover most of the basic results of ($[B-B-D]$) in this section and conclude with a decomposition theorem generalizing that of ($[B-B-D]$). We apply the above results in section 5 to study the intersection cohomology of algebraic stacks. Section 6 contains a brief discussion of the equivariant derived category. Some of the more technical results are left in an appendix. We also thank the referee and Angelo Vistoli for some helpful comments.

1. Introduction to algebraic stacks

(1.0). Throughout the paper we will restrict to *schemes and algebraic spaces of finite type over a noetherian separated base scheme B.* Let $(schemes/B)$ denote this category of schemes. We will (usually) provide $(schemes/B)$ with *the big etale-topology.* Recall this means the following: if X, Y are schemes, the morphisms $X \rightarrow Y$ are maps *locally of finite type* over B; the coverings of any X in $(schemes/B)$ are *the etale surjective* maps.

(1.1). We will assume the basic terminology on *algebraic spaces* from ([Knut], chapter 2). The category of algebraic spaces of finite type over B will be denoted $(alg.spaces/B)$. Observe that if X is a scheme of finite type over B, the associated functor $(schemes/B)^{op} \to$ (sets) represented by X is an algebraic space. Thus the category of schemes admits an imbedding as a full subcategory of the category of algebraic spaces.

(1.2.1). Next we consider the etale topology and topos associated to an algebraic stack. First recall the following (see [Knut] Prop 1.4, p.95) :

let A_1, A_2 be algebraic spaces, and $u_i : U_i \to A_i$, $i = 1, 2$, be representable etale coverings. Let g, h be maps so that in the diagram

$$\begin{array}{ccccc} U_1 \underset{A_1}{\times} U_1 & \xrightarrow{\pi_i} & U_1 & \xrightarrow{u_1} & A_1 \\ {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow \\ U_2 \underset{A_2}{\times} U_2 & \xrightarrow{\pi_i} & U_2 & \xrightarrow{u_2} & A_2 \end{array}$$

$h \circ \pi_i = \pi_i \circ g$, $i = 1, 2$. (Here $\pi_i : U_1 \underset{A_1}{\times} U_1 \longrightarrow U_1$, and similarly $\pi_i : U_2 \underset{A_2}{\times} U_2 \longrightarrow U_2$ is the projection to the $i - th$ factor for $i = 1$, 2.) Then there exists a unique map $f : A_1 \to A_2$ of algebraic spaces so that $u_2 \circ h = f \circ u_1$. Conversely, every map $f : A_1 \to A_2$ is induced in this way for some choice of u_1, u_2, g and h.

(1.2.2). Let $f : A_1 \to A_2$ be a map of algebraic spaces. We will say f is *locally of finite type (etale, smooth)* if for some representable etale coverings $u_i : U_i \to A_i$, $i = 1, 2$, there exists a lifting $h : U_1 \to U_2$ which is locally of finite type (etale, smooth, respectively). The class of maps between algebraic spaces that are locally of finite type (etale, smooth) is stable under composition and base-change. Most local properties of algebraic spaces are given in terms of the corresponding properties of a representable etale covering.

(1.2.3) The *big* or *global* etale topology on the category of algebraic spaces is the category whose morphisms are maps locally of finite type and where the coverings are etale surjective maps. The category of algebraic spaces with the big etale topology will be often denoted merely *(alg. spaces)*.

(1.3.1) Let C denote a category with finite limits and let S be a category fibered in groupoids over C as in ([D-M] (section 4)). Assume that for every $\varphi : U \to V$ in C, and for every $y \in Ob(S_V)$ a map $f : x \to y$ lifting φ has been chosen. Then x will be denoted $\varphi^*(y)$. Now $\varphi^* : S_V \to S_U$ is a functor so that $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ if $\varphi : U \to V$ and $\psi : Z \to U$ are in C.

(1.3.2) Let C denote a category with finite limits and provided with a Grothendieck topology. A *stack in groupoids over* C is a category fibered in groupoids over C satisfying the conditions in ([D-M] Definition (4.1)). Observe that the stacks in groupoids over C (denoted $(stacks/C)$) forms a 2-category: the 1-morphisms are functors from one stack to another and the 2-morphisms are morphisms of such functors. Let C also denote the 2-category having the same objects and morphisms as C and where the 2-morphisms are all the identities; thus C may be identified as a sub 2-category of the the 2-category $(stacks/C)$. If K is an object of C, K provides a stack, namely the category whose sections over U ϵ C is the discrete category of morphisms in C from U to K. Such a stack is said to be *represented* by the object K.

(1.3.3). A 1-morphism $F : \mathfrak{S}_1 \to \mathfrak{S}_2$ of stacks over C is *representable* if for every X in C and any 1-morphism $x : X \to \mathfrak{S}_2$, the fiber-product $X \underset{\mathfrak{S}_2}{\times} \mathfrak{S}_1$ is a representable stack. (Recall that if A ϵ C,

$$X \underset{\mathfrak{S}_2}{\times} \mathfrak{S}_1(A) = \{(f : A \to X,\ u \ \epsilon\ Ob(\mathfrak{S}_1(A)) | F(u) = f^*(x),\ x \text{ regarded as an object of } \mathfrak{S}_2(X)\}$$

regarded as a category in the trivial manner (ie. all morphisms are the identities); the above condition says that the functor $f \to Ob(X \underset{\mathfrak{S}_2}{\times} \mathfrak{S}_1(A))$ is representable by some $g : Y \to X$.)

(1.3.4). Let P be a property of morphisms in C, stable under base-change and of a local nature on the target. *A representable map* $F : \mathfrak{S}_1 \to \mathfrak{S}_2$ of stacks over C has the property P if the map $F' : X \underset{\mathfrak{S}_1}{\times} \mathfrak{S}_2 \longrightarrow X$ induced by base change for any 1-morphism $x : X \longrightarrow \mathfrak{S}_2$, $X \ \epsilon\ C$, has the same property P.

(1.3.5). Finally observe the following: let $\mathfrak{S}$ be a stack over C. Then the diagonal map $\mathfrak{S} \to \mathfrak{S} \times \mathfrak{S}$ is representable if and only if for every X, $Y \ \epsilon\ C$, and 1-morphisms $x : X \to \mathfrak{S}$, $y : Y \to \mathfrak{S}$, the fiber-product $X \underset{\mathfrak{S}}{\times} Y$ is representable.

From now on C will denote either the category $(schemes/B)$ or (alg. $spaces/B$) provided with the big etale topology, with B a noetherian separated scheme as in (1.0).

(1.4). **Definition**. An *algebraic stack* $\mathfrak{S}$ is a stack in groupoids over the category (alg. spaces/B) so that

(a). $\Delta : \mathfrak{S} \to \mathfrak{S} \underset{B}{\times} \mathfrak{S}$ is representable and

(b). there exists a representable *smooth and surjective map* $x : X \to \mathfrak{S}$ with X an algebraic space. (ie. for every $Y \to \mathfrak{S}$, the fibered product $X \underset{\mathfrak{S}}{\times} Y$ is a representable stack represented by an algebraic space and the obvious induced map $X \underset{\mathfrak{S}}{\times} Y \to Y$ is smooth and surjective.) We will often refer to $x : X \to \mathfrak{S}$ as an *atlas* or a *smooth atlas.* (Most local properties of algebraic stacks are expressed in terms of the corresponding properties of an atlas.)

(1.4)' An algebraic stack is *Deligne-Mumford* if the map $x : X \to \mathfrak{S}$ is *étale surjective.* A general stack as in (1.4) will often be referred to as an *Artin* stack.

(1.5) *Examples of algebraic stacks.* (i). Observe from ([D-M] example (4.9)) that an algebraic space X itself may be regarded as an algebraic stack with an etale atlas $x : \tilde{X} \longrightarrow X$, where $\tilde{X}$ is a *scheme.* In this case we will *define the dimension of* X *to be the dimension of the scheme* $\tilde{X}$.

(ii). Let X be a scheme over B; let G denote a group-scheme, smooth separated and of finite type over B acting on X. In this situation let $[X/G]$ denote the stack whose category of sections over a B-scheme T is is the category of principal homogeneous spaces E over T with structure group G. The principal homogeneous space $G \times X$ over X (with G acting on the first-factor by translation) is a section of $[X/G]$ over X. The corresponding map $x : X \to [X/G]$ is smooth and surjective; hence $[X/G]$ is an algebraic stack. (See [D-M] (eg. 4.8) for the case G is finite.)

(iii). Let B denote a base scheme and let M denote a category fibered over the category $(schemes/B)$ provided with one of the moduli-topologies as in [Mum-2]. One readily verifies that M is fibered in groupoids over (schemes) and that M has products - see ([Mum-2] p.50). Therefore M will be an algebraic stack provided there exists an atlas $x : \tilde{M} \to M$, which is smooth. (See [Ar] corollary (5.2) for conditions that ensure the existence of such an atlas.)

(iv). *Q-varieties.* (See [Gil] (9.1) for a good definition.) As shown in ([Gil](9.1)) these are algebraic spaces that may be interpreted as the 'coarse moduli space' for certain etale algebraic groupoids.

(1.6). *Conventions.* Throughout the rest of the paper $\mathfrak{S}$ will denote an algebraic stack with a smooth atlas $x : X \to \mathfrak{S}$, which is of finite type over a field k (ie. X is of finite type over k) satisfying:

(i) k has finite cohomological dimension and

(ii) for each prime l different from the characteristic of k, each $H^n(Gal(\bar{k}/k); \mathbb{Z}/l^\nu)$ is finite for all n, $\nu \geq 1$. Here $\bar{k}$ is the separable closure of k.

(For eg. k could be a finite field or be algebraically closed. If either of the above two conditions fails, one may observe readily that the terms $Ext^n(K_\nu, L_\nu)$ in (3.4.2) *need not be finite any longer.* As argued in the lines following (3.4.3), the finiteness of the above groups is essential even to provide a triangulated category structure on $D^b_c(Et(B\mathfrak{S}.), Q_l)$.) *We will let* $char(k) = p \geq 0$ *throughout the rest of the paper.*

2. The smooth and étale topoi

In this section we consider the étale and smooth topologies on algebraic stacks and sheaves on such topologies. *Convention*: if $\mathfrak{S}$ is Deligne-Mumford we will consider both the (local) *smooth* and *étale* *topologies* on $\mathfrak{S}$; however if $\mathfrak{S}$ is Artin we will only consider the (local) *smooth* topology on $\mathfrak{S}$. These are defined as follows.

(2.1) Let $\mathfrak{S}$ denote an algebraic stack as above with an atlas $x : X \to \mathfrak{S}$ as in (1.4). The *smooth* (local) topology (the (local)*étale* topology) of $\mathfrak{S}$ will be denoted $\mathfrak{S}_{smt}(\mathfrak{S}_{et}$, respectively). The open sets of $\mathfrak{S}_{smt}$ ($\mathfrak{S}_{et}$) will be representable smooth (étale, respectively) maps $Y \to \mathfrak{S}$, where Y *is an algebraic space.* (ie. $Y \underset{\mathfrak{S}}{\times} Z \to Y$ is smooth (étale, respectively) for any algebraic space Z provided with a map $Z \longrightarrow \mathfrak{S}$).

(2.2.0) Let $\mathfrak{S}$ denote a smooth stack as before with $x : X \to \mathfrak{S}$ its atlas. Now one may obtain a simplicial object $B\mathfrak{S}.$ in the category of algebraic spaces, called the *classifying simplicial groupoid*, by letting $B\mathfrak{S}_0 = X$, $B\mathfrak{S}_1 = X \underset{\mathfrak{S}}{\times} X$,..., $(B\mathfrak{S})_n = \overbrace{X \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X ... \underset{\mathfrak{S}}{\times} X}^{(n+1)-times}$, with the structure maps induced from the two projections pr_1, $pr_2 : X \underset{\mathfrak{S}}{\times} X \to X$ and the diagonal $X \to X \underset{\mathfrak{S}}{\times} X$. (This is merely $cosk_{0\mathfrak{S}}(X)$.) Observe that all the face maps $\{d_i\}$ of this simplicial object, being induced by pr_1 or pr_2 are *smooth* maps. The (local) étale topology and the (local) smooth topology of such a simplicial object may be defined in the usual manner — see (A.1.0). We will denote these by $Et(B\mathfrak{S}.)$ and $Smt(B\mathfrak{S}.)$. There is also an alternate étale site $SEt(B\mathfrak{S}.)$ defined in (A.4).

(2.2.0.*) Let $\bar{x}. : B\mathfrak{S}. \longrightarrow \mathfrak{S}$ denote the map given in degree n as $\bar{x}_n = x \circ d_0 \circ ... \circ d_0 : B\mathfrak{S}_n \longrightarrow \mathfrak{S}$.

(2.2.1).Observe from (1.5)(i) that an algebraic space Y may be regarded as an algebraic stack in the obvious manner; the representable étale cover $\tilde{Y} \to Y$ now provides an atlas for the associated stack, which will be denoted by Y itself. The classifying simplicial groupoid will now be denoted $BY.$. Since $\tilde{Y}$ is a scheme it follows that $BY.$ is a *simplicial scheme.*

(2.2.2). If $f : \mathfrak{S} \longrightarrow \bar{\mathfrak{S}}$ is a representable map of algebraic stacks the induced map $B\mathfrak{S}. \longrightarrow B\bar{\mathfrak{S}}.$ ($B\mathfrak{S}_n \longrightarrow B\bar{\mathfrak{S}}_n$) will be denoted $f.$ (f_n, respectively). The induced map of sites $Et(B\bar{\mathfrak{S}}.) \longrightarrow Et(B\mathfrak{S}.)$ will be denoted merely $f.$ while the corresponding map of sites $SEt(B\bar{\mathfrak{S}}.) \longrightarrow SEt(B\mathfrak{S}.)$ (as in (A.4.0)) will be often denoted $\bar{f}$.

(2.3.1) Assume the above situation. If R denotes a commutative noetherian ring with a unit and of torsion prime to p (= the characteristic of k), (for eg. $R = \mathbb{Z}/l^{\nu}$ where l is prime to p), the category of sheaves of R-modules on $Et(B\mathfrak{S}.)$ and $Et(B\mathfrak{S}_n)$ for each $n \geq 0$ will be defined in the usual manner. (See (A.1.1) for the *basic terminology* that will be adopted throughout this paper.) These will be denoted Absh($Et(B\mathfrak{S}.)$; R) (Absh($Et(B\mathfrak{S}_n)$; R), respectively). Observe that these are abelian categories. Next observe (from [Knut] p.116 which is for quasi-coherent sheaves, but the same proof works for abelian sheaves) that if X is an algebraic space, then there are enough injectives in the category of sheaves of R-modules on the étale and smooth sites of X. Now the arguments in ([Fr] p.15) showing the existence of enough injectives on the étale site of a simplicial scheme readily apply in this more general context and show Absh($Et(B\mathfrak{S}.)$; R) and $Absh(Smt(B\mathfrak{S}.)$; R) have enough injectives. That the obvious restriction functor $(\ )_n$:Absh($Et(B\mathfrak{S}.)$; R) $\rightarrow$ $Absh(Et(B\mathfrak{S}_n)$; R) preserves injectives and is exact may be shown similarly.

(2.3.2) Next let R denote a local ring of dimension 1 with maximal ideal m so that the residue field R/m is of characteristic l which is prime to p (= char(k) as in (1.6)) and R is complete in the m-adic topology. For eg. $R = \mathbb{Z}_l$ and $m = (l\mathbb{Z})_l$ or E is a finite extension of Q_l and R is the integral closure of $\mathbb{Z}_l$ in E. Let $\nu \geq 1$ be an integer. Let J be an open ideal in R, for eg. $J = m^\nu$. The category of J-adic sheaves on $Et(B\mathfrak{S}.)$ will now be defined to be the category $J - ad(Absh(Et(B\mathfrak{S}.)$; R) in the sense of ([Jou-1] p. 219). Recall that this means a J-adic sheaf on $Et(B\mathfrak{S}.)$ consists of an inverse system $\{{}^nF|n \geq 0\}$ of sheaves of R-modules on $Et(B\mathfrak{S}.)$ so that (i) $J^{n+1}.{}^nF = 0$ and (ii) for every pair of integers m, n with $m \geq n \geq 0$, the map $R/m^{\nu n+1} \underset{R/m^{\nu m+1}}{\otimes} {}^{m+1}F \rightarrow^n F$ induced by the map ${}^mF \rightarrow^n F$ is an isomorphism.

(2.3.3). If $J = m$ is the maximal ideal of R, the category of m-adic sheaves will be denoted by R-Absh($Et(B\mathfrak{S}.)$). If we let $R = Z_l$, the l-adic integers, and m = the ideal (lZ_l), the resulting category of m-adic sheaves will be the category of l-adic sheaves on $Et(B\mathfrak{S}.)$

(2.4)**Remarks**. (i)Throughout the paper a superscript to the left of a sheaf on $Et(B\mathfrak{S}.)$ will denote it is the $n - th$ stage of an inverse system of sheaves as above; a subscript to the right will denote it is the restriction to $Et(B\mathfrak{S}_n)$ of a sheaf on $Et(B\mathfrak{S}.)$. (*The only exception to this rule* is that in *section 4* a superscript o to the left of an object *indicates it is defined over a finite field k*. See (4.9.0).)

(ii) It is clear that similar statements hold for sheaves on $Smt(B\mathfrak{S}.)$

(iii) *The term sheaf will from now on denote generically an abelian sheaf, a sheaf of R-modules or a J-adic sheaf as in (2.3.2)* (*unless made explicit*) *on either one of the smooth or the étale topologies.*

(2.5.1) Next let $\mathfrak{S}$ denote an algebraic stack as in (1.6). Let $\alpha : \mathfrak{S}_{smt} \rightarrow \mathfrak{S}_{et}$ denote the obvious map of sites. Similarly let $\alpha. : Smt(B\mathfrak{S}.) \rightarrow Et(B\mathfrak{S}.)$ denote the obvious map of sites. One may readily verify that $R^i\alpha_* = 0$ for $i > 0$ and that if F is a sheaf on $\mathfrak{S}_{et}$, $\alpha_*\alpha^*F \cong F$; similarly $R^i\alpha_{.*} = 0$ for $i > 0$ and that if $F^\cdot$ is a sheaf on $Et(B\mathfrak{S}.)$, then $F^\cdot \cong \alpha_{.*}\alpha^*_{.}F^\cdot$. (Both assertions are of a local nature on the étale topology of $\mathfrak{S}$ and hence one may reduce them to the corresponding assertions on algebraic spaces and finally to schemes, where they are well-known. See [Mil] pp.111-112 and also Chapter I, Proposition (3.26).). It follows therefore that α^* (as a functor from sheaves on $\mathfrak{S}_{et}$ to sheaves on $\mathfrak{S}_{smt}$) is *fully-faithful.*

(2.5.2) Assume the above situation. Let $F = \{F_n|n\}$ be either an abelian sheaf, a sheaf of R-modules or a J-adic sheaf on $Et(B\mathfrak{S}.)$ or on $Smt(B\mathfrak{S}.)$. F will be called a *sheaf with descent* if each of the maps $\alpha^* F_m \to F_n$ are isomorphisms in the appropriate category for any structure map $\alpha : (B\mathfrak{S}_n) \to (B\mathfrak{S}_m)$. (Equivalently there exists an isomorphism $\varphi : d_0^* F_0 \to d_1^* F_0$ satisfying the usual cocycle conditions (see for eg. [Mum-1] p.30) when pulled back to $B\mathfrak{S}_2$ by d_i, $i = 0, 1$ or 2 and φ pulled back to $B\mathfrak{S}_0$ by s_0 is the identity.) Let ($\bar{x}^*$: (sheaves on $\mathfrak{S}_{et}$) $\to$ (sheaves on $Et(B\mathfrak{S}.)$ with descent) denote the functor given by $\bar{x}^* K = \{\bar{x}_n^* K | n \geq 0\}$ where $\bar{x}_n : B\mathfrak{S}_n \to \mathfrak{S}$ is the structure map in (2.2.0.*). A functor ${}_s\bar{x}^*$: (sheaves on $\mathfrak{S}_{smt}$) $\to$ (sheaves on $Smt(B\mathfrak{S}.)$ with descent) is defined similarly. The functors ${}_s\bar{x}^*$ and $\bar{x}^*$ have right adjoints which are given by: $F = \{F^n | n \geq 0\} \to \ker(\delta^0 - \delta^1 : x_{0*}F^0 \to x_{1*}F^1)$, where δ^i is the map $x_{0*}F^0 \to x_{1*}F^1$ induced by d_i. (ie. $\Gamma(U, \bar{x}_* F) = \ker(\delta^0 - \delta^1 : \Gamma(U \underset{\mathfrak{S}}{\times} X,\ F^0) \to \Gamma(U \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X, F^1))$, $U \,\epsilon\, \mathfrak{S}_{et}$. The functor ${}_s\bar{x}_*$ is defined similarly.) The right-adjoint to ${}_s\bar{x}^*$ ($\bar{x}^*$) will be denoted ${}_s\bar{x}_*$ ($\bar{x}_*$, respectively).

(2.5.3). **Proposition**. Assume the above situation. Now the functors:

(2.5.3.1) ${}_s\bar{x}^*$: (sheaves on $\mathfrak{S}_{smt}$) $\to$ (sheaves on $Smt(B\mathfrak{S}.)$ with descent) and

(2.5.3.2) $\bar{x}^*$: (sheaves on $\mathfrak{S}_{et}$) $\to$ (sheaves on $Et(B\mathfrak{S}.)$ with descent)

are equivalences of categories.

Proof. Observe that if F is a sheaf on $\mathfrak{S}_{smt}$,

(2.5.3.3) $R^i\ {}_s\bar{x}_*\ {}_s\bar{x}^* F = 0$ for all $i > 0$ and ${}_s\bar{x}_*\ {}_s\bar{x}^* F = F$.

This follows readily since $\Gamma(U, F) \cong \ker(\delta^0 - \delta^1 : \Gamma(U \underset{\mathfrak{S}}{\times} X,\ F^0) \to \Gamma(U \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X, F^1))$, $U \,\epsilon\, \mathfrak{S}_{smt}$ by the sheaf-axiom for F on $\mathfrak{S}_{smt}$. Now it follows readily that ${}_s\bar{x}^*$ is a fully-faithful functor. The assumption that $x : X \to \mathfrak{S}$ is smooth shows ${}_s\bar{x}^*$ is essentially surjective onto the subcategory of sheaves with descent on $Smt(B\mathfrak{S}.)$ (Given a sheaf F on $Smt(B\mathfrak{S}.)$ with descent, the descent data makes it possible to obtain a sheaf K on $\mathfrak{S}_{smt}$ so that ${}_s\bar{x}^* K = F$.) It follows that

${}_s\bar{x}^*$: (sheaves on $\mathfrak{S}_{smt}$) $\to$ (sheaves on $Smt(B\mathfrak{S}.)$ with descent)

is an equivalence of categories. This proves (2.5.3.1). Let $(\alpha_.)_* = \{\alpha_{n*} | n \geq 0\}$ and $(\alpha_.)^* = \{\alpha_n^* | n \geq 0\}$, where $\alpha_n : Smt(B\mathfrak{S}_n) \to Et(B\mathfrak{S}_n)$ is the obvious map. Now (2.5.1) and (2.5.3.1) show the composite functor $(\alpha_.)^* \circ \bar{x}^* = {}_s\bar{x}^* \circ \alpha^*$: (sheaves on $\mathfrak{S}_{et}$) $\to$ (sheaves on $Smt(B\mathfrak{S}.)$ with descent) is fully faithful;

since $(\alpha.)^*$ is also fully-faithful, it follows $\bar{x}^*$: (sheaves on $\mathfrak{S}_{et}$) $\rightarrow$ (sheaves on $Et(B\mathfrak{S}.)$ with descent) is also fully faithful.

Now let K be a sheaf on $Et(B\mathfrak{S}.)$ with descent; clearly $(\alpha.)^*K$ is a sheaf on $Smt(B\mathfrak{S}.)$ with descent. Using (2.5.3.1), there exists a sheaf L on $\mathfrak{S}_{smt}$ so that $(\alpha.)^*K \cong {}_s\bar{x}^*L$. Now $K \cong (\alpha.)_* \circ (\alpha.)^*K \cong (\alpha.)_* \circ {}_s\bar{x}^*L$ which may be readily shown to be isomorphic to $\bar{x}^* \circ \alpha_* L$. It follows therefore that

$\bar{x}^*$: (sheaves on $\mathfrak{S}_{et}$) $\rightarrow$ (sheaves on $Et(B\mathfrak{S}.)$ with descent)

is essentially surjective; since it is already fully-faithful $\bar{x}^*$ is an equivalence of categories.

(2.5.4). The inverse to the above equivalence (given by $\bar{x}^*$) is in fact given by the functor $\bar{x}_*$ as in (2.5.2). ie.

$F \cong \bar{x}_*\bar{x}^*F$, $F \in Absh(\mathfrak{S}_{et})$

Recall (2.5.3.3) proves the corresponding assertion for the smooth topology. One may prove this for the étale topology as follows. Let F denote a sheaf on $\mathfrak{S}_{et}$; now if $\alpha : \mathfrak{S}_{smt} \longrightarrow \mathfrak{S}_{et}(\alpha. : Smt(B\mathfrak{S}) \longrightarrow Et(B\mathfrak{S}))$ is the obvious map of sites as before, then the natural map $F \longrightarrow \alpha_*\alpha^*(F)$ is an isomorphism as observed in (2.5.1). Therefore, in order to show that $F \xrightarrow{\simeq} \bar{x}_*(\bar{x}^*F)$, it suffices to show the natural map $\alpha_*\alpha^*F \longrightarrow \alpha_*\alpha^*(\bar{x}_*\bar{x}^*F)$ is an isomorphism. Next one verifies the isomorphism:

(2.5.4.1) $\alpha^* \circ \bar{x}_*\bar{x}^*F \xrightarrow{\simeq} {}_s\bar{x}_* \circ (\alpha.)^*\bar{x}^*F$

([Mil] Chapter 1, Proposition (3.26)) shows that the étale neighborhoods are cofinal in the system of smooth neighborhoods of 'points' on schemes; clearly the same holds for 'points' on algebraic spaces. (See [Knut] chapter II, theorem (6.4).) Since each of the maps $\bar{x}_n$ is smooth, one may apply this observation to show the natural map

$\alpha^* \circ x_{n*}\bar{x}_n^*F \xrightarrow{\simeq} ({}_sx)_{n*} \circ (\alpha.)^*\bar{x}_n^*F$

is an isomorphism.

Now the definition of $\bar{x}_*\bar{x}^*F$ and similarly that of $\bar{x}_{s*} \circ (\alpha.)^* {}_s\bar{x}^*F$ as in (2.5.2) completes the proof of (2.5.4.1). Therefore we obtain the isomorphisms

$\alpha_*\alpha^*(\bar{x}_*\bar{x}^*F) \xleftarrow{\simeq} \alpha_* {}_s\bar{x}_*(\alpha.)^*\bar{x}^*F \xleftarrow{\simeq} \alpha_* {}_s\bar{x}_* {}_s\bar{x}^*\alpha^*F \xleftarrow{\simeq} \alpha_*\alpha^*F$

which completes the proof of (2.5.4). (Here the last isomorphism follows from (2.5.3.3) applied to α^*F.)

(2.6.1) **Definition**. Assume the above situation. Let F denote a sheaf on either of the two sites $Et(B\mathfrak{S}.)$ or $Smt(B\mathfrak{S}.)$. We will now define $H^n(B\mathfrak{S}.; F)$ to be the n-th right derived functor of the functor $F \rightarrow \ker(\delta^0 - \delta^1 : \Gamma(B\mathfrak{S}_0; F_0) \rightarrow \Gamma(B\mathfrak{S}_1; F_1))$ where δ^i is the map induced by d^i.

(2.6.2). Let $\mathfrak{S}$ denote an algebraic stack with $x : X \rightarrow \mathfrak{S}$ an atlas. One observes that the categories Absh($\mathfrak{S}_{smt}$) and Absh($\mathfrak{S}_{et}$) have enough injectives in the following manner. Let F be an abelian sheaf on $\mathfrak{S}_{smt}$. Now imbed ${}_s\bar{x}^*F$ in an injective sheaf I on $Smt(B\mathfrak{S}.)$. (See (2.3.1).) It is clear from (2.5.3.3) that $F \cong {}_s\bar{x}_* \circ {}_s\bar{x}^*F$ and that the obvious map $F \cong {}_s\bar{x}_* \circ {}_s\bar{x}^*F \rightarrow {}_s\bar{x}_*I$ is an injection. (Recall that since ${}_s\bar{x}_*$ has an exact left-adjoint ${}_s\bar{x}^*$, it preserves injections and injectives.) If F is now a sheaf on $\mathfrak{S}_{et}$, one may use the isomorphism $F \cong \alpha_*\alpha^*F$ (see (2.5.1)) (and the fact that α_* has an exact left-adjoint α^* and hence preserves injectives and injections) to imbed F in an injective sheaf. Therefore one may define the i-th cohomology of $\mathfrak{S}$ with respect to an abelian sheaf F on $\mathfrak{S}_{smt}$ or $\mathfrak{S}_{et}$ to be $Ext^i(\mathbb{Z}_{\mathfrak{S}}, F)$ where $\mathbb{Z}_{\mathfrak{S}}$ is the obvious constant sheaf.

(2.6.3)**Lemma**. Assume the situation of (2.6.2). If F is an abelian sheaf on $\mathfrak{S}_{et}$, the cohomology of $\mathfrak{S}$ with respect to $F \cong$ *the* cohomology of α^*F on $\mathfrak{S}_{smt} \cong$ *the* cohomology of $B\mathfrak{S}.$ with respect $({}_s\bar{x})^* \circ \alpha^*F = (\alpha_.)^*(\bar{x})^*F$ on $Smt(B\mathfrak{S}.) \cong$ *the* cohomology of $B\mathfrak{S}.$ with respect to $\bar{x}^*F$ on $Et(B\mathfrak{S}.)$.

Proof. The first and last isomorphism in the lemma follow from (2.5.1). Since $X \rightarrow \mathfrak{S}$ is a smooth covering, it follows that $Hom(\mathbb{Z}_{\mathfrak{S}}, \alpha^*F) = \ker(\delta^0 - \delta^1 : \Gamma(X, (\bar{x}^* \circ \alpha^*F)_0) \rightarrow \Gamma(X \underset{\mathfrak{S}}{\times} X, (\bar{x}^* \circ \alpha^*F)_1)$, where δ^i is the map induced by the projection to the i-th factor $X \underset{\mathfrak{S}}{\times} X \rightarrow X$. The definition of the cohomology of $B\mathfrak{S}.$ with respect to $(\alpha_.)^* \circ \bar{x}^*F$, as in (2.6.1) now provides the second isomorphism.

(2.6.4) The results above identify the cohomology of an algebraic stack with the cohomology of its classifying simplicial groupoid. We will use this identification now to give a finiteness result for the cohomology of algebraic stacks as in (1.6). Let $\mathfrak{S}$ denote an algebraic stack satisfying the finiteness conditions as in (1.6) and let $B\mathfrak{S}.$ denote its associated classifying simplicial groupoid. Let l denote a fixed prime different from the residue characteristics.

(2.6.5)**Definition**. Let $\mathfrak{S}$ be an algebraic stack as in (1.6) and let $F = \{F_n | n \geq 0\}$ be an abelian sheaf (sheaf of R-modules, where R is a commutative ring as in (2.3.1)) on $Et(B\mathfrak{S}.)$. F is *constructible* if each F_n is a constructible sheaf (of R-modules, respectively) on $Et(B\mathfrak{S}_n)$. If F is an abelian sheaf on $\mathfrak{S}_{et}$, F is constructible if $\bar{x}^*F$ on $Et(B\mathfrak{S}.)$ is. (Here $\bar{x}^* : Absh(\mathfrak{S}_{et}) \rightarrow Absh(Et(B\mathfrak{S}.))$ is the functor defined in (2.5.2).)

(2.7.1)**Proposition**. (i) Assume the above situation. Let F denote a constructible sheaf with finite torsion stalks (finite l-torsion stalks where l is different from the characteristic p) on $Et(B\mathfrak{G}.)$. Then each of the cohomology groups $H^n(B\mathfrak{G}.; F)$ is finite (and l-torsion, respectively).

(ii). Assume that $\varphi : \mathfrak{G} \rightarrow \bar{\mathfrak{G}}$ is a quasi-compact map of smooth algebraic stacks as above and let $\varphi.$ denote the induced map $B\mathfrak{G}. \rightarrow B\bar{\mathfrak{G}}.$ If F is a constructible sheaf (with finite tor dimension) on $Et(B\mathfrak{G}.)$ with finite l-torsion stalks where l is away from the characteristics, then $R\varphi._*F = \{R(\varphi)_{n*}F_n|n\}$ also has the same property. If R is also regular (for eg. $R = \mathbb{Z}_l$ or its integral closure in a finite extension E of Q_l) and F has finite tor dimension, then so does $R\varphi._*F = \{R(\varphi)_{n*}F_n|n\}$

Proof. (i) First observe the existence of the first quadrant spectral sequence

$$E_1^{p,q} = H^q(B\mathfrak{G}_p; F_p) \Rightarrow H^{p+q}(B\mathfrak{G}.; F)$$

(See (A.2.1).) Recall each $B\mathfrak{G}_p$ is an algebraic space; hence its étale cohomology is defined in (2.6.2) and in view of (2.6.3) may be identified with the étale cohomology of $B(B\mathfrak{G}_p)$, where $B(B\mathfrak{G}_p)$ is the classifying simplicial scheme associated to $B\mathfrak{G}_p$ as in (2.2.1). This reduces the problem to showing that the cohomology of an algebraic space as in (1.6) with respect to a constructible sheaf with finite torsion (l-torsion) stalks is finite (and l-torsion, respectively). In view of (2.2.1) a similar spectral sequence reduces the problem to a similar statement for schemes which is clear by the assumptions in (1.6).

(ii). Recall that the induced map $\varphi^*_. : Et(B\bar{\mathfrak{G}}.) \rightarrow Et(B\mathfrak{G}.)$ of sites is given by $U \longrightarrow U \underset{(B\bar{\mathfrak{G}})_n}{\times} (B\mathfrak{G})_n$, $U \in Et(B\bar{\mathfrak{G}}_n)$. It follows that the right-derived functor $R\varphi._* = \{R\varphi_{n*}|n \geq 0\}$. Therefore we reduce to showing: let $f : X \rightarrow Y$ denote a map of algebraic spaces. If F is a constructible sheaf on $Et(X)$, then so is $R^m f_*F$ for each m. Similarly if F has finite tor dimension, then so does $R^m f_*F$ for each m, provided R is regular.

Let $x : \tilde{X} \rightarrow X$, $y : \tilde{Y} \rightarrow Y$ denote representable étale coverings and let $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ denote the induced map. Now observe (as in (2.5.4)) that if F is as above, $F \cong \bar{x}_*\bar{x}^*F$, where $\bar{x} : BX \rightarrow X$, $\bar{x}^*$ and $\bar{x}_*$ are as in (2.5.1) Let $\bar{y} : BY \rightarrow Y$ denote the corresponding map and $\bar{y}^*$, $\bar{y}_*$ the corresponding functors associated to Y. Let $f. : BX \rightarrow BY$ denote the induced map. Then we obtain the isomorphisms:

$$R^m f_*F \cong \bar{y}_*\bar{y}^*R^m f_*F \cong \bar{y}_*R^m f._*\bar{x}^*F$$

The first isomorphism follows from (2.5.4) applied to $R^m f_*F$; the last isomorphism follows from the isomorphism $R^m f._*\bar{x}^*F \cong \bar{y}^*R^m f_*F$ which is clear by smooth base-change (recall x and y are étale) applied to the pull-back square:

$$
\begin{array}{ccc}
BY. & \xrightarrow{f.} & BX. \\
{\scriptstyle \bar{y}}\downarrow & & {\scriptstyle \bar{x}}\downarrow \\
Y & \xrightarrow{f} & X
\end{array}
$$

Since $f. : BX \to BY$ is a map of simplicial schemes, observe that $R^m f._* \bar{x}^* F = \{R^m f._n \bar{x}_n^* F_n | n \geq 0\}$; since $\bar{x}_n^* F_n$ is constructible (and of finite tor dimension) if F is, it is clear that $R^m f._* \bar{x}^* F$ also has the corresponding property. Therefore, in order to prove (ii), it suffices to prove that if G is a constructible sheaf on Et(BY.) (with finite tor dimension when R is regular), and *with descent*, then $\bar{y}_* G$ is constructible (and has finite tor dimension, respectively). Since G has descent, it follows from (2.5.3.2) applied to Y that there exists a $K \in$ Absh($Et(Y)$) such that $\bar{y}^* K \cong G$; hence $K \cong \bar{y}_* \bar{y}^* K \cong \bar{y}_* G$ (as in (2.5.4) since $y : \tilde{Y} \to Y$ is étale). Therefore it suffices to show K is constructible (and of finite tor dimension) if G is. Let $G = \{G_n | n \geq 0\}$. Now the assumption that G has descent and is constructible, shows there exists a filtration $V_1 \subseteq V_2 \subseteq ... \subseteq V_{n+1} = Y$ by Zariski open algebraic subspaces, so that if $U_i = (V_i) \underset{Y}{\times} \tilde{Y}$, then $G_{0|U_i - U_{i-1}}$ is locally constant on the étale site of $U_i - U_{i-1}$. It follows that K is locally constant on the étale topology of $V_i - V_{i-1}$. The case of finite tor dimension follows readily using the projection-formula, the assumption that R is regular and the assumptions of (1.6) that show the functor $R\phi._*$ has finite l-cohomological dimension. This concludes the proof of the proposition.

(2.7.2). Let $f : \mathfrak{S} \to \bar{\mathfrak{S}}$ be a representable map of algebraic stacks and let $f. : B\mathfrak{S}. \to B\bar{\mathfrak{S}}.$ denote the induced map. Suppose in addition it is quasi-compact. Let $F = \{F_n\}$ ($K = \{K_n\}$) be a sheaf with descent on $B\mathfrak{S}.$($B\bar{\mathfrak{S}}.$, respectively). Since the face maps of $B\mathfrak{S}.$ are all smooth one may show (using smooth base change) that $R^m f._* F$ is a sheaf with descent on $Et(B\bar{\mathfrak{S}})$. for any $m \geq 0$. (One observes readily that $f.^* K$ is a sheaf with descent on $Et(B\mathfrak{S}.)$ if K is a sheaf with descent on $Et(B\bar{\mathfrak{S}}.)$.)

(2.7.3). **Corollary**. (i) Assume that $\mathfrak{S}$ is an algebraic stack satisfying the finiteness conditions of (1.6). If F is a constructible sheaf on $\mathfrak{S}_{et}$ with finite torsion stalks (finite l-torsion stalks where l is different from the characteristic p), each of the cohomology groups $H^n(\mathfrak{S}; F)$ is finite (and l-torsion, respectively).

(ii). Assume that $\varphi : \mathfrak{S} \to \bar{\mathfrak{S}}$ is a quasi-compact map of smooth algebraic stacks as above. If F is a constructible sheaf on $Et(B\mathfrak{S}.)$ with finite l-torsion stalks where l is different from the characteristics, then $R^n \varphi_* F$ also has the same property for each $n \geq 0$. If R is a regular ring and F has finite tor dimension, then so does $R^n \varphi_* F$ for each $n \geq 0$.

Proof is clear in view of (2.7.1) and (2.7.2).

(2.7.4) Let $\mathfrak{S}$ denote an algebraic stack as in (1.6) with a smooth atlas $x : X \to \mathfrak{S}$. Let $j : \bar{\mathfrak{S}} \to \mathfrak{S}$ ($i : \tilde{\mathfrak{S}} \to \mathfrak{S}$) be a representable map of algebraic substacks so that j (i) is an open (closed, respectively) immersion. Let $j. = Bj$ and $i. = Bi$ denote the induced maps. Observe that for each m, $j_m(i_m)$ is a open (closed, respectively) immersion, and that the induced map $j_{.!} : Absh(Et(B\bar{\mathfrak{S}})) \to Absh(Et(B\mathfrak{S}))$ is given by $j_{.!}(K)_m = j_{m!}(K_m)$, $K = \{K^m | m\} \,\epsilon\, Absh(Et(B\bar{\mathfrak{S}}))$. Similarly $(Ri_.)^!(L))_m = Ri^!_m(L_m)$, $L \,\epsilon\, Absh(Et(B\mathfrak{S}.))$. (Once again one may prove readily that the above functors send sheaves with *descent* to sheaves with *descent*; moreover considering sheaves of R-modules, where R is a regular ring (as in (2.7.1)), one may also show that the above functors preserve the property of having finite tor-dimension as well.

3. The derived categories and t-structures

(3.0) Throughout this section $\mathfrak{S}$ will denote an algebraic stack as in(1.6) and $B\mathfrak{S}.$ will denote the corresponding simplicial groupoid defined in (2.2.0). In this section we consider various derived categories associated to $Smt(B\mathfrak{S}.)$ and to $\mathfrak{S}_{mt}$.

(3.0.*). *All our results apply equally well to the étale topologies*, $Et(B\mathfrak{S}.)$ and $\mathfrak{S}_{et}$ for Deligne-Mumford stacks. However we state our results explicitly only for the étale topology.

(3.1) Let l be a prime number different from the residue characteristics. For each $\nu > 0$ we let $C^b_c(Smt(B\mathfrak{S}.), \mathbb{Z}/l^\nu)$ denote the category of bounded complexes of sheaves of $\mathbb{Z}/l^\nu$-modules on $Smt(B\mathfrak{S}.)$ with *constructible cohomology sheaves*, *constructible* as in (A.2.3). $C^b_{ctf}(Smt(B\mathfrak{S}.), \mathbb{Z}/\ell^\nu)$ will denote the full sub-category $C^b_c(Smt(B\mathfrak{S}.); \mathbb{Z}/l^\nu)$ of complexes that are *of finite tor dimension.* If E is a finite extension of Q_ℓ and R is the integral closure of $\mathbb{Z}_\ell$ in E, we obtain the categories $C^b_c(Smt(B\mathfrak{S}.), R/m^\nu)$ and $C^b_{ctf}(Smt(B\mathfrak{S}.), R/m^\nu)$ in a similar manner- see (2.3.2). A map $f : K^\cdot \longrightarrow L^\cdot$ of complexes in the above categories is a quasi-isomorphism if it induces an isomorphism of the cohomology sheaves $\underline{H}^n(K^\cdot|_{Smt(B\mathfrak{S}_m)}) \longrightarrow \underline{H}^n(L^\cdot|_{Smt(B\mathfrak{S}_m)})$ for all $m \geq 0$, all n. We obtain the derived categories $D^b_c(Smt(B\mathfrak{S}.), \mathbb{Z}/\ell^\nu)$, $D^b_c(Smt(B\mathfrak{S}.), R/m^\nu)$, $D^b_{ctf}(Smt(B\mathfrak{S}.), \mathbb{Z}/\ell^\nu)$, $(D^b_{ctf}(Smt(B\mathfrak{S}.), R/m^\nu)$ by inverting the quasi-isomorphisms. The derived categories $D^b_c(\mathfrak{S}_{et}, \mathbb{Z}/\ell^\nu)$, $D^b_c(\mathfrak{S}_{et}, R/m^\nu)$, $D^b_{ctf}(\mathfrak{S}_{et}, \mathbb{Z}/\ell^\nu)$, $D^b_{ctf}(\mathfrak{S}_{et}, R/m^\nu)$ may also be defined in a similar manner.

(3.2). Assume the situation of (3.1). Observe that the full abelian sub-category of sheaves with *descent* on $Smt(B\mathfrak{S}.)$ is closed under extensions in the category of all sheaves on $Smt(B\mathfrak{S}.)$. Therefore (see [Hart] p.47) we may let $D^{des}_c(B\mathfrak{S}.; R/m^\nu)$ denote the full subcategory of $D_c(Smt(B\mathfrak{S}.); R/m^\nu)$ consisting of complexes $K^\cdot$ so that each of the cohomology sheaves $H^i(K^\cdot)$ is a sheaf with *descent.*

The category $D_{ctf}^{b,des}(B\mathfrak{S}.;\ R/m^{\nu})$ will be defined to be the *full subcategory* of $D_{ctf}^{b}(Smt(B\mathfrak{S}.);\ R/m^{\nu})$ satisfying a similar condition.

(3.3) One defines $D_c^b(Smt(B\mathfrak{S}.),\ \mathbb{Z}_\ell) = 2 - \lim\limits_{\leftarrow \atop \nu} D_{ctf}^b(Smt(B\mathfrak{S}.),\ \mathbb{Z}/\ell^{\nu})$ $(D_c^b(Smt(B\mathfrak{S}.),\ R)$ $= 2 - \lim\limits_{\leftarrow \atop \nu} D_{ctf}^b(Smt(B\mathfrak{S}.),\ R/m^{\nu}))$ (see [Del-2] p. 148.) (Recall this means the objects of $D_c^b(Smt(B\mathfrak{S}.),$ $R)$ are inverse systems $\{_\nu K^\cdot\}$, with $_\nu K^\cdot \epsilon\ D_{ctf}^b(Smt(B\mathfrak{S}.),\ R/m^{\nu})$ so that $R/m^{\nu+1} \overset{L}{\underset{R/m^{\nu}}{\otimes}} ({_\nu K^\cdot}) \simeq$ $_{\nu+1}K^\cdot$. Given two such inverse systems $\{_\nu K^\cdot\}$, $\{_\nu L^\cdot\}$,

(3.3.*) $Hom(\{_\nu K^\cdot\},\ \{_\nu L^\cdot\}) = \lim\limits_{\longleftarrow \atop \nu} Hom(_\nu K^\cdot,\ _\nu L^\cdot)$.).

$D_c^b(Smt(B\mathfrak{S}.),\ Q_\ell)$ $(D_c^b(Smt(B\mathfrak{S}.),\ E))$ is the quotient of $D_c^b(Smt(B\mathfrak{S}.),\ \mathbb{Z}_\ell)$ $(D_c^b(Smt(B\mathfrak{S}.),\ R)$, respectively) by the *thick* subcategory of torsion sheaves. Finally we define the derived category $D_c^b(Smt(B\mathfrak{S}.),\ \bar{Q}_\ell)$ as $2 - \lim\limits_{\to \atop E} D_c^b(Smt(B\mathfrak{S}.),\ E)$ where the colimit is over all finite extensions E of Q_ℓ. (Recall this means the objects of $D_c^b(Smt(B\mathfrak{S}.),\ \bar{Q}_\ell)$ are direct systems $\{K_E|E\}$, where $K_E\ \epsilon\ D_c^b(Smt(B\mathfrak{S}.),$ $E))$ and that given two such systems $K = \{K_E|E\}$ and $L = \{L_E|E\}$,

(3.3.**) $Hom(K,\ L) = \lim\limits_{\to \atop E} Hom(K_E,\ L_E)$

(3.3)' Assuming the above situation one defines $D_{ctf}^{b,des}(B\mathfrak{S}.;\ R)$ $(D_c^{b,des}(Smt(B\mathfrak{S}.),\ E)$, $D_c^{b,des}(Smt(B\mathfrak{S}.),$ $\bar{Q}_\ell))$ to be the *full-subcategory* of $D_{ctf}^b(B\mathfrak{S}.;\ R)$ $(D_c^b(Smt(B\mathfrak{S}.),\ E)$, $D_c^b(Smt(B\mathfrak{S}.),\ \bar{Q}_\ell)$ respectively) consisting of complexes *whose cohomology sheaves have descent.* One may similarly define $D_c^b(\mathfrak{S}_{et}; R) =$ $2 - \lim\limits_{\leftarrow \atop \nu} D_{ctf}^b(\mathfrak{S}_{et};\ R/m^{\nu})$, $D_c^b(\mathfrak{S}_{et};\ E) =$ the quotient of $D_c^b(\mathfrak{S}_{et}; R)$ by the thick subcategory of torsion sheaves and $D_c^b(\mathfrak{S}_{et};\ \bar{Q}_l) = 2 - \lim\limits_{\to \atop E} D_c^b(\mathfrak{S}_{et};\ E)$.

(3.4) Let $\nu > 0$ be a fixed integer and let R be the integral closure of $\mathbb{Z}_\ell$ in a finite extension E of Q_ℓ. We now observe the existence of spectral sequences (as in (A.2.1))

(3.4.1) $E_1^{p,q}(\nu) = Ext^q(_\nu K_p^\cdot,\ _\nu L_p^\cdot) \Rightarrow Ext^{p+q}(_\nu K^\cdot\ ,\ _\nu L^\cdot)$

where Ext^n is the n-th right derived functor of Hom in $C_c^b(Smt(B\mathfrak{S}_p),\ R/m^{\nu})$.

(3.4.2) Observe that (3.4.1) is a *right-half-plane* spectral sequence. Hence if $_\nu K^\cdot$ and $_\nu L^\cdot$ are *bounded* complexes with constructible cohomology sheaves (with l-torsion, l as always different from the residue characteristics) it follows (as in (2.7.1)) that each $Ext^n(_\nu K^\cdot,\ \ _\nu L^\cdot)$ is *finite* (with l-torsion) since $\mathfrak{S}$

satisfies the finiteness conditions as in (1.6). Therefore taking the inverse limit of the spectral sequences in (3.4.1) over $\nu > 0$ provides strongly-convergent spectral sequences

(3.4.3) $E_1^{p,q} = \lim_{\overleftarrow{\nu}} Ext^q({}_\nu K^\cdot_p,\ {}_\nu L^\cdot_p) \Rightarrow \lim_{\overleftarrow{\nu}} Ext^{p+q}({}_\nu K^\cdot\ ,\ {}_\nu L^\cdot)$

The finiteness of the Ext-groups in (3.4.1) shows (in view of $[B-B-D]$ Proposition 2.2.15) that the categories $D^b_c(Smt(B\mathfrak{S}.),\ \mathbb{Z}_\ell)$ and $D^b_c(Smt(B\mathfrak{S}.),\ R)$ (and $D^{b,des}_c(Smt(B\mathfrak{S}.),\ \mathbb{Z}_\ell)$ and $D^{b,des}_c(Smt(B\mathfrak{S}.),\ R)$ are *triangulated categories* where the distinguished triangles are inverse systems of distinguished triangles in $D^b_{ctf}(Smt(B\mathfrak{S}.),\ \mathbb{Z}/\ell^\nu)$ and $D^b_{ctf}(Smt(B\mathfrak{S}.),\ R/m^\nu))$ respectively.

(3.4.4) We next observe that the functor ${}_s\bar{x}^*$ (as in (2.5.2)) induces equivalences:

${}_s\bar{x}^* : D^b_c(\mathfrak{S}_{smt};\ R/m^\nu) \rightarrow D^{b,des}_c(Smt(B\mathfrak{S}.);\ R/m^\nu)$ and

${}_s\bar{x}^* : D^b_{ctf}(\mathfrak{S}_{smt};\ R/m^\nu) \rightarrow D^{b,des}_{ctf}(Smt(B\mathfrak{S}.);\ R/m^\nu)$

Since ${}_s\bar{x}^*$ already induces an equivalence of the *hearts* ie. between the category of sheaves of R/m^ν-modules on $\mathfrak{S}_{smt}$ and the category of sheaves of R/m^ν-modules on $Smt(B\mathfrak{S})$ and with descent, and since it preserves the property of having finite tor-dimension, it suffices to show that ${}_s\bar{x}^*$ is fully-faithful. (See [Beil] Lemma 1.4 for a proof of this.) This follows from (2.5.3.1).

In the case of Deligne-Mumford stacks one obtains a similar equivalence:

$\bar{x}^* : D^b_c(\mathfrak{S}_{ett};\ R/m^\nu) \rightarrow D^{b,des}_c(Et(B\mathfrak{S}.);\ R/m^\nu)$ and

$\bar{x}^* : D^b_{ctf}(\mathfrak{S}_{et};\ R/m^\nu) \rightarrow D^{b,des}_{ctf}(Et(B\mathfrak{S}.);\ R/m^\nu)$

To see this one may proceed as follows. Let $K^\cdot,\ L^\cdot \epsilon\ D_c(\mathfrak{S}_{et}; R/m^\nu)$. Let ${}_s\bar{x}^*$ denote the functor: $D_c(\mathfrak{S}_{smt}; R/m^\nu) \rightarrow D^{des}_c(Smt(B\mathfrak{S}; R/m^\nu))$ (and induced by ${}_s\bar{x}$ as in (2.5.2)), let $(\alpha_.) : Smt(B\mathfrak{S}.) \rightarrow Smt(B\mathfrak{S}.)$ and $\alpha : \mathfrak{S}_{smt} \rightarrow \mathfrak{S}_{et}$ be the functors as before. Now recall (from (2.5.1)) that $(\alpha_.)^*$ and α^* are fully-faithful also at the level of derived categories. Moreover $(\alpha_.)^* \circ \bar{x}^* = {}_s\bar{x}^* \circ \alpha^*$ and ${}_s\bar{x}^*$ is fully-faithful -see (2.5.3.3). Therefore

$$Hom_{D^{des}_c(Et(B\mathfrak{S}.))}(\bar{x}^*K,\ \bar{x}^*L) \cong Hom_{D^{des}_c(Smt(B\mathfrak{S}.))}((\alpha_.)^* \circ \bar{x}^*K,\ (\alpha_.)^* \circ \bar{x}^*L)$$
$$\cong Hom_{D^{des}_c(Smt(B\mathfrak{S}.))}(\ {}_s\bar{x}^* \circ \alpha^*K,\ \ {}_s\bar{x}^* \circ \alpha^*L) \cong Hom_{D_c(\mathfrak{S}_{smt})}(\alpha^*K,\ \alpha^*L) \cong Hom_{D_c(\mathfrak{S}_{et})}(K,\ L).$$

where the last-but-one isomorphism follows from the fact ${}_s\bar{x}^*$ is fully-faithful- see (2.5.3.3). It is clear that $\bar{x}^*$ preserves the property of having finite tor dimension.

(3.4.5).) In view of the definitions in (3.3) and (3.3'), (3.3.*), (3.3.**) and (3.4.4) show that *the functor* ${}_s\bar{x}^*$ *induces the equivalences*:

$D^b_c(\mathfrak{S}_{smt};R) \simeq D^{b,des}_c(Smt(B\mathfrak{S}.);R)$,

$D^b_c(\mathfrak{S}_{smt};E) \simeq D^{b,des}_c(Smt(B\mathfrak{S}.);E)$ and

$D^b_c(\mathfrak{S}_{smt};\bar{Q}_l) \simeq D^{b,des}_c(Smt(B\mathfrak{S}.);\bar{Q}_l)$.

(3.4.6) Finally observe that by taking first the inverse limit over ν and then the tensor product of the spectral sequence in (3.4.1) with E and finally taking the direct limit over all such finite extensions E of Q_l one obtains a spectral sequence (recall E is flat over R)

(3.4.6.*) $E_1^{p,q} = Ext^q(K^\cdot_p,\, L^\cdot_p) \Rightarrow Ext^{p+q}(K^\cdot\;,\, L^\cdot)$

where $K^\cdot = \{K^\cdot_p|p \geq 0\}$, $L^\cdot = \{L^\cdot_p|p \geq 0\}$ ϵ $D^{b,des}_c(Smt(B\mathfrak{S}.);\, \bar{Q}_l)$.

(3.4.7). **Remark.**Observe also that as a result, Hom in the derived category $D^{b,des}_c(\mathfrak{S}_{smt};\, \bar{Q}_l)$ is readily computable by means of the spectral sequence in (3.4.6.*). It follows that Hom in the category of perverse sheaves on algebraic stacks (see the next section for the definition) is also computable similarly, since Hom in the category of perverse sheaves is Hom in the derived category when the perverse sheaves are viewed as complexes. (In (4.12) it becomes essential to be able to compute the Hom (and the Ext^i, $i > 0$) in this category of perverse sheaves.) Moreover the spectral sequence (3.4.6.*) (the identification in (3.4.5)) is *important* (see (3.4.2)) even in providing a triangulated category structure to $D^b_c(Smt(B\mathfrak{S}.)$, $\bar{Q}_l)$ and $(D^b_c(\mathfrak{S}_{smt};\, \bar{Q}_l)$, respectively). It follows therefore that to be able to define a derived category of $\ell - adic$ sheaves with a t-structure whose heart will be the category of perverse $\ell - adic$ sheaves on algebraic stacks (see the next section) it seems essential to adopt our approach.

Another reason for adopting our approach is the following. Clearly one needs to relate the derived category on an algebraic stack $\mathfrak{S}$ with the derived category on its atlas X. However the category of sheaves with descent on X is not closed under extensions in the category of all sheaves on X; this makes it difficult to consider the category of complexes of sheaves on X whose cohomology sheaves have descent. On the other hand the category of sheaves with descent on the simplicial algebraic space $B\mathfrak{S}.$ is closed under extensions in the category of all sheaves on $B\mathfrak{S}.$; hence it becomes possible to consider the derived categories as in (3.2) and (3.3).

(3.5.1) We next introduce *standard t-structures* on the above derived categories. Let R denote the integral closure of $\mathbb{Z}_\ell$ in some finite extension E of Q_ℓ and let m denote the maximal ideal of R. First observe that the standard t-structures on $D^b_c(\mathfrak{S}_{smt};\, R/m^\nu)$ may be defined in the usual manner. We may define standard t-structures on $D^b_c(Smt(B\mathfrak{S}.),\, R/m^\nu)$ as follows. For each $q \geq 0$, let $D^{b,\leq q}_c(Smt(B\mathfrak{S}.),\, R/m^\nu)(D^{b,\geq q}_c(Smt(B\mathfrak{S}.),\, R/m^\nu))$ denote the full subcategory of $D^b_c(Smt(B\mathfrak{S}.),\, R/m^\nu)$ consisting of complexes ${}_\nu K^\cdot$ so that $\mathcal{H}^n({}_\nu K^\cdot) = 0$ for all $n > q$(*all* $n < q$, respectively). One readily proves that the inclusion $D^{b,\leq q}_c(Smt(B\mathfrak{S}.),\, R/m^\nu) \rightarrow D^b_c(Smt(B\mathfrak{S}.),\, R/m^\nu)$ has a right adjoint $\tau_{\leq q}$ which is given as follows.

Let ${}_\nu K^\cdot = \{{}_\nu K^\cdot_n | n \geq 0\} \epsilon\ D^b_c(Smt(B\mathfrak{S}.),\, R/m^\nu)$. We let $\tau_{\leq q}({}_\nu K^\cdot)$ denote the complex given on $Smt(X_n)$ by $\tau_{\leq q}({}_\nu K^\cdot_n)$. If $\alpha : B\mathfrak{S}_p \rightarrow B\mathfrak{S}_n$ denotes a structure map of the simplicial object $B\mathfrak{S}.$, there exist a map $\alpha^{-1}(\tau_{\leq q}\ {}_\nu K^\cdot_n) \xrightarrow{\simeq} \tau_{\leq q}(\alpha_\nu^{-1} K^\cdot_n) \rightarrow \tau_{\leq q}({}_\nu K^\cdot_p)$; the naturality of the first isomorphism shows that so defined $\tau_{\leq q}({}_\nu K^\cdot)$ is a complex on $Smt(B\mathfrak{S}.)$. Similarly one shows that the inclusion $D^{b,\geq q}_c(Smt(B\mathfrak{S}.), R/m^\nu) \rightarrow D^b_c(Smt(B\mathfrak{S}.), R/m^\nu)$ has a left adjoint $\tau_{\geq q}$ which may be defined similarly. It follows readily that $(D^{b,\leq 0}_c(Smt(B\mathfrak{S}.),\, R/m^\nu),\ D^{b,\geq 0}_c(Smt(B\mathfrak{S}.),\, R/m^\nu))$ forms *a t-structure* on $D^b_c(Smt(B\mathfrak{S}.),\, R/m^\nu)$. (See $[B-B-D]p.$ 29 for the definition of t-structures.). This will be referred to as *the standard t-structure* on $D^b_c(Smt(B\mathfrak{S}.),\, R/m^\nu)$.

(3.5.2) Observe that the functor $\bar{x}^*$ in (3.4.4) is exact; therefore *it preserves the standard t-structures.*

(3.5.3) The standard t-structure on the derived categories $D^b_c(Smt(B\mathfrak{S}.),\, \mathbb{Z}_\ell)$ and $(D^b_c(Smt(B\mathfrak{S}.),\, R)$ will be now defined in the obvious manner. Nevertheless the truncation functors defined in (3.5.1) do not preserve the property of having finite tor dimension as observed in ([Del-2] p.149.). Hence we may define new truncation functors $\tau'_{\leq q}$ and $\tau'_{\geq q}$ following ([Del-2] p.149). We skip the detailed definitions. One shows readily that $\tau'_{\leq q}(\tau'_{\geq q})$ is right adjoint (left adjoint) to the inclusion

$$D^{b,\leq q}_c(Smt(B\mathfrak{S}.),\, R) \rightarrow D^b_c(Smt(B\mathfrak{S}.),\, R)(D^{b,\geq q}_c(Smt(B\mathfrak{S}.),\, R) \rightarrow D^b_c(Smt(B\mathfrak{S}.),\, R),$$

respectively). Clearly this induces a t-structure on the category $D^b_c(Smt(B\mathfrak{S}.),\, E)$ where E is a finite extension of Q_ℓ and R is the integral closure of $\mathbb{Z}_\ell$ in E.

(3.5.4) As $D^b_c(Smt(B\mathfrak{S}.),\, \bar{Q}_\ell) = 2 - \lim\limits_{\overrightarrow{E}} D^b_c(Smt(B\mathfrak{S}.),\, E)$ (with the colimit over all finite extensions E of Q_ℓ) $D^b_c(Smt(B\mathfrak{S}.),\, \bar{Q}_\ell)$ is a triangulated category. The t-structures on the various $D^b_c(Smt(B\mathfrak{S}.),\, E)$ define a t-structure on $D^b_c(Smt(B\mathfrak{S}.),\, \bar{Q}_\ell)$ which we call *the standard t-structure.* The standard t-structure on $D^b_c(\mathfrak{S}_{smt};\, R)$ and $D^b_c(\mathfrak{S}_{smt};\, \bar{Q}_l)$ may be defined similarly.

(3.5.5) Clearly the derived categories $D^{b,des}_c(Smt(B\mathfrak{S}.;\, R/m^\nu))$, $(D^{b,des}_c(Smt(B\mathfrak{S}.;\, R))$, $D^{b,des}_c(Smt(B\mathfrak{S}.;\, \bar{Q}_l)))$ inherit the standard t-structure from $D^b_c(Smt(B\mathfrak{S}.;\, R/m^\nu))(D^b_c(Smt(B\mathfrak{S}.;\, R))$, $D^b_c(Smt(B\mathfrak{S}.;$

$\bar{Q}_l$)), respectively). Moreover it is clear that the functor $\bar{x}^*$ in (3.4.4) preserves the standard t-structures we have defined above.

(3.5.6). Let n denote a fixed integer. If $K^\cdot$ belongs to any one of the above derived categories $K^\cdot[n]$ will denote the obvious complex obtained by shifting $K^\cdot$ n-times to the left. The standard t-structure on any of the above derived categories also gets shifted by this functor.

Next we consider *non-standard* t-structures on the derived categories considered earlier. We begin with the following observations:

(3.6.1) Let $\underline{S}$ denote a stratification of $\mathfrak{S}$ ie. a partition of $\mathfrak{S}$ into a finite number of locally-closed algebraic substacks S^α so that $S^\alpha \underset{(Spec\ k)}{\times} (Spec\ \bar{k})$ is smooth. (Observe that one may obtain such a stratification of $\mathfrak{S}$ as follows beginning with a similar stratification $\underline{T} = \{T\}$ of X; let T be such a stratum of X and let $\mathfrak{S}_T$ be the locally-closed stratum of $\mathfrak{S}$ defined by

$$\mathfrak{S}_T(Y) = \{g : Y \to \mathfrak{S} |\text{ the induced map } g' : Y \underset{\mathfrak{S}}{\times} X \to X \text{ factors through the given map } T \to X\}.$$

for any algebraic space Y as in (1.6). One may verify readily that $\mathfrak{S}_T$ is an algebraic stack with atlas given by T.) In this situation we will let $S^\alpha_.$ denote the subsimplicial object of $B\mathfrak{S}.$ given in degree n by $(S^\alpha)_n = S^\alpha \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} ... \underset{\mathfrak{S}}{\times} X$. Now let $\{U^i\}$ denote the filtration of $\mathfrak{S}$ by open substacks given by $U^i - U^{i-1} = \cup\{S^\alpha | \dim(X \underset{\mathfrak{S}}{\times} (S^\alpha)) = n - i + 1\}$ where $n = \dim_k\ X$ which is the dimension of its atlas as in (1.5)(i). (The dimension $dim(X \underset{\mathfrak{S}}{\times} (S^\alpha)$ is the dimension of the algebraic space $X \underset{\mathfrak{S}}{\times} (S^\alpha)$ as in (1.5)(i).) Let $U^i_.$ denote the corresponding subsimplicial algebraic space of $B\mathfrak{S}.$. Observe that the filtration $U^0_. \subseteq U^1_. \subseteq ... \subseteq U^n_. \subseteq U^{n+1}_. = B\mathfrak{S}.$ has the following properties:

(i) each of the maps $U^\alpha_m \to U^{\alpha+1}_m$ is an open immersion.(The corresponding map of simplicial algebraic spaces will be denoted $j^\alpha_.$)

(ii) each $(U^{\alpha+1}_m - U^\alpha_m)$ is smooth. The simplicial algebraic space $(U^{\alpha+1}_. - U^\alpha_.)$ will be called the $\alpha - th$ *stratum* of the simplicial algebraic space $B\mathfrak{S}.$.

(3.6.2) A *perversity* is a function p : (non-negative integers) $\to \mathbb{Z}$ so that (i) p is non-increasing and (ii) $0 \leq p(n) - p(n+1) \leq 1$. Let $\bar{p} : \{S^\alpha_. | 0 \leq \alpha \leq n+1\} \longrightarrow \mathbb{Z}$ be the function associated to a perversity p by letting $\bar{p}(S^\alpha) = p(dim_k(S_\alpha \underset{\mathfrak{S}}{\times} X))$. For eg. the middle perversity m is defined by $m(k) = -k$ and $\bar{m}(S_\alpha) = -dim_k(S_\alpha \underset{\mathfrak{S}}{\times} X)$. Given a perversity p, the dual perversity p^* is given by $p^*(k) = -p(k) - 2k$. Assume the situation of (3.1). Starting with the standard t-structures on each $D^b_{ctf}(Smt(S^\alpha_.),\ R/m^\nu)$ shifted by $p(S^\alpha_.)$ (as in (3.5.6)) we will define a (non-standard) t-structure on $D^b_{ctf}(Smt(B\mathfrak{S}.),\ R/m^\nu)$

following ($[B-B-D]$ chapter 2); this will be referred to as the *t-structure obtained by gluing and using the stratification* $\underline{S}$. We may define a t-structure on $D^b_{ctf}(\mathfrak{S}_{smt},\, R/m^\nu)$ in a similar manner by gluing the t-structures on $D^b_{ctf}(S^\alpha_{smt},\, R/m^\nu)$.

(3.6.3)**Definition**. For each stratum S let $i_{S.}$ denote the immersion $S. \to B\mathfrak{S}.$. We let $D^{b,\,\leq 0}_{ctf}(Smt(B\mathfrak{S}.),\, \underline{S};\, R/m^\nu) =$ *the* full subcategory of $D^b_{ctf}(Smt(B\mathfrak{S}.),\, R/m^\nu)$ consisting of complexes ${}_\nu K^\cdot$ so that $\mathcal{H}^n(i^{-1}_{S.}({}_\nu K^\cdot)) = 0$ for all $n > p(S.)$ and all strata $S.$; $D^{b,\,\geq 0}_{ctf}(Smt(B\mathfrak{S}.),\, \underline{S};\, R/m^\nu) =$ *the* full subcategory of $D^b_{ctf}(Smt(B\mathfrak{S}.),\, R/m^\nu)$ consisting of complexes ${}_\nu K^\cdot$ so that $\mathcal{H}^n(i^!_{S.}({}_\nu K^\cdot)) = 0$ for all $n < p(S.)$ and for all strata S. The fact that this is a t-structure follows exactly as in ($[B-B-D]p$. 67) in view of the above observations.

(3.6.3)'. Observe that if $\mathfrak{S}$ itself is *smooth* with the *obvious trivial* stratification $\underline{S}$, then

$$D^{b,\leq 0}_{ctf}(Smt(B\mathfrak{S}.),\, \underline{S};\, R/m^\nu) \bigcap D^{b,\geq 0}_{ctf}(Smt(B\mathfrak{S}.),\, \underline{S};\, R/m^\nu)$$

consists of complexes $K^\cdot$ so that $H^i(K^\cdot) = 0$ unless $i = p(dim(X))$.

(3.6.4). Clearly the above t-structures induce similar t-structures on $D^{b,des}_c(Smt(B\mathfrak{S}.);\, R/m^\nu)$ $(D^{b,des}_c(Smt(B\mathfrak{S}.);\, R)$, $D^{b,des}_{ctf}(Smt(B\mathfrak{S}.);,\, R/m^\nu)$, $(D^{b,des}_{ctf}(Smt(B\mathfrak{S}.);\, R)$, $D^{b,des}_{ctf}(Smt(B\mathfrak{S}.);\, E)$ and $D^{b,des}_c(Smt(B\mathfrak{S}.);\, \bar{Q}_l))$ as well as on the corresponding derived categories associated to $\mathfrak{S}_{smt}$. These t-structures will from now on be referred to as *the t-structures obtained by gluing and the stratification* $\underline{S}$. We will often use $D^{b,des}_c(Smt(B\mathfrak{S}.))$ generically denote any one of these categories. In each case the truncation functor $\tau_{\leq n} : D^{b,des}_c(Smt(B\mathfrak{S}.)) \to^{0\leq} D^{b,des}_c(Smt(B\mathfrak{S}.);\, \underline{S}.)$ $(\tau_{\geq n} : D^{b,des}_c(Smt(B\mathfrak{S}.)) \to D^{b,des}_c(Smt(B\mathfrak{S}.);\, \underline{S})^{\geq 0})$ will be denoted $\tau^{des}_{\leq n}(\tau^{des}_{\geq n}$, respectively).

(3.6.5) Since $x : X \to \mathfrak{S}$ is smooth, one may readily show that the functor ${}_s\bar{x}^*$ preserves the t-structures; now (3.4.4) shows it induces an equivalence:

$$D^{b,\leq 0}_{ctf}(\mathfrak{S}_{smt},\, \underline{S},\, R/m^\nu) \bigcap D^{b,\geq 0}_{ctf}(\mathfrak{S}_{smt},\, \underline{S},\, R/m^\nu)$$
$$\simeq D^{b,des\leq 0}_{ctf}(Smt(B\mathfrak{S}.),\, \underline{S},\, R/m^\nu) \bigcap D^{b,des\geq 0}_{ctf}(Smt(B\mathfrak{S}.),\, \underline{S},\, R/m^\nu)$$

and similarly for the other derived categories in (3.6.4).

(3.6.6). Recall the map of sites $\alpha : \mathfrak{S}_{et} \to \mathfrak{S}_{smt}$, $\alpha. : Smt(B\mathfrak{S}.) \to Et(B\mathfrak{S}.)$ as in (2.5.1). Next observe as in (2.5.1) that $R^i\alpha_* = 0$, $R^i\alpha_{.*} = 0$ and that $\alpha_*\alpha^*(K) \overset{\simeq}{\leftarrow} K$, $\alpha_{.*}\alpha^*_.(F) \overset{\simeq}{\leftarrow} F$, if $K \,\epsilon\, Absh(\mathfrak{S}_{smt})$ with $\mathfrak{S}$ a Deligne-Mumford stack and $F \,\epsilon\, Absh(Et(B\mathfrak{S}))$ in general. It follows that the functors α^*

and $\alpha^*_\cdot$ are fully-faithful at the level of the appropriare derived categories. The observation that étale neighborhoods are cofinal in the system of all smooth neighborhoods (ie. neighborhoods in the smooth topology) (see (2.5.1) or [Mil] pp.111-112 and chapter I, proposition (3.26)) once again readily shows that $\alpha^*\alpha_* F \overset{\simeq}{\to} F$, if $F \epsilon Absh(\mathfrak{S}_{smt})$ with $\mathfrak{S}$ a Deligne-Mumford stack. Similarly $\alpha^*_\cdot \alpha_{\cdot *} F^\cdot \overset{\simeq}{\to} F^\cdot$, if $F^\cdot \epsilon Absh(Et(B\mathfrak{S}.))$ in general. *It follows that*

$$(3.6.6.*)\alpha^* : D^b_c(\mathfrak{S}_{smt}) \overset{\simeq}{\to} D^b_c(\mathfrak{S}_{smt}) \text{ and } \alpha^*_\cdot : D^{b,des}_c(Et(B\mathfrak{S}.)) \overset{\simeq}{\to} D^{b,des}_c(Smt(B\mathfrak{S}.))$$

are *equivalences* of categories, where the derived categories denote any one of categories as in (3.6.4) and *preserve the standard t-structures as well as the ones obtained by gluing.* (Recall the second equivalence is only for Deligne-Mumford stacks.) *This observation plays an important role in the proof of (4.2).*

(3.7.1) We conclude this section with a brief discussion of *Verdier duality* for algebraic stacks. First this involves the definition of the functors $Rf_!$ and $Rf^!$ for a 'compactifiable' representable morphism $f : \mathfrak{S} \longrightarrow \tilde{\mathfrak{S}}$ of algebraic stacks. Let $f : \mathfrak{S} \to \tilde{\mathfrak{S}}$ be a representable map of algebraic stacks; we say f is *compactifiable* if there exists a factorisation $f = p \circ j$, with $j : \mathfrak{S} \to \bar{\mathfrak{S}} (p : \bar{\mathfrak{S}} \to \tilde{\mathfrak{S}})$ a representable open immersion (p a representable proper map, respectively). In this case one may define $Rf_!$ to be $Rp_* \circ j_!$; if $f_\cdot = p_\cdot \circ j_\cdot$ is the induced map $B\mathfrak{S}. \to B\tilde{\mathfrak{S}}. \to B\bar{\mathfrak{S}}$ we may also define $Rf_{\cdot !}$ to be $Rp_{\cdot *} \circ j_{\cdot !}$, making use of the site defined in (A.1.0). (One may show as usual that these are independent of the factorisation of f.) Finally $Rf^!(Rf^!_\cdot)$ may be defined to be right adjoint to $Rf_!(Rf_{\cdot !}$, respectively). Observe that $Rf_{\cdot !} = \{Rf_{n!}|n\}$ and $Rf^!_\cdot = \{Rf^!_n|n\}$. (*Alternatively* one may define these functors in full generality for maps of algebraic stacks of finite type over a field as in (1.6) in the obvious manner making use of the *dualising complexes* defined below.)

(3.7.2). Next we show the existence of *dualising complexes.* Let X denote an algebraic space as in (1.6) and let $\tilde{x} : \tilde{X} \to X$ denote a representable étale cover. Our assumptions as in (1.6) show that $\tilde{X}$ is a scheme of finite type over k where k satisfies the conditions as in (1.6). Let $D_k =$ the constant sheaf $\bar{Q}_l$ on Spec k; if $\tilde{p} : \tilde{X} \to Spec\ k$ is the structure map of $\tilde{X}$, we will let $D_{\tilde{X}} = R\tilde{p}^! D_k = the$ dualising complex on $Et(\tilde{X})$. We will first show that this descends to a complex D_X on $Et(X)$.

Let $\pi_i : \tilde{X} \underset{X}{\times} \tilde{X} \to \tilde{X}$ for $i = 1, 2$ denote projection to the i-th factor. Next observe that the constant sheaf $\bar{Q}_l$ on $Et(\tilde{X})$ clearly has descent ie. there is an isomorphism $\varphi : \pi_1^* \bar{Q}_l \simeq \pi_2^* \bar{Q}_l$ satisfying the obvious cocycle conditions. Taking duals, we obtain a quasi-isomorphism

$$R\pi_1^! D_{\tilde{X}} = R\pi_1^! D(\bar{Q}_l) \simeq D(\pi_1^* \bar{Q}_l) \simeq D(\pi_2^* \bar{Q}_l) \simeq R\pi_2^! D(\bar{Q}_l) = R\pi_2^! D_{\tilde{X}}$$

satisfying similar conditions. (Observe that $\tilde{X} \underset{X}{\times} \tilde{X}$ is a scheme of finite type over Spec(k); therefore the above quasi-isomorphisms are clear.) Now recall that each of the maps π_i ($i = 1, 2$) is étale; therefore $R\pi_{i!} = \pi_{i*}$; since $R\pi_i^!$ is right-adjoint to $R\pi_{i!}$ and π_i^* is right-adjoint to π_{i*}, it follows that $R\pi_i^! \simeq \pi_i^*$. It follows that $D_{\tilde{X}}$ descends to a complex D_X (see (3.4.5)) which we call the dualising complex on $Et(X)$. (One may verify this is indeed a dualising complex, by working locally on the étale topology of X.)

(3.7.3) Now let $f : X \to Y$ denote a representable map of algebraic spaces as in (1.3.3). Let K denote a bounded complex of $\bar{Q}_l$-sheaves on $Et(Y)$. We will now define $Rf^!K$ to be $D_X(f^*D_Y K)$, where $D_Y K = Rhom(K, D_Y)$, $D_X M = Rhom(K, D_X)$, M a complex of $\bar{Q}_l$ sheaves on $Et(X)$.. Clearly similar arguments hold for the smooth topology.

(3.7.4). Next assume $\mathfrak{S}$ is an algebraic stack as in (1.6) over a field k as before and $x : X \to \mathfrak{S}$ is a smooth atlas. We will now show that the dualising complex D_X (obtained as in (3.7.2)) descends to a complex $D_{\mathfrak{S}}$ on $\mathfrak{S}_{smt}$. Let $\pi_i : X \underset{\mathfrak{S}}{\times} X \to X$ for $i = 1, 2$ denote projection to the i-th factor. Next observe that the constant sheaf $\bar{Q}_l$ on $Smt(X)$ clearly has descent i.e. there is an isomorphism $\varphi : \pi_1^*\bar{Q}_l \simeq \pi_2^*\bar{Q}_l$ satisfying the obvious co-cycle conditions. Taking duals, we obtain a quasi- isomorphism

$$R\pi_1^! D_X = R\pi_1^! D(\bar{Q}_l) \simeq D(\pi_1^*\bar{Q}_l) \simeq D(\pi_2^*\bar{Q}_l) \simeq R\pi_2^! D(\bar{Q}_l) = R\pi_2^! D_X$$

satisfying similar conditions. Now recall that both the maps π_i are smooth of relative dimension d; therefore one may show readily that $R\pi_i^! \simeq \pi_i^*[2d]$. See [SGA]4. Expose XVIII, Theorem 3.2.5 for Poincare-duality for compactifiable maps between schemes; one may generalize this to algebraic spaces using the classifying simplicial scheme associated to an algebraic space as in (2.2.1). It follows (see (3.4.5)) that D_X descends to a complex of sheaves $D_{\mathfrak{S}}$ which we call the dualising complex on $\mathfrak{S}_{smt}$. (One may verify that this is indeed a dualising complex, by working locally on the smooth topology of $\mathfrak{S}$.)

4. Perverse sheaves on algebraic stacks

(4.0) In this section we study *perverse sheaves on algebraic stacks.* Assume that $\mathfrak{S}$ is an algebraic stack as in (1.6) and that $\underline{S}$ is a stratification of $\mathfrak{S}$ as in (3.6.1). A stratification $\underline{T}$ of $\mathfrak{S}$ is a *refinement* of $\underline{S}$ if (a) each stratum T of $\underline{T}$ is contained in some stratum S of $\underline{S}$ and (b) each stratum S of $\underline{S}$ is a union of strata belonging to $\underline{T}$. One may readily verify that the category of all stratifications of $\mathfrak{S}$ as in (3.6.1) is small (in an appropriate universe) and is filtered under refinement.

(4.0.1)If $\underline{S}$ is a stratification of $\mathfrak{S}$ as in (3.6.1), we let $C_c(\mathfrak{S}_{smt}; \underline{S})$ ($C_c^{des}(Smt(B\mathfrak{S}.); \underline{S})$) = the heart

$D_c^{b,0\leq}(\mathfrak{S}_{smt};\ \underline{S},\ \bar{Q}_l)\bigcap D_c^{b,\geq 0}(\mathfrak{S}_{smt};\ \underline{S},\ \bar{Q}_l)$ $(D_c^{b,des,0\leq}(Smt(B\mathfrak{S}.);\ \underline{S},\ \bar{Q}_l)\bigcap D_c^{b,des,\geq 0}(Smt(B\mathfrak{S}.);\ \underline{S},\ \bar{Q}_l)$, respectively). Similarly $C_c(\mathfrak{S}_{et};\ \underline{S})$, $C_c^{des}(Et(B\mathfrak{S}.);\ \underline{S})$ will denote the corresponding categories defined using the étale topologies. Now one may show exactly as in ([B-B-D] p. 62) that if $\underline{T}$ is a refinement of the stratification $\underline{S}$ and $K^\cdot$ ($L^\cdot$) belongs to $C_c(\mathfrak{S}_{smt};\ \underline{S})$ ($L^\cdot$ belongs to $C_c^{des}(Smt(B\mathfrak{S}.);\ \underline{S})$) then $K^\cdot$ ($L^\cdot$) also belongs to $C_c(\mathfrak{S}_{smt};\ \underline{T})$ ($C_c^{des}(Smt(B\mathfrak{S}.);\ \underline{T})$, respectively .) Evidently the corresponding assertion holds for the smooth topologies as well. Therefore we make the following definition.

(4.0.2). **Definition**. (i) $C_c(\mathfrak{S}_{et}) = \lim\limits_{\underset{\underline{S}}{\rightarrow}} C_c(\mathfrak{S}_{et};\ \underline{S})$. ($C_c(\mathfrak{S}_{smt}) = \lim\limits_{\underset{\underline{S}}{\rightarrow}} C_c(\mathfrak{S}_{smt};\ \underline{S})$.) This category will be called *the category of perverse sheaves* on $\mathfrak{S}_{et}$ ($\mathfrak{S}_{smt}$, respectively).

(ii). $C_c^{des}(Et(B\mathfrak{S}.)) = \lim\limits_{\underset{\underline{S}}{\rightarrow}} C_c^{des}(Et(B\mathfrak{S}.);\ \underline{S})$. ($C_c^{des}(Smt(B\mathfrak{S}.)) = \lim\limits_{\underset{\underline{S}}{\rightarrow}} C_c^{des}(Smt(B\mathfrak{S}.);\ \underline{S})$). This category will be called the category of *perverse sheaves with descent* on $Smt(B\mathfrak{S}.)$ ($Smt(B\mathfrak{S}.)$, respectively) where the direct limit is over all stratifications as in (3.6.1).

(4.0.3)Observe from (3.6.6.*) that the functor α^* ($\alpha_.^*$) induces an equivalence (the first holding only for Deligne-Mumford stacks):

$C_c(\mathfrak{S}_{et}) \simeq C_c(\mathfrak{S}_{smt})$ ($C_c(Et(B\mathfrak{S}.)) \simeq C_c(Smt(B\mathfrak{S}.))$, respectively .)

(4.1). Now (Theorem 1.3.6, $[B-B-D]$) shows that $C_c^{des}(Smt(B\mathfrak{S}.))$, ($C_c(\mathfrak{S}_{smt})$) is an abelian category and the functor $\underline{H}^0 = \tau_{\leq 0}\circ\tau_{\geq 0} : D_c^{b,des}(Smt(B\mathfrak{S}.);\ \bar{Q}_l) \rightarrow C_c^{des}(Smt(B\mathfrak{S}.))$ ($\underline{H}^0 = \tau_{\leq 0}\circ\tau_{\geq 0} : D_c^b(\mathfrak{S}_{smt};\ \bar{Q}_l) \rightarrow C_c(\mathfrak{S}_{smt})$) is a cohomological functor. Similar statements hold for the categories defined using the smooth topologies.

(4.2) **Proposition**. Let $\mathfrak{S}$ denote an algebraic stack as in (1.6) with a smooth atlas $x : X \rightarrow \mathfrak{S}$. Assume the above situation. Now the functor $\bar{x}^*$ (defined as in (2.5.2)) provides the equivalences of categories:

(perverse sheaves on $\mathfrak{S}_{smt}$) $\simeq$ (perverse sheaves on $Smt(B\mathfrak{S}.)$ with descent)

$\simeq$ (perverse sheaves on $Smt(X)$ with descent).

Here the last category consists of perverse sheaves F on $Smt(X)$ provided with an isomorphism $\varphi : \pi_1^*F \rightarrow \pi_2^*F$ (in the appropriate category of perverse sheaves) satisfying the usual conditions; here $\pi_i : X\underset{\mathfrak{S}}{\times}X \rightarrow X$ is projection to the i-th factor. If F and K are two such objects, a morphism $f : F \rightarrow K$ in the above category will mean a map of perverse sheaves on $Smt(X)$ commuting with the above extra structure. The corresponding result holds with the étale site in the place of the smooth site for all Deligne-Mumford stacks.

Proof. Observe that the first equivalence follows from (3.6.5). Therefore it suffices to prove that the functor

$${}_s\bar{x}^*\text{:(perverse sheaves on } \mathfrak{S}_{smt}) \simeq \text{(perverse sheaves on } Smt(X) \text{ with descent).}$$

induces an equivalence. Moreover to prove the last assertion for Deligne-Mumford stacks, in view of the equivalence in (4.0.3), it suffices to prove the equivalence:

$$\bar{x}^* \text{ (perverse sheaves on } \mathfrak{S}_{et}) \simeq \text{(perverse sheaves on } Et(X) \text{ with descent).}$$

Locally on $\mathfrak{S}_{smt}$ we may assume that the atlas $x : X \to \mathfrak{S}$ has a section s; therefore it follows readily (see [SGA 4] Expose V^{bis}) that if F is a perverse sheaf on $Smt(X)$ with descent, then there exists an open *smooth* covering U_i of $\mathfrak{S}$ and perverse sheaves K_i on $Smt(U_i)$ so that $x^*K_i \cong F$ on $X \underset{\mathfrak{S}}{\times} U_i$ where $\cong$ denotes isomorphism in the appropriate category of perverse sheaves. The fact that each K_i is perverse enables us (see $[B - B - D]$ Theorem (3.2.4)) to glue them together to obtain a perverse sheaf K on $\mathfrak{S}_{smt}$ so that $x^*K \cong F$. Similarly if $\alpha : F \to F'$ is a map in the category on the right, one may show it descends to a map of perverse sheaves on $\mathfrak{S}_{smt}$.

Note: The remaining results of this section and the next also hold for Deligne-Mumford stacks with the étale topology in place of the smooth topology. We do not state these explicitly.

(4.3). **Corollary**. Assume the above situation. Then the category (perverse sheaves on $\mathfrak{S}_{smt}$) is both artinian and noetherian; every perverse sheaf on $\mathfrak{S}_{smt}$ has finite length.
Proof. Observe (from (4.2)) that it suffices to show that the category (perverse sheaves on $Smt(X)$) has the above property. Regarding X as an algebraic stack, and applying (4.2) to a representable étale cover $\tilde{X} \to X$, it suffices to show (perverse sheaves on $Smt(\tilde{X})$) has the required properties. This is clear in view of ($[B - B - D]$ Theorem (4.3.1).

(4.4).**Proposition**. Let $f : \mathfrak{S} \to \bar{\mathfrak{S}}$ denote a representable smooth map of algebraic stacks as in (1.6) with *connected non-empty geometric fibers*; let $x : X \to \mathfrak{S}$, $y : Y \to \bar{\mathfrak{S}}$ denote the smooth atlases. Let the relative dimension of the induced map $\bar{f} : X \to Y$ be d. Then $f^*[d]$:(perverse sheaves on $\bar{\mathfrak{S}}_{smt}$) $\to$ (perverse sheaves on $\mathfrak{S}_{smt}$) is fully-faithful.

Proof. In view of the equivalence in (4.2) we first reduce to showing that

(4.4.1) $\bar{f}^*[d]$: (perverse sheaves on $Smt(Y)$ with descent) $\to$ (perverse sheaves on $Smt(X)$ with descent)

is fully-faithful.

The main step in the proof will be to show (4.4.1) assuming that

(4.4.2) $\bar{f}^*[d]$: (perverse sheaves on $Smt(Y)$) $\rightarrow$ (perverse sheaves on $Smt(X)$)

is fully-faithful for all algebraic spaces X, Y and smooth maps $\bar{f} : X \rightarrow Y$ with connected geometric fibers of relative dimension d.

First observe that $\bar{f}^*[d]$ preserves the property of having descent. Therefore it suffices to prove the following: let $\alpha : K \rightarrow L$ denote a map of perverse sheaves on $Smt(X)$ with descent (as in (4.4.1)). Then there exists a map $\beta : K' \rightarrow L'$ of perverse sheaves on $Smt(Y)$ with descent so that $\bar{f}^*[d](\beta) = \alpha$.

By (4.4.2), clearly there exists a map $\beta : K' \rightarrow L'$ of perverse sheaves on $Smt(Y)$ so that $\bar{f}^*[d](\beta) = \alpha$. It suffices to show that the map β has descent. The assumption that α has descent shows the commutativity of the square:

$$\begin{array}{ccc} \pi_1^*(K) & \xrightarrow{\pi_1^*(\alpha)} & \pi_1^*(L) \\ \simeq\downarrow & & \simeq\downarrow \\ \pi_2^*(K) & \xrightarrow{\pi_2^*(\alpha)} & \pi_2^*(L) \end{array}$$

Since $\alpha = \bar{f}^*[d](\beta)$, by the assumption, it follows that the above square is merely:

$$\begin{array}{ccc} \bar{f}_1^*[d](\pi_1^*(K')) = \pi_1^*(\bar{f}^*[d](K')) & \xrightarrow{\bar{f}_1^*[d]\pi_1^*(\beta)} & \pi_1^*(\bar{f}^*[d](L')) = \bar{f}_1^*[d](\pi_1^*(L')) \\ \simeq\downarrow & & \simeq\downarrow \\ \bar{f}_1^*[d](\pi_2^*(K')) = \pi_2^*(\bar{f}^*[d](K')) & \xrightarrow{\bar{f}_1^*[d]\pi_2^*(\beta)} & \pi_2^*(\bar{f}^*[d](L')) = \bar{f}_1^*[d](\pi_2^*(L')) \end{array}$$

Here $\bar{f}_1 : Y \underset{\bar{\mathfrak{S}}}{\times} X \rightarrow Y \underset{\bar{\mathfrak{S}}}{\times} Y$ is the map induced by $\bar{f}$. Observe this is is also of relative dimension d and with connected geometric fibers. Therefore the assumption in (4.4.2) shows that

$\bar{f}_1^*[d]$: (perverse sheaves on $Smt(Y \underset{\bar{\mathfrak{S}}}{\times} Y)$ $\rightarrow$(perverse sheaves on $Smt(Y \underset{\bar{\mathfrak{S}}}{\times} X)$

is fully-faithful. Hence the square

$$\begin{array}{ccc} \pi_1^*(K') & \xrightarrow{\pi_1^*(\beta)} & \pi_1^*(L') \\ \simeq\downarrow & & \simeq\downarrow \\ \pi_2^*(K') & \xrightarrow{\pi_2^*(\beta)} & \pi_2^*(L') \end{array}$$

commutes. It follows that the map β has descent. This proves (4.4.1) assuming (4.4.2).

One may prove (4.4.2) as follows. Let $\tilde{x} : \tilde{X} \to X$, $\tilde{y} : \tilde{Y} \to Y$ be atlases as in (1.5)(i) and $\tilde{f} : \tilde{X} \to \tilde{Y}$, $\tilde{f}_1 : \tilde{Y} \underset{X}{\times} \tilde{X} \to \tilde{Y} \underset{X}{\times} \tilde{Y}$ denote the induced maps, then these are both smooth maps between schemes of relative dimension d and with connected geometric fibers. Therefore, one may invoke ([B-B-D] Proposition (4.2.5)) to observe that the functors:

$\tilde{f}^*[d]$: (perverse sheaves on $Smt(\tilde{Y})$) $\to$ (perverse sheaves on $Smt(\tilde{X})$) and

$\tilde{f}_1^*[d]$: (perverse sheaves on $Smt(\tilde{Y} \underset{X}{\times} \tilde{Y})$) $\to$(perverse sheaves on $Smt(\tilde{Y} \underset{X}{\times} \tilde{X})$)

are fully-faithful. The same proof as above now shows that this implies

$\tilde{f}^*[d]$: (perverse sheaves on $Smt(\tilde{Y})$ with descent) $\to$ (perverse sheaves on $Smt(\tilde{X})$ with descent)

is fully-faithful. By (4.2) this shows that the functor

$\bar{f}^*[d]$: (perverse sheaves on $Smt(Y)$) $\to$ (perverse sheaves on $Smt(X)$)

is fully-faithful. Clearly this completes the proof of (4.4.2) and hence that of (4.4).

(4.5). **Corollary**. Assume the above situation. Then $f^*[d]$ identifies (perverse sheaves on $\bar{\mathfrak{S}}_{smt}$) with a thick (ie. closed under subquotients) subcategory of (perverse sheaves on $\mathfrak{S}_{smt}$).

Proof. Clearly $f^*[d]$ is an exact fully-faithful functor and every object of (perverse sheaves on $\bar{\mathfrak{S}}_{smt}$) has finite length. Moreover the characterization of the simple objects as in (4.8) shows it sends simple objects to simple objects. Now apply (4.2.6) and lemma (4.2.6.1) of $[B-B-D]$.

(4.6) **Proposition**. Let $\mathfrak{S}$ denote an algebraic stack as in (1.6) with a smooth atlas $x : X \to \mathfrak{S}$ whose geometric fibers are *non-empty and connected*. Let K be a perverse sheaf with descent on $Smt(X)$. Then every subquotient of K *in the category of all perverse sheaves on* $Smt(X)$ also is a perverse sheaf with descent.

Proof. Let K be a perverse sheaf with descent on X and let L be one of its subquotients. Let π_i : $X \underset{\mathfrak{S}}{\times} X \to X$, $i = 1, 2$ denote the projection to the i-th factor. Observe that each of these is smooth of relative dimension = the relative dimension of $x : X \to \mathfrak{S} = d$. Now $\pi_1^* L[d]$ is a subquotient of $\pi_1^* K[d] = \pi_2^* K[d]$. (4.5) shows there exists a perverse sheaf M on X so that $\pi_1^* L[d] \cong \pi_2^* M[d]$. Now pulling back by the diagonal $\Delta : X \to X \underset{\mathfrak{S}}{\times} X$, we see that $L = M$. Thus L has descent. The equivalence of categories in (4.2) now implies the corresponding statement when K is a perverse sheaf on $Smt(B\mathfrak{S}.)$ with descent.

Next we derive the following results as formal consequences of the existence of the t-structure obtained by gluing as in (3.6.3) on $D_c^{b,des}(Smt(B\mathfrak{S}.); \underline{S}, \bar{Q}_l)$ and on $D_c^b(\mathfrak{S}_{smt}; \underline{S}, \bar{Q}_l)$, where $\underline{S}$ is a fixed stratification of $\mathfrak{S}$ as in (3.6.1). Let $\{U^i|i\}$, $\{U.^i|i\}$ denote the filtration of $\mathfrak{S}$ ($B\mathfrak{S}.$, respectively) as in (3.6.1).

(4.7.1). If $K^\cdot \in D_c^{b,des}(Smt(U.^1), \bar{Q}_l)$ and $F^\cdot \in D_c^{b,des}(Smt(B\mathfrak{S}.); \bar{Q}_l)$, we will say $F^\cdot$ is an *extension* of $K^\cdot$ if $j.^*(F^\cdot) \simeq K^\cdot$, where $j. : U.^1 \to B\mathfrak{S}.$ is the obvious map.

(4.7.2). Moreover if $K^\cdot \in D_c^{b,des}(Smt(U.^1), \bar{Q}_l)$ there is a *unique extension* $F^\cdot \in D_c^{b,des}(Smt(B\mathfrak{S}.), \bar{Q}_l)$ so that if $i_S. : S. \to B\mathfrak{S}.$ is the inclusion of a stratum, then,

(i) $\mathcal{H}^i(i_S.^*(F^\cdot)) = 0$ for $i > p(S.) - 1$ and

(ii) $\mathcal{H}^i(Ri_S.^!(F^\cdot)) = 0$ for $i < p(S.) + 1$

One may obtain this from (Proposition 1.4.14, [B-B-D]) by taking U (Y) in ([B-B-D]) to be $U^1.$ ($B\mathfrak{S}. - U^1.$, respectively) where $D_c^{b,des}(Smt(B\mathfrak{S}. - U^1.), \bar{Q}_l)$ has the induced t-structure and by applying induction on the number of strata in $B\mathfrak{S}. - U^1..$ (Observe that in ([B-B-D] Proposition 1.4.14, it is not required that the strata U and Y be smooth.)

(4.7.3). One may prove similar assertions about the derived categories $D_c^b(\mathfrak{S}_{smt}, \bar{Q}_l)$ and $D_c^b(U^1_{smt}, \bar{Q}_l)$.

(4.7.4). Assume the above situation; let $Y = \mathfrak{S} - U^1$ and $Y. = (B\mathfrak{S} - U^1).$. Let$C^{des}_{U^1.}$ ($C^{des}_{Y.}$, C^{des}) denote the heart of the t-category $D_c^{b,des}(Smt(U.^1); \bar{Q}_l)$ ($D_c^{b,des}(Smt(Y.); \bar{Q}_l)$, $D_c^{b,des}(Smt(B\mathfrak{S}.), \bar{Q}_l)$, respectively) with respect to the t-structure induced from the t-structure on $D_c^{b,des}(Smt(B\mathfrak{S}.); \bar{Q}_l)$ obtained by gluing as in (3.6.3).

Assuming the above situation we will let $i_. : Y. \to B\mathfrak{S}.$ denote the obvious map. If T denotes any one of the functors $j._!$, $j.^*$, $Rj._*$, $i.^*$, $i._*$ or $Ri.^!$, we let ${}^pT$ denote the functor $H^0 \circ T \circ \epsilon$, where ϵ is the inclusion of the heart into the appropriate derived category. (ie. H^0 denotes taking 'perverse-cohomology'.) We now obtain the following Proposition exactly as in ([B-B-D] p.52.)

(4.7.5). **Proposition**. The functor ${}^pj.^*$ identifies C^{des}_U, with the quotient of C^{des} by the thick subcategory $\bar{C}^{des}_Y$ which is the image of C^{des}_Y of the functor ${}^pi._*$.

(4.7.6) Now one may define the functor $j._{!*} : C^{des}_U \to C^{des}$ as the image of ${}^pj._! \to^p j._*$ and show that the simple objects of the category C^{des} are obtained as either ${}^pi._*F$, where F is simple in C^{des}_Y or as $j._{!*}K$, where K is simple in C^{des}_U.

(4.8) **Proposition**. Let $j : V \to \mathfrak{S}$ be the open immersion of an irreducible substack and let

$j. : V. \to B\mathfrak{S}.$ denote the open immersions of the corresponding simplicial objects. Assume further that $V \underset{k}{\times} \bar{k}$ is smooth, where $\bar{k}$ is the algebraic closure of k. Let L be an irreducible and lisse Q_l-sheaf on V. with descent. (ie. L corresponds to an irreducible representation of $\pi_1((V.)_{smt})\hat{}$; here $(V.)_{smt}$ denotes the étale topological type of the simplicial algebraic space $V.$ which may be defined as in [Fr]p.43, in view of (A.5.1) and $\hat{}$ is the profinite completion.) If $j. : V. \to B\mathfrak{S}.$ is the induced map, then $j._{!*}(L[\dim V \underset{\mathfrak{S}}{\times} X])$ is a simple object in the category $C^{des}(B\mathfrak{S}.; \bar{Q}_l)$. Moreover every simple object in $C^{des}(B\mathfrak{S}.; \bar{Q}_l)$ is obtained in this manner for some choice of V.

Proof. follows by a double application of (4.7.4) and (4.7.5) (ie. apply (4.7.5) to the open immersion $V \to \bar{V}$ and the closed immersion $\bar{V} \to X$) and the observation that if L is an irreducible lisse sheaf of $\bar{Q}_l$-modules on $V.$ with descent, then $L[\dim\ V \underset{\mathfrak{S}}{\times} X]$ is a simple object in $C^{des}(V; \bar{Q}_l)$. (For further details see $[B-B-D]$4.3.)

(4.9.0) Let k denote a finite field with q elements and let $\bar{k}$ denote its algebraic closure. Let ${}^0\mathfrak{S}$ denote an algebraic stack of finite type over k as in (1.6) and let $\mathfrak{S}$ denote ${}^0\mathfrak{S} \underset{Spec(k)}{\times} Spec(\bar{k})$. Let ${}^0x : {}^0X \longrightarrow {}^0\mathfrak{S}$ and $x : X \longrightarrow \mathfrak{S}$ denote atlases as in (1.4); recall ${}^0X$ is an algebraic space of finite type over k. Let ${}^0\tilde{x} : {}^0\tilde{X} \longrightarrow {}^0X$ and $\tilde{x} : \tilde{X} \longrightarrow X$ denote atlases for these algebraic spaces as in (1.5)(i). (Recall these are in fact schemes.) In the rest of this section we develop the *yoga of weights* for $\bar{Q}_l$-sheaves on $\mathfrak{S}_{smt}$.

(4.9.1) Let w be an integer. A $\bar{Q}_l$-sheaf ${}^0F$ on ${}^0\mathfrak{S}_{smt}$ will be said to be *(exactly) pure of weight* w if the sheaf ${}^0\tilde{x}^* \circ {}^0x^*({}^0F)$ on the scheme ${}^0\tilde{X}$ (of finite type over k) is pure of weight w ie. it satisfies the condition as in ($[B-B-D]p.126$). The induced sheaf on $\mathfrak{S}_{smt}$ will be denoted F. *Mixed* Sheaves (*Mixed* complexes of $\bar{Q}_l$-sheaves) with weights $\leq w$ or $\geq w$ on ${}^0\mathfrak{S}_{smt}$ may now be defined in the obvious manner. (See ($[B-B-D]p.126$). (Recall a mixed complex of $\bar{Q}_l$-sheaves ${}^0K$ on ${}^0\mathfrak{S}_{smt}$ is pure of weight w (has weights $\leq w$, $\geq w$) if each of the cohomology sheaves $\underline{H}^i({}^0K)$ is pure of weight $w+i (\leq w+i$, $\geq w+i$, respectively).) A complex of $\bar{Q}_l$-sheaves ${}^0K = \{{}^0K_n | n \geq 0\}$ on $B{}^0\mathfrak{S}.$ with *descent* will be said to be mixed of weights $\leq w (\geq w)$ if ${}^0K_0$ on $Smt({}^0X)$ is mixed of weights $\leq w (\geq w$, respectively). (Observe that since ${}^0K$ is assumed to have descent and is mixed, it follows that if ${}^0K_0$ has weights $\leq w (\geq w)$, then so does each ${}^0K_n$).

As an immediate consequence of our definition, it follows that if ${}^0L$ is a complex of $\bar{Q}_l$-sheaves on ${}^0\mathfrak{S}$ which is mixed of weights $\leq w (\geq w)$, then so is $\bar{x}^*\ {}^0L$, where $\bar{x}^*$ is the functor as in (2.5.2). *For the rest of this section we will only consider the middle perversity.*

(4.9.2)**Theorem**. Assume the above situation. Then every perverse sheaf ${}^0F$ on ${}^0\mathfrak{S}_{smt}$ *which is* *mixed and simple is pure.*

Proof. (4.8) identifies the simple objects in the category of perverse sheaves on $\mathfrak{S}$ with the perverse sheaves $j_{\cdot !*}L[\dim V \underset{\mathfrak{S}}{\times} X]$, where L is an irreducible and lisse $\bar{Q}_l$-sheaf on a locally-closed substack V of $\mathfrak{S}$ so that $V \underset{Spec(k)}{\times} Spec(\bar{k})$ is smooth. Let ${}^0x : {}^0X \longrightarrow {}^0\mathfrak{S}$ denote an atlas for ${}^0\mathfrak{S}$. Similarly let ${}^0\tilde{x} : {}^0\tilde{X} \longrightarrow {}^0X$ be an atlas for the algebraic space ${}^0X$. It is clear that the above perverse sheaf is pure if and only if ${}^0\tilde{x}^* \, {}^0x^*(j_{\cdot !*}L[\dim V \underset{\mathfrak{S}}{\times} X]$ is pure. This is clear since the above perverse sheaf is nothing but $\tilde{j}_{!*} \, {}^0\tilde{x}^* \, {}^0x^*(L[\dim V \underset{\mathfrak{S}}{\times} X])$, where $\tilde{j} : V \underset{\mathfrak{S}}{\times} \tilde{X} \to \tilde{X}$ is the induced open immersion; this is clearly pure by ([B-B-D] corollary (5.3.4)).

(4.10.0). Assume ${}^0\mathfrak{S}$ is an algebraic stack with the atlases ${}^0x : {}^0X \longrightarrow {}^0\mathfrak{S}$ and ${}^0\tilde{x} : {}^0\tilde{X} \longrightarrow {}^0X$ as in the proof of (4.9.2). Let $\mathfrak{S}$ denote the induced algebraic stack of finite type over $\bar{k}$ and let $x : X \longrightarrow \mathfrak{S}$, $\tilde{x} : \tilde{X} \longrightarrow X$ denote the induced atlases. The remaining main result we need to establish is that if ${}^0F$ is a perverse sheaf on ${}^0\mathfrak{S}_{smt}$ as in (4.9.1) which is also *pure* and if F is the induced perverse sheaf on $\mathfrak{S}_{smt}$, then F is *semi-simple.* In order to prove this it is not enough to show $\tilde{x}^* \circ x^*F$ is semi-simple because of the following: that $\tilde{x}^* \circ x^*F$ is semi-simple shows the terms of its composition series split up as its summands. However this splitting may not descend to a similar splitting of the corresponding terms of the composition series of F. In order to obtain such a splitting for the terms of the composition series of F it becomes necessary to be able to perform the full yoga of weights for perverse sheaves with descent on *the simplicial algebraic space* $B^0\mathfrak{S}$. We digress to establish these next.

(4.10.1)Next observe from our definition (see (4.9.1)) and the corresponding property for $\bar{Q}_l$-sheaves on schemes of finite type over k that the category of $\bar{Q}_l$-sheaves of exact weight w *is closed under extensions as well as sub- and quotient objects.*

Let ${}^0x : {}^0X \longrightarrow {}^0\mathfrak{S}$ and ${}^0\tilde{x} : {}^0\tilde{X} \longrightarrow {}^0X$ denote atlases. Let $x : X \longrightarrow \mathfrak{S}$ and $\tilde{x} : \tilde{X} \longrightarrow X$ denote the corresponding atlases for $\mathfrak{S}$ and ${}^0X$ respectively .

Next let ${}^0K$ and ${}^0L$ denote two mixed bounded complexes of $\bar{Q}_l$-sheaves with constructible cohomology sheaves on ${}^0\mathfrak{S}_{smt}$. Let ${}^0\bar{x}^*({}^0K) = \{ {}^0K_n | n \geq 0\}$ and ${}^0\bar{x}^*({}^0L) = \{ {}^0L_n | n \geq 0\}$ denote the induced complexes on $Smt(B^0\mathfrak{S})$. One may now readily verify the following (local assertions):

(4.10.2.1) If ${}^0K$ has weights $\leq w'$ and ${}^0L$ has weights $\geq w$, then $\underline{Rhom}({}^0K, {}^0L)$ has weights $\geq w - w'$.

(4.10.2.2). If instead ${}^0K$ has weights $\geq w'$ and ${}^0L$ has weights $\leq w$, then $\underline{Rhom}({}^0K, {}^0L)$ has weights $\leq w - w'$.

(4.10.2.3) If ${}^0K$ has weights $\leq w'$ and ${}^0L$ has weights $\leq w$, then ${}^0K \otimes {}^0L$ has weights $\leq w + w'$.

(4.10.2.4) Let $f : {}^0\mathfrak{S} \longrightarrow {}^0\bar{\mathfrak{S}}$ denote a representable map of algebraic stacks and let ${}^0K$ $({}^0L)$ denote a mixed bounded complex of $\bar{Q}_l$-sheaves on ${}^0\mathfrak{S}_{smt}$ with weights $\geq w$ ($\leq w$, respectively). Then $Rf_* {}^0K$ $(Rf_! {}^0L)$ is also mixed and has weights $\geq w$ ($\leq w$, respectively). *Therefore if in addition f is also proper, and* ${}^0K$ *is pure of weight* w*, then so is* $Rf_* {}^0K$. (Observe that this assertion is local on ${}^0\bar{\mathfrak{S}}$; hence one may reduce this to the corresponding statement when f is a map of schemes of finite type over k.)

(4.10.3). Assume the situation of (4.10.1). Let ${}^0\bar{x} : B({}^0\mathfrak{S}) \longrightarrow {}^0\mathfrak{S}$ and ${}^{\bar{0}}\tilde{x} : B({}^0\tilde{X}) \longrightarrow {}^0X$ denote the maps in (4.10.1). Let $\sigma_{{}^0\mathfrak{S}} : {}^0\mathfrak{S} \longrightarrow Spec(k)$, $\sigma_{{}^0X} : {}^0X \longrightarrow Spec(k)$, $\sigma_{{}^0\tilde{X}} : {}^0\tilde{X} \longrightarrow Spec(k)$ denote the structure maps. Similarly let $\sigma_{(B^0\mathfrak{S})_n} : (B^0\mathfrak{S})_n \longrightarrow Spec(k)$, $\sigma_{(B^0X)_n} : (B^0X)_n \longrightarrow Spec(k)$ denote the structure maps. Now one readily verifies the following:

(4.10.3.1). $\sigma_{{}^0\mathfrak{S}*} \circ {}^0\bar{x}_*(F) = ker(\sigma_{(B^0\mathfrak{S})_0*}(F_0) \rightarrow \sigma_{(B^0\mathfrak{S})_1*}F_1)$

where $F = \{F_n | n \geq 0\}$ is a bounded complex of $\bar{Q}_l$-sheaves on $Smt(B^0\mathfrak{S})$.

(4.10.4) **Lemma**. Assume the above situation. Let w be an integer. Now the functor

$R\sigma_{{}^0\mathfrak{S}*} : D^c({}^0\mathfrak{S}_{smt}; \bar{Q}_l) \longrightarrow D^c(Smt(Spec(k)); \bar{Q}_l)$

sends complexes with weight $\geq w$ to complexes with weight $\geq w$.

Proof. Let ${}^0K \epsilon D^c({}^0\mathfrak{S}_{smt}; \bar{Q}_l)$. Now (2.5.4) shows that the natural map ${}^0K \longrightarrow {}^0\bar{x}_* \circ {}^0\bar{x}^*({}^0K)$ is a quasi-isomorphism.

Next assume that ${}^0\mathfrak{S}$ is an algebraic space regarded as an algebraic stack as in ((1.5)(i)). Apply (4.10.3.1) to ${}^0\bar{x}^*({}^0K)$; now the right hand side of (4.10.3.1) has weights $\geq w$ by ($[B-B-D]$(5.1.14)) and (4.10.1) above. This proves the assertion when ${}^0\mathfrak{S}$ is an algebraic space of finite type over k.

Next consider the general case. Once again apply (4.10.3.1) to ${}^0\bar{x}^*({}^0K)$; since each $(B^0\mathfrak{S})_n$ is an algebraic space of finite type over k the right hand side of (4.10.3.1) has weights $\geq w$ by what we have already established in the above paragraph and by (4.10.1) above. This completes the proof the lemma.

(4.11.1). Let k be a finite field with $q(= p^n)$ elements and let ${}^0\mathfrak{S}$ denote an algebraic stack of finite type over k. For each $n \geq 1$, let Fr_{q^n} denote the geometric Frobenius 'raising the coordinates to the q^n-th power'. One verifies that this induces a representable map ${}^0\mathfrak{S} \longrightarrow {}^0\mathfrak{S}$ of algebraic stacks. Let ${}^0F$ denote a Q_l-sheaf on ${}^0\mathfrak{S}_{smt}$; if F denotes the induced sheaf on $\mathfrak{S}_{smt}$, then one may readily verify that there exists an isomorphism $(Fr_{q^n})^*F \rightarrow F$. (See $[B-B-D]$(5.1.1).)

(4.11.2). **Proposition**. The functor ${}^0F \longrightarrow (F, (Fr_q)^*)$ from the category of perverse sheaves on ${}^0\mathfrak{S}_{smt}$ to the the category of perverse sheaves F on $\mathfrak{S}_{smt}$ provided with an isomorphism $(Fr_q)^*F \overset{\simeq}{\rightarrow} F$ is fully-faithful. Moreover the 'image' of the above functor is a subcategory that is closed under extensions and sub-quotients.

Proof. Let ${}^0x : {}^0X \longrightarrow {}^0\mathfrak{S}$ denote an atlas of ${}^0\mathfrak{S}$, while ${}^0\tilde{x} : {}^0\tilde{X} \longrightarrow {}^0X$ denote an atlas for the algebraic space X. Let $x : X \longrightarrow \mathfrak{S}$ and $\tilde{x} : \tilde{X} \longrightarrow X$ denote the corresponding maps for $\mathfrak{S}$ and X respectively . Proposition (4.2) enables one to identify the category of perverse sheaves on ${}^0\mathfrak{S}_{smt}$ ($\mathfrak{S}_{smt}$, respectively) with the category of perverse sheaves with descent on $Smt({}^0X)$ ($Smt(X)$, respectively). Now (4.11.2) follows readily from the two observations:

(i) a perverse sheaf F on $\mathfrak{S}_{smt}$ has the property that the map $Fr_q^*F \longrightarrow F$ is an isomorphism if and only if the induced perverse sheaf on $Smt(\tilde{X})$ has the same property and

(ii) a perverse sheaf ${}^0F$ on $Smt({}^0X)$ ($Smt({}^0\tilde{X})$) has descent if and only if the induced perverse sheaf F on $Smt(X)$ ($Smt(\tilde{X})$, respectively) has descent. Similarly a map ${}^0\alpha : {}^0F \rightarrow {}^0K$ of perverse sheaves on $Smt({}^0X)$ ($Smt({}^0\tilde{X})$) has descent if and only if the induced map $\alpha : F \rightarrow K$ of perverse sheaves on $Smt(X)$ ($Smt(\tilde{X})$, respectively) has descent.

We skip the remaining details of the proof.

(4.11.3). Now let ${}^0\mathfrak{S}$ denote an algebraic stack of finite type over a field k as before and let ${}^0M$ denote a bounded complex of $\bar{Q}_l$-sheaves on $Smt(Spec\ k)$. Let $\sigma_{{}^0\mathfrak{S}} : {}^0\mathfrak{S} \longrightarrow Spec(k)$ denote the obvious structure map. One verifies readily that one obtains the spectral sequence:

$$E_2^{p,q} = H^p_{smt}((Spec\ k); \underline{H}^q({}^0M^\cdot)) \Rightarrow H^{p+q}R\Gamma({}^0\mathfrak{S}, {}^0M^\cdot))$$

Observe that k is a finite field with q-elements and hence $Gal(\bar{k}/k) \cong \hat{\mathbb{Z}}$; therefore $E_2^{p,q} = 0$ if $p \neq 0$ or 1. Hence one obtains a short-exact sequence

$$(4.11.3.1)\ 0 \longrightarrow E_\infty^{1,n-1} \longrightarrow H^nR\Gamma({}^0\mathfrak{S}, {}^0M^\cdot) \longrightarrow E_\infty^{0,n} \longrightarrow 0$$

(4.11.3.2) Now let ${}^0K$ and ${}^0L$ denote two bounded complexes of $\bar{Q}_l$-sheaves with constructible cohomology sheaves on ${}^0\mathfrak{S}_{smt}$ and let ${}^0M = \underline{Rhom}({}^0K, {}^0L)$. Let $Rhom(K, L)$ denote the complex $R\sigma_{{}^0\mathfrak{S}*}\underline{Rhom}({}^0K, {}^0L)$ and let $Hom({}^0K, {}^0L) = H^0Rhom({}^0K, {}^0L)$ as in ([B-B-D] section 5.1). The short-exact sequence in (4.11.3.1) now becomes:

$$(4.11.3.3)\ 0 \longrightarrow (Ext^{n-1}(K, L))_{Fr} \longrightarrow Ext^n({}^0K, {}^0L) \longrightarrow Ext^n(K, L)^{Fr} \longrightarrow 0$$

where $(Ext^{n-1}(K,L))_{Fr}$ $(Ext^n(K,L)^{Fr})$ denotes the co-invariants (the invariants, respectively) under the action of the Galois group $Gal(\bar{k}/k)$ or equivalently under the Frobenius Fr_q.

Next let $\{K_p|p \geq 0\} = \bar{x}^*K$ and let $\{L_p|p \geq 0\} = \bar{x}^*L$ be the induced complexes on the simplicial algebraic space $B\mathfrak{S}$. Next observe the existence of the spectral sequence-see (A.2.1),

$$E_1^{p,q} = Ext^q(K_p, L_p) \Rightarrow Ext^{p+q}(K,L)$$

If $B(B\mathfrak{S}_p)$ denotes the classifying simplicial scheme for the algebraic space $(B\mathfrak{S})_p$, and $\{(K_p)_u|u \geq 0\}$ and $\{(L_p)_u|u \geq 0\}$ denote the corresponding induced complexes on $B(B\mathfrak{S}_p)$, there is a similar spectral sequence with $E_1^{u,v} = Ext^v((K_p)_u,(L_p)_u)$ converging to $Ext^{u+v}(K_p,L_p)$.

Making use of (3.4.5), the above spectral sequences and the corresponding assertion for perverse sheaves on schemes one may now verify that if K and L are perverse sheaves on $\mathfrak{S}_{smt}$ then $Ext^i(K,L) = 0$ if $i < 0$. (This is also a formal consequence of the fact that the perverse sheaves form the heart of the t-structure we have defined in (3.6.3).) Therefore taking n=0, (4.11.3.3) now provides the isomorphism

(4.11.3.4) $Hom(\ {}^0K,\ {}^0L) \cong Hom(K,L)^{Fr}$

provided ${}^0K$ and ${}^0L$ are perverse sheaves on the algebraic stack ${}^0\mathfrak{S}$. Taking n=1, (4.11.3.3) also provides the short-exact sequence

$$0 \longrightarrow (Hom(K,L))_{Fr} \longrightarrow Ext^1(\ {}^0K,\ {}^0L) \longrightarrow Ext^1(K,L)^{Fr} \longrightarrow 0$$

Finally we make the following important observation: *assume in addition that* ${}^0K$ *and* ${}^0L$ *are pure of the same weight w. Then*

(4.11.3.5)$Ext^1(K,L)^{Fr} = 0$

To see this, first observe from (4.11.3.2) that $Ext^1(K,L) = \underline{H}^1(R\sigma_{{}^0\mathfrak{S}*} \circ \underline{Rhom}(\ {}^0K,\ {}^0L))$. By (4.10.4), this has strictly positive weights; hence (4.11.3.5) follows. We now conclude this section with the following important result.

(4.12) **Theorem**. Assume the situation of (4.11.1). Let ${}^0F$ denote a perverse sheaf on ${}^0\mathfrak{S}_{smt}$ which is also *pure*. If F denotes the induced perverse sheaf on $\mathfrak{S}_{smt}$, then F is *semi-simple.*

Proof. The proof parallels the original proof in ($[B-B-D]$ Theorem (5.3.8)) making use of the results of (4.10) and (4.11). Let F' denote the largest sub-object of F in the abelian category of all perverse sheaves on $\mathfrak{S}_{smt}$ which is also semi-simple; it suffices to show $F' = F$. Now observe that F'

is stable under the action of the Frobenius; this is a local assertion and one may reduce to verifying $\tilde{x}^* \circ x^*(F')$ on the scheme $\tilde{X}$ is stable under the Frobenius - see (4.11.2) above. Therefore (by (4.11.2)) there exists a perverse sheaf ${}^0F'$ on ${}^0\mathfrak{S}_{smt}$ which induces the perverse sheaf F' on $\mathfrak{S}_{smt}$.

Now consider the short-exact sequence:

$$0 \to {}^0F' \to {}^0F \to ({}^0F/\,{}^0F') \to 0$$

This represents an element of $Ext^1({}^0F', ({}^0F/\,{}^0F'))$. (Recall once again that the category of perverse sheaves is abelian. Moreover it follows as in ($[B-B-D]$ Remarque (3.1.17)(ii)) that the $Yoneda-Ext^1$ computed in the abelian category of perverse sheaves is isomorphic to the Ext^1 computed in the usual derived category of complexes of sheaves.) By (4.11.3.5) and (4.11.2) the image of this term in $Ext^1(F', (F/F'))$ is 0 ie. the short exact sequence

$$0 \to F' \to F \to F/F' \to 0$$

splits. Therefore $F \cong F' \oplus (F/F')$; if F/F' is non-null it admits a non-null *simple sub-object.* Taking the sum of F' and this sub-object provides a larger semi-simple sub-object of F, contradicting the original choice of F'. It follows therefore that F/F' is null or $F = F'$; this completes the proof of the theorem.

(4.13).**Corollary**. (*Decomposition theorem*). Let ${}^0p : {}^0\mathfrak{S} \to {}^0\bar{\mathfrak{S}}$ denote a proper representable map of algebraic stacks over k as in (4.9.0) and let $p : \mathfrak{S} \to \bar{\mathfrak{S}}$ denote the induced map over $\bar{k}$. Let ${}^0F$ denote a perverse sheaf on ${}^0\mathfrak{S}$ which is *pure* and let F denote the induced perverse sheaf on $\mathfrak{S}$. Then Rp_*F is *semi-simple.*

Proof. Observe that $R\,{}^0p_*\,{}^0F$ is a perverse sheaf which is pure by (4.10.2.4). Now one shows that so is each perverse cohomology $\mathcal{H}^i(R\,{}^0p_*\,{}^0F)$ and that $R\,{}^0p_*\,{}^0F\Sigma_i\mathcal{H}^i(R\,{}^0p_*\,{}^0F)[-i]$. Now apply (4.12) to each $\mathcal{H}^i(R\,{}^0p_*\,{}^0F)$.

5. Intersection cohomology of algebraic stacks

In this section we study the *Intersection cohomology of algebraic stacks.*

(5.0).Let $\mathfrak{S}$ denote an algebraic stack and let $x : X \to \mathfrak{S}$ denote a smooth atlas as in (1.4). Let $\underline{S}$ denote a stratification of $\mathfrak{S}$ as in (3.6.1); since the atlas $x : X \to \mathfrak{S}$ is assumed to be smooth, the inverse images of these strata define a similar stratification $\underline{S}'$ of X. We will assume throughout the rest of this section that the dimension of X is n.

(5.1). Now one may obtain compatible filtrations $U_1 \subseteq U_2 \subseteq ... \subseteq U_n \subseteq U_{n+1} = X$, $V_1 \subseteq V_2 \subseteq ... \subseteq V_n \subseteq V_{n+1} = \mathfrak{S}$ as follows: let $U_i - U_{i-1}$ = the union of strata $\underline{S}'$ (of X) of dimension $n-i+1$. $V_i - V_{i-1}$ is defined similarly. Observe that $U_i = the$ inverse image of V_i; therefore if $B\mathfrak{S}_n = X \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X ... \underset{\mathfrak{S}}{\times} X$ is the n-th stage of the classifying simplicial groupoid of $\mathfrak{S}$, we obtain a similar filtration of $B\mathfrak{S}_n$ by letting $U^i_n = (V_i) \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X ... X \underset{\mathfrak{S}}{\times} X$. One readily verifies that so defined $U^i_.$ defines a sub-simplicial object of $B\mathfrak{S}.$ so that the obvious map $U^i_. \to B\mathfrak{S}.$ is an open immersion in each degree. The open immersion $V_i \to V_{i+1} (U_i \to U_{i+1}, U^i_. \to U^{i+1}_., U^i_n \to U^{i+1}_n)$ will be denoted $\bar{j}_i (j_i, j^i_., j^i_n$ respectively).

(5.2) A *perversity* is a function p : (non-negative integers) $\to \mathbb{Z}$ defined as in (3.6.2).

(5.3.1) Let $\underline{S}$ denote a fixed stratification of $\mathfrak{S}$ as above and let $\{U^i.|i\}$ denote the induced filtration of $B\mathfrak{S}..$ Let F' denote a fixed *lisse* (see (A.2.4)) sheaf of Q_l-modules on the étale topology $Smt(U^1_.)$ of the simplicial object $(U^1)_..$ (Observe from the very definition of lisse sheaves that F' descends to a sheaf F on $Smt(V^1)$.) Let F'_n denote the restriction of F' to $Smt(U^1_n)$.

(5.3.2)We will now construct a perverse sheaf of Q_l-modules on $Smt(B\mathfrak{S}.)$, denoted $IC^S_p(F')$, and called the *intersection cohomology complex* for the perversity p associated to F'. On $C_{U^2_.}$, this is merely $\sigma_{<\bar{p}(U_1)} Rj_{1*}(F')$, where $\sigma_{<\bar{p}(U_1)}$ is the cohomology truncation defined in (3.5.1). Assume that we have extended F' to a cosimplicial object $IC^S_p(F')_{|U^k_.}$, for $k \leq n$. Now we will extend this to the complex $IC^S_p(F')_{|U^{k+1}_.} = \sigma_{<\bar{p}(U_{k+1}-U_k)} Rj^k_*(IC_p(F')_{|U^k_.}))$ of sheaves on $Smt(U^{k+1}_.)$. The perverse sheaf of Q_l-modules on $Smt(B\mathfrak{S}.)$ obtained in this manner will be denoted $IC^S_p(F')..$ Clearly this has descent; in fact if $IC^S_p(F)$ denotes the perverse sheaf of Q_l-modules on $\mathfrak{S}_{smt}$ obtained using the filtration $\{V_i\}$ in a similar manner, it is clear that $\bar{x}^* IC^S_p(F) \cong IC^S_p(F')$. (Use smooth base change on the open immersions j_i.)

(5.3.3) One may verify readily (see $[B-B-D]$ Proposition (2.1.11)) that $IC^S_p(F)$ is quasi-isomorphic to the unique extension $j_{.!*}F$ as in (4.7.6), where $j. : U^1_. \longrightarrow B\mathfrak{S}.$ is the open immersion.

(5.3.4)The hyper-cohomology of $B\mathfrak{S}.$ with respect to $IC^S_p(F')$ will be called the *intersection cohomology of* $B\mathfrak{S}.$ *with respect to* F' *and the stratification* $\underline{S}$ *and the perversity* p. This will be denoted $IH^*_{S,p}(B\mathfrak{S}.; F')$. By (3.4.6), this is isomorphic to $IH^*_{S,p}(\mathfrak{S}.; IC^S_p(F))$.

(5.3.5) Observe that $IH^*_{S,p}(B\mathfrak{S}.; F)$ is *independent* of the stratification; this follows from the observation in (5.3.3) identifying $IC^S_p(F)$ with $j_{!*}F$.

(5.3.6). For the rest of the paper we will only consider the case where F' = the constant $l-adic$ sheaf $\underline{Q_l}$; ie. $IC^S_p(Q_l)$ will denote the intersection cohomology complex on $\mathfrak{S}_{smt}$ obtained by starting with the constant $l - adic$ sheaf Q_l on the smooth-open stratum of the stratification $\underline{S}$. The corresponding hypercohomology groups will be denoted $IH^*_p(\mathfrak{S}; Q_l)$.

(5.3.7)The complex $IC^S_p(Q_l)$ on $\mathfrak{S}_{smt}$ is characterized by the following axioms. Let $x : X \to \mathfrak{S}$ denote an atlas for $\mathfrak{S}$ as in (1.4)(b) which is an algebraic space and let $\tilde{x} : \tilde{X} \to X$ denote an atlas for X as in (1.5)(i) which is a scheme of finite type over k . Then

(i). dim support $\mathcal{H}^i(\tilde{x}^* x^* IC^S_p(Q_l)) \leq n - p^{-1}(n+i)$ and

(ii) dim support $\mathcal{H}^i(D(\tilde{x}^* x^* IC^S_p(Q_l))) \leq n - (p^*)^{-1}(n+i)$

where $p^{-1}(l) = min\{c|p(c) = l\}$, p^* is the complimentary perversity defined in ([B-B-D] p. 63), $(p^*)^{-1}$ is defined similarly and D denotes taking the Verdier-dual. This follows from the easy observation as in (5.3.2) that $\tilde{x}^* x^* IC^S_p(Q_l)$ is the intersection cohomology complex on $\tilde{X}$ with perversity p if and only if $IC^S_p(Q_l)$ is the intersection cohomology complex on $\mathfrak{S}$ with perversity p; the former is characterized by the above axioms. (See [G-M-2] p.107). Observe that our terminology is chosen to agree with that of [B-B-D]; this means a perversity p_{G-M} in the terminology of [G-M-2] is obtained by $p_{G-M}(i) = p(i) + (2dim_k(X))$.)

(5.3.8). *Pairings and Poincare-Verdier duality.* One may readily show (as in [G-M-2] pp.112-113) that if p is a fixed perversity, p^* is its complimentary perversity and $\underline{S}$ is a stratification of $\mathfrak{S}$ as in (3.6.1), then there exists a pairing:

$$IC^S_p(Q_l) \otimes IC^S_{p^*}(Q_l) \longrightarrow D_{\mathfrak{S}}$$

where $D_{\mathfrak{S}}$ denotes the dualising complex on $\mathfrak{S}_{smt}$. Next one obtains

$$D(IC^S_p(Q_l)) \simeq IC^S_{p^*}(Q_l),$$

Observe that this readily follows from the above axiomatic characterization of the complex $IC^S_p(Q_l)$ and $IC^S_{p^*}(Q_l)$.

(5.3.9). Assume the situation of (5.3.1). Let F' and F be lisse sheaves as in (5.3.1); let $IC^S_p(F')$ ($IC^S_p(F)$) denote the intersection cohomology complex on $\mathfrak{S}_{smt}$ ($Smt(B\mathfrak{S}.)$, respectively) constructed as in (5.3.2). Let $IC^S_p(F')_n$ denote the restriction of the complex $IC^S_p(F')$ to $Smt(B\mathfrak{S}_n)$. One may now readily verify (from the construction) that this is isomorphic to the intersection cohomology complex of $B\mathfrak{S}_n$ constructed using the filtration $U^n_1 \subseteq U^n_2 \subseteq .... \subseteq U_n \subseteq U_{n+1} = B\mathfrak{S}_n$. Therefore we obtain the hypercohomology spectral sequence:

$$E_1^{p,q} = IH^q((B\mathfrak{S}_p);\, F'_p) \Rightarrow IH^{p+q}(B\mathfrak{S};\, F')$$

(5.3.10). Assume the above situation. Let $j : U \to \mathfrak{S}$ denote the open immersion of an open substack; smooth-base-change for j shows

$$j^* IC_p^S(F') \simeq IC_p^{S'}(F'_{|U \bigcap U_1})$$

where S' is the induced stratification of U.

(5.3.11)Assume in addition to the above that U, V denote open sub-stacks of $\mathfrak{S}$. Let U., V., $(U \cap V)$., $(U \cup V)$. denote the associated simplicial objects. Then (5.3.10) provides Mayer-Vietoris sequences:

$$\ldots \to IH_p^n(B(U \cup V).;\, F') \to IH_p^n(BU.;\, F') \oplus IH_p^n(BV.;\, F') \to IH_p^n(B((U \cap V).;\, F') \to \ldots$$

in the obvious manner.

(5.3.12) **The decomposition theorem**. Assume ${}^0 f : {}^0\mathfrak{S} \to {}^0\mathfrak{S}$ is a representable (see (1.3.3)) proper map of algebraic stacks as in (4.11.1). Let $f : \mathfrak{S} \to \bar{\mathfrak{S}}$ be the induced map of stacks over $\bar{k}$. Let $IC_m(\mathfrak{S};\, \bar{Q}_l)$ denote the intersection cohomology complex for the middle perversity on $\mathfrak{S}$, obtained by starting with the constant sheaf $\bar{Q}_l$-sheaf F on the smooth stratum in a stratification of $\mathfrak{S}$ as in (5.3.1). Then

$$Rf_* IC_m(\mathfrak{S};\, F) = \bigoplus_{\alpha} IC_m(V_\alpha;\, F_\alpha)[d_\alpha];$$

here V_α are locally closed substacks of $\bar{\mathfrak{S}}$, F_α is an irreducible lisse $\bar{Q}_l$-sheaf on the smooth stratum of a stratification of V_α, d_α is an integer and IC_m is the intersection cohomology with the middle perversity.

Proof. This follows readily from (4.13) and (4.8).

(5.4.0) *For the rest of this section we will restrict to Deligne-Mumford stacks.* Recall a stack $\mathfrak{S}$ is Deligne-Mumford if the atlas $x : X \to \mathfrak{S}$ is étale. By composing x with the map $\tilde{x} : \tilde{X} \to X$ with $\tilde{X}$ a scheme of finite type over k, we may assume without loss of generality that the given atlas X is itself a scheme. Next observe that the support of any constructible sheaf on the stack $\mathfrak{S}$ (in fact for any algebraic stack in general) is a locally-closed algebraic sub-stack. When $\mathfrak{S}$ has an étale atlas as here the dimension of any locally-closed algebraic sub-stack may be defined to be the dimension of its atlas. Therefore we may observe that that the intersection cohomology complex $IC_p^S(Q_l)$ on $\mathfrak{S}_{et}$ is characterized by the two axioms:

(i). dim support $\mathcal{H}^i(IC_p^S(Q_l)) \leq n - p^{-1}(n+i)$ and

(ii) dim support $\mathcal{H}^i(D(IC^S_p(Q_l)) \leq n - p^{*-1}(n+i)$

(5.5.0) Let $\mathfrak{M}$ denote a coarse moduli-space for $\mathfrak{S}$. i.e. $\mathfrak{M}$ is a scheme along with a proper map $\pi : \mathfrak{S} \to \mathfrak{M}$ so that if Ω is an algebraically closed field, then $\pi_0(\mathfrak{S}(\Omega)) \cong \mathfrak{M}(\Omega)$, where π_0 denotes the set of connected components for the action of the étale groupoid G associated to $\mathfrak{S}$ as in (1.5)(ii). *See [Gil] (3.10).)* Recall that the groupoid G is merely $B\mathfrak{S}.$ where the all the face-maps are étale. In this case we will let $\epsilon. : B\mathfrak{S}. \to \mathfrak{S}$ denote the obvious augmentation given by ϵ_n = the composition of $d_0 \circ ... \circ d_0 : B\mathfrak{S}_n \to \ X$ and the map $x : X \to \mathfrak{S}$. (Recall this was denoted $\bar{x}$ in (2.5.2); for the rest of this section we have changed this notation.) We will let $\pi. : B\mathfrak{S}. \to \mathfrak{M}$ denote the composition of $B\mathfrak{S}. \xrightarrow{\epsilon.} \mathfrak{S}$ and $\pi : \mathfrak{S} \to \mathfrak{M}$.

(5.5.1) For the proof of (5.5.2) it is essential to use the simplicial étale site defined in (A.4.0). Any scheme, algebraic space or stack Y will be considered as the associated obvious simplicial object which will be denoted Y itself. Functors defined on the site Et as in (A.1.0) will have the superscript $'et'$ to indicate the site used, while functors on the site SEt (as in (A.4.0)) will denoted in the usual manner. (For eg. $\pi._* : Absh(SEt(B\mathfrak{S}.); Q_l) \to Absh(SEt(\mathfrak{M}), Q_l)$ while ${}^{et}\pi._* : Absh(Et(B\mathfrak{S}.); Q_l) \to Absh(Et(\mathfrak{M}); Q_l)$ will denote the obvious direct-image functors associated to π.

(5.5.2).**Theorem**. Let p denote a fixed perversity. Assuming the above situation, $R\pi._* IC_p(B\mathfrak{S}.;$ $Q_l) \simeq IC_p(\mathfrak{M};\ Q_l)$ where IC_p denotes the intersection cohomology complexes with perversity p and $\pi._*$ is the direct-image functor $Absh(SEt(B\mathfrak{S}.)) \to Absh(SEt(K(\mathfrak{M},0)))$ where $K(\mathfrak{M},0)$ is the obvious constant simplicial scheme associated to $\mathfrak{M}$ and where SEt is the site defined as in (A.4.0). Hence $IH^*_p(\mathfrak{S};\ Q_l) \cong IH^*_p(\mathfrak{M};\ Q_l)$.

Proof. First observe that the map $X \longrightarrow \mathfrak{S} \longrightarrow \mathfrak{M}$ is finite and hence π itself is finite. Let $n = dim_k(X) = dim_k(\mathfrak{M})$

Let $U_1 \subseteq U_2 \subseteq ... \subseteq U_n \subseteq U_{n+1} = \mathfrak{M}$ be the filtration of $\mathfrak{M}$ associated to a smooth stratification as in (5.1); let $V^i = U_i \underset{\mathfrak{M}}{\times} \mathfrak{S}$, $V^i_. = U_i \underset{\mathfrak{M}}{\times} B\mathfrak{S}$ and let p denote a fixed perversity. Let $j_i(\bar{j}^i,\ \bar{j}^i_.)$ denote the open immersion $U_i \to U_{i+1}(V^i \to V^{i+1},\ V^i_. \to V^{i+1}_.$, respectively). *Observe that* V^i *and* $V^i_.$ *need not be smooth in any degree.* Let $IC_p(\mathfrak{S};\ Q_l)$ $(IC_p(B\mathfrak{S}.;\ Q_l))$ denote the intersection cohomology complex on $\mathfrak{S}$ ($B\mathfrak{S}.$, respectively) constructed using some smooth stratification (and the constant sheaf $\underline{Q_\ell}$ on the smooth stratum) by applying the construction of (5.3.2).

Next one may readily verify using smooth base-change (on the open immersions $j^i_.$) that

(5.5.2.1) ${}^{et}\pi^*_. IC_p(\mathfrak{M};\, Q_l) \simeq$ the complex $\sigma_{<\bar{p}(U_{n+1}-U_n)} R\ {}^{et}\bar{j}^n_{.*} \sigma_{<\bar{p}(U_n-U_{n-1})} \cdots\ \sigma_{<\bar{p}(U_1)} R\ {}^{et}\bar{j}^1_{.*} \underline{Q}_\ell$,

where $\underline{Q}_\ell$ now denotes the constant sheaf on $Et(V^1_.)$. Similarly one may show that

(5.5.2.1)' ${}^{et}\pi^* IC_p(\mathfrak{M};\, Q_l) \simeq$ the complex ${}_{<\bar{p}(U_{n+}-U_n)} R\ {}^{et}\bar{j}^n_* \sigma_{<\bar{p}(U_n-U_{n-1})} \cdots\ \sigma_{<\bar{p}(U_1)} R\ {}^{et}\bar{j}^1_* \underline{Q}_\ell$,

where $\underline{Q}_\ell$ is the constant sheaf on $Et(V^1)$.

(5.5.2.2) Next we will show that the above complex is quasi-isomorphic to the intersection cohomology complex $IC_p(\mathfrak{S},\, \underline{Q}_\ell)$. We will achieve this by showing that the complex ${}^{et}\pi^* IC_p(\mathfrak{M};\, Q_l)$ satisfies the two axioms:
(5.5.2.3) (i) dim support $(\mathcal{H}^i(\ {}^{et}\pi^* IC_p(\mathfrak{M};\, Q_l))) \leq n - p(n+i)$
(5.5.2.3)(ii) dim support$(\mathcal{H}^i D(\ {}^{et}\pi^* IC_p(\mathfrak{M};\, Q_l))) \leq n - (p^*)^{-1}(n+i)$

characterizing the complex $IC_p(\mathfrak{S}, Q_l)$ where $D(\ {}^{et}\pi^* IC_p(\mathfrak{M};\, Q_l))$ denotes the $Verdier-dual$. The first axiom is readily verified since π is finite. In order to verify the second axiom one may proceed as follows.

Let the functor $R^{et}\pi.^! : {D_c}^b(Et(\mathfrak{M}); Q_l) \to {D_c}^b(Et(B\mathfrak{S}.); Q_l)$ be right-adjoint to the functor $R\ {}^{et}\pi._* :$ $D^b_c(Et(B\mathfrak{S}.); Q_l) \to D^b_c(Et(\mathfrak{M}); Q_l)$; observe that $R\ {}^{et}\pi.^! = \{R\ {}^{et}\pi^!_n|n\}$. Similarly define the functor $R\ {}^{et}\pi^! : {D_c}^b(Et(\mathfrak{M}); Q_l) \to {D_c}^b(Et(\mathfrak{S}); Q_l)$ to be right adjoint to the obvious functor $R\ {}^{et}\pi_*$. Let $D_{\mathfrak{M}}$ denote the dualising complex on $Et(\mathfrak{M})$, and $D_{\mathfrak{S}}$ denote the dualising complex on $\mathfrak{S}_{et}$ (whose existence was shown in (3.7.1)-(3.7.4)); one may now verify that $R\ {}^{et}\pi^!(D_{\mathfrak{M}}) \simeq D_{\mathfrak{S}}$. Now let $D_{B\mathfrak{S}.} = \phi(R\ {}^{et}\pi.^!(D_{\mathfrak{M}}))$, where ϕ is the functor in (A.6.3). We will presently show

(5.5.2.4) $R\pi._*(D_{B\mathfrak{S}.}) \simeq \phi(D_{\mathfrak{M}})$

(5.5.2.5) Observe that $R\pi._* D_{B\mathfrak{S}.} = R\pi._*\phi(\underline{Rhom}(Q_l,\, R\ {}^{et}\pi.^! D_{\mathfrak{M}}) \simeq \phi(R\ {}^{et}\pi._*\underline{Rhom}(Q_l,\, R\ {}^{et}\pi.^!(D_{\mathfrak{M}}))$
$\simeq \phi(\underline{Rhom}(R\ {}^{et}\pi._*(Q_l),\, (D_{\mathfrak{M}})) \simeq \underline{Rhom}(\phi(R\ {}^{et}\pi._*(Q_l),\, \phi(D_{\mathfrak{M}}))$

Here the first isomorphism (denoted $\simeq$) is by (A.6.4)(ii), while the second one follows from the fact that $R\ {}^{et}\pi.^!$ is right-adjoint to $R\ {}^{et}\pi._*$. The last isomorphism follows from (A.6.4)(vii). Finally observe that the geometric fiber of the map $\pi. : B\mathfrak{S}. \longrightarrow \mathfrak{M}$ over any fixed geometric point $\bar{x}$ in $\mathfrak{M}$ may be observed to be the classifying space for the finite group that stabilizes $\bar{x}$. Hence one may compute (see (A.8)) $R^i(\pi.)_*(\phi(Q_l)) = 0$ if $i > 0$ and $= Q_l$ if $i = 0$. Now $\phi(R\ {}^{et}\pi._*(Q_l)) \simeq R\pi._*(\phi(Q_l))$ Therefore one may now show that the last term in (5.5.2.5) is quasi-isomorphic to $\underline{Rhom}(\phi(Q_l), \phi(D_{\mathfrak{M}})) \simeq \phi(D_{\mathfrak{M}})$.

Next we will show that

(5.5.2.6) $\ ^{et}\pi_* D_{\mathfrak{S}} \simeq D_{\mathfrak{M}}$

where $\ ^{et}\pi_* = R\ ^{et}\pi_*$ (recall π is finite) : $D_c{}^b(\mathfrak{S}_{et}; Q_l) \rightarrow D_c{}^b(Et(\mathfrak{M}); Q_l)$ is the derived functor of the direct-image functor of π. Now observe that

$$\phi(D_{\mathfrak{M}}) \simeq R\pi_{.*} D_{B\mathfrak{S}.} = \pi_* R\epsilon_{.*}(\phi(R\ ^{et}\pi_.^!(D_{\mathfrak{M}}))) \simeq \pi_*\phi(R\ ^{et}\epsilon_{.*}(R\ ^{et}\pi_.^!(D_{\mathfrak{M}})))$$
$$\simeq \pi_*(\phi(R\ ^{et}\epsilon_{.*}\ ^{et}\epsilon_.^*(R\ ^{et}\pi^!(D_{\mathfrak{M}}))))$$

Here the first isomorphism is by (5.5.2.5) and the observation that $\pi_. = \pi \circ \epsilon_.$, while the second one follows from (A.6.4)(ii). . The third follows from the assumption that each of the maps ϵ_n is *étale*. Next observe that, since $\mathfrak{M}$ is a scheme, all the geometric points of the associated constant simplicial scheme as in (A.4.2) are merely the usual geometric points of the scheme. Therefore the system of simplicial étale neighborhoods $U.$ of any such geometric point $\bar{x}$ as in (A.4.3) with $U.$ the constant simplicial schemes associated to an étale neighborhood of $\bar{x}$ of U in the usual sense are *cofinal* in the system of all simplicial étale neighborhoods of $\bar{x}$ as in (A.4.3). Therefore the stalks of the last term above may be readily shown to be isomorphic to the stalks of $\ ^{et}\pi_*(\bar{x}_*\bar{x}^*(R\ ^{et}\pi^!(D_{\mathfrak{M}}))$ where the functors $\bar{x}^*$ and $\bar{x}_*$ are defined as in (2.5.4). Therefore we obtain the quasi-isomorphism:

$$\ ^{et}\pi_*\phi(R\ ^{et}\epsilon_{.*}\ ^{et}\epsilon_.^*(R\ ^{et}\pi^!(D_{\mathfrak{M}}))) \simeq\ ^{et}\pi_*\phi(\bar{x}_*\bar{x}^*(R\ ^{et}\pi^!(D_{\mathfrak{M}}))).$$

Since $R\ ^{et}\pi^!(D_{\mathfrak{M}})$ is a complex of l-adic sheaves on $Et(B\mathfrak{S}.)$ with *descent* the last term is readily seen to be quasi-isomorphic to $\ ^{et}\pi_*(\phi(R\ ^{et}\pi^!(D_{\mathfrak{M}}))) \simeq \phi(\ ^{et}\pi_* D_{\mathfrak{S}})$ by (2.5.4) and the observation that $R\ ^{et}\pi^! D_{\mathfrak{M}} \simeq D_{\mathfrak{S}}$.

Now one may readily observe that

$$\ ^{et}\pi_* D(\ ^{et}\pi^* IC_p(\mathfrak{M}; Q_l)) = \ ^{et}\pi_* \underline{Rhom}(\ ^{et}\pi^* IC_p(\mathfrak{M}; Q_l);\ D_{\mathfrak{S}})$$
$$\simeq \underline{Rhom}(IC_p(\mathfrak{M}; Q_l);\ \ ^{et}\pi_* D_{\mathfrak{S}}) \simeq \underline{Rhom}(IC_p(\mathfrak{M}; Q_l);\ D_{\mathfrak{M}})$$

Here the first isomorphism follows from the fact that $R\ ^{et}\pi_* = \ ^{et}\pi_*$ is right-adjoint to $\ ^{et}\pi^*$, while the last isomorphism follows from (5.5.2.6). Since $dim(\mathfrak{M}) = n$, clearly $D(IC_p(\mathfrak{M}; Q_l)) = \underline{Rhom}(IC_p(\mathfrak{M}; Q_l);$ $D_{\mathfrak{M}})$ satisfies the axiom:

dim support $\mathcal{H}^i(D(IC_p(\mathfrak{M}; Q_l))) \leq n - (p^*)^{-1}(n+i)$

Since π is finite, it follows that $D(\ ^{et}\pi^*(IC_p(\mathfrak{M};\ Q_l)))$ satisfies the second axiom in (5.5.2.3). This completes the proof of (5.5.2.2).

Next we will show using ascending induction on k that if $\bar{\pi}^k_{.} : V^k_{.} \rightarrow \mathfrak{M}$ is the map induced by $\pi.$, then

(5.5.2.7) $R\pi.^k_*(\pi^{k*}_{.}(\phi(IC_p(\mathfrak{M};\, Q_l)_{|V^k_{.}}))) \simeq \phi(IC(\mathfrak{M};\, Q_l)_{|U^k})$,

where $R\pi^k{}_{.*}$ is the derived functor of the direct-image-functor defined above using the alternate étale site in (A.4.0). Now observe first that for each k, $\pi^*_k(IC_p(\mathfrak{M}; Q_l)_{|U_k})$ satisfies the axioms characterizing the complex $IC_p(\mathfrak{S}; Q_l)_{|U_k}$ as shown above and hence is quasi-isomorphic to the latter. Since the geometric fibers (as in A. 8)) of the map $\pi. : B\mathfrak{S}. \rightarrow \mathfrak{M}$ are the classifying simplicial schemes of the finite groups that are the stabilizers of the corresponding geometric points, proper-base-change (see (A.8)) shows that $R^i\pi^k{}_{.*} = 0$ for $i > 0$ and hence $\pi^k{}_{.*}$ is an exact functor for each k. Moreover if $\underline{Q}_\ell$ is the constant sheaf on $Et(V^1_{.})$, it is clear that $\pi^1{}_{.*}(\phi\underline{Q}_\ell) \simeq (\phi\underline{Q}_\ell)$ the constant sheaf on $SEt(U_1)$. Therefore let $m \geq 1$, be an integer for which (5.5.2.7) is true; we will show (5.5.2.7) is also true for $m+1$. To see this observe that

$$R\pi^{m+1}{}_{.*}(\pi^{m+1}{}_{.}{}^*\phi(IC_p(\mathfrak{M};\, Q_l))_{|V^{m+1}_{.}}) \simeq \pi^{m+1}{}_{.*}(\pi^{m+1}_{.})^*(\phi(IC_p(\mathfrak{M};\, Q_l))_{|V^{m+1}_{.}})$$
$$\simeq \pi^{m+1}{}_{.*}\phi(\sigma_{<\bar{p}(U_{m+1}-U_m)}R\ {}^{et}\bar{j}^m_{.*}\sigma_{<\bar{p}(U_m-U_{m-1})}\cdots\ \sigma_{<\bar{p}(U_1)}R\ {}^{et}\bar{j}^1_{.*}\underline{Q}_\ell)$$
$$\simeq \pi^{m+1}{}_{.*}(\sigma_{<\bar{p}(U_{m+1}-U_m)}\phi(R\ {}^{et}\bar{j}^m_{.*}\sigma_{<\bar{p}(U_m-U_{m-1})}\cdots\ \sigma_{<\bar{p}(U_1)}R\ {}^{et}\bar{j}^1{}_{.*}\underline{Q}_\ell))$$
$$\simeq (\sigma_{<\bar{p}(U_{m+1}-U_m)}\pi^{m+1}_{.*}\phi(R\ {}^{et}\bar{j}^m_{.*}\sigma_{<\bar{p}(U_m-U_{m-1})}\cdots\ \sigma_{<\bar{p}(U_1)}R\ {}^{et}\bar{j}^1_{.*}\underline{Q}_\ell))$$
$$\simeq (\sigma_{<\bar{p}(U_{m+1}-U_m)}\pi^{m+1}_{.*}(R\bar{j}^m_{.*}\phi(\sigma_{<\bar{p}(U_m-U_{m-1})}\cdots\ \sigma_{<\bar{p}(U_1)}R\ {}^{et}\bar{j}^1_{.*}\underline{Q}_\ell))$$
$$\simeq \sigma_{<\bar{p}(U_{m+1}-U_m)}Rj^m_*\pi^m_{.*}\phi(\sigma_{<\bar{p}(U_m-U_{m-1})}\cdots\ \sigma_{<\bar{p}(U_1)}R\ {}^{et}\bar{j}^1_{.*}\underline{Q}_\ell))$$
$$\simeq \sigma_{<\bar{p}(U_{m+1}-U_m)}Rj^m_*\phi(IC(\mathfrak{M};\, Q_\ell)_{|U_m})$$
$$\simeq \phi(IC(\mathfrak{M};\, Q_\ell)_{|U_{m+1}}).$$

(The fourth $\simeq$ follows from the fact that $\pi^{m+1}_{.*}$ is an exact functor as observed above; the last-but-one $\simeq$ follows from the inductive assumption. The other isomorphisms are clear.). Observe that the use of the alternate étale site and the functor $\pi_{.*}$ as in (A.4.0) is essential to the above proof.

(5.5.3) **Theorem**. Assume the situation of (5.5.0). Now the spectral sequence

$E_1^{p,q} = IH^q((B\mathfrak{S}_p);\, Q_\ell) \Rightarrow IH^{p+q}(\mathfrak{S};\, Q_\ell)$

degenerates i.e. $E_2^{p,q} = 0$ for $p > 0$. Hence $H^0(\{IH^q(B\mathfrak{S}_p;\, (Q_\ell))|p\}) \cong IH^q(\mathfrak{S};\, Q_\ell) \cong IH^q(\mathfrak{M};\, Q_\ell)$.

Proof. The proof parallels the proof of Theorem (5.1) in [Gil]; recall Gillet's theorem is for K'-theory which in this case already has a transfer map, namely the one induced by the direct image functor. We will employ the trace-map associated to étale maps; observe that the trace-map behaves like a transfer-map. Let $\pi : \mathfrak{S} \rightarrow \bar{\mathfrak{S}}$ denote a representable étale-surjective map of algebraic stacks as in (1.6). Now one may readily define a functor

$$R\pi^! : D^b_c(\bar{\mathfrak{S}}_{et};\, Q_l) \to D^b_c(\mathfrak{S}_{et};\, Q_l)$$

as right-adjoint to the functor $R\pi_*$. Now one may verify that $R\pi^! \simeq \pi^*$ since π is étale; this is a local assertion, local on $\bar{\mathfrak{S}}$ and therefore one may readily verify this. Therefore one obtains the *trace-map*

$$(5.5.3.1)\ tr(\pi) : \pi_*\pi^*\underline{Q}_\ell \to \underline{Q}_\ell$$

so that the composition $\underline{Q}_\ell \to \pi_*\pi^*\underline{Q}_\ell \to \underline{Q}_\ell$ is an *isomorphism* and $tr(\pi)$ is natural with *respect* to base-change i.e. if

$$\begin{array}{ccc} V' & \xrightarrow{g} & V \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ U' & \xrightarrow{f} & U \end{array}$$

is a pull-back square (of representable maps between algebraic stacks) with π étale surjective, then the square

$$\begin{array}{ccccc} f^*\pi_* \circ \pi^*\underline{Q}_l & \longrightarrow & \pi'_*\pi'^*(f^*\underline{Q}_l) & \longrightarrow & \pi'_*\pi'^*f^*\underline{Q}_l \\ {\scriptstyle f^*(tr(\pi))}\downarrow & & \downarrow & & {\scriptstyle tr(\pi')}\downarrow \\ Q_{l|U'} = f^*\underline{Q}_l & \xrightarrow{identity} & f^*\underline{Q}_l & \xrightarrow{identity} & f^*\underline{Q}_l \end{array}$$

commutes, where $\underline{Q}_l$ denotes the constant $l-adic$ sheaf on the respective space.

Now let $\pi : X \to Y$ denote a representable étale surjective map of possibly singular algebraic stacks as in (1.6); let $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq U_{n+1} = Y$ be the filtration of Y associated to some stratification as in (3.6.1) and let $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq V_{n+1} = X$ denote the induced filtration of X. Let p denote a fixed perversity and let $IC_p(X;\, \underline{Q}_\ell)$, $IC_p(Y;\, \underline{Q}_\ell)$ denote the corresponding intersection cohomology complexes on X and Y. One may now use ascending induction on k and smooth-base change to obtain the quasi-isomorphism: $\pi^*IC_p(Y;\, Q_l)_{|U_k} \simeq IC_p(X;\, Q_l)_{|V_k}$ for all k. Next one may similarly use ascending induction on k and smooth base-change to define a transfer (see $[J-5]$ for a more general case when π is assumed to be proper and smooth instead of étale as in the present situation)

$$tr(\pi) : \pi_*IC_p(X;\, \underline{Q}_\ell) \to IC_p(Y : \underline{Q}_\ell)$$

so that (i) the composition $IC_p(Y;\underline{Q}_\ell) \to \pi_*IC_p(X;\, \underline{Q}_\ell) \to IC_p(Y;\underline{Q}_\ell)$ is an isomorphism (where the first map $IC_p(Y;\, Q_l) \to \pi_*IC_p(X;\, Q_l)$ is adjoint to the natural quasi-isomorphism $\pi^*IC_p(Y;\, Q_l) \xrightarrow{\simeq} IC_p(X;\, Q_l)$) and

(ii) the transfer $tr(\pi)$ is natural with respect to base change i.e. if

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a pull-back square with π étale surjective and f smooth, then the square

$$\begin{array}{ccc} (f^*\pi_*(IC_p(X;\underline{Q_l})))_{|U'_k} \simeq (\pi'_* g^*(IC_p(X;\underline{Q_l}))) & \longrightarrow & (\pi'_*(IC_p(X';\underline{Q_l}))_{|U'_k}) \\ {\scriptstyle f^*(tr(\pi))_{|U_k}}\downarrow & & {\scriptstyle (tr(\pi')_{|U'_k})}\downarrow \\ (f^*IC_p(Y;\underline{Q_l}))_{|U'_k} & \longrightarrow & (IC_p(Y';\underline{Q_l}))_{|U'_k} \end{array}$$

commutes for all k, where $U'_k = U_k \underset{Y}{\times} Y'$. One proves this using ascending induction k and proper-smooth base change-recall π is étale. Therefore the square

$$(5.5.3.2)\quad \begin{array}{ccc} IH^n_p(X';Q_l) & \xleftarrow{g^*} & IH^n_p(X;Q_l) \\ {\scriptstyle tr(\pi)}\uparrow & & {\scriptstyle tr(\pi)}\uparrow \\ IH^n_p(Y';Q_l) & \xleftarrow{f^*} & IH^n_p(Y;Q_l) \end{array} \quad \text{commutes.}$$

Next assume the hypotheses of the theorem. We will first observe that $\{IH^n_p(B\mathfrak{S}_k;Q_l)|k\}$ is a cosimplicial abelian group. The face maps d^i of the simplicial object $B\mathfrak{S}.$ are all étale; therefore one may readily verify that if $IC_p(B\mathfrak{S}_k;Q_l)$ $(IC_p(B\mathfrak{S}_{k-1};Q_l))$ is the intersection cohomology complex on $B\mathfrak{S}_k$ ($B\mathfrak{S}_{k_1}$, respectively) then $d_i^*(IC_p(B\mathfrak{S}_{k-1};Q_l) \simeq IC_p(B\mathfrak{S}_k;Q_l)$ for any face-map $d_i : B\mathfrak{S}_k \to B\mathfrak{S}_{k-1}$. Since $d_i \circ s_i =$ the identity, it follows that $s_i^*(IC_p(B\mathfrak{S}_k;Q_l)) \simeq IC_p(B\mathfrak{S}_{k-1};Q_l)$ as well. It follows that

$\{IH^n_p(B\mathfrak{S}_k;Q_l)|k\}$ is a cosimplicial abelian group.

for each fixed integer $n \geq 0$. Let $n \geq 0$ be a fixed integer. We will next define a sequence of maps

$$\pi_k : IH^n_p(B\mathfrak{S}_k;\, \underline{Q}_\ell) \to H^0(\{IH^n_p(B\mathfrak{S}_k;\, \underline{Q}_\ell)|k \geq 0\}) \to IH^n_p(B\mathfrak{S}_k;\, \underline{Q}_\ell),\ k \geq 0$$

as follows. First observe that each face map $d_i : B\mathfrak{S}_{k+1} \to B\mathfrak{S}_k$ of the simplicial scheme $B\mathfrak{S}.$ is *étale surjective*; therefore d_i induces a map $d_i^* : IH^n_p(B\mathfrak{S}_k;\, \underline{Q}_\ell) \to IH^n_p(B\mathfrak{S}_{k+1};\, \underline{Q}_\ell)$ and a transfer $tr(d_i) : IH^n_p(B\mathfrak{S}_{k+1};\, \underline{Q}_\ell) \to IH^n_p(B\mathfrak{S}_k;\, \underline{Q}_\ell)$.

Now let $\pi_0 = tr(d_1) \circ d_0^*$; $\pi_k = [d_{k+1}^* \circ d_k^* \circ ... \circ d_1^* \circ (tr(d_1) \circ d_0^*) \circ tr(d_0)^k]$. Now the pull-back squares

$$\begin{array}{ccc} B_k\mathfrak{S} & \xrightarrow{d_{i-1}} & B_{k-1}\mathfrak{S} \\ {\scriptstyle d_i}\downarrow & & {\scriptstyle d_{i-1}}\downarrow \\ B_{k-1}\mathfrak{S} & \xrightarrow{d_{i-1}} & B_{k-2}\mathfrak{S} \end{array}$$

(where $B\mathfrak{S}_{-1} = \mathfrak{S}$) and (5.5.3.2) above show

(5.5.3.4) $d_{i-1}^* \circ tr(d_{i-1}) = tr(d_i) \circ d_{i-1}^*$.

Now $d_1^* \circ tr(d_1) \circ d_0^* = d_1^* \circ d_0^* \circ tr(d_0) = d_0^* \circ d_0^* \circ tr(d_0) = d_0^* \circ tr(d_1) \circ d_0^*$; it follows that the maps π_k map into $H^0(\{IH_p^n(B\mathfrak{S}_k;\, \underline{Q}_\ell)|k \geq 0\}) = ker(d_0^* - d_1^* : IH_p^n(B\mathfrak{S}_0; Q_l) \rightarrow IH_p^n(B\mathfrak{S}_1; Q_l))$. Moreover one may readily verify that $\pi. = \{\pi_k|k\}$ is a *map of cosimplicial abelian groups.*

We will now define explicitly a cosimplicial homotopy between $\pi.$ and the identity map of the cosimplicial abelian group $\{IH_p^n(B\mathfrak{S}_k; Q_l)|k\}$. Let $h_1 = (tr(d_k) \circ d_{k-1}^*) \circ (tr(d_{k-1}) \circ d_{k-2}^*) \circ ... \circ tr(d_1)$, $h_i = (tr(d_k) \circ d_{k-1}^*) \circ (tr(d_{k-1}) \circ d_{k-2}^*) \circ ... \circ tr(d_i)$ for $1 < i < k$, and $h_k = tr(d_k)$. By making repeated use of the formulae: $d_{i-1}^* \circ tr(d_{i-1}) = tr(d_i) \circ d_{i-1}^*$, one may now verify that $h_1 \circ d_0^* = \pi_{k-1}$, $h_i \circ d_i^* = h_{i+1} \circ d_i^*$ and $h_k \circ d_k^* =$ the identity. It follows that the the maps $\{h_i|i\}$ provides the required homotopy between the identity map of the simplicial abelian group $\{IH_p^n(B\mathfrak{S}_k; \underline{Q}_\ell)|k \geq 0\}$ and the map $\pi. : \{IH_p^n(B\mathfrak{S}_k;\, \underline{Q}_\ell)|k \geq 0\} \rightarrow H^0(\{IH_p^n(B\mathfrak{S}_k;\, \underline{Q}_\ell)|k \geq 0\}) \rightarrow \{IH_p^n(B\mathfrak{S}_k;\, \underline{Q}_\ell)|k \geq 0\}$. Hence

$E_2^{r,s} \cong 0$ for $r > 0$ and $\cong H^0(\{IH_p^n(B\mathfrak{S}_k;\, \underline{Q}_\ell)|k \geq 0\})$ in the spectral sequence in the statement of the theorem. This proves the theorem.

6. The equivariant derived category and equivariant perverse sheaves

(6.0) Let k denote a field as in (1.6) and let X (G) denote a algebraic space (smooth group-scheme, respectively) of finite type over k; assume G acts on X. (1.5)(ii) shows that in this case one obtains an algebraic stack X/G in the sense of Artin; therefore this situation is merely a special case of the general case dealt with in the previous sections. Now observe that the simplicial groupoid constructed in (2.2.0) is now merely the simplicial algebraic space $EG\underset{G}{\times}X$ given by the familiar bar-construction as in [Fr] p.9. Stratifications of the algebraic stack X/G as in (3.6.1) now correspond to *G-invariant stratifications of* X. A sheaf with *descent* on $Et(EG\underset{G}{\times}X)$ will from now on be referred to as *an equivariant sheaf*, or a G-equivariant sheaf. We will now summarize the results of the earlier sections as applied to this particular case.

(6.0.1) Let $D^G(X; Q_l)$ $(D^G(X; \bar{Q}_l))$ denote the derived category $D_c^{b,des}(EG\underset{G}{\times}X; Q_l)$ $(D_c^{b,des}(EG\underset{G}{\times}X; \bar{Q}_l)$, respectively) as in (3.3). If p is a fixed perversity, one defines non-standard t-structures on the above categories by starting with a G-invariant stratification of X as in (3.6.3). We will assume a fixed perversity p for the time being. The heart of the above t-structure will be denoted

$C^G(X; Q_l)$ ($C^G(X; \bar{Q}_l)$, respectively) and called the category of *equivariant perverse sheaves.* Now we restate the results of section 4 as applied to equivariant perverse sheaves.

(6.0.2). There exists an equivalence of categories:

$C^G(X; \bar{Q}_l) \simeq$ (perverse sheaves on $Et(X)$ with descent)

where the category on the right is defined as in (4.2).

(6.0.3) $C^G(X : \bar{Q}_l)$ is both Artinian an Noetherian; every object in this category has finite length.

(6.0.4) Let X, Y be algebraic spaces as in (6.0) acted on by a smooth group-scheme G again as in (6.0) and let $f : X \to Y$ denote a G-equivariant *smooth* map with *connected nonempty geometric fibers.* Assume the relative dimension of f is d. Now the functor $f^*[d] : C^G(Y; \bar{Q}_l) \to C^G(X; \bar{Q}_l)$ is *fully-faithful* and identifies $C^G(Y; \bar{Q}_l)$ with a *thick* subcategory of $C^G(X; \bar{Q}_l)$.

(6.0.5) Assume in addition to the hypotheses of (6.0) that G is connected and non-empty. Then if $K^\cdot$ belongs to $C^G(X : \bar{Q}_l)$, every sub-quotient of $K^\cdot$ in the category $C(X; \bar{Q}_l)$ of all perverse-sheaves also belongs to $C^G(X; \bar{Q}_l)$

(6.0.6) Assume the hypotheses of (6.0). Now the simple objects in the category $C^G(X; \bar{Q}_l)$ are of the form $\epsilon^*(j_{!*}L[dimV])$, where $j : V \to X$ is the inclusion of a locally-closed G-invariant sub-algebraic space of X, L is a lisse, $\bar{Q}_l$-sheaf on $Et(V)$ corresponding to an irreducible representation of $\hat{\pi_1(V_{et})}$. Here $\epsilon : EG\underset{G}{\times}X \to X$ is the obvious map, which in degree n is $(d_0)^n$.

(6.0.7) Assume the situation of (6.0). Let $p : EG\underset{G}{\times}X \to BG$ denote the obvious map of simplicial algebraic spaces. (Here BG is the 'classifying simplicial algebraic space for G' defined as in [Fr] p.8.) If $K^\cdot \epsilon\ D^G(X;\ \bar{Q}_l)$ one obtains the pairing $p^*(\bar{Q}_l) \otimes K^\cdot \to K^\cdot$ of complexes of sheaves on $Et(EG\underset{G}{\times}X)$ which induces a pairing of the corresponding hypercohomology spectral sequences in (3.4.6.*) (with $K^\cdot$ = the constant sheaf $\bar{Q}_l$) compatible with the pairing

$H^*(BG;\ Q_l) \otimes H^*(EG\underset{G}{\times}X\ ;\ K^\cdot) \to H^*(EG\underset{G}{\times}X\ ;\ K^\cdot)$.

(6.0.8) Let $M^\cdot$, $N^\cdot \epsilon\ D^G(X;\ \bar{Q}_l)$ ($D^G(X;\ \bar{Q}_l)$) and let $K^\cdot = \underline{Rhom}(M^\cdot, N^\cdot)$ denote the derived functor of the internal hom. Observe that now $H^*(EG\underset{G}{\times}X\ ;\ K^\cdot) \simeq \underset{n \geq 0}{\oplus}$ Hom $(M^\cdot,\ N^\cdot[n])$, where Hom denotes hom in the derived category. It follows that $\underset{n \geq p}{\oplus}$ Hom $(M^\cdot,\ N^\cdot[n])$ is a module over $H^*(BG;\ \bar{Q}_l)$.

(6.0.9). Recall that the stratifications of the stack X/G now correspond to G-invariant stratifications of X. If $\underline{S}$ is such a G-invariant stratification of X, and p is a fixed perversity, the complex $IC^S_p(Q_l)$

now corresponds to the *equivariant intersection cohomology complex* defined in [J-2]. (We will therefore denote the complex $IC_p^S(Q_l)$ on the stack X/G by $IC_{p,G}(Q_l)$ in conformity with [J-2].) Therefore

$$IH_p^*(X/G;\, Q_l) \simeq IH_{G,p}^*(X;\, Q_l)$$

where the right hand side is the *equivariant intersection cohomology groups of* X again as in [J-2].

(6.0.10) *Pairings and Poincare-Verdier duality.* Assume the above situation. Now (5.3.8) shows that one obtains pairings:

$$IC_{p,G}(Q_l) \otimes IC_{p^*,G}(Q_l) \to D_{EG \underset{G}{\times} X}$$

so that $D(IC_{p,G}(Q_l)) \simeq IC_{p^*}(Q_l)$. Taking hypercohomology of $EG\underset{G}{\times}X$ this provides the *Poincare duality isomorphism*:

$$IH^i_{p,G}(X; Q_l) \cong Ext^{-i}(IC_{p,G}(Q_l), D_{EG\underset{G}{\times}X}).$$

(6.1) Throughout the rest of this section we assume $\bar{k}$ is an algebraically closed field of arbitrary characteristic $p \geq 0$. Next assume in addition to the hypotheses of (6.0) that G is a connected algebraic group acting (on the left) on an algebraic space X of finite type over $\bar{k}$. In this context we proceed to establish *induction and restriction* functors with respect to connected algebraic subgroups H of G. Such functors have many applications for eg. to Lusztig's character sheaves- see [Lusy]. The results established here also find application in [J- 3]. Now we make the following observations.

(6.2.1) Let $i : H \to G$ denote the closed immersion of a connected subgroup of G and let $\bar{i} : EH\underset{H}{\times}X \to EG\underset{G}{\times}X$ denote the induced map.

(6.2.2). Let H act on $G \times X$ by $h.(g,\, x) = (g.h^{-1},\, \text{hx})$, $h \in H$, $g \in G$ and $x \in X$. Then a geometric quotient $G\underset{H}{\times}X$ exists for this action and the map $s : G \times X \to G\underset{H}{\times}X$ is smooth with fibers isomorphic to H.

(6.2.3). Now G has an action on $G \times X$ induced from its action by translation on the first factor of $G \times X$; this induces a G-action on $G\underset{H}{\times}X$ as well. One verifies that the map s is equivariant for these actions of G.

(6.2.4). Let $p : G\underset{H}{\times}X \to X$ denote the map induced by the map $G \times X \to X$ which is defined by $(g,\, x) \to (g.x)$. One verifies that p is G-equivariant for the G-action on $G\underset{H}{\times}X$ as in (6.2.3) and the G-action on X. It follows that p defines a map $\bar{p} : EG\underset{G}{\times}(G\underset{H}{\times}X) \to EG\underset{G}{\times}X$.

(6.2.5) Let $r : G \times X \to G\underset{G}{\times}X = X$ denote the projection to the second factor.

(6.2.6). Next let $G \times H$ act on $G \times X$ by $(g_1, h_1).(g, x) = (g_1 g h_1^{-1}, h_1 x)$, $g_1, g \in G$, $h_1 \in H$ and $x \in X$. We observe that the maps r and s are such that we obtain the commutative squares:

$$\begin{array}{ccc} (G \times H) \times (G \times X) & \longrightarrow & G \times X \\ {\scriptstyle pr_1 \times s}\downarrow & & {\scriptstyle s}\downarrow \\ G \times (G\underset{H}{\times}X) & \longrightarrow & G\underset{H}{\times}X \end{array}$$

and
$$\begin{array}{ccc} (G \times H) \times (G \times X) & \longrightarrow & G \times X \\ {\scriptstyle pr_2 \times r}\downarrow & & {\scriptstyle r}\downarrow \\ H \times X & \longrightarrow & X \end{array}$$

It follows that r and s induce maps $\bar{r} : E(G \times H) \underset{G\times H}{\times} (G \times X) \to EH\underset{H}{\times}X$ and $\bar{s} : E(G \times H) \underset{G\times H}{\times} (G \times X) \to EG\underset{G}{\times}(G\underset{H}{\times}X)$.

(6.2.7). Let $\Delta : H \to G \times H$ denote the diagonal and let $j : X \to G \times X$ denote the map $x \to (e, x)$ where e is the identity element of G. We now observe that the square

$$\begin{array}{ccc} H \times X & \longrightarrow & X \\ {\scriptstyle \Delta \times j}\downarrow & & {\scriptstyle j}\downarrow \\ (G \times H) \times (G \times X) & \longrightarrow & G \times X \end{array}$$

commutes. It follows that j and Δ induce a map $\bar{j} : EH\underset{H}{\times}X \to E(G \times H)\underset{G}{\times}(G \times X)$; one checks readily that $\bar{r} \circ \bar{j} =$ the identity; also $\bar{p} \circ \bar{s} \circ \bar{j} = \bar{i}$.

(6.3) **Definition**. Let $D^G(X)(D^H(X))$ denote the derived category of bounded complexes of Q_l-sheaves with constructible cohomology sheaves having descent on $Et(EG\underset{G}{\times}X)$ ($Et(EH\underset{H}{\times}X)$, respectively) as in (6.0.1). We define the restriction functor $Res_H^G : D^G(X) \to D^H(X)$ to be $\bar{i}^*$.

(6.3') Suppose in addition X is also smooth; now observe that $R\bar{i}^! = \bar{i}^*[2(dim(G) - dim(H))](d)$ since, for each n, $\bar{i}_n : (EH\underset{H}{\times}X)_n \to (EG\underset{G}{\times}X)_n$ is a regular immersion of relative dimension $= ((dim(G) - (dim(H))))$. Therefore, in this case we may use $R\bar{i}^![-2(dim(G) - dim(H))](-d)$ as a restriction functor.

(6.4) **Theorem**. Assume in addition to the hypotheses of (6.1) that G, H are connected and let m be the middle perversity. Then we obtain the equivalences of categories:

$$D^H(X) \xrightarrow{\bar{r}^*} D^{G\times H}(G \times X) \text{ and } D^G(G\underset{H}{\times}X) \xrightarrow{\bar{s}^*} D^{G\times H}(G \times X)$$

Proof. Observe first that each r_n and s_n have connected geometric fibers, each being isomorphic to a finite product of H with itself. Therefore observe that the geometric fibers of $\bar{r}$ and $\bar{s}$ are isomorphic to the simplicial algebraic space EG and EH respectively; hence they have trivial cohomology with respect to any locally constant abelian sheaf with torsion prime to p. Since $(EH)_0 = H$ any constructible H-equivariant abelian sheaf on Et(EH) is locally constant as in (A.2.2); if $F(K)$ is a constructible H-equivariant abelian sheaf on $Et(EH\underset{H}{\times}X)(Et(EG\underset{G}{\times}(G\underset{H}{\times}X)))$, $\bar{r}^*(F)(\bar{s}^*(K)$, respectively) is an H-equivariant constructible sheaf on $Et(E(G \times H) \underset{G\times H}{\times} (G \times X))$. It follows that that the cohomology sheaves of $\bar{r}^*(F)$ $(\bar{s}^*(K))$ are locally constant on the geometric fibers of $\bar{r}(\bar{s}$, respectively); recall these geometric fibers were observed to be $\cong EH$. Therefore, the geometric fibers of $\bar{r}$ $(\bar{s})$ are acyclic with respect to $\bar{r}^*(F)(\bar{s}^*(K)$, respectively).

Hence $\bar{r}^*(F)$ $(\bar{s}^*(K))$ satisfies the conditions in (A.10) and hence the conditions (A.9). Hence the natural map

$F = \{F_n|n\} \rightarrow \{Rr_{n*}r_n^*F_n|n\}, \;\; (K = \{K_n|n\} \rightarrow \{Rs_{n*}s_n^*K_n|n\})$

induces an isomorphism

(6.4.1.1) $H^t(EH\underset{H}{\times}X;\, F) \simeq H^t(E(G \times H) \underset{G\times H}{\times} (G \times X);\, \bar{r}^*F)$

((6.4.1.2) $(H^t(EG\underset{G}{\times}(G\underset{H}{\times}X);\, K) \simeq H^t(E(G \times H) \underset{G\times H}{\times} (G \times X);\, \bar{s}^*K))$, respectively .)

As these isomorphisms are natural in F and K they induce a map of the hypercohomology spectral sequences proving thereby that such an isomorphism holds for any $F^{\cdot} \epsilon\, D^H(X)(K^{\cdot} \epsilon\, D^G(G\underset{H}{\times}X)$, respectively). (Recall that the above derived categories consist of bounded complexes.)

Now we show that $\bar{r}^*$ and $\bar{s}^*$ are fully faithful. Let $M^{\cdot}$, $N^{\cdot} \epsilon\, D^H(X)$ and let $F^{\cdot} = \underline{Rhom}(M^{\cdot}, N^{\cdot})$; observe that $F^{\cdot} \epsilon\, D^H(X)$. Now the left-side of (6.4.1.1) is $Ext^t(M^{\cdot}, N^{\cdot})$ while the right-side is $Ext^t(\bar{r}^*M^{\cdot}$, $\bar{r}^*N^{\cdot})$. This proves that the functor $\bar{r}^*$ is fully faithful; the proof for $\bar{s}^*$ is similar.

Finally to prove the surjectivity of $\bar{r}^*(\bar{s}^*)$, observe that it suffices to establish that $\bar{r}^*$ $(\bar{s}^*)$ provides the equivalence:

(6.4.2.1) (H-equivariant sheaves of Q_l-modules on $Et(EH\underset{H}{\times}X)$
$\simeq$ (GxH-equivariant sheaves of Q_l-modules on $Et(E(G \times H) \underset{(G\times H)}{\times} (G \times X)))$

((6.4.2.2) (G-equivariant sheaves of Q_l-modules on $Et(EG\underset{G}{\times}(G\underset{H}{\times}X))) \simeq$
($(G \times H)$-equivariant sheaves of Q_l-modules on $Et(E(G \times H) \underset{(G\times H)}{\times} (G \times X)))$, respectively .)

One may readily show that (6.4.2.1) ((6.4.2.2)) is equivalent to

(6.4.2.1)' ($H-$equivariant sheaves of Q_l-modules on $Et(X)$) $\simeq$ (($G \times H$)$-$equivariant sheaves of Q_l-modules on $Et(G \times X)$)

((6.4.2.2)' ($G-$equivariant sheaves of Q_l-modules on $Et(G \underset{H}{\times} X)$ where G acts by left-translation on the left-factor G)
$\simeq$ (($G \times H$)$-$equivariant sheaves of Q_l-modules on $(G \times X)$), respectively .)

One verifies (6.4.2.1) as follows. First define automorphisms of $G \times G$ and $G \times G \times G$ by $(h, k) \to (h.k, k)$ and $(g, h, k) \to (g.h.k, h.k, k)$ respectively where $g, h, k \epsilon G$. These maps induce an automorphism of the schemes $(G \times X)$ and $(G \times \times G \times X)$. Under these automorphisms, now one verifies that descent data for the projection $G \times X \to X$ correspond to the conditions for a sheaf on $Et(G \times X)$ to be G-equivariant when G acts by left-translation on $G \times X$. It follows that one obtains the equivalence:

(G-equivariant sheaves of Q_l-modules on $Et(G \times X)$)
$\simeq$ (sheaves of Q_l-modules on $Et(G \times X)$ with descent data for the map $G \times X \to X$)
$\simeq$(sheaves of Q_l-modules on $Et(X)$).

where the last equivalence follows from the fact that the projection $G \times X \to X$ has a section. Now on adding the action of H, one obtains (6.4.2.1)'.

The proof of (6.4.2.2)' is similar except one has to use the isomorphism:

$H \times G \times X \overset{\cong}{\to} (G \times X) \underset{(G \underset{H}{\times} X)}{\times} (G \times X)$ given by $(h, g, x) \to ((g, x), (gh^{-1}, hx))$ and the isomorphism:

$H \times H \times G \times X \overset{\cong}{\to} (G \times X) \underset{(G \underset{H}{\times} X)}{\times} (G \times X) \underset{(G \underset{H}{\times} X)}{\times} (G \times X)$
given by $(h_1, h_2, g, x) \to ((g, x), (gh_1^{-1}, h_1 x), (gh_2^{-1}, h_2.x))$ to obtain the equivalence:

(H-equivariant sheaves of Q_l-modules on $Et(G \times X)$ for the H-action on $G \times X$ given by $h.(g, x) = (gh^{-1}, hx)$)
$\simeq$ (sheaves of Q_l-modules on $Et(G \times X)$ with descent data for the map $G \times X \to G \underset{H}{\times} X$)
$\simeq$ (sheaves of Q_l-modules on $Et(X)$)

where the last equivalence follows from the fact the map $G \times X \to G \underset{H}{\times} X$ is smooth and locally has a section. On adding the G-action as well one obtains (6.4.2.2)'. This completes the proof of (6.4).

(6.5) **Definition**. Assuming the above theorem, we will define induction functors $Ind_H^G : D^H(X) \to D^G(X)$ to be right adjoint to Res_H^G in the following manner. Let $F \epsilon D^H(X)$; theorem (6.4) shows that there exists a $K \epsilon D^G(G \underset{H}{\times} X)$ so that $\bar{r}^*(F) = \bar{s}^*(K)$. We let $Ind_H^G(F) = R\bar{p}_*(K)$, where

$\bar{p} : EG \underset{G}{\times} (G \underset{H}{\times} X) \to EG \underset{G}{\times} X$ is the map induced by $G \underset{H}{\times} X \to X$ as before.

To see Ind_H^G so defined is right adjoint to Res_H^G, we proceed as follows. Let $K \epsilon D^G(X)$, $F \epsilon D^H(X)$. Now observe that $Res_H^G(K) = \bar{i}^*(K) = \bar{j}^* \circ \bar{s}^* \circ \bar{p}^*(K)$. As $\bar{r}^*$ is an equivalence so is its section $\bar{j}^*$. Therefore there exists an $L \epsilon D^G(G \underset{H}{\times} X)$ so that $\bar{j}^* \circ \bar{s}^*(L) = F$. Therefore we obtain the isomorphisms

$$Hom_{D^H(X)}(Res_H^G(K), F) = Hom_{D^H(X)}(\bar{j}^* \circ \bar{s}^* \circ \bar{p}^*(K), \bar{j}^* \circ \bar{s}^*(L))$$
$$\cong Hom_{D^G(G \underset{H}{\times} X)}(\bar{p}^*(K), L) \cong Hom_{D^G(X)}(K, R\bar{p}_* L) = Hom_{D^G(X)}(K, Ind_H^G(F)).$$

Appendix.

(A.0) Throughout the paper we will be primarily concerned with the following situation: B is a noetherian scheme (serving as a 'base') and X. is a simplicial object in the category of algebraic spaces of finite type over B (i.e. a functor $\Delta^{op} \rightarrow$ algebraic spaces of finite type over B) .(See section 1 and [Knut] for basic properties of algebraic spaces.)

(A.1.0) *The étale site associated to a simplicial algebraic spaces.* (See [Del-1] p.10 or [Fr] p.11). Let X.denote a simplicial algebraic space. Et(X.) will denote the category whose objects are étale maps $U \rightarrow X_n$, for some $n \geq 0$ with U an algebraic space (see (1.2.2)); a morphism$(U \rightarrow X_n) \rightarrow (V \rightarrow X_m)$ is a map $U \rightarrow V$ lying over some structure map $X_n \rightarrow X_m$ of the simplicial algebraic space X. A covering of a given object $U \rightarrow X_n$ in the site is a family of étale maps $U_i \rightarrow U$ over X_n so that $\sqcup U_i \rightarrow U$ is étale surjective. We will let $Et(X_n)$ denote the corresponding étale site of each X_n. Evidently there exists a restriction functor $()_n : Et(X.) \rightarrow Et(X_n)$ for each n. Using the *smooth* topology one may similarly define the sites $Smt(X_n)$, for each $n \geq 0$ and $Smt(X.)$

(A.1.1) We next consider sheaves on the site $Et(X.)$. A sheaf F on $Et(X.)$ is a collection $\{F_n|n\}$ where F_n is a sheaf on $Et(X_n)$ so that for each structure map $\alpha : X_n \rightarrow X_m$, there exists a natural map $\alpha^* F_m \rightarrow F_n$ and these maps satisfy an obvious compatibility condition as in ([Fr] p.14.) See (2.3.2), (2.3.3) and (2.4) for more details.

(A.2.1). *A basic spectral sequence.* Let $K = \{K^{\cdot}_n|n\}$, $L = \{L^{\cdot}_n|n\}$ be two bounded complexes of abelian sheaves on $Et(X.)$. Then there exists a right-half-plane spectral sequence

$$E_1^{u,v} = Ext^v(K^{\cdot}_u, L^{\cdot}_u) \Rightarrow Ext^{u+v}(K^{\cdot}, L^{\cdot})$$

Observe that (2.3.1) shows the existence of enough injectives in the category of abelian sheaves on $Et(X.)$. Therefore the construction of a similar spectral sequence when $X.$ is a simplicial scheme as in ([Fr] pp.16-17) carries over. (Strictly speaking the construction in [Fr] pp. 16- 17 is only for the special case $K^{\cdot} =$ the constant sheaf $\mathbb{Z}_{X.}$. Observe from [Fr]pp. 16-17 that the restriction-functor $()_n$ has a left-adjoint L_n; now one may show that the natural map $Tot \underset{n}{\oplus} L_n(K^{\cdot}_n) \rightarrow K^{\cdot}$ is a quasi-isomorphism. The above spectral sequence is merely the spectral sequence associated to the double complex $Hom(\underset{n}{\oplus} L_n(K^{\cdot}_n), L^{\cdot})$ obtained by filtering the above double complex by $\{Hom(\underset{n \leq N}{\oplus} L_n(K^{\cdot}_n), L^{\cdot})|N\}$. The identification of the E_1-terms follow from the fact that L_n is left-adjoint to $()_n$.

(A.2.2) Next we define locally constant, lisse and constructible sheaves on Et(X.). Let R and J be as in (2.3.2). A sheaf $F = \{F_n|n \geq 0\}$ of R/m^{ν}-modules on Et(X.) is *locally constant* if F_0 is locally constant on $Et(X_0)$ and for every structure map $\alpha : X_m \to X_n$ the induced map $\alpha^*(F_n) \to F_m$ is an isomorphism of sheaves of R/m^ν-modules. If $F = \{{}^\nu F\}$ is a J-adic sheaf on Et(X.), we say $\mathfrak{S}$ is *lisse* if each ${}^\nu F$ is a locally constant sheaf of R/m^ν-modules.

(A.2.3) A sheaf of R/m^ν-modules $F = \{F_n | n \geq 0\}$ on Et(X.) is *constructible* if each F_n is a constructible sheaf on $Et(X_n)$. An m^ν-adic sheaf $F = \{{}^\nu F_n | \nu,\, n \geq 0\}$ on Et(X.) is constructible if each ${}^\nu F_n$ is a constructible sheaf of R/m^ν-modules on $Et(X_n)$. Let $(m^\nu R) - adn$ -(Absh(Et(X.); R)) denote the full sub-category of $(m^\nu R) - ad$-(Absh(Et(X.); R)) consisting of m^ν-adic sheaves $F = \{{}^\nu F_n | \nu,\, n \geq 0\}$ so that each ${}^\nu F_n$ is a noetherian object in the category of sheaves of R/m^ν-modules on $Et(X_n)$. In view of the fact that each X_n is noetherian, we obtain the equivalence of categories

$$((m^\nu R) - adn\text{-}(\text{Absh}(Et(X.)))) \simeq (\text{Constructible } \; m^\nu\text{-adic sheaves on Et(X.)})$$

Observe (see [Jou-1] Proposition 5.2.1, 5.2.2 and 5.2.3) that the above category is abelian. If $R = \mathbb{Z}_l$ and $m = (l\mathbb{Z}_l)$, the resulting category will be denoted R-Absh$_c(Et(X.))$; in the case of l-adic sheaves the resulting category is thus denoted Z_l-Absh(Et(X.)).

(A.2.4) The category of constructible Q_ℓ-sheaves on Et(X.) will be defined to be the quotient of the category Z_l-Absh(Et(X.)) by the thick full subcategory of torsion sheaves. Similarly if E is a finite extension of Q_ℓ and R is the integral closure of Z_l in E, we let the category of constructible E-sheaves be the quotient of the category R-Absh$_c(Et(X.))$ by the thick full subcategory of torsion sheaves. This category will be denoted E-Absh$_c(Et(X.))$. A constructible E-sheaf is *lisse* if it is locally of the form $L \underset{R}{\otimes} E$ with L a lisse sheaf in R-Absh$_c(Et(X.))$. Finally the category of constructible $\bar{Q}_\ell$-sheaves is defined as follows: the objects are sheaves of the form $F \underset{E}{\otimes} \bar{Q}_\ell$ where F is a constructible E-sheaf for some finite extension E of Q_ℓ. The morphisms of this category are defined in the obvious manner- see ([Del-2] pp. 146-147) for details. A constructible $\bar{Q}_\ell$-sheaf is lisse if it is of the form $F \underset{E}{\otimes} \bar{Q}_\ell$ where F is a lisse sheaf in E-Absh$_c(Et(X.))$.

(A.2.5). Obviously all of the discussion carries over to other topologies for eg. Smt(X.) defined as in (A.1.0).

(A.3.0) We will assume now that X. is a connected simplicial algebraic space. Now we observe (see $[Fr]p.$ 49) that for each $\nu \geq 1$, there exists an equivalence of categories

(locally constant constructible Z/ℓ^ν-sheaves on $Et(X.)$)
$\to$ (ℓ^ν-torsion abelian groups provided with an action by $\pi_1((X., x)_{et})\hat{}$

where $\bar{x}$ is a fixed geometric point of X_0 and ˆ denotes the pro-finite completion. It follows that we obtain the equivalence of categories:

(A.3.1) (lisse and constructible ℓ-adic sheaves on $Et(X.)) \to ((\ell Z) - ad - Ab_f(\pi_1((X, x)_{et})\hat{\ })$

where $Ab_f(\pi_1((X., x)_{et})\hat{\ })$ is the category of finite abelian groups with a continuous action by $\pi_1((X., x)_{et})\hat{\ }$. Finally observe that taking the inverse limit of a system $\{E_n | n \geq 0\}$ in $(\ell Z) - ad - (Ab_f(\pi_1((X., x)_{et})\hat{\ })$ provides a pro-finite abelian group which is a Z_ℓ-module provided with a continuous action by the pro-finite group $\pi_1((X., x)_{et})\hat{\ }$. It follows as in ([Jou-2] pp. 258-259.) that we obtain the equivalence

(A.3.2) (constructible lisse ℓ-adic sheaves on $Et(X.)) \simeq$ (Z_ℓ-modules of finite type provided with a continuous action by the pro-finite group $\pi_1((X., x)_{et})\hat{\ }$)

If E is a finite extension of Q_ℓ and R is the integral closure of Z_ℓ in E, we obtain a similar equivalence

(A.3.3) (constructible lisse R-sheaves on $Et(X.)) \simeq$ (R-modules of finite type (provided with a continuous action by the pro-finite group $\pi_1((X., x)_{et})\hat{\ }$ and

(constructible lisse $\bar{Q}_\ell$-sheaves on $Et(X.)) \simeq$ (finite dimensional vector spaces V_ℓ (over $\bar{Q}_l$) provided with a continuous action by $\pi_1((X., x)_{et})\hat{\ }$ that is induced by an action of $\pi_1((X., x)_{et})\hat{\ }$ on V_E where E is some finite extension of Q_ℓ)

(A.4) Now we define a different étale site for any simplicial algebraic space, different from the one in (A.1.0). This site has often several computational advantages over the site in (A.1.0) as (5.5.2) and (6.2) show. Most of this material is taken from the author's $Ph.D$ thesis ([$J-T$] appendix C) and also appears in ([$J-1$]). As before we fix a noetherian base scheme B and will henceforth only consider algebraic spaces of finite type over B. Let X. be a simplicial algebraic space over B.

(A.4.0) **Definition**. We let SEt(X.) denote the (small) site associated to X. whose objects are maps $\alpha. : U. \to X.$ of simplicial algebraic spaces so that each $\alpha_n : U_n \to X_n$ is *étale*. A morphism (U. $\xrightarrow{\alpha} X.) \to (V. \xrightarrow{\beta} X.)$ is a commutative triangle over X. The coverings of any $\alpha. : U. \to X.$ are given by $\beta. : V. \to U.$ in SEt(X.) with each β_n surjective.

(A.4.1)We may define hypercoverings in SEt(X.) to be simplicial objects V. in SEt(X.) (i.e. bisimplicial algebraic spaces over X.) so that for each $t \geq 0$, the map $V_t \to (\mathrm{cosk}_{t-1}^{X.} V.)_t$ is a surjection. (Here $(\mathrm{cosk}_{-1}^{X.} V.)_0 = X.$) One readily observes that so defined hypercoverings in SEt(X.) and Et(X.) (see [Fr] $p.$23) for this) are identical.

(A.4.2)**Definition.**(Friedlander) Let X. be a simplicial algebraic space as above and let $n \geq 0$ be an integer. A *simplicial geometric point (or simply, geometric point)* of X. is a map

$\bar{x}. : (\text{Spec } \Omega) \otimes \Delta[n] \to X.$

of simplicial algebraic spaces, where Ω is a separably closed field and $\otimes$*has* the meaning as in ([Fr] *p*.7).

Observe that if $\bar{x} : (\text{Spec } \Omega) \to X_n$ is a geometric point of X_n (for any $n \geq 0$), we may associate to it a simplicial geometric point $\bar{x}.$ (in the obvious manner) so that the map $\bar{x}$ factors through $\bar{x}_n$. Observe also that with this definition $(\text{Spec } \Omega) \otimes \Delta[n]$ are all acyclic with respect to any abelian sheaf on SEt(X.) — for now the global section section functor coincides with a stalk. One may also observe that $(\text{Spec } \Omega) \otimes \Delta[n]$ is acyclic with respect to any locally constant abelian sheaf F on $\text{Et}(\text{Spec } \Omega \otimes)\Delta[n]$; this follows readily by considering the étale homotopy type of $(\text{Spec } \Omega \otimes)\Delta[n]$ which is $\Delta[n]$. (See [Fr] *p*. 40.)

(A.4.3).**Definition**(Friedlander). Let $\bar{x}. : (\text{Spec } \Omega) \otimes \Delta[n] \to X.$ be a geometric point of X. as above. A (simplicial) étale neighborhood of $\bar{x}.$ is a commutative triangle

$$\begin{array}{ccc} & & U. \\ & \nearrow & \downarrow \\ (Spec\ \Omega) \otimes \Delta[n] & \longrightarrow & X. \end{array}$$

where U. is in SEt(X.).

(A.5.0) If X. is a simplicial algebraic space, we let Absh(SEt(X.)) denote the category of abelian sheaves on the site SEt(X.). (One may readily observe that *there are enough (simplicial) geometric points on the site* SEt(X.).) As SEt(X.) is a small site, we may readily construct a generator for Absh(SEt(X.)); it follows that Absh(SEt(X.)) is a Grothendieck category and therefore has enough injectives. Now one may consider the cohomology of X. with respect to any abelian sheaf or a (bounded-below) complex of sheaves on SEt(X.) in the obvious manner. The category of complexes in Absh(SEt(X.)) that are bounded below (trivial in negative dimensions, bounded) will be denoted C_+(Absh(SEt(X.))) (C_0(Absh(SEt(X.))), C_b(Absh(SEt(X.))), respectively.)

(A.5.0.1). *Convention.* We will adopt the following convention for the rest of the appendix. If $f. : X. \to Y.$ is a map of simplicial algebraic spaces as above, we let $\bar{f}._*$ denote the induced map Absh(SEt(X.)) $\to$ *Absh*(*SEt*(*Y*.)). If $F^\cdot \epsilon\ C_+$(Absh(SEt(X.))), $R\bar{f}._* F^\cdot$ also may now be defined in the standard manner.

(A.5.1).**Lemma**. Let X. be a simplicial algebraic space as before and let F be an abelian sheaf on SEt(X.). Then

$$H^*_{SEt(X.)}(X.;\, F) \simeq \varinjlim_{K^{\cdot}} H^*(\Gamma(K.., F))$$

where the direct limit is taken over the (filtered) homotopy category of hypercoverings and the left-side is cohomology of X. computed using SEt(X.).

Proof. To prove the lemma, it suffices to show that the functor sending F to the right hand side of (A.5.1) is an effaceable δ-functor. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian sheaves on SEt(X.). Since cohomology with respect to an abelian sheaf vanishes locally (see the Proof of Theorem 8.16 in $[A-M]$), it follows that we obtain the following short exact sequence of complexes

$$0 \to \varinjlim_{K^{\cdot}} \Gamma(K.., A) \to \varinjlim_{K^{\cdot}} \Gamma(K.., B) \to \varinjlim_{K^{\cdot}} \Gamma(K.., C) \to 0$$

of abelian groups, where the direct limit is as before. We obtain the associated long-exact sequence on taking the cohomology groups; this proves that the above functor is a δ-functor. To show it is effaceable one merely has to show that $\varinjlim_{K^{\cdot}} H^i(\Gamma(K.., I)) = 0$, for all $i > 0$, when I is an injective abelian sheaf on SEt(X.). This follows essentially from the fact that for each hypercovering K.., $Z_{K..} \to Z_{X.}$ is a resolution and hence the spectral sequence for the double complex $Hom(Z_{K..}, \mathrm{I})$ degenerates. (Here $Z_{K..}$, $Z_{X.}$ have the obvious meaning.)

Finally observe that as a corollary we obtain:

(A.5.2) $H^*_{SEt(X.)}(X.;\, F^{\cdot}) \simeq \varinjlim_{K^{\cdot}} H^*(\Delta\Gamma(K.., F^{\cdot}))$, $F^{\cdot} \epsilon\ C_+(\mathrm{Absh(SEt(X.))})$.

(A.6.0) Let X. be a simplicial algebraic space as before. Next we define a functor

$\Phi : \mathrm{Absh(Et(X.))} \to C_0(\mathrm{Absh(SEt(X.))})$

as follows. (The obvious induced map $C_0(\mathrm{Absh(Et(X.))}) \to C_0(\mathrm{Absh(SEt(X.))})$ will also be denoted by the same symbol.) Let $F = \{F_n | n\} \epsilon$ Absh(Et(X.)) and U. SEt(X.). We let $\bar{\Phi}(F)^{\cdot}$ be the cosimplicial abelian sheaf whose $m-th$ term is given by

$\Gamma(U., \bar{\Phi}(F)^m) = \Gamma(U_m, F_m)$;

for each structure map $\alpha : \Delta[n] \to \Delta[m]$, we obtain an induced map $\Gamma(U., \bar{\Phi}(F)^n) = \Gamma(U_n, F_n) \to \Gamma(U_n, \alpha_*(F_m)) = \Gamma(X_m \underset{X_n}{\times} U_n, F_m) \to \Gamma(U_m, F_m)$, where the last map is induced by the map $U_m \to X_m \underset{X_n}{\times} (U_n)$. Therefore $\{\bar{\Phi}(F)^m | m\}$ is a cosimplicial abelian sheaf on SEt(X.); normalizing this in the obvious manner provides a complex which we denote by $\Phi(F)^{\cdot}$.

(A.6.1). Let $\eta_k : Et(X_k) \to SEt(X.)$ denote the map of sites given by $U. \to U_k$, if $U. \epsilon\ SEt(X.)$. Now one may readily verify that if F is an abelian sheaf on Et(X.), then

$\Phi(F) =$ the normalization of the cosimplicial sheaf $\{\eta_{k*}(F_k) | k\}$.

where $F = \{F_k|k\}$ is an abelian sheaf on $Et(B\mathfrak{S}.)$.

(A.6.2) Let X. be as before and let $F = \{F_n|n\} \epsilon$ Absh(Et(X.)) be such that for each structure map $\alpha : X_n \to X_m$, the map $\alpha^*(F_m) \to F_n$ is an isomorphism. *Recall that such sheaves are called sheaves with descent and the full subcategory of such sheaves is denoted* $Absh^{des}(Et(X.))$. Then $\underline{H}^t(\Phi(F)) = 0$ for all $t > 0$ and $\underline{H}^0(\Phi(F)) \cong \eta_{0*}(F_0)$. (Hence $H^*_{SEt(X.)}(X.;\ H^0(\Phi(F))) \simeq H^*_{SEt(X.)}(X.;\ \Phi(F))$.) To see this we may argue as follows. Let $\bar{x}. : (\text{Spec } \Omega \otimes \Delta[n]) \to X.$ be a geometric point of X.. Now $\underline{H}^t(\Phi(F))_{\bar{x}.}$ is the $t-th$ cohomology of the cosimplicial abelian group

$\Gamma((\text{Spec } \Omega \otimes \Delta[n]_0);\ F_0) \to \Gamma((\text{Spec } \Omega \otimes \Delta[n]_1);\ F_1) \to ...$

As F satisfies the above hypotheses, we observe that this cosimplicial abelian group is merely the cosimplicial abelian group

$\Delta[n]_0 \otimes F_{\bar{x}_0} \overset{\to}{\longrightarrow} \Delta[n]_1 \otimes F_{\bar{x}_0}.....$

where $F_{\bar{x}_0}$ is the stalk of F_0 at the geometric point $\bar{x}_0$ of X_0. Clearly this complex is contractible; hence (A.6.2) follows.

(A.6.3). In view of the above result we may define a functor

$\phi : Absh^{des}(Et(X.)) \to Absh(SEt(X.)$

by $\phi(F) = H^0(\Phi(F))$.

(A.6.4)**Lemma**. The functors Φ and ϕ have the following properties.

(i) Φ and ϕ are exact functors preserving quasi-isomorphisms.

(ii). Let $f. : X. \to Y.$ be a map of simplicial algebraic spaces and let $F \ \epsilon\ C_0$(Absh(Et(X.))). If $\bar{f}._*$ denotes the induced functor (as in (A.5.0.1)),

$\Phi(f._* F^\cdot) = \bar{f}._* \Phi(F^\cdot)$. and $\phi(f._* F^\cdot) = \bar{f}._* \phi(F^\cdot)$.

(iii). If $F^\cdot \epsilon\ C_0$(Absh(Et(X.))), $H^*_{Et(X.)}(X.;\ F^\cdot) \simeq H^*_{SEt(X.)}(X.;\ \Phi(F^\cdot))$.

(iv)The functor

$F \to \{\eta_k^* F|k\}$, $F \epsilon$ Absh(SEt(X.))

will be denoted $\bar{\eta}^*$. The natural map

$\bar{\eta}^*(\phi(F = \{F_k|k\})) \to F = \{F_k|k\}$

is an isomorphism whenever F is a sheaf with descent on $Et(X.)$

(v) The functors Φ and ϕ both preserve injections and injectives.

(vi) Let K, L be two complexes in $D_c^{b,des}(Absh(Et(X.))$ (=the full subcategory of $D_c^b(Absh(Et(X.)))$ of complexes whose cohomology sheaves all have descent). If $\underline{RHom}$ denotes the internal hom in this category,

$$\phi\underline{RHom}(K,\, L) \simeq \underline{RHom}(\phi(K),\, \phi(L)).$$

Proof. (i). Let $\bar{x}.$ be a simplicial geometric point of X. and let $F \in$ Absh(Et(X.)). Observe now that $(\bar{\Phi}(F)^m)_{\bar{x}} = \lim_{\overrightarrow{U.}} \Gamma(U.,\, \bar{\Phi}(F)^m)$, where the filtered direct limit is over all simplicial étale neighborhoods of $\bar{x}.$; observe that $\Gamma(U.,\, \bar{\Phi}(F)^m) = \Gamma(U_m,\, F_m)$ and that for every étale neighborhood U of $\bar{x}_m$, one may find (see [Fr] $p.12.$) an étale neighborhood U. of $\bar{x}.$ so that the map $U_m \to X_m$ factors through the given map $U \to X_m$. Therefore $(\bar{\Phi}(F)^m)_{\bar{x}.} \simeq (F^m)_{\bar{x}_m}$. The exactness of Φ is now clear; now (i) follows from (A.6.2) ad (A.6.3). (ii) follows readily from the definition of the functors $f._*$ and $\bar{f}._*$ and by (A.6.1). Finally (iii) follows from the fact that hypercohomology on Et(X.) and SEt(X.) may be computed using hypercoverings (see (A.5.1) and [Fr] p.27) and the observation that hypercoverings on the two sites coincide.

To prove (iv) we begin with the following observation. Let $\bar{x}.$ denote a simplicial geometric point of $X.$ of the form in (A.4.3). and let $\bar{x}_n : Spec\ \ \Omega \otimes i_n \to X.$ its term in degree n. (Here i_n is the generator of $\Delta[n]$.) Let $F = \{F_n | n\} \in Absh(Et(X.))$. Now

$$\eta_{n*}(F_n)_{\bar{x}.} \cong (F_n)_{\bar{x}_n}$$

This follows readily from the observation that given any étale neigborhood V of $\bar{x}_n$, there exists a simplicial étale neighborhood $V.$ of $\bar{x}.$ so that the map $V_n \to X_n$ factors through V. It follows readily that, under the same hypothesis:

$$\eta_n^*(\eta_{n*}(F_n)_{\bar{x}_n} \cong \eta_{n*}(F_n)_{\bar{x}.} \cong (F_n)_{\bar{x}_n}$$

Observe that there exists a natural map $\eta_{0*}(F_0) \to \eta_{n*}(F_n)$ for all n. (For this observe that if $\alpha : X_n \to X_0$ is any structure map of the simplicial space $X.$, there exists natural maps $\eta_{0*}(F_0) \to \eta_{0*}(\alpha_*(F_n)) \to \eta_{n*}(F_n)$.) Now assume F has descent so that $\Phi(F) \simeq \phi(F) \simeq \eta_{0*}(F_0)$. Therefore it suffices to show that the map $\eta_{0*}(F_0) \to \eta_{n*}(F_n)$ is an isomorphism.

Now $\eta_{0*}(F_0)_{\bar{x}.} \cong colim\Gamma(V., \eta_{0*}(F_0)) \cong colim\Gamma(V_0, F_0)$, where the colimit is over all $V.$ in $SEt(X.)$ so that of V_n is an étale neighborhood of $\bar{x}_n$. Let $\bar{x}_0 = d_\alpha(\bar{x}_n)$ for some structure map $d_\alpha : X_n \to X_0$

. Observe that if V is any étale neighborhood of $\bar{x}_0$, $d_\alpha^{-1}(V) = X_n \underset{X_0}{\times} V$ is an étale neighborhood of $\bar{x}_n$. One may readily apply the construction in ([Fr] p. 12) to conclude that if V is an étale neighborhood of $\bar{x}_0$, there exists a $W. \epsilon\ SEt(X.)$ so that (i) W_n is an étale neighborhood of $\bar{x}_n$ and (ii) W_0 is an étale nighborhood of $\bar{x}_0$ that dominates the given V. It follows that $\eta_{0*}(F_0)_{\bar{x}.} \cong (F_0)_{\bar{x}_0}$. Since F has descent, $\eta_{n*}(F_n)_{\bar{x}.} \cong (F_n)_{\bar{x}_n} \cong (F_0)_{\bar{x}_0}$. This proves (iv).

Next observe that Φ has the required property since the restriction functor $()_k : Absh(Et(X.)) \rightarrow Absh(Et(X_k))$ has the property of preserving injections and injectives. Clearly ϕ also has the same property-see (A.6.3).

(vi). Observe that Φ has a left adjoint denoted $\bar{\eta}^*$ (as in (iv) that preserves the tensor-product which is left-adjoint to the internal hom <u>*hom*</u>. Now the assertion of (vi) follows readily in view of (vi).

(A.6.5).**Corollary**. Let the functor

$D_c^{b,des}(Absh(Et(X.))) \longrightarrow D_c^b(Absh(SEt(X.)))$

induced by the functor ϕ in (A.6.2) be also denoted ϕ. Then ϕ is fully-faithful.

Proof. Let $\bar{\eta}^*$ denote the left-adjoint of the functor ϕ as in (A.6.4)(iv). Given two complexes $K^\cdot$, $L^\cdot$ in $D_c^{b,des}(Absh(Et(X.)))$, one observes that

$Hom(\phi(K^\cdot), \phi(L^\cdot)) \cong Hom(\bar{\eta}^*(\phi(K^\cdot)), L^\cdot) \cong Hom(K^\cdot, L^\cdot)$

Observe that one may identify ϕ with Φ. This may be identified with $\{\eta_{n*}|n\}$. Therefore the left-adjoint to Φ (at the level of the appropriate derived categories) is $\bar{\eta}^*$. Now the second isomorphism follows by (A.6.4)(iv). This completes the proof.

(A.7).**Definition**. Let $f : X \rightarrow Y$ be a map of algebraic spaces and $F \ \epsilon$ Absh $(Et(X))$. Then (F, f) is cohomologically proper if for every map $g : Y' \rightarrow Y$, the canonical base-change $g^* Rf_* F \rightarrow Rf'_* g'^* F$ is a quasi-isomorphism, where f', g' are defined by the pull-back square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

(A.7.1)*Examples.*(i). Let f be proper and F torsion; now the proper-base-change theorem (see [Mi] chapter 4 or [SGA]4 Expose XVI) shows (F, f) is cohomologically proper.

(ii). Let f be a smooth map with connected non-empty geometric fibers. Let $F = f^*K$, for an abelian sheaf K on $Et(Y)$, with torsion prime to the residue characteristics. Then (F, f) is cohomologically proper.

Proof of(ii). First we will obtain the isomorphisms:

$F \longrightarrow f_* f^* F$ and $g^* F \longrightarrow {f'}_* {f'}^* g^*(F)$.

The first one is a local assertion on Y while the second one is local on Y'. Moreover it is clear that the first one implies the second. Next we may assume without loss of generality (by restricting to étale open neighborhoods on Y) that f factors as an étale map $i : X \to A^n_Y$ followed by the projection $A^n_Y \to Y$. However since the geometric fibers of f are nonempty and connected, one may readily verify that the map $X \underset{(Spec\ k)}{\times} (Spec\ \bar{k}) \to A^n_Y \underset{(Speck)}{\times} (Spec\ \bar{k})$ is an isomorphism. Now it is clear that the natural map $F \to f_* f^* F$ is an isomorphism.

Now observe that $R(g^* \circ f_* \circ f^*)F \simeq g^*(Rf_*)f^*F$, while $R({f'}_* {f'}^* g^*)F = R{f'}_*({f'}^* g^*)(F)$; however the composite functor $g^* f_* f^* \cong$ the composite functor ${f'}_* {f'}^* g^*$, since $g^* f_* f^* F \cong g^* F \cong {f'}_* {f'}^* g^* F$ as observed above. It follows, therefore, that the natural maps

$$g^*(Rf_*)f^*F \overset{\simeq}{\longrightarrow} R{f'}_*({f'}^* g^*)F \overset{\simeq}{\longrightarrow} R{f'}_* {g'}^*(f^*F)$$

are isomorphisms. This shows ($K = f^*F$, f) is cohomologically proper.

(A.8). **Theorem**. (Joshua: see [J-T] appendix C or [J-1].) Let $f. : X. \to Y.$ be a map of simplicial algebraic spaces. If $K^{\cdot} \epsilon\ C_0(\text{Absh(SEt(X.))})$, we obtain a Leray spectral sequence:

$$E_2^{p,q} = H^p(Y.; R^q \bar{f}._* K^{\cdot}) \Rightarrow H^{p+q}(X.; K^{\cdot}).$$

If in addition $K^{\cdot} = \Phi(F^{\cdot})$, $F^{\cdot} \epsilon\ C_0(\text{Absh(Et(X.))})$ and each (F_n, f_n) is cohomologically proper, then we obtain the identification of the stalks:

$$(R^q \bar{f}._* K^{\cdot})_{\bar{x}.} \simeq H^q(X. \underset{Y.}{\times} \bar{x}.; K^{\cdot} | X. \underset{Y.}{\times} \bar{x}.).$$

Proof. We begin with the hypercohomology spectral sequence

$$E_2^{p,q} = H^p(Y.; \underline{H}^q(R\bar{f}._* K^{\cdot})) \Rightarrow H^{p+q}(Y.; R\bar{f}._* K^{\cdot}) \simeq H^{p+q}(X.; K^{\cdot}).$$

This clearly provides the required spectral sequence. We proceed to identify the stalks of $R^q \bar{f}._* K^{\cdot}$. Let $\bar{y}.$ denote a simplicial geometric point of Y. Observe that

$$(R^q \bar{f}._* K^{\cdot})_{\bar{y}.} = \lim_{\overrightarrow{U.}} H^q(U. \underset{Y.}{\times} X.; K^{\cdot})$$

where the colimit is over all (simplicial) étale neighborhoods of $\bar{y}.$ as in (A.4.3) and the cohomology is computed on the site $SEt(U. \underset{Y.}{\times} X.)$. Next recall the isomorphism

$$H^q(U. \underset{Y.}{\times} X.; K^{\cdot}) \simeq H^q(U. \underset{Y.}{\times} X.; F^{\cdot})$$

since $K^{\cdot} = \Phi(F^{\cdot})$. Therefore we are able to make use of the spectral sequence

$$E_1^{r,s} = H^s(U_r \underset{Y_r}{\times} X_r;\ F_r^\cdot) \Rightarrow H^{r+s}(U. \underset{Y.}{\times} X.;\ F^\cdot)$$

as in (A.2.1). Taking the direct limit over all such étale neighborhoods U. of $\bar{y}.$, we obtain the spectral sequence

$$\text{(A.8.1)} \ \underset{\overrightarrow{U.}}{\lim} E_1^{r,s} = \underset{\overrightarrow{U.}}{\lim} H^s(U_r \underset{Y_r}{\times} X_r;\ F_r^\cdot) \Rightarrow \underset{\overrightarrow{U.}}{\lim} H^{r+s}(U. \underset{Y.}{\times} X.;\ F^\cdot)$$

Now we use the assumption that each $(F_n,\ f_n)$ is cohomologically proper to identify the left-side as $H^s(\bar{y}_r \underset{Y_r}{\times} X_r;\ F_r^\cdot|\bar{y}_r \underset{Y_r}{\times} X_r)$. Therefore we may identify the right side of (A.8.1) as $H^{r+s}(\bar{y}. \underset{Y.}{\times} X.; F^\cdot|\bar{y}. \underset{Y.}{\times} X.)$. This proves the theorem.

(A.9) **Corollary**. Assume the following in addition to the hypotheses of (A.8).

(i) $F^\cdot$ is a complex concentrated in one degree (i.e. it is the obvious complex associated to an abelian sheaf $F = \{F_n|n\}$ on Et(X.)).

(ii) each $(F_n,\ f_n)$ is cohomologically proper

(iii) $H^t(f.^{-1}(\bar{y}.);\ F|f.^{-1}(\bar{y}.)) = 0$ for all $t > 0$, $f.^{-1}(\bar{y}.)$ denotes the geometric fiber at $\bar{y}.$ and $\bar{y}.$ is a simplicial geometric point of $Y.$ as in (A.4.2).

Then the natural map $F = \{F_n|n\} \rightarrow \{Rf_{n*}f_n^*F_n|n\}$ induces an isomorphism $H^t(X.;\ F) \simeq H^t(Y.;\ \{f_{n*}F_n\})$ for all t.

Proof. First observe using (A.8) and the third assumption that $R^t\bar{f}._*(\Phi(F))_{\bar{y}.} \simeq 0$ for all $t > 0$. This shows

$$H^t_{Et(X.)}(X.;\ F) \simeq H^t_{SEt(X.)}(X.;\ \Phi(F)) \simeq H^t_{SEt(Y.)}(Y.;\ R^0\bar{f}._*\Phi(F))$$
$$\simeq H^t_{SEt(Y.)}(Y.;\ \bar{f}._*\Phi(F)) \simeq H^t_{SEt(Y.)}(Y.;\ \Phi(\bar{f}._*(F))) \simeq$$
$$H^t_{Et(Y.)}(Y.; (\bar{f}._*(F))).$$

This proves the corollary.

(A.10) **Remark**. Assume that $f. : X. \rightarrow Y.$ is a map of simplicial algebraic spaces so that for each n, the geometric fibers of f_n are connected, nonempty and f_n is smooth. Assume in addition that $M = \{M_n|n\}$ is an abelian sheaf on Et(Y.) with torsion prime to the residue characteristics and $F = f.^*(M)$. Now the second example in (A.7.1) shows each (F_n, f_n) is *cohomologically proper*. If $H^t(f.^{-1}(\bar{y}.);\ F_n) = 0$ for all $t > 0$ and l prime to the residue characteristics, $(F.,\ f.)$ satisfies the assumptions of (A.9).

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