Bredon-style homology, cohomology and Riemann–Roch for algebraic stacks

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Abstract

One of the main obstacles for proving Riemann–Roch for algebraic stacks is the lack of cohomology and homology theories that are closer to the K-theory and G-theory of algebraic stacks than the traditional cohomology and homology theories for algebraic stacks. In this paper we study in detail a family of cohomology and homology theories which we call *Bredon-style* theories that are of this type and in the spirit of the classical Bredon cohomology and homology theories defined for the actions of compact topological groups on topological spaces. We establish Riemann–Roch theorems in this setting: it is shown elsewhere that such Riemann–Roch theorems provide a powerful tool for deriving formulae involving *virtual fundamental classes* associated to dg-stacks, for example, moduli stacks of stable curves provided with a virtual structure sheaf associated to a perfect obstruction theory. We conclude the present paper with a brief application of this nature.

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1. Introduction

Quotient stacks form a good class of algebraic stacks which are rather easily understood even without involving stack-theoretic terminology: these correspond to actions of smooth (affine) group-schemes on schemes. The traditional cohomology theories for studying such group-actions are the Borel-style equivariant cohomology theories. For simplicity, let X denote a scheme with the action of a finite group G. Then

$$G \times X \underset{\mu}{\overset{pr_2}{\rightrightarrows}} X$$

defines a presentation of the quotient stack $[X/G]$ (with μ (pr_2) denoting the group action (the projection to the second factor, respectively)) so that the étale cohomology of the stack $[X/G]$ with respect to an equivariant abelian sheaf F identifies with the cohomology of the simplicial scheme $EG \times_G X$ with respect to the pull-back of F. (Here $EG \times_G X$ may be identified with $cosk_0^{[X/G]}(X)$.) Thus the traditional cohomology theories for quotient stacks identify with the Borel-style equivariant cohomology theories: such theories were originally introduced by Borel in the context of actions of compact groups on topological spaces. (See [8].) Even for schemes with finite cohomological dimension, these cohomology theories need not vanish in infinitely many degrees. Moreover, the module structure of these cohomology theories over the representation ring of the group, factors through the completion of the representation ring at the augmentation ideal. There are further issues with Borel-style cohomology theories as far as Riemann–Roch is considered: these are discussed below.

In addition to the Borel-style equivariant theories, there is another class of equivariant theories originally due to Bredon (see [10,33]) considered so far for compact group actions on topological spaces. Let X denote a G-space where G is a compact topological group. In the Bredon-style theories, one defines the *G-topology on X* with the closed subsets of X given by G-stable closed sub-spaces of X. The points in this topology therefore correspond to the orbits of G on X, all of which are closed since the group G is compact. One may readily see that, therefore, the G-topology on X is equivalent to the topology on the quotient space X/G. In contrast, in Borel style theories, one defines a simplicial space $EG \times_G X$, then takes its realization, $|EG \times_G X|$, to obtain a space and defines the topology to be the topology on the above realization.

The difference between the two is clearly seen in the definition of equivariant K-theory. The Atiyah–Segal equivariant K-theory of X is the Grothendieck group of the category of all G-equivariant vector bundles on X. This is a Bredon-style theory, since it is defined only

on G-stable subsets of X and a map $f : X \to Y$ between two G-spaces induces an isomorphism on Atiyah–Segal G-equivariant K-theory, in general, only if there is a G-equivariant map $g : Y \to X$ and G-equivariant homotopy equivalences $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. On the other hand, one may consider $K^0(|EG \times_G X|)$. This is a Borel style equivariant cohomology theory. A G-equivariant map $f : X \to Y$ induces an isomorphism on these groups, if there is a map $g : Y \to X$, not necessarily G-equivariant, so that the compositions $f \circ g \simeq id_X$ and $g \circ f \simeq id_Y$ by homotopies that are once again not necessarily G-equivariant. Moreover, one knows that the Borel-style equivariant K-theory of X is the completion of the Atiyah–Segal equivariant K-theory of X (see [1]) and is therefore a coarser invariant of X.

Bredon-style equivariant cohomology in the sense of Bredon may be defined concisely as follows. (The definitions in [10] and [33] are essentially equivalent to this, though the definitions seem a bit more complicated as they are not stated in terms of sheaf cohomology.) First, define a presheaf $\mathcal{R}^G : (G\text{-topology of } X) \to (\text{abelian groups})$ by $\Gamma(U, \mathcal{R}^G) = K_G^0(U) = $ the G-equivariant Atiyah–Segal K-theory of U. One may observe that if G/H is a point on the above topology of X, the stalk $\mathcal{R}_{G/H}^G \cong R(H)$, at least for suitably nice X. Given an abelian presheaf P on the G-topology of X, one defines the Bredon equivariant cohomology of X, $H_{G,Br}^*(X; P) = R\Gamma(X, (P \otimes \mathcal{R}^G)\tilde{\ })$ where $\tilde{\ }$ denotes the functor sending a presheaf to its associated sheaf and $R\Gamma(X, \ )$ denotes the derived functor of the global section functor computed on the G-topology of X. So defined, $H_{G,Br}^*(X; P)$ is a module over $K_G^0(X)$ and hence over $R(G)$.

The philosophy for defining Bredon-style equivariant cohomology may therefore be summarized as follows: define a topology where the open sets are G-stable open sets. Then compute the cohomology on this topology with respect to abelian presheaves or sheaves that contains information on the representations of G. (For example, one may start with any abelian presheaf or sheaf P and consider the presheaf $P \otimes \mathcal{R}^G$.)

Finally consider the case where G is a group scheme acting on a scheme X. One runs into various difficulties, if one tries to define a Bredon-style equivariant étale cohomology in this setting. Some of the main difficulties are in the definition of a suitable site or Grothendieck topology corresponding to the G-topology above; this was rectified in our earlier work, [25].

Amplifying on the techniques developed there, we define and study in detail in this paper, cohomology and homology theories for algebraic stacks generalizing simultaneously Bredon-style equivariant cohomology for group actions and Bloch–Ogus-style theories for schemes and algebraic spaces: see [30].

One big motivation for introducing these Bredon-style theories is the observation that Riemann–Roch problems for algebraic stacks seem much more tractable by using these class of theories. To see this, observe that the K-theory and G-theory for algebraic stacks are in fact closer to Bredon-style theories: this should be clear for quotient stacks where the definition of these theories is similar to that of the Atiyah–Segal equivariant K-theory which we observed is a Bredon-style theory (see [21]. In fact, in [25, Theorem 1.1], we constructed a spectral sequence converging to (rational) G-theory of the stack and where the E_2-terms in fact form a Bredon-style theory as discussed above. Moreover, the following example should suffice to show that a crucial issue with Riemann–Roch for non-representable morphisms of stacks, is the incompatibility of K-theory (which is a Bredon-style theory) with the usual cohomology of stacks (which are Borel-style theories).

Let G denote a finite group, viewed as a group scheme over a field k: we assume the order of G is prime to the characteristic of k. Now the Grothendieck group of vector bundles on the stack $[Spec\, k/G]$ may be identified with the representation ring of the finite group, namely $R(G)$ or equivalently $K_G^0(Spec\, k)$. Moreover, $H_{et}^*([Spec\, k/G]; \mathbb{Q}) \cong H_{et}^*(BG; \mathbb{Q})$ where BG denotes the

classifying simplicial space for the group G. Though $R(G)$ is far from being trivial (even when tensored with $\mathbb{Q}$), the cohomology ring $H^*(BG; \mathbb{Q}) \cong \mathbb{Q}$. Therefore, the diagram

$$
\begin{array}{ccc}
K_G^0(Spec\, k) & \xrightarrow{\ ch^G\ } & H_{et}^*(BG; \mathbb{Q}) \\
p_* \downarrow & & \downarrow p_* \\
K^0(Spec\, k) & \xrightarrow{\ ch\ } & H_{et}^*(Spec\, k; \mathbb{Q})
\end{array}
\tag{1.0.1}
$$

fails to commute, where $p : [Spec\, k/G] \to Spec\, k$ is the obvious (non-representable) map of algebraic stacks. (The top row is the G-equivariant Chern character, whereas the bottom row is the usual Chern character which one may identify with the rank map. One may identify the left most column with the map, sending a representation of G to its G-invariant part.) The first example in Examples 1.3 shows how to resolve this issue using Bredon-style equivariant cohomology in the place of $H_{et}^*(BG; \mathbb{Q})$.

The following theorem summarizes some of the main properties of the Bredon–Bredon-style cohomology and homology theories we define.

Throughout this theorem, we will assume that whenever a coarse moduli space is assumed to exist, it exists as a *quasi-projective scheme*. Moreover, we will assume that, in the equivariant case, provided with the action of a smooth group scheme, it is G-quasi-projective, i.e. it admits a G-equivariant locally closed immersion into a projective space on which the group G acts linearly (see [39]). There are two *distinct* versions of Bredon-style cohomology and homology considered here, *one in general and the second when a coarse moduli space exists*. The *first version*, which is defined in general, uses hyper-cohomology on the isovariant étale site of the stack. The *second version* uses hyper-cohomology on the étale site of the coarse moduli space when it exists. The two are different in general, but agree when the stack is a gerbe over its coarse moduli space. $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ will denote complexes of sheaves on the big isovariant étale site of algebraic stacks or the big étale site of algebraic spaces as in Section 3. (Strictly speaking these complexes need not be contravariant for arbitrary maps, but for the sake of this introduction one may assume they are. See Section 3 for more precise details. The isovariant étale site of algebraic stacks is recalled below, in the second section, following [25, Section 3].) The Bredon cohomology (homology) $H_{Br}^s(\mathcal{S}, \Gamma(t))$ ($H_s^{Br}(\mathcal{S}, \Gamma(t))$) is defined by first defining certain presheaves $K\Gamma(\bullet)$ and $K\Gamma^h(\bullet)$ using the complexes $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$. These are presheaves on the isovariant étale site of the given stack or on the étale site of its coarse moduli space: see Section 5 for details. Ideally one would like to define the Bredon cohomology (homology) groups to be the hyper-cohomology on the isovariant étale site of the stack or the étale site of its coarse moduli space with respect to these presheaves. While such a definition is meaningful, the property (v) in Theorem 1.1 will fail in general with this definition. Therefore, we adopt a variant of this as in Definitions 5.5 and 5.7 in general: *one could interpret these definitions as first computing hyper-cohomology on the isovariant étale site of the stack* (*or the étale site of its coarse moduli space*) *with respect to the complexes $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ and then modifying it with K-theoretic data to obtain a finer invariant of the stack*. (The approach above we do not pursue in detail could be viewed as doing these in a different order.) We also consider local Bredon cohomology groups, which are defined in Definition 5.11.

All algebraic stacks considered in this paper are dg-stacks in the sense of Definition 2.7 and the dg-structure sheaf on a stack $\mathcal{S}$ will usually be denoted $\mathcal{A}_{\mathcal{S}}$ or simply $\mathcal{A}$. One motivation for considering such dg-stacks is the possibility of deriving various formulae for the virtual funda-

mental classes from Riemann–Roch. Throughout the following theorem we will assume that $\mathcal{S}$ is a dg-stack provided with a dg-structure sheaf $\mathcal{A}$. $K(\mathcal{S}, \mathcal{A})$ $(G(\mathcal{S}, \mathcal{A}))$ will denote the K-theory (G-theory, respectively) spectra of the dg-stack $(\mathcal{S}, \mathcal{A})$: these are discussed in Section 2. See also (1.0.3) for our conventions regarding coarse moduli spaces.)

Theorem 1.1 *(Existence of Bredon-style theories with good properties). In statements* (i) *through* (iv) *and* (vi) *the Bredon homology and cohomology are defined using the presheaves in Definitions* 5.4, 5.7 *or* 5.8.

(i) *Assume that* $f : \mathcal{S}' \to \mathcal{S}$ *is an arbitrary map of algebraic stacks. Then* f^* *defines a map* $H^s_{Br}(\mathcal{S}, \Gamma(t)) \to H^s_{Br}(\mathcal{S}', \Gamma(t))$ *making Bredon-style cohomology a contravariant functor* $(alg.stacks/S) \to (graded\ rings)$. *Both Bredon-style cohomology and Bredon-style local cohomology are provided with ring structures.*

(ii) *If, in addition,* f *is proper, one obtains a map* $f_* : H^{Br}_s(\mathcal{S}'; \Gamma(t)) \to H^{Br}_s(\mathcal{S}; \Gamma(t))$ *making Bredon-style homology a covariant functor for proper maps* $(alg.stacks) \to (abelian\ groups)$.

(iii) $H^{Br}_*(\mathcal{S}; \Gamma(\bullet))$ *is a module over* $H^*_{Br}(\mathcal{S}; \Gamma(\bullet))$ *and the latter is a module over* $\pi_*(K(\mathcal{S}, \mathcal{A}_\mathcal{S}))$.

(iv) *Projection formula. Let* $f : \mathcal{S}' \to \mathcal{S}$ *denote a proper map of algebraic stacks. Now the following diagram commutes*:

$$
\begin{array}{ccccc}
H^*(\mathcal{S}; \Gamma(s)) \otimes H_*(\mathcal{S}'; \Gamma(t)) & \xrightarrow{f^* \otimes id} & H^*(\mathcal{S}'; \Gamma(s)) \otimes H_*(\mathcal{S}'; \Gamma(t)) & \longrightarrow & H_*(\mathcal{S}'; \Gamma(t-s)) \\
{\scriptstyle id \otimes f_*} \downarrow & & & & \downarrow {\scriptstyle f_*} \\
H^*(\mathcal{S}; \Gamma(s)) \otimes H_*(\mathcal{S}; \Gamma(t)) & \longrightarrow & & & H_*(\mathcal{S}; \Gamma(t-s)).
\end{array}
$$

(v) *Here we use the presheaves in Definitions* 5.4, 5.7. *In case the algebraic stack* $\mathcal{S}$ *is a separated algebraic space of finite type over the base scheme, one obtains an isomorphism* $H^*_{Br}(\mathcal{S}, \Gamma(\bullet)) \cong H^*_{et}(\mathcal{S}, \Gamma(\bullet))$ *where the right-hand side is the étale hyper-cohomology of* $\mathcal{S}$ *defined with respect to the complex* $\Gamma(\bullet)$. *Under the same hypothesis, one obtains an isomorphism* $H^{Br}_*(\mathcal{S}, \Gamma(\bullet)) \cong H^{et}_*(\mathcal{S}, \Gamma(\bullet)) \cong \mathbb{H}^*_{et}(\mathcal{S}, \Gamma^h(\bullet))$. *(The corresponding statements hold generically if the algebraic stack* $\mathcal{S}$ *is a separated Deligne–Mumford stack which generically is an algebraic space, i.e. if the stack* $\mathcal{S}$ *is an orbifold.)*

(vi) *There exists a multiplicative homomorphism* $ch : \pi_* K(\mathcal{S}, \mathcal{A}) \to H^*_{Br}(\mathcal{S}; \Gamma(\bullet))$ *called the Chern character.*

    For the remaining properties we will assume the following: *a coarse moduli space* $\mathfrak{M}$ *exists as a quasi-projective scheme associated to the algebraic stack* $\mathcal{S}$ *and that the natural map* $p : \mathcal{S} \to \mathfrak{M}$ *is of finite cohomological dimension. (This hypothesis is always satisfied in characteristic* 0 *by Artin stacks with quasi-finite diagonal*: *see the discussion in* 1.0.3 *below.) Moreover, we will assume that either* $\mathfrak{M}$ *is smooth or that the Bredon homology and cohomology theories are the ones defined using the presheaves in Definition* 5.8 *and with respect to a fixed closed immersion* $\mathfrak{M} \to \tilde{\mathfrak{M}}$ *into a smooth quasi-projective scheme.*

(vii) *The Riemann–Roch transformation and the fundamental class. In this case there exists a Riemann–Roch transformation*:

$$\tau : \pi_* G(\mathcal{S}, \mathcal{A}) \to H_*^{Br}(\mathcal{S}; \Gamma(\bullet)).$$

Moreover, the Chern character and τ are compatible in the usual sense:

i.e. $\quad \tau(\alpha \circ \beta) = \tau(\alpha) \circ ch(\beta), \quad$ *where $\alpha \in \pi_0(G(\mathcal{S}, \mathcal{A}_{\mathcal{S}}))$ and $\beta \in \pi_0(K(\mathcal{S}, \mathcal{A}_{\mathcal{S}}))$.*

(viii) *Assume the stack $\mathcal{S}$ is defined over a field. Then there exists a fundamental class $[\mathcal{S}] \in H_*^{Br}(\mathcal{S}, \Gamma^h(\bullet))$ such that cap-product with this class induces a map*:

$$\bigcap [\mathcal{S}] : H_{Br}^*(\mathcal{S}, \Gamma(\bullet)) \to H_*^{Br}(\mathcal{S}, \Gamma(\bullet)).$$

The fundamental class $[\mathcal{S}]$ is defined to be the term of highest weight (and degree $=$ twice the weight) in $\tau(\mathcal{A}_{\mathcal{S}})$. (Classes in $H_n^{Br}(\mathcal{S}, \Gamma(t))$ have degree n and weight t.)

(ix) *Let $\mathcal{S}$ denote a non-dg-stack and let $\pi : \mathcal{S} \times \mathbb{A}^1 \to \mathcal{S}$ denote the obvious projection. Now $\pi^* : H_{Br}^*(\mathcal{S}; \Gamma(\bullet)) \cong H_{Br}^*(\mathcal{S} \times \mathbb{A}^1; \Gamma(\bullet))$ provided the stack $\mathcal{S}$ is smooth. A corresponding assertion holds for Bredon local cohomology when the moduli space is also smooth.*

(x) *Let $\tilde{\mathcal{E}}$ denote a vector bundle on $\tilde{\mathfrak{M}}$ and let $\mathcal{E}$ denote its pull-back to the stack $\mathcal{S}$. Let $\mathbb{P}(\mathcal{E})$ be the associated projective space with the dg-structure sheaf $\pi^*(\mathcal{A})$ where $\mathcal{A}$ is the dg-structure sheaf on $\mathcal{S}$ and $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{S}$ is the obvious projection. Then*:

$$H_{Br}^*(\mathbb{P}(\mathcal{E}), \Gamma(\bullet)) \simeq \bigoplus_{i=0}^{i=n} H_{Br}^*(\mathcal{S}, \Gamma(\bullet)).$$

The induced map in Bredon homology $\pi_* : H_*^{Br}(\mathbb{P}(\mathcal{E}), \Gamma(\bullet)) \to H_*^{Br}(\mathcal{S}, \Gamma(\bullet))$ factors as $H_*^{Br}(\mathbb{P}(\mathcal{E}), \Gamma(\bullet)) \to \bigoplus_{i=0}^{i=n} H_*^{Br}(\mathcal{S}, \Gamma(\bullet)) \to H_*^{Br}(\mathcal{S}, \Gamma(\bullet))$ with the second map being the obvious projection to the 0th summand.

Remark 1.2. One could extend the homotopy property in (ix) to non-smooth stacks and to situations where the moduli spaces may not exist if one uses *homotopy K-theory* (see, for example, [17]) throughout. This is a variant of K-theory having the homotopy property for non-smooth objects as well. For a given stack $\mathcal{S}$, one may define this to be $KH(\mathcal{S}) = hocolim_\Delta \{ K(\mathcal{S} \times \Delta[n]) | n \}$. Moreover, for smooth stacks $KH(\mathcal{S}) \simeq G(\mathcal{S}) \simeq K(\mathcal{S})$.

The following examples, discussed more fully later on (see Examples 5.6 and 5.9), should convey a flavor of the theories considered here and some of the applications.

Examples 1.3.

- Here we will consider the case of *trivial* actions by diagonalizable group schemes G on quasi-projective schemes X over an algebraically closed field k. The dg-structure sheaf will be the usual structure sheaf. In this case, $H_{Br}^*([X/G], \mathbb{Q}_l) \cong R(G) \otimes H^*(X, \mathbb{Q}_l)$ and if X is also smooth, $H_*^{Br}([X/G], \mathbb{Q}_l) \cong \mathcal{H}om(R(G), H_*(X, \mathbb{Q}_l))$. (Here l is a prime different from the residue characteristics and $H^*(X, \mathbb{Q}_l)$ and $H_*(X, \mathbb{Q}_l)$ denote l-adic étale cohomology

and homology.) A similar result holds with $\mathbb{Q}$ in the place of $\mathbb{Q}_l$ if $k = \mathbb{C}$. In particular, this example shows that if one uses Bredon-style cohomology for the Riemann–Roch problem considered in (1.0.1), the corresponding map in the top (and bottom row) would be the identity, thereby making the corresponding square commute.

- Next we will consider the case of *Galois* actions by a finite group on smooth quasi-projective schemes X (over an algebraically closed field k) (with X/G also smooth). Again the dg-structure sheaf will be the usual structure sheaf. In this case $H^*_{Br}([X/G], \mathbb{Q}_l) \cong H^*(X/G, \mathbb{Q}_l)$ and $H^{Br}_*([X/G], \mathbb{Q}_l) \cong H_*(X/G, \mathbb{Q}_l)$. Here X/G denotes the geometric quotient of X by G.

- Dg-moduli-stacks of stable curves. The basic example of a dg-stack that we consider will be an algebraic stack (typically of the form $\mathfrak{M}_{g,n}(X, \beta)$) provided with a *virtual structure sheaf* provided by a *perfect obstruction theory*. Here X is a projective variety, β is a one-dimensional algebraic cycle on X and $\mathfrak{M}_{g,n}(X, \beta)$ denotes the stack of n-pointed stable maps of genus g and class β: see [11, p. 169]. The virtual structure sheaf $\mathcal{O}^{virt}$ is the corresponding sheaf of dgas. One nice feature of our set-up is that it is able to handle such dg-stacks also with ease. See 2.11 for more details. This situation is considered in Theorem 1.5 (and also in more detail in [27]).

Moreover, observe that one of the main difficulties with Riemann–Roch for algebraic stacks is the fact that G-theory for algebraic stacks is essentially a Bredon-style homology theory; it does not behave well functorially with respect to other homology theories that are not of Bredon type. By considering Bredon-style homology theories we study in this paper, we show in Section 8 that the Riemann–Roch problem for algebraic stacks that admit coarse moduli spaces (observe this includes also some Artin stacks) can be solved fairly easily. The following is typical of the Riemann–Roch theorems we establish in Section 8. (The notion of a map being strongly of finite cohomological dimension is defined in Definition 8.4.)

Theorem 1.4 (*Riemann–Roch: first form*). *Let $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a proper map strongly of finite cohomological dimension between algebraic dg-stacks. Assume that a coarse moduli space $\mathfrak{M}'$ ($\mathfrak{M}$) exists for the stack $\mathcal{S}'$ ($\mathcal{S}$, respectively) as in 1.0.3 below which is also quasi-projective. Now one obtains the commutative square*:

$$
\begin{array}{ccc}
\pi_* G(\mathcal{S}', \mathcal{A}') & \xrightarrow{\tau_{\mathcal{S}'}} & H^{Br}_*(\mathcal{S}', \Gamma^h(*)) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\pi_* G(\mathcal{S}, \mathcal{A}) & \xrightarrow{\tau_{\mathcal{S}}} & H^{Br}_*(\mathcal{S}, \Gamma^h(*)).
\end{array}
$$

The above theorem applies to any of the Bredon-style homology theories considered in the paper. For example, by taking $\Gamma^h(*) =$ the sheafification of the higher cycle complex on the étale site of all quasi-projective schemes over fields, we obtain Riemann–Roch theorems with values in a variant of motivic homology. In other approaches to Riemann–Roch problems on algebraic stacks, much of the difficulty lies in the case of non-representable proper maps. The use of Bredon homology essentially circumvents this problem as may be seen in Example 8.8. In addition, we are able to handle stacks that are not smooth and not necessarily Deligne–Mumford: for example, the machinery here seems to apply readily to the *tame Artin stacks* considered in [2].

Applications to virtual structure sheaves and virtual fundamental classes make it necessary that we work throughout in the general context of dg-stacks. Further more, our Riemann–Roch makes full use of the existing Riemann–Roch at the level of the moduli spaces.

We consider compatibility of the above theories with other cohomology–homology theories in Theorem 6.15. It is shown in Theorem 6.15(i) that there is natural map from Bredon homology to the smooth homology of the underlying (non-dg-)stack. It is also shown in Theorem 6.15(iii), that our theory admits a variant that is closely related to the theory of [41] and [12].

One of the main applications of the theory developed in the present paper is as a machine for producing various formulae for virtual fundamental classes. Most of these are discussed at length in the forthcoming paper [27]. However, to give a small sample of what one can expect in this direction, we derive a form of the formula for the virtual fundamental classes that was conjectured in [29, p. 9]. There (and again in [5]) it was conjectured that the usual formalism of expressing the fundamental class of a smooth algebraic variety in terms of the Riemann–Roch transformation applied to the structure sheaf and the Todd class of its tangent bundle extends to the virtual setting. A full form of this formula, very likely involves working out everything in terms of derived moduli stacks, but we consider only the simpler situation where the virtual structure sheaf is defined in terms of an obstruction theory.

Let $\mathcal{S}$ denote a Deligne–Mumford stack over a field, with quasi-projective coarse moduli space and provided with a perfect obstruction theory $E^\bullet = E^{-1} \to E^0$. Observe (see [23, Section 4]) that since we are considering étale cohomology, there is no need to assume the existence of global resolutions for the perfect complexes E^i, $i = 0, -1$, to be able to define Chern classes: we define the Todd class $Td(P)$ for any perfect complex P on the stack $\mathcal{S}$ by the Todd polynomial in the Chern classes of P with values in $\mathbb{H}^*_{et}(\mathcal{S}; \Gamma(\bullet)) \otimes \mathbb{Q}$. Since the stack $\mathcal{S}$ is Deligne–Mumford and we are considering étale cohomology with rational coefficients, it follows readily (see 7.0.15 below) that the Todd class $Td(P)$ is a *unit* for any perfect complex P. Let $[\mathcal{S}]^{virt}_{Br}$ denote the fundamental class of the dg-stack $(\mathcal{S}, \mathcal{O}^{virt}_\mathcal{S})$ in Bredon homology and let $[\mathcal{S}]^{virt} = \sigma_*([\mathcal{S}]^{virt}_{Br})$ denote the image of the above class in the étale homology under the map to étale homology considered in Theorem 6.15. We define the *virtual Todd class* of the obstruction theory $E^\bullet$ as $Td(E_0).Td(E_1)^{-1}$ where $E_i = (E^i)^\vee$. *We also call this the Todd class of the virtual tangent bundle and denote it by* $Td(T\mathcal{S}^{virt})$. We define the Todd homomorphism:

$$\tau^{et} : \pi_0\big(K\big(\mathcal{S}, \mathcal{O}^{virt}_\mathcal{S}\big)\big) \to \mathbb{H}^{et}_*\big(\mathcal{S}, \Gamma^h(\bullet)\big) \otimes \mathbb{Q} \tag{1.0.2}$$

by $\tau^{et}(\mathcal{F}) = (\sigma_*(ch^{Br}(\mathcal{F}) \cap [\mathcal{S}]^{virt}_{Br})) \cap Td(T\mathcal{S}^{virt})$, where ch^{Br} denotes the Chern character map into Bredon cohomology. (See (6.0.6).) (Observe that if $\mathcal{S}$ is a smooth scheme with the obstruction theory defined by $\Omega^1_\mathcal{S}$, then τ^{et} identifies with the usual Riemann–Roch transformation to étale hyper-homology.)

Theorem 1.5 (*A form of Kontsevich's conjectural formula for the virtual fundamental class*). *Assume the above situation. Then the Todd class $Td(T\mathcal{S}^{virt})$ is invertible in $\mathbb{H}^*_{et}(\mathcal{S}; \Gamma(\bullet)) \otimes \mathbb{Q}$ and we obtain*:

$$[\mathcal{S}]^{virt} = \tau^{et}\big(\mathcal{O}^{virt}_\mathcal{S}\big) \cap Td\big(T\mathcal{S}^{virt}\big)^{-1}.$$

Here is an outline of the layout of the paper. In Section 2, we recall the main results on the isovariant étale site from [25] and also briefly discuss the rudiments of dg-stacks. (Full details can be found in [26].) All stacks we consider in this paper will be dg-stacks in the sense of

Definition 2.7. The next three sections are devoted to a detailed study of the cohomology and homology theories we define: we call these Bredon-style theories since they incorporate many of the nice features of the equivariant theories of Bredon (for compact group actions see [10]). We define these by beginning with homology–cohomology theories already defined on algebraic spaces in the sense of Bloch and Ogus. (See [7].) These are axiomatized in Sections 3 and 4 discusses several examples of such theories, for example, continuous étale cohomology, De Rham cohomology, cohomology based on Gersten complexes, etc. Then we define several variants of Bredon-style cohomology and homology theories in detail in Section 5. This is followed by a detailed proof of Theorems 1.1, 6.15 and 1.5. The next section is devoted to Riemann–Roch theorems. We have devoted a couple of appendices to discuss some of the technical details.

To keep things simple, we do *not* consider the equivariant situation where a smooth group scheme acts on a dg-stack in any explicit detail (except for a couple of basic definitions): the equivariant theory will be discussed in detail elsewhere.

We will adopt the following terminology throughout the paper.

1.0.3. Basic frame work

Let S denote a Noetherian separated *smooth* scheme which will serve as the base scheme. All objects we consider will be *locally finitely presented over S, and locally Noetherian*. (Whenever we require these to be generically smooth, the base scheme will be assumed to be a field.) In particular, all objects we consider are locally quasi-compact. However, our main results are valid, for the most part, only for objects that are finitely presented over the base scheme S.

We will adopt the following conventions regarding moduli spaces. A *coarse moduli space* for an algebraic stack $\mathcal{S}$ will be a *proper map* $p : \mathcal{S} \to \mathfrak{M}_{\mathcal{S}}$ (with $\mathfrak{M}_{\mathcal{S}}$ an algebraic space) which is a uniform categorical quotient and a uniform geometric quotient in the sense of [28, 1.1 Theorem]. In particular, p is universal with respect to maps from $\mathcal{S}$ to algebraic spaces. (Note: this may be different from the notion adopted in [43].) It is shown in [28] that if the stack $\mathcal{S}$ is a separated Deligne–Mumford stack, of finite type over k and the obvious map $I_{\mathcal{S}} \to \mathcal{S}$ is finite (where $I_{\mathcal{S}}$ is the inertia stack), then a coarse moduli space exists with all of the above properties. Moreover, for purposes of defining the Riemann–Roch transformation, we will assume that p has *finite cohomological dimension. We say that a map $f : \mathcal{S}' \to \mathcal{S}$ of algebraic stacks has finite cohomological dimension if there exists an integer $N \gg 0$ so that $R^i f_*(M) = 0$ for all $i > N$ and all $\mathcal{O}_{\mathcal{S}'}$-modules M. (Observe that this hypothesis is satisfied if the order of the residual gerbes are prime to the residue characteristics, for example in characteristic 0 for all Artin stacks with quasi-finite diagonal. Proposition 5.14(i) of [25] shows that in characteristic 0, generically one may assume the stack is a neutral gerbe. When the stack has quasi-finite diagonal, the stabilizer groups are finite.)

Given a presheaf of (Ω)-spectra P, $P_{\mathbb{Q}}$ will denote its localization at $\mathbb{Q}$. (Observe that then $\pi_*(P_{\mathbb{Q}}) = \pi_*(P) \otimes \mathbb{Q}$.)

2. The isovariant étale site of algebraic stacks and dg-stacks: a quick review

The basic reference for this section is [25], especially Sections 3 and 4. Let $\mathcal{S}$ denote an algebraic stack finitely presented over the base scheme S. *All stacks we consider in this section will be of this type.* Recall the *inertia stack* $I_{\mathcal{S}}$ associated to $\mathcal{S}$ is defined by the fibered product $\mathcal{S} \times_{\Delta, \mathcal{S} \times_S \mathcal{S}, \Delta} \mathcal{S}$. Since $\Delta : \mathcal{S} \to \mathcal{S} \times_S \mathcal{S}$ is representable, so is the obvious induced map $I_{\mathcal{S}} \to \mathcal{S}$.

Definitions 2.1. (i) Let $f : \mathcal{S}' \to \mathcal{S}$ be a map of algebraic stacks. We say f is *isovariant* if the natural map $I_{\mathcal{S}'} \to I_{\mathcal{S}} \times_S \mathcal{S}'$ is a 1-isomorphism, where $I_{\mathcal{S}'}$ ($I_{\mathcal{S}}$) denotes the inertia stack of $\mathcal{S}'$ ($\mathcal{S}$, respectively).

(ii) *The smooth, lisse-étale and étale sites.* Given an algebraic stack $\mathcal{S}$, we let $\mathcal{S}_{smt}$ ($\mathcal{S}_{lis\text{-}et}$) denote the site whose objects are smooth maps $u : \mathcal{S}' \to \mathcal{S}$ of algebraic stacks (smooth maps $u : U \to \mathcal{S}$ with U an algebraic space). Given two such objects $u : \mathcal{S}' \to \mathcal{S}$ and $v : \mathcal{S}'' \to \mathcal{S}$, a morphism $u \to v$ is a commutative triangle of stacks

$$
\begin{array}{ccc}
\mathcal{S}' & \xrightarrow{\phi} & \mathcal{S}'' \\
& \searrow u \quad v \swarrow & \\
& \mathcal{S}. &
\end{array}
$$

(I.e. there is *given* a 2-isomorphism $\alpha : u \to v \circ \phi$.) The site $\mathcal{S}_{et}$ is the full sub-category of $\mathcal{S}_{smt}$ consisting of étale representable maps $u : \mathcal{S}' \to \mathcal{S}$, where $\mathcal{S}'$ is an algebraic stack. Finally, when $\mathcal{S}$ is a Deligne–Mumford stack, $\mathcal{S}_{et}$ will denote the full sub-category of $\mathcal{S}_{et}$ consisting of étale maps $u : U \to \mathcal{S}$ with U an algebraic space as objects. (The coverings of any object in $\mathcal{S}_{smt}$ are smooth surjective maps whereas in $\mathcal{S}_{et}$ and $\mathcal{S}_{lis\text{-}et}$ they are étale surjective maps.)

(iii) *The isovariant étale and smooth sites.* If $\mathcal{S}$ is an algebraic stack, $\mathcal{S}_{iso.et}$ will denote the full sub-category of $\mathcal{S}_{et}$ consisting of (representable) maps $u : \mathcal{S}' \to \mathcal{S}$ that are also *isovariant*. $\mathcal{S}_{iso.smt}$ is defined similarly as a full sub-category of $\mathcal{S}_{smt}$ and coverings are defined to be isovariant étale surjective (smooth surjective, respectively) maps. For the most part we will only consider the site $\mathcal{S}_{iso.et}$. (It follows from the lemma below that these indeed define pre-topologies (or sites) in the sense of Grothendieck.)

(iv) We will consider sheaves on any of the above sites with values in the category of abelian groups, or modules over a ring, etc. If $\mathcal{C}$ is any one of the above sites, we will denote the corresponding category of sheaves on $\mathcal{C}$ by $Sh(\mathcal{C})$.

Lemma 2.2. (See [25, Section 3].)

 (i) *Isovariant maps are representable.*
(ii) *Isovariant maps are stable by base-change and composition.*

Example 2.3 (*Quotient stacks*). Let G denote a smooth group scheme acting on an algebraic space X. Now the objects of $[X/G]_{iso.et}$ may be identified with maps $u : U \to X$ where U is an algebraic space provided with a G-action so that u is étale and induces an isomorphism on the isotropy groups. Observe that any representable map $\mathcal{S}' \to [X/G]$ of algebraic stacks may identified with a G-equivariant map $u : U \to X$. The iso-variance forces isomorphism of the isotropy sub-groups.

The following results are the keys to understanding and working with the isovariant sites.

Theorem 2.4. (See [25, Theorem 3.13].) *Assume that a coarse moduli space $\mathcal{M}$ exists (as an algebraic space) for the stack $\mathcal{S}$ and that $\mathcal{S}$ is a (faithfully flat) gerbe over $\mathcal{M}$. Now the functor $V \mapsto V \times_{\mathcal{M}} \mathcal{S}$, $\mathcal{M}_{et} \to \mathcal{S}_{iso.et}$ is an equivalence of sites. Therefore one obtains an equivalence*

of the following categories of sheaves: $Sh(\mathcal{S}_{iso.et})$ *and* $Sh(\mathcal{M}_{et})$, *where the sheaves are either sheaves of sets or sheaves with values in any abelian category.*

Theorem 2.5. (See [25, Theorem 3.27].) *Let* $\mathcal{S}$ *denote an algebraic stack with* $x : X \rightarrow \mathcal{S}$ *an atlas. Now there exists a finite filtration of* $\mathcal{S}$

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_n = \mathcal{S} \tag{2.0.4}$$

by locally closed algebraic sub-stacks so that each $\mathcal{S}_i - \mathcal{S}_{i-1}$ *is a gerbe over its coarse moduli space* $\mathfrak{M}_i$ *and* $Sh((\mathcal{S}_i - \mathcal{S}_{i-1})_{iso.et})$ *is equivalent to the topos of sheaves on the étale site of* $\mathfrak{M}_i$: *here sheaves mean sheaves of sets or sheaves with values in any abelian category. The isovariant étale site has a conservative family of points and the points correspond to the geometric points of the coarse moduli space of* $\mathfrak{M}_i$ *for all* i.

Corollary 2.6. (See [25, Propositions 4.3 and 4.4].)

(i) *Let* $\mathcal{S}$ *denote an algebraic stack over* S. (*Recall by our hypotheses, this is required to be Noetherian.*) *If* $\{F_\alpha | \alpha\}$ *is a filtered direct system of presheaves of abelian groups or spectra on* $\mathcal{S}_{iso.et}$, *one obtains a natural quasi-isomorphism* $colim_\alpha \, \mathbb{H}(\mathcal{S}_{iso.et}, F_\alpha) \simeq \mathbb{H}(\mathcal{S}_{iso.et}, colim_\alpha \, F_\alpha)$. (*Here the hyper-cohomology is defined using the Godement resolution as in* (9.0.4).)

(ii) *If* $f : \mathcal{S}' \rightarrow \mathcal{S}$ *is a map of algebraic stacks and* $Rf_* = f_*\mathcal{G}$, *with* $\mathcal{G} = holim_\Delta \, G^\bullet$ *computed on the isovariant étale site of* $\mathcal{S}'$, *one obtains a similar quasi-isomorphism* $colim_\alpha \, Rf_*F_\alpha \simeq Rf_*(colim_\alpha \, F_\alpha)$ *for a filtered direct system of presheaves* $\{F_\alpha | \alpha\}$. (*Here* $G^\bullet$ *denotes the cosimplicial object defined by the Godement resolution—see* (9.0.4).)

(iii) *Finite cohomological dimension with respect to sheaves of* $\mathbb{Q}$-vector spaces: *under the above hypotheses, the site* $\mathcal{S}_{iso.et}$ *has finite cohomological dimension with respect to all sheaves of* $\mathbb{Q}$-vector spaces.

Proof. Recall that all the stacks above are required to be Noetherian. The first two theorems are proved in [25]. The last corollary follows readily from Theorems 2.4 and 2.5. □

2.1. Dg-stacks

Definition 2.7. We define a sheaf of $\mathcal{O}_\mathcal{S}$-modules on $\mathcal{S}_{lis\text{-}et}$ to be *quasi-coherent* if its restriction to the étale sites of *all* atlases for $\mathcal{S}$ are quasi-coherent. Coherent sheaves and locally free coherent sheaves are defined similarly. (Observe that this is slightly different from the usage in [32, Chapter 13], where a quasi-coherent sheaf also is assumed to be Cartesian as in [32, Definition 12.3].) An $\mathcal{O}_\mathcal{S}$-module will always mean a sheaf of $\mathcal{O}_\mathcal{S}$-modules on $\mathcal{S}_{lis\text{-}et}$. $Mod(\mathcal{S}, \mathcal{O}_\mathcal{S})$ (or $Mod(\mathcal{S}_{lis\text{-}et}, \mathcal{O}_\mathcal{S})$ to be more precise) will denote this category.

A dg-stack is an algebraic stack $\mathcal{S}$ of Artin type which is also Noetherian provided with a sheaf of commutative dgas, $\mathcal{A}$, in $Mod(\mathcal{S}, \mathcal{O}_\mathcal{S})$, so that $\mathcal{A}^i = 0$ for $i > 0$ or $i \ll 0$, each $\mathcal{A}^i$ is a coherent $\mathcal{O}_\mathcal{S}$-module and the cohomology sheaves $\mathcal{H}^i(\mathcal{A})$ are all Cartesian. (Observe that our hypotheses imply that $\mathcal{H}^*(\mathcal{A})$ is a sheaf of graded Noetherian rings.) (The need to consider such stacks should be clear in view of the applications to virtual structure sheaves and virtual fundamental classes: see Example 2.11 and Section 7. See [26] for a comprehensive study of such stacks from

a K-theory point of view.) For the purposes of this paper, we will define a dg-stack $(\mathcal{S}, \mathcal{A})$ to have property P if the associated underlying stack $\mathcal{S}$ has property P: for example, $(\mathcal{S}, \mathcal{A})$ is *smooth* if $\mathcal{S}$ is smooth. Often it is convenient to also include disjoint unions of such algebraic stacks into consideration (see also [42]).

2.1.1. Morphisms of dg-stacks

A 1-morphism $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ of dg-stacks is a morphism of the underlying stacks $\mathcal{S}' \to \mathcal{S}$ together with a map $\mathcal{A} \to f_*(\mathcal{A}')$ of sheaves compatible with the map $\mathcal{O}_\mathcal{S} \to f_*(\mathcal{O}_{\mathcal{S}'})$. Such a morphism will have property P if the associated underlying 1-morphism of algebraic stacks has property P. Clearly dg-stacks form a 2-category. If $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{S}', \mathcal{A}')$ are two dg-stacks, one defines their *product* to be the product stack $\mathcal{S} \times \mathcal{S}'$ endowed with the sheaf of dgas $\mathcal{A} \boxtimes \mathcal{A}'$.

2.1.2.

A left $\mathcal{A}$-module is a complex of sheaves M of $\mathcal{O}_\mathcal{S}$-modules, *bounded above* and so that M is a sheaf of left modules over the sheaf of dgas $\mathcal{A}$ (on $\mathcal{S}_{lis\text{-}et}$) and so that the cohomology sheaves $\mathcal{H}^i(M)$ are all Cartesian. The category of all left $\mathcal{A}$-modules and morphisms will be denoted $Mod_l(\mathcal{S}, \mathcal{A})$. We define a map $f : M' \to M$ in $Mod_l(\mathcal{S}, \mathcal{A})$ to be a quasi-isomorphism if it is a quasi-isomorphism of $\mathcal{O}_\mathcal{S}$-modules: observe that this is equivalent to requiring that $\mathcal{H}^*(Cone(f)) = 0$ in $Mod(\mathcal{S}, \mathcal{O}_\mathcal{S})$. This is in view of the fact that the mapping cone of the given map $f : M' \to M$ of $\mathcal{A}$-modules taken in the category of $\mathcal{O}_\mathcal{S}$-modules has an induced $\mathcal{A}$-module structure. A diagram $M' \xrightarrow{f} M \to M'' \to M[1]$ in $Mod_l(\mathcal{S}, \mathcal{A})$ is a *distinguished triangle* if there is a map $M'' \to Cone(f)$ in $Mod_l(\mathcal{S}, \mathcal{A})$ which is a quasi-isomorphism. Since we assume $\mathcal{A}$ is a sheaf of commutative dgas, there is an equivalence of categories between left and right modules; therefore, henceforth we will simply refer to $\mathcal{A}$-modules rather than left or right $\mathcal{A}$-modules. An $\mathcal{A}$-module M is *perfect* if the following holds: there exists a non-negative integer n and distinguished triangles $F_i M \to F_{i+1} M \to \mathcal{A} \otimes^L_{\mathcal{O}_\mathcal{S}} P_{i+1} \to F_i M[1]$ in $Mod(\mathcal{S}, \mathcal{A})$, for all $0 \leqslant i \leqslant n-1$ and so that $F_0 M \simeq \mathcal{A} \otimes^L_{\mathcal{O}_\mathcal{S}} P_0$ with each P_i a perfect complex of $\mathcal{O}_\mathcal{S}$-modules and there is given a quasi-isomorphism $F_n M \to M$ of $\mathcal{A}$-modules. The morphisms between two such objects will be just morphisms of $\mathcal{A}$-modules. This category will be denoted $Perf(\mathcal{S}, \mathcal{A})$. M is coherent if $\mathcal{H}^*(M)$ is bounded and finitely generated as a sheaf of $\mathcal{H}^*(\mathcal{A})$-modules. Again morphisms between two such objects will be morphisms of $\mathcal{A}$-modules. This category will be denoted $Coh(\mathcal{S}, \mathcal{A})$. A left $\mathcal{A}$-module M is *flat* if $M \otimes_\mathcal{A} - : Mod(\mathcal{S}, \mathcal{A}) \to Mod(\mathcal{S}, \mathcal{A})$ preserves quasi-isomorphisms. If $\mathcal{S}'$ is a given closed sub-algebraic stack of $\mathcal{S}$, $Perf_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ will denote the full subcategory of $Perf(\mathcal{S}, \mathcal{A})$ consisting of objects with supports contained in $\mathcal{S}'$.

Definition 2.8. The categories $Coh(\mathcal{S}, \mathcal{A})$, $Perf(\mathcal{S}, \mathcal{A})$ and $Perf_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ form Waldhausen categories with fibrations and weak-equivalences where the *fibrations* are defined to be maps of $\mathcal{A}$-modules that are degree-wise surjections (i.e. surjections of $\mathcal{O}_\mathcal{S}$-modules) and the *weak-equivalences* are defined to be maps of $\mathcal{A}$-modules that are quasi-isomorphisms. To see this one defines the structure of a Waldhausen category, see [40]: observe that it suffices to verify the fibrations and weak-equivalences are stable by compositions and satisfy a few easily verified extra properties as in [40, Section 2]. We will let $Coh(\mathcal{S}, \mathcal{A})$ ($Perf(\mathcal{S}, \mathcal{A})$, $Perf_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$) denote the above category with this Waldhausen structure. The K-theory (G-theory) spectra of $(\mathcal{S}, \mathcal{A})$ will be defined to be the K-theory spectra of the Waldhausen category $Perf(\mathcal{S}, \mathcal{A})$ ($Coh(\mathcal{S}, \mathcal{A})$, respectively) and denoted $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$, respectively). When $\mathcal{A} = \mathcal{O}_\mathcal{S}$, $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$) will be denoted $K(\mathcal{S})$ ($G(\mathcal{S})$, respectively). $K(\mathcal{S}, \mathcal{A})_0$ ($G(\mathcal{S}, \mathcal{A})_0$) will denote the space forming the 0th term of the spectrum $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$, respectively). Let $Perf_{fl}(\mathcal{S}, \mathcal{A})$ denote the

full sub-category of *Perf*$(\mathcal{S}, \mathcal{A})$ consisting of flat $\mathcal{A}$-modules. This sub-category inherits a Waldhausen category structure from the one on *Perf*$(\mathcal{S}, \mathcal{A})$.

Proposition 2.9.

(i) *If M is perfect, it is coherent.*

(ii) *Let $M \in Perf(\mathcal{S}, \mathcal{A})$. Then there exists a flat $\mathcal{A}$-module $\tilde{M} \in Perf(\mathcal{S}, \mathcal{A})$ together with a quasi-isomorphism $\tilde{M} \to M$.*

(iii) *Let $M' \to M \to M'' \to M'[1]$ denote a distinguished triangle of $\mathcal{A}$-modules. Then, if two of the modules M', M and M'' are coherent (perfect) $\mathcal{A}$-modules, so is the third.*

(iv) *Let $\phi : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a map of dg-stacks. Then one obtains an induced functor $\phi^* : Perf_{fl}(\mathcal{S}, \mathcal{A}) \to Perf_{fl}(\mathcal{S}', \mathcal{A}')$ of Waldhausen categories with fibrations and weak-equivalences.*

(v) *Assume in addition to the situation in* (iii) *that $\mathcal{S}' = \mathcal{S}$ and that the given map $\phi : \mathcal{A}' \to \mathcal{A}$ is a quasi-isomorphism. Then $\phi_* : Perf(\mathcal{S}, \mathcal{A}) \to Perf(\mathcal{S}, \mathcal{A}')$ defines a functor of Waldhausen categories with fibrations and weak-equivalences. Moreover, the compositions $\phi_* \circ \phi^*$ and $\phi^* \circ \phi_*$ are naturally quasi-isomorphic to the identity.*

(vi) *There exists natural pairing $(\ ) \otimes_{\mathcal{A}}^{L} (\ ) : Perf(\mathcal{S}, \mathcal{A}) \times Perf(\mathcal{S}, \mathcal{A}) \to Perf(\mathcal{S}, \mathcal{A})$ so that $\mathcal{A}$ acts as the unit for this pairing.*

Proof. In view of the results in Appendix B, one may replace the stack by the simplicial scheme $B_x \mathcal{S}$ where $x : X \to \mathcal{S}$ is an atlas and $B_x \mathcal{S}$ is the corresponding classifying simplicial space. To simplify the discussion, we will, however, pretend $B_x \mathcal{S}$ is just $\mathcal{S}$ itself.

(i) follows readily. Given any $M \in Mod(\mathcal{S}, \mathcal{A})$, one may find a flat $\mathcal{A}$-module $\tilde{M}$ together with a quasi-isomorphism $\tilde{M} \to M$: this follows readily since we are considering all $\mathcal{O}_{\mathcal{S}}$-modules and not just quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules. Given the $\mathcal{A}$-modules $F_i M$ associated to M, one may define $F_i \tilde{M}$ by the canonical homotopy pull-back: $F_i \tilde{M} = F_i M \times_M^h \tilde{M}$—see the definition of the latter in [40, (1.1.2.5)]. Since the obvious map $F_i \tilde{M} \to F_i M$ is a quasi-isomorphism, it follows that $\tilde{M} \in Perf(\mathcal{S}, \mathcal{A})$. This proves (ii).

To prove (iii), it suffices to show that if M' and M'' are coherent (perfect) then so is M. The coherence of M is clear and to see that M is perfect, one may proceed as follows. One may start with the $\{F_i M'' | i = 0, \ldots, n''\}$, $\{F_j M' | j = 0, \ldots, n'\}$ and define $F_{i+n'+1} M = M \times_{M''}^h F_i M''$. $F_{n'} M = M \times_{M''}^h 0 = M'$; now one may continue this by defining $F_j M = F_j M'$, $j = 0, \ldots, n'$. Therefore, it is clear that $M \in Perf(\mathcal{S}, \mathcal{A})$. This proves (iii).

(ii) shows how to define the functor ϕ^*. Since ϕ^* identifies with $L\phi^*$, it is clear it sends quasi-isomorphisms (distinguished triangles) of $\mathcal{A}$-modules to quasi-isomorphisms (distinguished triangles, respectively) of $\mathcal{A}'$-modules. Since ϕ^* is defined by tensor product, it clearly preserves surjections and hence fibrations. This proves (iv).

The obvious map $\mathcal{A}' \to \mathcal{A}$ defines the functor ϕ_* that sends an $\mathcal{A}$-module M to the same $\mathcal{O}_{\mathcal{S}}$-module M, but viewed as an $\mathcal{A}'$-module via the map $\mathcal{A}' \to \mathcal{A}$. Therefore the distinguished triangle $F_i M \to F_{i+1} M \to \mathcal{A} \otimes_{\mathcal{O}_{\mathcal{S}}}^L P \to F_i M[1]$ is sent to the same distinguished triangle; since $\mathcal{A}' \to \mathcal{A}$ is a quasi-isomorphism, it follows that $\mathcal{A}' \otimes_{\mathcal{O}_{\mathcal{S}}}^L P \to \mathcal{A} \otimes_{\mathcal{O}_{\mathcal{S}}}^L P$ is also a quasi-isomorphism for any complex of $\mathcal{O}_{\mathcal{S}}$-modules P. Therefore, ϕ_* sends $Perf(\mathcal{S}, \mathcal{A})$ to $Perf(\mathcal{S}, \mathcal{A}')$ preserving quasi-isomorphisms and surjections which are the fibrations. Assuming the existence of functorial flat resolutions (which follows since the smooth sites of algebraic stacks locally of

finite type are essentially small: see 10.2), one shows readily that the two compositions $\phi_* \circ \phi^*$ and $\phi^* \circ \phi_*$ are naturally quasi-isomorphic to the identity functors.

(vi) Making use of the functorial flat resolution in 10.2 one may define a pairing $(\ ) \otimes_{\mathcal{A}}^L$ $(\ ): Mod(\mathcal{S}, \mathcal{A}) \times Mod(\mathcal{S}, \mathcal{A}) \to Mod(\mathcal{S}, \mathcal{A})$. It is straightforward from the definition to verify that this induces a pairing $Perf(\mathcal{S}, \mathcal{A}) \times Perf(\mathcal{S}, \mathcal{A}) \to Perf(\mathcal{S}, \mathcal{A})$ preserving weak-equivalences and fibrations in each argument. It is also clear that $\mathcal{A}$ acts as the *unit* for this pairing. $\quad\square$

Remarks 2.10. (1) Observe that the above K-theory spectra, $K(Perf(\mathcal{S}, \mathcal{O}_\mathcal{S}))$ and $K(Perf(\mathcal{S}, \mathcal{A}))$ are in fact E^∞-ring spectra and the obvious augmentation $\mathcal{O}_\mathcal{S} \to \mathcal{A}$ makes $K(\mathcal{S}, \mathcal{A})$ a $K(\mathcal{S})$-algebra. Given two modules M and N over $\mathcal{A}$, one may compute $\mathcal{H}^*(M \otimes_{\mathcal{A}}^L N)$ using the spectral sequence

$$E_2^{s,t} = Tor_{s,t}^{\mathcal{H}^*(\mathcal{A})}\big(\mathcal{H}^*(M), \mathcal{H}^*(N)\big) \quad \Rightarrow \quad \mathcal{H}^*\big(M \otimes_{\mathcal{A}}^L N\big).$$

Since the above spectral sequence is strongly convergent, it follows that if M and N are coherent, so is $M \otimes_{\mathcal{A}}^L N$ provided it has bounded cohomology sheaves. It follows from this observation that $G(\mathcal{S}, \mathcal{A})$ is a module spectrum over $K(\mathcal{S}, \mathcal{A})$ as well.

(2) Assume $f: (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ is a *proper map* of dg-stacks so that Rf_*: $D_+(Mod(\mathcal{S}', \mathcal{O}_{\mathcal{S}'})) \to D_+(Mod(\mathcal{S}, \mathcal{O}_\mathcal{S}))$ has finite cohomological dimension. Now Rf_* induces a map $Rf_*: G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$.

(3) Assume that the dg-structure sheaf $\mathcal{A}$ is in fact the structure sheaf $\mathcal{O}$ and the stack $\mathcal{S}$ is *smooth*. Then it is shown in [23, (1.6.2)] that the obvious map $K(\mathcal{S}) \to G(\mathcal{S})$ is a weak-equivalence. If $\mathcal{S}'$ is a closed sub-stack of $\mathcal{S}$, then the obvious map $K_{\mathcal{S}'}(\mathcal{S}) \to G(\mathcal{S})$ is also a weak-equivalence where $K_{\mathcal{S}'}(\mathcal{S})$ denotes the K-theory of the Waldhausen category $Perf_{\mathcal{S}'}(\mathcal{S})$.

Example 2.11 *(Algebraic stacks provided with virtual structure sheaves).* The basic example of a dg-stack that we consider will be an algebraic stack (typically of the form $\mathfrak{M}_{g,n}(X, \beta)$) provided with a *virtual structure sheaf* provided by a *perfect obstruction theory* in the sense of [5]. Here X is a projective variety, β is a one-dimensional cycle and $\mathfrak{M}_{g,n}(X, \beta)$ denotes the stack of stable curves of genus g and n-markings associated to X. The virtual structure sheaf $\mathcal{O}^{virt}$ is the corresponding sheaf of dgas. Since this is the key-example of dg-stacks we consider, we will discuss this in some detail. We will fix a base-scheme B, which could be a field or more generally a Noetherian excellent scheme of pure dimension b.

Let $\mathcal{S}$ denote a Deligne–Mumford stack (over B) with $u: U \to \mathcal{S}$ an atlas and let $i: U \to M$ denote a closed immersion into a smooth scheme. Let $C_{U/M}$ ($N_{U/M}$) denote the normal cone (normal bundle, respectively) associated to the closed immersion i. (Recall that if $\mathcal{I}$ denotes the sheaf of ideals associated to the closed immersion i, $C_{U/M} = Spec \bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1}$ and $N_{U/M} = Spec\, Sym(\mathcal{I}/\mathcal{I}^2)$. Now $[C_{U/M}/i^*(T_M)]$ ($[N_{U/M}/i^*(T_M)]$) denotes the *intrinsic normal cone* denoted $\mathcal{C}_\mathcal{S}$ (the *intrinsic abelian normal cone* denoted $\mathcal{N}_\mathcal{S}$, respectively).

Let $E^\bullet$ denote a complex of $\mathcal{O}_\mathcal{S}$-modules so that it is trivial in positive degrees and whose cohomology sheaves in degrees 0 and -1 are coherent. Let $L_\mathcal{S}^\bullet$ denote the *relative cotangent complex* of the stack $\mathcal{S}$ over the base B. A morphism $\phi: E^\bullet \to L_\mathcal{S}^\bullet$ in the derived category of complexes of $\mathcal{O}_\mathcal{S}$-modules is called an *obstruction theory* if ϕ induces an isomorphism (surjection) on the cohomology sheaves in degree 0 (in degree -1, respectively). We call the obstruction theory $E^\bullet$ *perfect* if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$ (i.e. locally on the étale site of the stack, it is quasi-isomorphic to a complex of vector bundles concentrated in degrees 0 and -1). In this case, one may define the *virtual dimension* of $\mathcal{S}$ with respect to the obstruction

theory $E^\bullet$ as $rank(E^0) - rank(E^{-1}) + b$: this is a locally constant function on $\mathcal{S}$, which we will assume (as is customary, see [5, Section 5]) is, in fact, constant. Moreover, in this case, we let $\mathcal{E}_\mathcal{S} = h^1/h^0(E^\bullet) = [\mathcal{E}_1/\mathcal{E}_0]$ where $\mathcal{E}_i = Spec \, Sym(E^{-i})$. We will denote $\mathcal{E}_i$ also by $C(E^{-i})$.

Now the morphism ϕ defines a closed immersion $\phi^\vee : \mathcal{N}_\mathcal{S} \to \mathcal{E}_\mathcal{S}$. Composing with the closed immersion $\mathcal{C}_\mathcal{S} \to \mathcal{N}_\mathcal{S}$ one observes that $\mathcal{C}_\mathcal{S}$ is a closed cone sub-stack of $\mathcal{E}_\mathcal{S}$. Let the zero section of $\mathcal{S}$ in $\mathcal{E}_\mathcal{S}$ be denotes $0_\mathcal{S}$. Now we define the *virtual structure sheaf* $\mathcal{O}_\mathcal{S}^{virt}$ with respect to the given obstruction theory to be $L0_\mathcal{S}^*(\mathcal{O}_{\mathcal{C}_\mathcal{S}})$. We proceed to show (at least, in outline) that then $(\mathcal{S}, \mathcal{O}_\mathcal{S}^{virt})$ is a dg-stack in the sense of 2.7.

Locally on the stack $\mathcal{S}$, $0_\mathcal{S}$ is the zero section imbedding into a vector bundle, so that one may see readily that it has finite tor dimension. Therefore, one may define the functor $L0_\mathcal{S}^*$ as in Appendix B. As argued there, by making use of a classifying simplicial space associated to the given stack, we may assume the stack $\mathcal{S}$ is a scheme X and that $0_\mathcal{S} = 0_X : \mathcal{S} = X \to F$ is the zero section imbedding into a vector bundle F over X and that C is a closed sub-scheme of F. Therefore one may now invoke the functorial flat resolution as in Appendix B, 10.2.1 with $\mathcal{S}$ there being $\mathcal{E}_\mathcal{S}$, A there being $\mathcal{O}_\mathcal{C}$ and consider $\Delta \mathfrak{F}_\bullet(\mathcal{O}_\mathcal{C}) = \mathfrak{F}_\bullet(\mathcal{O}_\mathcal{C})$: this will be a functorial flat resolution of $\mathcal{O}_\mathcal{C}$ by $\mathcal{O}_{\mathcal{E}_\mathcal{S}}$-modules so that it is also a sheaf of commutative dgas. It follows that $0_\mathcal{S}^*(\mathfrak{F}(\mathcal{O}_\mathcal{C}))$ is a commutative dga in $Mod(\mathcal{S}, \mathcal{O}_\mathcal{S})$ and trivial in positive degrees.

Since this complex has bounded cohomology, we have obtained a sheaf of commutative dgas that is trivial in positive degrees and with bounded cohomology to represent $L0_\mathcal{S}^*(\mathcal{O}_\mathcal{C})$. Call this sheaf of dgas $\mathcal{B}$. Suppose n is chosen so that $\mathcal{H}^i(\mathcal{B}) = 0$ for all $i < n$. Let $\tau_{\geqslant n}$ denote the functor that kills cohomology in degrees lower than n. Then the canonical pairing $\tau_{\geqslant n}(\mathcal{B}) \otimes \tau_{\geqslant n}(\mathcal{B}) \to \tau_{\geqslant 2n}(\mathcal{B}) \to \tau_{\geqslant n}(\mathcal{B})$ shows that we may replace $\mathcal{B}$ by $\tau_{\geqslant n}(\mathcal{B})$ and assume $\mathcal{B}$ is bounded. Observe that the quasi-coherator $RQ : D_{qcoh,cart}^b(Mod(\mathcal{S}, \mathcal{O})) \to D_{cart}^b(QCoh(\mathcal{S}, \mathcal{O}))$ may be defined explicitly as follows which will show it is functorial at the level of complexes (and not merely at the level of derived categories). Let $x : X \to \mathcal{S}$ denote an atlas for the stack. We may choose X to be affine and Noetherian; now the resulting classifying simplicial space $B_x\mathcal{S}$ is a separated Noetherian scheme in each degree. Given an $M \in Mod(\mathcal{S}_{lis\text{-}et}, \mathcal{O})$, $\bar{x}^*(M) \in Mod(B_x\mathcal{S}_{et}^+, \mathcal{O})$, where $\bar{x}^*(M)$ denotes the pull-back to $B_x\mathcal{S}_{lis\text{-}et}^+$ followed by restriction to $B_x\mathcal{S}_{et}^+$. Next assume $\mathcal{S}$ is separated; now each $B_x\mathcal{S}_n$ is affine. In this case, the quasi-coherator on $B_x\mathcal{S}^+$ is the functor sending $\bar{x}^*(M)$ to $\{\epsilon^*(\Gamma(B_x\mathcal{S}_n, \epsilon_* x_n^*(M))\check{\,}) | n\} = $ the associated quasi-coherent sheaf on $B_x\mathcal{S}_{et}^+$, where $\epsilon : B_x\mathcal{S}_{et}^+ \to B_x\mathcal{S}_{Zar}^+$ is the obvious map of sites. (See 10.0.4 in Appendix B for more details).

Now we may let $RQ(M) = \bar{x}_*(\{\epsilon^*(\Gamma(B_x\mathcal{S}_n, \epsilon_*\mathcal{G}(x_n^*(M)))\check{\,}) | n\})$. (Here $\bar{x}^*$ and $\bar{x}_*$ are defined as in the proof of Proposition 10.3 in Appendix B.) In general, one considers an étale surjective map $U_\bullet \to B_x\mathcal{S}$ (i.e. surjective in each degree) and with each U_n affine and Noetherian. Now $U_{\bullet\bullet} = cosk_0^{B_x\mathcal{S}}(U_\bullet)$ is a bi-simplicial scheme, affine and Noetherian in each bi-degree. Let $v_\bullet : V_\bullet = \Delta(U_{\bullet\bullet}) \to \mathcal{S}$ denote the obvious map. Given $M \in Mod(\mathcal{S}_{lis\text{-}et}, \mathcal{O})$ one lets $RQ(M) = \bar{v}_*(\{\epsilon^*(\Gamma(V_n, \epsilon_*\mathcal{G}(v_n^*(M)))\check{\,}) | n\})$, where $\bar{v}^*$ ($\bar{v}_*$) is defined just like $\bar{x}^*$ ($\bar{x}_*$, respectively).

Now one may readily verify that this quasi-coherator is compatible with tensor products. This shows that one may replace $\tau_{\geqslant n}(\mathcal{B})$ by a sheaf of commutative dgas that consists of quasi-coherent $\mathcal{O}_\mathcal{S}$-modules in each degree. Therefore we have produced a representative for $L0_\mathcal{S}^*(\mathcal{O}_\mathcal{C})$ that satisfies all the required properties except that it consists of quasi-coherent $\mathcal{O}_\mathcal{S}$-modules in each degree. Finally using the observation that every quasi-coherent $\mathcal{O}_\mathcal{S}$-module is the filtered colimit of its coherent sub-sheaves, one may replace the above sheaf of dgas upto quasi-isomorphism by a sheaf of dgas which satisfies all the required properties. (The replace-

ment upto quasi-isomorphism by a complex of coherent sub-sheaves is clear. The assertion that it leads to a sheaf of commutative dgas is left as an easy exercise.)

Remark 2.12. The dg-structure sheaf $\mathcal{O}_{\mathcal{S}}^{virt}$ may also be defined as $LO_{\mathcal{S}}^{!}(\mathcal{O}_{\mathcal{C}}) = O_{\mathcal{C}} \otimes_{\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}} K(\mathcal{O}_{\mathcal{S}})$ where $K(\mathcal{O}_{\mathcal{S}})$ is the canonical Koszul-resolution of $\mathcal{O}_{\mathcal{S}}$ by $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$-modules provided by the obstruction theory. This has the *disadvantage* that it will not be a complex of $\mathcal{O}_{\mathcal{S}}$-modules but only $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$-modules. $\mathcal{E}_{\mathcal{S}}$ with this sheaf of commutative dgas will be a dg-stack. In fact both definitions provide the same class in the ordinary G-theory of the stack $\mathcal{S}$: see [27, Theorem 1.2]. However, for purposes of Riemann–Roch, it is necessary to have the dg-structure sheaf defined as a complex of sheaves of $\mathcal{O}_{\mathcal{S}}$-modules, since the given map of algebraic stacks will be proper only with source $\mathcal{S}$ and *not* with $\mathcal{E}_{\mathcal{S}}$.

Proposition 2.13. *Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack in the above sense and let $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a map of dg-stacks.*

  (i) *An $\mathcal{A}$-module M is coherent in the above sense if and only if it is pseudo-coherent (i.e. locally on $\mathcal{S}_{lis\text{-}et}$ quasi-isomorphic to a bounded above complex of locally free sheaves of $\mathcal{O}_{\mathcal{S}}$-modules) with bounded coherent cohomology sheaves.*
 (ii) *One has an induced map $f^* : K(\mathcal{S}, \mathcal{A}) \to K(\mathcal{S}', \mathcal{A}')$ and if f is proper and of finite cohomological dimension an induced map $f_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$.*
(iii) *If $\mathcal{H}^*(\mathcal{A}')$ is of finite tor dimension over $f^{-1}(\mathcal{H}^*(\mathcal{A}))$, then one obtains an induced map $f^* : G(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S}', \mathcal{A}')$.*
(iv) *If f_* sends $Perf(\mathcal{S}', \mathcal{A}')$ to $Perf(\mathcal{S}, \mathcal{A})$, then it induces a direct image map $f_* : K(\mathcal{S}', \mathcal{A}') \to K(\mathcal{S}, \mathcal{A})$.*

Proof. In view of the hypotheses on $\mathcal{A}$, one may observe that if M is coherent as an $\mathcal{A}$-module, then the cohomology sheaves $\mathcal{H}^*(M)$ are bounded and coherent over the structure sheaf $\mathcal{O}_{\mathcal{S}}$. Therefore, if M is coherent as an $\mathcal{A}$-module, then M is pseudo-coherent as a complex of $\mathcal{O}_{\mathcal{S}}$-modules. Conversely suppose that M is an $\mathcal{A}$-module, so that, when viewed as a complex of $\mathcal{O}_{\mathcal{S}}$-modules, it is pseudo-coherent with bounded coherent cohomology sheaves. Now the cohomology sheaves $\mathcal{H}^*(M)$ are bounded and coherent $\mathcal{O}_{\mathcal{S}}$-modules. It follows that $\mathcal{H}^*(M)$ is finitely generated over $\mathcal{H}^*(\mathcal{A})$ and hence that M is coherent as an $\mathcal{A}$-module. This proves (i). The remaining statements are clear from the last proposition. $\square$

Convention 2.14. Henceforth a stack will mean a dg-stack. Dg-stacks whose associated underlying stack is of Deligne–Mumford type will be referred to as Deligne–Mumford dg-stacks.

Often we also need to include the action of an affine smooth group scheme, which may be defined as follows (see [25, Section 5] for more details):

Definition 2.15. Let $\mathcal{S}$ denote an algebraic stack and let G denote an affine smooth group scheme (both over the base scheme S). An action of G on $\mathcal{S}$ is given by the following: representable maps $G \times \mathcal{S} \xrightarrow{\mu} \mathcal{S}$ and $G \times \mathcal{S} \xrightarrow{pr_2} \mathcal{S}$, along with a common section $s : \mathcal{S} \to G \times \mathcal{S}$ satisfying the usual relations when $G \times G \times \mathcal{S}$, $G \times \mathcal{S}$ and $\mathcal{S}$ are viewed as lax functors from schemes to sets.

An *action* of a group scheme G on a dg-stack $(\mathcal{S}, \mathcal{A})$ will mean morphisms $\mu, pr_2 : (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A}) \to (\mathcal{S}, \mathcal{A})$ and $e : (\mathcal{S}, \mathcal{A}) \to (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A})$ satisfying the relations as above.

Remark 2.16. It follows from the discussion in [25, Appendix] that the quotient stack $[\mathcal{S}/G]$ exists in the above situation. Now a G-equivariant quasi-coherent $\mathcal{O}_\mathcal{S}$-module identifies canonically with a quasi-coherent $\mathcal{O}_{[\mathcal{S}/G]}$-module. Therefore, our general discussion of dg-stacks incorporates a corresponding discussion for dg-stacks with G-action.

2.1.3. K-theory and G-theory presheaves

For each scheme X, X_{et} will denote the small étale site of X. Assume that a coarse moduli space $\mathfrak{M}$ exists for the given stack $\mathcal{S}$. We let $p: \mathcal{S} \to \mathfrak{M}$ denote the obvious proper map. (In general, we will also let $p: \mathcal{S}_{smt} \to \mathcal{S}_{iso.et}$ denote the obvious map of sites.)

In this situation, we let $\mathbf{K}(\ )_\mathfrak{M}$ denote the presheaf of spectra defined on $\mathfrak{M}_{et}$ by $V \to K(V) =$ the K-theory spectrum of the Waldhausen category of vector bundles on V. ($\mathbf{G}(\ )_\mathfrak{M}$ will denote the corresponding presheaf of spectra defined by the Waldhausen K-theory of the category of coherent sheaves.) Next assume that $i: \mathfrak{M} \to \tilde{\mathfrak{M}}$ is a fixed closed immersion into a smooth quasi-projective scheme. Let $\mathbf{K}(\ )_{\tilde{\mathfrak{M}}}$ denote the presheaf defined on $\tilde{\mathfrak{M}}_{et}$ by $\tilde{V} \mapsto K(\tilde{V})$. Now $i^{-1}\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})$ will denote the obvious presheaf of graded rings on $\mathfrak{M}_{et}$.

Next consider the general situation where a coarse moduli space need not exist. Then we let $\bar{\mathbf{K}}(\ )_\mathcal{S} =$ the presheaf of spectra on the isovariant étale site of $\mathcal{S}$ defined by $U \to \bar{K}(U) =$ the K-theory spectrum of the Waldhausen category of vector bundles that are locally trivial on $U_{iso.et}$. $\mathbf{K}(\ , \mathcal{A})_\mathcal{S}$ will denote the corresponding presheaf of spectra defined on the smooth site of $\mathcal{S}$ as in Definition 2.8, where $\mathcal{A}$ is the given dg-structure sheaf. When the dg-structure sheaf $\mathcal{A} = \mathcal{O}_\mathcal{S}$, we will denote this simply by $\mathbf{K}(\ )_\mathcal{S}$. (The subscripts in all of these will be omitted often if there is no cause for confusion.) Observe that the map $p^*: \mathbf{K}(\ )_\mathfrak{M} \to \bar{\mathbf{K}}(\ )_\mathcal{S}$ is a map of presheaves of ring-spectra.

Proposition 2.17. *The functor p^* sending a vector bundle on $\mathfrak{M}$ to a vector bundle on the stack $\mathcal{S}$ induces a weak-equivalence $\mathbf{K}(V)_\mathfrak{M} \to \bar{\mathbf{K}}(V \times_\mathfrak{M} \mathcal{S})$ when the stack $\mathcal{S}$ is a gerbe over its coarse moduli space $\mathfrak{M}$.*

Proof. Recall the functor $p^{-1}: \mathfrak{M}_{et} \to \mathcal{S}_{iso.et}$ sending $V \to V \times_\mathfrak{M} \mathcal{S}$ is an equivalence of sites: see Theorem 2.4. $\quad\square$

3. Cohomology and homology theories for algebraic spaces

In order to define cohomology and homology theories on algebraic stacks the basic strategy adopted in this paper is the following: we begin with cohomology and homology theories defined on algebraic spaces in the setting of Bloch–Ogus (see [7]). We will assume these theories are defined by complexes of sheaves defined on the étale site of all algebraic spaces. (Strictly speaking, one cannot really say these are defined on the big étale site of algebraic spaces as they may not be contravariant for arbitrary maps.) By suitably modifying these using K-theoretic information, we are able to incorporate data about the isotropies at each point and therefore obtain cohomology and homology theories that are more suitable for algebraic stacks. If the complexes of abelian sheaves we start out with extend to the big isovariant étale site of algebraic stacks, we are able to define cohomology and homology theories for algebraic stacks in general using these; otherwise, we will only obtain cohomology and homology theories when the algebraic stacks have coarse moduli spaces.

Therefore, we begin this section by considering the key properties of these cohomology and homology theories on algebraic spaces we and recall the standard construction of higher Chern

classes. We then consider the Chern character. In view of applications in later sections, we try to extend as much of the discussion as possible to the isovariant étale site of algebraic stacks. In the next section, we provide a listing of standard examples of such theories.

Definition 3.1 *(Basic hypotheses on cohomology–homology theories).* Let S denote a base-scheme and let (schemes/S) ((alg.spaces/S), (alg.stacks/S), respectively) denote the category of all locally Noetherian schemes over S (the category of all locally Noetherian algebraic spaces over S, the category of all locally Noetherian algebraic stacks over S, respectively). (We will provide the first two with the étale topology and the last with the isovariant étale topology to make them into sites.) We will denote any of these categories *generically* by $\mathcal{C}$. A *duality theory* on the category $\mathcal{C}$ is given by a collection of complexes $\{\Gamma_Z(r)|r\}$ and $\{\Gamma_Z^h(r)|r\}$ for each object Z in the site so that the following axioms hold. $\Gamma_Z(r)$ ($\Gamma_Z^h(r)$) is a complex of abelian sheaves on Z_{et} ($Z_{iso.et}$) if Z is an algebraic space (Z is an algebraic stack, respectively). (The subscript Z will be often dropped.)

(i) Each $\Gamma_Z(r)$ ($\Gamma_Z^h(r)$) is required to be trivial in negative degrees (in positive degrees, respectively). Moreover, $\Gamma_Z(r)$ ($\Gamma_Z^h(r)$) is trivial for r outside of the interval $[0, \infty)$ (the interval $[-\infty, d]$, respectively) where $d = dim(Z)$. There exist pairings $\Gamma_Z(r) \otimes_{\mathbb{Z}}^L \Gamma_Z(s) \to \Gamma_Z(r + s)$, $\Gamma_Z(r) \otimes_{\mathbb{Z}}^L \Gamma_Z^h(s) \to \Gamma_Z^h(s - r)$ for each Z. These pairings are associative with unit (i.e. $\Gamma_Z(0)$ in degree 0 is a commutative ring with unit) and the first pairing is graded commutative.

Remark 3.2. In later parts of the paper it will be particularly convenient to replace the complexes $\Gamma(\bullet)$ ($\Gamma^h(\bullet)$) by the presheaves of spectra $Sp(\Gamma(\bullet))$ ($Sp(\Gamma^h(\bullet))$, respectively). These are defined in Appendix A: see Lemmas 9.1 and 9.2.

(ii) If X is a scheme or an algebraic space over S, we let

$$H^i\big(X, \Gamma_X(r)\big) = \mathbb{H}_{et}^i\big(X, \Gamma_X(r)\big) \quad \text{and} \quad H_i\big(X, \Gamma_X(r)\big) = \mathbb{H}_{et}^{-i}\big(X, \Gamma_X^h(r)\big).$$

(The right-hand sides are the étale hyper-cohomology groups.) Moreover, under the same hypotheses, if Y is a closed sub-scheme (algebraic sub-space) of X, we let

$$H_Y^i\big(X, \Gamma_X(r)\big) = \mathbb{H}_{et,Y}^i\big(X, \Gamma_X(r)\big).$$

In case X is an algebraic stack over S (with Y a closed algebraic sub-stack), we let

$$H^i\big(X, \Gamma_X(r)\big) = \mathbb{H}_{iso.et}^i\big(X, \Gamma_X(r)\big)$$

($H_i(X, \Gamma_X(r)) = \mathbb{H}_{iso.et}^{-i}(X, \Gamma_X^h(r))$, $H_Y^i(X, \Gamma_X(r)) = \mathbb{H}_{iso.et,Y}^i(X, \Gamma_X(r))$, respectively). We will let $\mathbb{H}^*(X, \Gamma_X(r))_{\mathbb{Q}}$ ($\mathbb{H}_Y^*(X, \Gamma_X(r))_{\mathbb{Q}}$, $\mathbb{H}^*(X, \Gamma_X^h(r))_{\mathbb{Q}}$) denote the corresponding hyper-cohomology objects tensored with $\mathbb{Q}$.

3.0.3.   One of the basic hypotheses we require is that for each fixed integer r, $\mathbb{H}^n(X, \Gamma_X(r))_{\mathbb{Q}}$ ($\mathbb{H}_Y^n(X, \Gamma_X(r))_{\mathbb{Q}}$, $\mathbb{H}^n(X, \Gamma_X^h(r))_{\mathbb{Q}}$) vanishes in all but a finite interval containing n depending on X, if $dim(X) < \infty$, and the choice of the complexes $\{\Gamma(r), \Gamma^h(s)|r, s\}$. (This is true for most cohomology–homology theories we consider; for motivic cohomology and homology, this is also

true modulo the Beilinson–Soulé vanishing conjecture.) The index denoting n in $\mathbb{H}^n(\ ,\Gamma(r))$ and $\mathbb{H}^n(\ ,\Gamma^h(r))$ will be called the *degree* while the index denoting r above will be called the *weight*.

(iii) For each fixed r and each map $f:Z'\to Z$ in the site $\mathcal{C}$, there is given a unique map $f^{-1}(\Gamma_Z(r))\to\Gamma_{Z'}(r)$ so that these are compatible with compositions and flat base-change. So defined, cohomology (and cohomology with supports in a closed sub-scheme (algebraic sub-space, algebraic sub-stack)) is contravariant. Homology is covariant for all proper maps (and contravariant for flat maps with constant relative dimension).

Stated more precisely this means the following: for each algebraic space or stack Z, we will let $\Gamma(r)_Z$ ($\Gamma^h(r)_Z$) denote the restriction of $\Gamma(r)$ ($\Gamma^h(r)$) to the étale site (isovariant étale site if Z is an algebraic stack, respectively) of Z. Given a map (a proper map) $f:X\to Y$ of algebraic spaces or stacks (proper over the base scheme S, respectively), we will require that there is given a map $\Gamma_Y(r)\to Rf_*\Gamma_X(r)$ ($Rf_*\Gamma_X^h(s)\to\Gamma_Y^h(s)$, respectively) which is compatible with compositions. Similarly if $f:X\to Y$ is a flat map of constant relative dimension c, we assume that we are given a map $\Gamma_Y^h(r)\to Rf_*\Gamma_X^h(r+c)[dc]$ (where d is a positive integer, depending on the duality theory), which is compatible with compositions and with the direct image maps so that for a Cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f'\ } & Y' \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

with g flat and f proper, the square

$$
\begin{array}{ccc}
H_*(X',\Gamma(\bullet)) & \xrightarrow{\ f'_*\ } & H_*(Y',\Gamma(\bullet)) \\
{\scriptstyle g'^*}\uparrow & & \uparrow{\scriptstyle g^*} \\
H_*(X,\Gamma(\bullet)) & \xrightarrow{\ f_*\ } & H_*(Y,\Gamma(\bullet))
\end{array}
$$

commutes.

(iii)$'$ Often we will also need to make the additional hypothesis that there exists a natural quasi-isomorphism $\Gamma_X^h(\bullet)\simeq Rf^!\Gamma_Y^h(\bullet)$ where $Rf^!$ is a right adjoint to Rf_* in the situation of (iii) with f proper. (This will be only in those situations where the right adjoint $Rf^!$ is known to exist.)

(iv) *Localization sequence*. Let $i:Y\to X$ denote a closed immersion of algebraic spaces with $j:U=X-Y\to X$ the corresponding open immersion. Now there exists a long exact sequence

$$
\cdots\to H_i\big(Y,\Gamma(j)\big)\to H_i\big(X,\Gamma(j)\big)\xrightarrow{\ j^*\ }H_i\big(U,\Gamma(j)\big)\to H_{i-1}\big(Y,\Gamma(j)\big)\to\cdots
$$

so that for all proper maps $f:X\to X'$, there exists a map from the long exact sequence above to the corresponding long exact sequence for $(f(Y),X')$.

(v) *Homotopy invariance property*. For any X and $p:\mathbb{A}_X^1\to X$ the natural map, the induced map $p^*:H_i(X,\Gamma(r))\to H_{i+d}(\mathbb{A}_X^1,\Gamma(r+1))$ is an isomorphism. (Here d is a positive integer depending on the duality theory.)

(vi) *Homology and cohomology of* $\mathbb{P}(\mathcal{E})$ where X is an algebraic space and $\mathcal{E}$ is a vector bundle on X. (Recall this means $\mathcal{E}$ is locally trivial on the étale topology of X.) In case X is an algebraic stack, let $\mathcal{E}$ denote a vector bundle on the stack $\mathcal{S}$ that is locally trivial on some isovariant étale cover of $\mathcal{S}$. In this case there exists a canonical class $c_1(\mathcal{E}) \in H^d(X; \Gamma(1))$ so that if $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the given map, the map π^* gives us isomorphisms

$$\sum_{i=0}^{n} \pi^*(\,) \cap c_1(\mathcal{E})^i : \bigoplus_{i=0}^{n} H_*\big(X; \Gamma(\bullet)\big) \to H_*\big(\mathbb{P}(\mathcal{E}); \Gamma(\bullet)\big)$$

$$\left(\sum_{i=0}^{n} \pi^*(\,) \cup c_1(\mathcal{E})^i : \bigoplus_{i=0}^{n} H^*\big(X; \Gamma(\bullet)\big) \to H^*\big(\mathbb{P}(\mathcal{E}); \Gamma(\bullet)\big) \right).$$

(vii) *Projection formula*. Let $f : X \to X'$ be a proper map so that

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
{\scriptstyle f_Y}\big\downarrow & & \big\downarrow{\scriptstyle f_X} \\
Y' & \longrightarrow & X'
\end{array}
$$

is Cartesian with $Y' \to X'$ a closed immersion. Now $f_*(\alpha) \cap z = f_*(\alpha \cap f^*(z))$, $\alpha \in H_i(X, \Gamma(r))$ and $z \in H^j_{Y'}(X', \Gamma(s))$ and the cap-product pairing is the one induced by the second pairing in (i) on taking hyper-cohomology.

(viii) *Fundamental class, cohomological semi-purity, purity and Poincaré–Lefschetz duality.* If X is a quasi-projective scheme of pure dimension n, we require that there exist a fundamental class $[X] \in H_{dn}(X, \Gamma_X(n))$ which restricts to a fundamental class in $H_{dn}(U, \Gamma_U(n))$ for each U in the étale site of X. Moreover, if $\tilde{i} : X \to \tilde{X}$ is a closed immersion of X into a smooth quasi-projective scheme $\tilde{X}$, there exists a pairing

$$H^i_X\big(\tilde{X}, \Gamma_X(r)\big) \otimes H_j\big(\tilde{X}, \Gamma_{\tilde{X}}(s)\big) \to H_{j-i}\big(X, \Gamma_X(s-r)\big).$$

This pairing defines an *isomorphism* when $[\tilde{X}] \in H_{dn}(\tilde{X}, \Gamma_{\tilde{X}}(n))$ is used: moreover, varying U over all neighborhoods of a point, we see that we obtain a quasi-isomorphism

$$Ri^!\Gamma_{\tilde{X}}(s)[dn] \to \Gamma_X^h(n-s).$$

In particular (taking $\tilde{X} = X$) when X itself is smooth, we see that

$$\Gamma_X(s)[dn] \simeq \Gamma_X^h(n-s).$$

For a quasi-projective scheme X of pure dimension imbedded in $\tilde{X}$ as above so that the codimension is c, we see that the fundamental class of X corresponds to a class in $H^{dc}_X(\tilde{X}, \Gamma_X(c)) = H^{dc}(X, R\tilde{i}^!\Gamma_X(c))$ which defines a similar class on restriction to any U in the étale site of X. We call this the *Koszul–Thom class* and denote it by $[T]$. Observe that now we have the formula: $[T] \cap [\tilde{X}] = [X]$. Moreover, taking cup-product with the class $[T]$ defines a map $\tilde{i}^*(\Gamma_{\tilde{X}}(r))[-dc] \to R\tilde{i}^!(\Gamma_{\tilde{X}}(r+c))$.

We will also require that cohomology satisfy a *cohomological semi-purity and purity* hypothesis as follows: if $i : X \to Y$ is a closed immersion (closed regular immersion) of pure codimension c, then $H^i_X(Y; \Gamma(\bullet)) = 0$ for all $i < d.c$ (and in addition, $H^{dc}_X(Y; \Gamma(c)) \neq 0$ and that X defines a class in $H^{dc}_X(Y; \Gamma(c))$, respectively).

(ix) *Excision.* Let $\pi : X' \to X$ denote an étale surjective map of algebraic spaces and Y a closed algebraic subspace of X so that π induces an isomorphism $Y' = Y \times_X X' \to Y$. Then the induced map $\pi^* : H^i_Y(X, \Gamma(\bullet)) \to H^i_{Y'}(X', \Gamma(\bullet))$ is an isomorphism.

(x) *Higher Chern classes.* If the complexes $\Gamma(r)$ and $\Gamma^h(s)$ are defined on the big étale site of S-schemes, they clearly extend to the big étale site of simplicial schemes over S. We will assume these are not the l-adic complexes, but the complexes defining any one of the other theories in Section 4. Let $\mathbf{K} = \mathbf{K}(\ )_S$ denote the K-theory presheaf of spectra on the big étale site of algebraic spaces, i.e. given an algebraic space S, $\Gamma(S, \mathbf{K}) = K(S)$. Let $\mathbf{K}_0$ denote the presheaf of fibrant simplicial sets forming the 0th term of this presheaf of spectra. Now we assume there exist *universal Chern classes* $C(i) \in \mathbb{H}^{di}(BGL_\bullet, \Gamma(di))$ where $BGL_\bullet$ denotes $lim_{N \to \infty} BGL_N$. These universal Chern classes may be viewed as maps of simplicial presheaves $\mathbf{K}_0 \simeq \mathbb{Z} \times \mathbb{Z}_\infty(BGL) \to Sp(\Gamma(i)[di])_0$ on the étale site of a given algebraic space S and define Chern classes $C(i)_n : \pi_n(K(S)) \to H^{di-n}(S, \Gamma(i))$ for each $n \geqslant 0$ and each i. (Here d is an integer depending on the given duality theory and $Sp(\Gamma(i)[di])_0$ is the 0th term of the presheaf of symmetric spectra $Sp(\Gamma(i)[di])$.)

Let $\tilde{Ch}(i)$ denote the ith Newton polynomial in the universal Chern classes $C(0), \ldots, C(i) \in H^*(BGL_\bullet, \Gamma(i))$. Now $Ch(i) = \tilde{Ch}(i)/i!$ is the component of degree di of the Chern character *Ch*. Then $Ch(i)$ defines a map $\mathbf{K}_0 = \mathbb{Z} \times \mathbb{Z}_\infty(BGL) \to \mathbb{H}(\ ; Sp(\Gamma(i)))_\mathbb{Q}$ on the étale site of a given algebraic space X and therefore induces a map $Ch(i)_n : \pi_n(K(X)) \to H^{di-n}(X, \Gamma(i))_\mathbb{Q}$. One may obtain a delooping of this Chern character as in Section 5.

To consider the l-adic case, we simply observe that the discussion on the l-adic case as in 5.0.8 applies here as well.

As an immediate consequence of the above axioms we derive the following corollary.

Corollary 3.3. *Assume the situation in* (vi). *Now there exist quasi-isomorphisms*

$$R\pi_*\big(\Gamma^h(\bullet)_{|\mathbb{P}(\mathcal{E})}\big) \simeq \bigoplus_{i=0}^{i=n} \Gamma^h(\bullet)_{|X} \quad and \quad R\pi_*\big(\Gamma(\bullet)\big)_{|\mathbb{P}(\mathcal{E})} \simeq \bigoplus_{i=0}^{i=n} \Gamma(\bullet)_{|X}$$

where $\mathcal{E}$ denotes a rank n vector bundle on the algebraic space (*stack*) *X.*

Proof. Both statements are clear on working locally on the appropriate site: in the case when X is an algebraic space (stack), one works locally on the étale site (isovariant étale site, respectively) of X. $\quad \square$

4. The main sources of Bredon-style cohomology–homology theories for algebraic stacks

In this section, we will consider typical examples of cohomology–homology theories on algebraic spaces that give rise to Bredon-style cohomology and homology theories on algebraic stacks. The first is continuous l-adic étale cohomology and homology (for any prime l different from the residue characteristics) which, we show extends to define continuous l-adic cohomology

and homology on the isovariant étale site of algebraic stacks (with finite l-cohomological dimension). Therefore, continuous l-adic étale cohomology and homology extends to define Bredon-style theories for all algebraic stacks (whose isovariant étale sites have finite l-cohomological dimension). The remaining cohomology and homology theories remain restricted to either algebraic spaces or quasi-projective schemes (often defined over a field) and therefore give rise to Bredon-style theories only for algebraic stacks that have coarse moduli spaces, for example those stacks that have a finite diagonal.

4.1. Continuous étale cohomology and homology

(See [20].) We prefer continuous étale cohomology as it is better behaved than étale cohomology. Given a complex of l-adic sheaves $K = \{K_{l^\nu} | \nu \geqslant 0\}$ on the étale site of a scheme or algebraic space X (with l different from the residue characteristics), we let $\mathbb{H}_{cont}(X, K) \otimes \mathbb{Q} = R(lim_{\infty \leftarrow \nu} \circ \Gamma)(X_{et}, K_{l^\nu}) \otimes_{\mathbb{Z}} \mathbb{Q}$. This defines continuous étale cohomology: $H^i_{cont}(X, \mathbb{Z}_l(r)) = H^i(\mathbb{H}_{cont}(X, \{\underline{\mathbb{Z}}/l^\nu(r)) | \nu\}))$ where each $\underline{\mathbb{Z}}/l^\nu$ is the obvious constant sheaf and r denotes the obvious Tate twist. We define continuous étale homology as the continuous étale hyper-cohomology with respect to the dualizing complex $\mathbb{D}(r) = \{R\pi^!(\underline{\mathbb{Z}}/l^\nu(r)) | \nu \geqslant 0\}$, i.e. $H^{cont}_i(X, \mathbb{Z}_l(r)) \otimes \mathbb{Q} = H^{-i}(\mathbb{H}_{cont}(X_{et}, \mathbb{D}(r))) \otimes \mathbb{Q}$. (Here $\pi : X \to S$ is the structure morphism and S is Noetherian, regular and of dimension at most 1.) Observe that $d = 2$ in this case.

Now we extend these to the isovariant étale site of algebraic stacks. Given a complex of l-adic sheaves $K = \{K_{l^\nu} | \nu \geqslant 0\}$ on the isovariant étale site of an algebraic stack $\mathcal{S}$, we let $\mathbb{H}_{cont}(\mathcal{S}, K)$ be defined exactly as in the case when $\mathcal{S}$ is an algebraic space. Observe that the functor $lim_{\infty \leftarrow \nu}$ sends injectives to objects that are acyclic for Γ. Therefore, in case K satisfied the Mittag-Leffler condition, one may identify $\mathbb{H}_{cont}(\mathcal{S}, K)$ with $lim_{\infty \leftarrow \nu} R\Gamma(\mathcal{S}_{iso.et}, K_\nu)$. Now $H^i_{cont}(\mathcal{S}, \mathbb{Z}_l(r))$ is defined exactly as in the case $\mathcal{S}$ is an algebraic space.

To be able to define homology in a similar manner, we will restrict to the category of algebraic stacks that are *proper* over the base scheme S. We will adopt the technique of compactly generated triangulated categories to first define a functor $f^!$ associated to any proper map $f : \mathcal{S}' \to \mathcal{S}$ of algebraic stacks. We begin by recalling the notion of *compact objects* from [34, p. 210]. We let $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ denote the derived category of bounded below complexes of $\mathbb{Z}/l^\nu$-modules, with l different from the residue characteristics. An object $K \in D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ is *compact* if for any collection $\{F_\alpha | \alpha\}$ of objects in $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$

$$Hom_{D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)}\left(K, \bigoplus_\alpha F_\alpha\right) \cong \bigoplus_\alpha Hom_{D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)}(K, F_\alpha). \qquad (4.1.1)$$

Proposition 4.1.

(i) *Every object of the form $j_{U!} j_U^*(\underline{\mathbb{Z}}/l^\nu[n])$ for $U \in \mathcal{S}_{iso.et}$ and n an integer is compact. (Here $j_{U!}$ is the extension by zero-functor left adjoint to j_U^*.)*

(ii) *The category $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ is compactly generated by the above objects as U varies among a cofinal set of neighborhoods of all the points, i.e. the above collection of objects is a small set T of compact objects in $D_+(\mathcal{S}_{iso.et}; \underline{\mathbb{Z}}/l^\nu)$, closed under suspension (i.e. under the translation functor [1]), so that $Hom_{D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)}(T, x) = 0$ for all T implies $x = 0$.*

Proof. (i) Observe that

$$Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}\big(j_{U!}\big(j_U^*\big(\underline{\mathbb{Z}}/l^\nu[n]\big)\big), F\big) \cong Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}\big(j_U^*\big(\underline{\mathbb{Z}}[n]\big), j_U^* F\big)$$

$$\cong Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}\big(\underline{\mathbb{Z}}/l_{|U}^\nu, j_{|U}^*(F)[-n]\big)$$

$$\cong R\Gamma\big(U, F[-n]\big).$$

Therefore, one now observes that

$$Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}\bigg(j_{U!}\big(j_U^*\big(\underline{\mathbb{Z}}/l^\nu[n]\big)\big), \bigoplus_\alpha F_\alpha\bigg) \cong R\Gamma\bigg(U, \bigoplus_\alpha F_\alpha[-n]\bigg)$$

$$\cong \bigoplus_\alpha R\Gamma(U, F_\alpha)[-n].$$

(Theorem 4.4 below shows that $R\Gamma$ commutes with filtered colimits.) This proves (i). Suppose $R\Gamma(U, F) = 0$ for all U that form a cofinal system of neighborhoods of all points in the site $\mathcal{S}_{iso.et}$. Now it follows immediately from the observation that one has *enough points* for the site $\mathcal{S}_{iso.et}$ that F is *acyclic* and therefore is isomorphic to 0 in the derived category $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$. This proves (ii). □

Definition 4.2 *(Compactly generated triangulated categories).* Let **S** denote a triangulated category. Suppose all small co-products exist in **S**. Suppose also that there exists a small set of objects S of **S** so that

 (i) for every $s \in S$, $Hom_\mathbf{S}(s, -)$ commutes with co-products in the second argument and
(ii) if $y \in \mathbf{S}$ is an object so that $Hom_\mathbf{S}(s, y) = 0$ for all $s \in S$, then $y = 0$.

Such a triangulated category is said to be *compactly generated*. An object s in a triangulated category **S** is called *compact* if it satisfies the hypothesis (i) above.

Theorem 4.3 *(Neeman: see [34, Theorems 4.1 and 5.1]). Let **S** denote a compactly generated triangulated category and let $F : \mathbf{S} \to \mathbf{T}$ denote a functor of triangulated categories. Suppose F has the following property*:

> *if $\{s_\lambda | \lambda\}$ is a small set of objects in **S**, the co-product $\bigsqcup_\lambda F(s_\lambda)$ exists in **T** and the natural map $\bigsqcup_\lambda F(s_\lambda) \to F(\bigsqcup_\lambda s_\lambda)$ is an isomorphism.*

> *Then F has a right adjoint G. Moreover, the functor G preserves co-products (i.e. if $\{t_\alpha | \alpha\}$ is a small set of objects in **T** whose sum exists in **T**, $G(\bigsqcup_\alpha t_\alpha) = \bigsqcup_\alpha G(t_\alpha)$) if for every s in a generating set S for **S**, $F(s)$ is a compact object in **T**.*

We will apply the above theorem in the following manner. (Recall that we have restricted to algebraic stacks that are quasi-compact and quasi-separated. It follows that the isovariant étale site of all the stacks we consider are *coherent* in the sense of [38, Exposé VI, Propositions 2.1, 2.2 and Corollary 4.7].)

Theorem 4.4. *Let $f : \mathcal{S}' \to \mathcal{S}$ denote a proper (not necessarily representable) map of algebraic stacks. Now f defines a right derived functor $Rf_* : D_+(\mathcal{S}'_{iso.et}; \mathbb{Z}/l^\nu) \to D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ that*

commutes with filtered colimits and therefore with sums. Therefore, Rf_* has a right adjoint which we will denote by $Rf^!$.

Moreover, if f has finite l-cohomological dimension on the isovariant étale sites, $Rf_*(j_{U!}(j_U^*(\mathbb{Z}/l^\nu[n])))$ is a compact object in $D_+(\mathcal{S}'_{iso.et}; \mathbb{Z}/l^\nu)$ for all objects $j_U : U \to \mathcal{S}$ in the site $\mathcal{S}_{iso.et}$ and all integers n and the functor $Rf^!$ preserves sums.

Proof. Evidently the derived categories $D_+(\mathcal{S}'_{iso.et}; \mathbb{Z}/l^\nu)$ and $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ are triangulated categories. Next we showed above in Proposition 4.1, that $\{j_{U!}(j_U^*(\mathbb{Z}/l^\nu[n]))|j_U : U \to \mathcal{S}$ in $\mathcal{S}_{iso.et}, n \in \mathbb{Z}/l^\nu\}$ is a small set of compact objects that generate the category $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$. Therefore, if Rf_* preserves sums, Theorem 4.3 shows it has an adjoint $Rf^!$. The functor Rf_* preserves all filtered colimits and finite sums, since the site is coherent: therefore it preserves all sums. The functor $Rf^!$ preserves sums, if $Rf_*(j_{U!}(j_U^*(\mathbb{Z}/l^\nu[n])))$ is a compact object in $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ for all objects $j_U : U \to \mathcal{S}'$ in the site $\mathcal{S}'_{iso.et}$ and all integers n. As shown in Theorem 2.5, one may filter the above stacks by locally closed algebraic sub-stacks $\{\mathcal{S}_i|i\}$ and $\{\mathcal{S}'_i|i\}$ so that the stacks $\mathcal{S}_{i+1} - \mathcal{S}_i$ and $\mathcal{S}'_{i+1} - \mathcal{S}'_i$ are gerbes over their coarse moduli spaces and that there is an equivalence of the corresponding isovariant étale sites with the étale sites of the corresponding coarse moduli spaces. Therefore, one may assume without loss of generality that the stacks under consideration are in fact algebraic spaces: it suffices to show that the functor $Rf_!$ sends the compact objects $j_{U!}(j_U^*(\mathbb{Z}/l^\nu[n]))$ to compact objects. The functor $Rf_!$ now corresponds to the derived direct image functor with compact supports of the induced map of the moduli spaces. Therefore it sends constructible sheaves to complexes with constructible bounded cohomology: now any bounded complex with constructible cohomology sheaves is a compact object in the derived category of sheaves of $\mathbb{Z}/l^\nu$-modules on the étale site of algebraic spaces. $\square$

Definition 4.5. (i) Let $f : \mathcal{S} \to S$ denote the structure map of the stack $\mathcal{S}$. *Assume this is proper and that the base scheme S is Noetherian, regular and of dimension at most 1.* Now we define the dualizing complex on $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ by $D_{\mathcal{S},\nu}(s) = Rf^!(\mathbb{Z}/l^\nu(s))$.

(ii) We define complexes $\mathbb{Z}_l(r)$ on (alg.stacks/S)$_{iso.et}$ by

$$\Gamma(U, \mathbb{Z}_l(r)) = R\left(\lim_{\infty \leftarrow \nu} \circ \Gamma\right)(U, \underline{\mathbb{Z}/l^\nu}(r))$$

and let $\mathbb{Z}_l^h(s)$ restricted to $\mathcal{S}_{iso.et}$ be defined by

$$\Gamma(U, \mathbb{Z}_l^h(s)) = R\left(\lim_{\infty \leftarrow \nu} \circ \Gamma\right)(U, D_{\mathcal{S},\nu}(s)).$$

(iii) We define

$$H_i^{cont}(\mathcal{S}, \mathbb{Z}_l(s)) = \mathbb{H}^{-i}\left(R\left(\lim_{\infty \leftarrow \nu} \circ \Gamma\right)(\mathcal{S}, D_{\mathcal{S},\nu}(s))\right).$$

4.1.2. Observe that $H_{cont}^i(\mathcal{S}, \mathbb{Z}_l(r)) = H^i(\mathbb{H}(\mathcal{S}_{iso.et}, \mathbb{Z}_l(r)))$ and that $H_i^{cont}(\mathcal{S}, \mathbb{Z}_l(s)) = H^{-i}(\mathbb{H}(\mathcal{S}_{iso.et}, \mathbb{Z}_l^h(s)))$. In this case the integer d (as in Definition 3.1(iii) and (viii)) is 2.

4.1.3. Basic hypothesis for isovariant étale cohomology

We will assume throughout the paper that, whenever we consider continuous étale or isovariant étale cohomology and homology as above, this will be only for objects of finite l-cohomological dimension, where l is a prime different from the residue characteristics.

The remaining cohomology and homology theories are defined only on algebraic spaces.

(ii), (iii) *Variants of the Gersten complex.* Here the integer d (as in Definition 3.1(iii) and (viii)) $= 1$. We may first of all define the complexes $\Gamma(r) = \pi_r(\mathbf{K}(\ ))$ for all r where $\mathbf{K}(\ )$ denotes the presheaf of K-theory spectra on the big étale site of all algebraic spaces. Similarly we may define $\Gamma^h(s) = \pi_s(\mathbf{G}(\ ))$ for all s, where $\mathbf{G}(\ )$ denotes the presheaf of G-theory spectra on the restricted big étale site of all algebraic spaces. (The site where the objects are all algebraic spaces, morphisms are only flat maps and coverings are étale coverings.)

We may also define complexes $\Gamma(r) = R^*(r) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\Gamma^h(s) = R_*(s) \otimes_{\mathbb{Z}} \mathbb{Q}$ of presheaves on the same restricted big étale site. For each integer p we define the presheaf $U \mapsto R^*(U, p)$ on the étale site of a stack $\mathcal{S}$ which is the complex:

$$\bigoplus_{x \in U^{(0)}} K_p\big(k(x)\big) \to \cdots \to \bigoplus_{x \in U^{(i)}} K_{p-i}\big(k(x)\big) \to \cdots \to \bigoplus_{x \in U^{(p)}} \mathbb{Z} \qquad (4.1.4)$$

and the presheaf $U \mapsto R_*(U, p)$ which is the complex:

$$\cdots \to \bigoplus_{x \in U^{(i)}} K_{p+i}\big(k(x)\big) \to \cdots \to \bigoplus_{x \in U^{(0)}} K_p\big(k(x)\big). \qquad (4.1.5)$$

(iv) *De Rham cohomology and homology.* (See [18].) Here $d = 2$ and we require that the base scheme S is the spectrum of a field of characteristic 0. If X is a smooth algebraic space, we let $\Gamma(q) = \Omega_X =$ the De Rham complex of $\mathcal{S}$ for all $q \geqslant 0$. We let $\Gamma^h(q) = \Gamma(q)$ in this case. In general, we define $\Gamma(q)$ and $\Gamma^h(q)$ only if X admits a closed immersion into a smooth algebraic space $\tilde{X}$. The complexes $\Gamma^h(q)$ (for all $q \geqslant 0$) are defined as $Ri^!\Omega_{\tilde{X}}^{\bullet}$, where $i : X \to \tilde{X}$ is the closed immersion into a smooth algebraic space. The De Rham homology of X is defined as the hyper-cohomology with respect to this complex. The complex $\Gamma(q)$ is defined in this case as the formal completion of the complex $\Omega_{\tilde{X}}^{\bullet}$ along X. The De Rham cohomology of X is the hyper-cohomology with respect to this complex.

(v) *Motivic cohomology and the higher Chow groups of Bloch.* (See [6].) Here we assume the base scheme S is the spectrum of a field k. Strictly speaking the higher Chow groups form a homology theory, since they are covariant for all proper maps. They are also contravariant for flat maps and Bloch shows (see [6]) that they are in fact contravariant for arbitrary maps between smooth schemes. However, the cycle complex itself is not contravariantly functorial, whereas the motivic complex is in fact contravariantly functorial for arbitrary maps between smooth schemes. Therefore, we let $\Gamma(r) = \mathbb{Z}(r) \otimes_{\mathbb{Z}} \mathbb{Q} =$ the codimension r (rational) motivic complex for all smooth schemes of finite type over k. (We do not define the complexes $\Gamma(\bullet)$ for non-smooth schemes.) We let $\Gamma^h(s)$ be defined by the *dimension s* rational higher cycle complex of Bloch. (See [14] for possible extensions and variations.) In this case $d = 2$ once again.

(vi) *Betti cohomology and homology.* In case the algebraic spaces (or schemes) are defined over $\mathbb{C}$, we may also consider the following. Let $\Gamma(q) = \underline{\mathbb{C}}[0]$ for all q viewed as complexes of sheaves on the transcendental site of complex of points of the algebraic space or scheme. ($\underline{\mathbb{C}}$ denotes the obvious constant sheaf.)

5. Bredon-style cohomology and homology: the different variations

In this section we define and study Bredon-style cohomology and homology theories in detail. The Chern character is crucial for this: therefore we begin by defining a Chern character map for the K-theory of vector bundles that are locally trivial on the isovariant étale site of algebraic stacks.

Proposition 5.1. *Let $(S, \mathcal{O})$ denote a locally ringed site with $\mathcal{O} = $ a sheaf of commutative rings with 1. Assume the site S has enough points. Let $Mod_{l.fr}(S, \mathcal{O})$ denote the category of all locally free and finite rank sheaves of $\mathcal{O}$-modules. Let $C_b(Mod_{l.fr}(S, \mathcal{O}))$ denote the category of all bounded chain complexes of such sheaves of $\mathcal{O}$-modules.*

(i) *For each U in the site S, $C_b(Mod_{l.fr}(U, \mathcal{O}_{|U}))$ has the structure of a complicial Waldhausen category with cofibrations and weak-equivalences: the cofibrations are maps of complexes that are degree-wise split injective and weak-equivalences are maps that are quasi-isomorphisms. Now $U \mapsto K(C_b(Mod_{l.fr}(U, \mathcal{O}_{|U})))_0$ defines a presheaf of spaces on the site S (denoted $K(C_b(Mod_{l.fr}(\ , \mathcal{O})))_0$).*

(ii) *Let $B_\bullet GL_n(\mathcal{O})$ denote the obvious simplicial presheaf on the site S and let $B_\bullet GL(\mathcal{O}) = lim_{n \to \infty} B_\bullet GL_n(\mathcal{O})$. Then there exists a natural map*

$$\underline{\mathbb{Z}} \times \mathbb{Z}_\infty\big(B_\bullet GL(\mathcal{O})\big) \to K\big(C_b\big(Mod_{l.fr}(\ , \mathcal{O})\big)\big)_0$$

of presheaves of spaces which is a weak-equivalence stalk-wise. (Here $\mathbb{Z}_\infty$ denotes the Bousfield–Kan completion.)

Proof. The assertions in (i) are all clear from [40, Section 1]. The second assertion may be obtained from the following observations. The continuity property of the K-theory functor (see [36, Section 2]) and the observation that the Quillen K-theory agrees with the Waldhausen style K-theory shows (see [40, (1.11.2)]) that the stalk of the presheaf $K(C_b(Mod_{l.fr}(\ , \mathcal{O})))_0$ at the point s may be identified with $K(C_b(Mod_{fr}(\mathcal{O}_s)))_0$. Now the telescope construction of Grayson (see [16]) provides the weak-equivalence in (ii). □

Remark 5.2. The main example to keep in mind is where the site is the isovariant étale site of an algebraic stack provided with the obvious structure sheaf. The last weak-equivalence enables us to produce higher Chern classes for vector bundles that are locally trivial on the isovariant étale site.

Finally it suffices to recall the definition of the functor GL for algebraic stacks. Evidently this is defined on the smooth site, $\mathcal{S}_{smt}$, of a given stack $\mathcal{S}$; however, we may extend it to a presheaf on the big smooth site of all algebraic stacks as follows.

5.0.6. The functor GL on $(alg.stacks)_{iso.et}$

Recall that the structure sheaf $\mathcal{O}$ on an algebraic stack $\mathcal{S}$ may be defined as follows. Let $x : X \to \mathcal{S}$ denote a smooth surjective map from an algebraic space. Now

$$\Gamma(\mathcal{S}, \mathcal{O}) = ker\big(\Gamma(X, \mathcal{O})\big) \rightrightarrows \Gamma(X \times_\mathcal{S} X, \mathcal{O}).$$

Next one defines the contravariant functor $GL_n(\mathcal{O})$ on the category (alg.stacks/S) by $\mathcal{S}' \mapsto GL_n(\Gamma(\mathcal{S}', \mathcal{O}))$. Letting $GL_{n,S}$ also denote the functor represented by the group scheme $GL_{n,S}$ on the category (alg.stacks/S), one obtains the natural isomorphism:

$$Hom_{\text{alg.stacks}/S}(\mathcal{S}', GL_{n,S}) = \Gamma(\mathcal{S}', GL_n) = ker\big(\Gamma(X', GL_{n,S}) \rightrightarrows \Gamma(X' \times_{\mathcal{S}'} X', GL_{n,S})\big)$$

where $X' \to \mathcal{S}'$ is an atlas for the stack $\mathcal{S}'$. One may similarly define the functors $B_k GL_{n,S}$ for all $k \geqslant 0$ so that $Hom_{\text{alg.stacks}}(\mathcal{S}', B_k GL_{n,S}) = B_k GL_n(\Gamma(\mathcal{S}', \mathcal{O}))$. (We will often omit the base-scheme S and simply denote $B_k GL_{n,S}$ as $B_k GL_n$.)

Let $\{\Gamma(r)|r\}$ denote a collection of complexes of sheaves on the big site (algebraic spaces/S)$_{et}$ so that they extend to a collection of presheaves on the big site (algebraic stacks/S)$_{iso.et}$ as in Definition 3.1. In view of the results in the last corollary and the proposition, we may observe that one obtains the Chern character

$$Ch_i : \pi_* \bar{\mathbf{K}}(\ ) \to \pi_* \mathbb{H}\big(\ , Sp\big(\Gamma(di)[di]\big)\big)_{\mathbb{Q}} \tag{5.0.7}$$

as a map of presheaves on the site (algebraic stacks/S)$_{iso.et}$. The above Chern character $Ch = \prod_i Ch_i$ provides $\prod_i \pi_* \mathbb{H}(\ , Sp(\Gamma(di)[di]))_{\mathbb{Q}}$ the structure of a presheaf of modules over $\pi_* \bar{\mathbf{K}}(\ )_{\mathbb{Q}}$.

5.0.8. The l-adic case

We pause to consider the l-adic situation here. Let $\mathcal{S}$ denote a given algebraic stack. We let $\{\mathbb{Z}/l^\nu(r)|\nu\}$ denote the obvious inverse system of l-adic sheaves on $\mathcal{S}_{iso.et}$ (or on $\mathfrak{M}_{et}$, if a coarse moduli space $\mathfrak{M}$ for $\mathcal{S}$ exists). One now forms the associated inverse system of presheaves of spaces $\{Sp_0(\mathbb{Z}/l^\nu(r)|\nu\}$, where $Sp_0(\mathbb{Z}/l^\nu(r))$ denotes the 0th term of the presheaf of spectra $Sp(\mathbb{Z}/l^\nu(r))$. Now one observes that the presheaf of spaces $holim_\nu \mathbb{H}(\ , Sp_0(\mathbb{Z}/l^\nu(r)))_{\mathbb{Q}}$ defines the continuous l-adic cohomology on taking the homotopy groups. One observes that the same computations as in [37] now define the l-adic Chern character $Ch : \bar{K}(\ )_0 \to holim_\nu \mathbb{H}(\ , Sp_0(\mathbb{Z}/l^\nu(r)))_{\mathbb{Q}}$ as a map of presheaves of spaces. We let Ch_i denote the ith component of the above Chern character.

Assume further that the algebraic stack $\mathcal{S}$ is of finite type over the base-scheme. We proceed to consider decompositions of $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ compatible with the Chern character considered above. First observe that if X is any scheme, the Adams operations ψ^k act on $\pi_*(\mathbf{K}(X))_{\mathbb{Q}}$ and are compatible with respect to pull-backs. Therefore, one obtains a decomposition of the presheaf $U \mapsto \pi_*(\mathbf{K}(U))_{\mathbb{Q}}$ into eigen-spaces for the action of the Adams operations: we will denote the eigen-space on which ψ^k acts by k^n as $\pi_*(\mathbf{K}(\ ))_{\mathbb{Q}}(i)$. When a coarse moduli space $\mathfrak{M}$ is assumed to exist (as before) for the stack $\mathcal{S}$, we therefore obtain a decomposition for each n:

$$\pi_n\big(\mathbf{K}(\ )_{\mathfrak{M}}\big)_{\mathbb{Q}} = \bigoplus_i \pi_n\big(\mathbf{K}(\ )_{\mathfrak{M}}\big)_{\mathbb{Q}}(i). \tag{5.0.9}$$

We would like a similar decomposition of $\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}})_{\mathbb{Q}}$. We proceed to consider this next.

Proposition 5.3. *Let $\mathcal{S}$ denote an algebraic stack as above. Then*

$$\pi_*\big(\mathbb{H}\big(\mathcal{S}_{iso.et}, \big(\mathbb{Z} \times \mathbb{Z}_\infty(B_\bullet GL)_{\mathbb{Q}}\big)\big)\big)$$

is a λ-ring. Therefore it has Adams operations defined on it. Moreover, it admits a decomposition

$$\pi_*\big(\mathbb{H}\big(\mathcal{S}_{iso.et}, \big(\mathbb{Z} \times \mathbb{Z}_\infty(B_\bullet GL)_\mathbb{Q}\big)\big)\big) \cong \bigoplus_i \pi_*\big(\mathbb{H}\big(\mathcal{S}_{iso.et}, \big(\mathbb{Z} \times \mathbb{Z}_\infty(B_\bullet GL)_\mathbb{Q}\big)\big)\big)(i)$$

into eigen-spaces for the Adams operations.

Proof. Given a presheaf P of spaces on $\mathcal{S}_{iso.et}$, there is a spectral sequence (see [25, Section 4]):

$$E_2^{s,t} = H_{iso.et}^s\big(\mathcal{S}, \pi_t(P_\mathbb{Q})\big) \Rightarrow \pi_{-s+t}\mathbb{H}_{iso.et}(\mathcal{S}, P_\mathbb{Q}).$$

The above spectral sequence converges *strongly* since the isovariant étale site has finite cohomological dimension with respect to sheaves of $\mathbb{Q}$-vector spaces. (See [25, Theorem 3.25].) One applies this to the natural map of presheaves $(\mathbb{Q} \times \mathbb{Z}_\infty(B_\bullet GL_N)_\mathbb{Q}) \to (\mathbb{Q} \times \mathbb{Z}_\infty(B_\bullet GL)_\mathbb{Q})$ and then take the colimit over $N \to \infty$ in the last spectral sequence to obtain the identification:

$$\underset{N \to \infty}{lim}\ \pi_*\big(\mathbb{H}\big(\mathcal{S}_{iso.et}, \big(\mathbb{Q} \times \mathbb{Z}_\infty(B_\bullet GL_N)_\mathbb{Q}\big)\big)\big) \simeq \pi_*\big(\mathbb{H}\big(\mathcal{S}_{iso.et}, \big(\mathbb{Q} \times \mathbb{Z}_\infty(B_\bullet GL)_\mathbb{Q}\big)\big)\big). \quad (5.0.10)$$

Next we will show one can define λ-operations as in [31, Section 4]. We will briefly recall this for the sake of completeness. Let $\rho : GL_N \to GL_M$ denote a representation of the group scheme GL_N. If $\mathbf{B}GL_N$ and $\mathbf{B}GL_M$ denote the associated simplicial sheaves on $\mathcal{S}_{iso.et}$, ρ induces a map $\mathbf{B}GL_N \to \mathbf{B}GL_M$. Recall these are presheaves of simplicial groups on $\mathcal{S}_{iso.et}$. Composing with the obvious map to $\mathbf{B}GL \to \mathbb{Z}_\infty \mathbf{B}GL$, ρ induces a map $\rho : \mathbf{Z} \times \mathbf{B}GL_N \to \mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL$, i.e. one obtains a map of abelian groups $R_\mathbb{Z}(GL_N) \to \pi_0(RMap(\mathbf{Z} \times \mathbf{B}GL_N, \mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL))$ where $RMap(\mathbf{Z} \times \mathbf{B}GL_N, \mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL)$ is defined to be $holim_\Delta\, Map(\mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL_N, G^\bullet(\mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL))$ and $R_\mathbb{Z}(GL_N)$ denotes the integral representation ring of the group scheme GL_N. (The functor $Map : (\text{simplicial presheaves on } \mathcal{S}_{iso.et}) \times (\text{simplicial presheaves on } \mathcal{S}_{et}) \to (\text{simplicial sets})$ is defined by $Map(F, K)_n = Hom_{\text{simplicial presheaves}}(F \times \Delta[n], K).$)

Therefore one obtains an *additive* homomorphism

$$r : \underset{\infty \leftarrow N}{lim}\ R_\mathbb{Z}(GL_N) \to \underset{\infty \leftarrow N}{lim}\ \pi_0\big(RMap(\mathbf{Z} \times \mathbf{B}GL_N, \mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL)\big)$$
$$\cong \underset{\infty \leftarrow N}{lim}\ \pi_0\big(RMap(\mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL_N, \mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL)\big)$$

where the last isomorphism follows from the universal property of the Bousfield–Kan completion $\mathbb{Z}_\infty$. Now $r(\{\lambda^n(id_{GL_N} - N) | N\})$ defines a compatible collection of homotopy classes of maps $\lambda^n : \mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL_N \to \mathbf{Z} \times \mathbb{Z}_\infty \mathbf{B}GL$ of presheaves. (Here $id_{GL_N}(N)$ denotes the identity representation (N times the trivial representation, respectively).) In view of the isomorphism in (5.0.10) above, it follows that on taking $\pi_*\mathbb{H}(\mathcal{S}_{iso.et},\ )$, the map λ^n induces the lambda operation $\lambda^n : \pi_*(\mathbb{H}(\mathcal{S}_{iso.et}, (\mathbb{Q} \times \mathbb{Z}_\infty(B_\bullet GL)_\mathbb{Q})))$. To prove that one obtains the usual relations on the λ^ns, one reduces to showing they hold on $lim_{\infty \leftarrow N}\ \pi_0(RMap(\mathbf{Z} \times \mathbf{B}GL_N^+, \mathbf{Z} \times \mathbf{B}GL^+))$: this follows readily from the fact they hold on the representation ring $lim_{\infty \leftarrow N}\ R_\mathbb{Z}(GL_N)$. The existence of the Adams operations is now a formal consequence. To obtain the last assertion it suffices to show that every class α in the above λ ring is nilpotent, i.e. for each class α, there exists an $n \gg 0$ so that $\lambda^n(\alpha) = 0$. In view of the isomorphism in (5.0.10), one may assume $\alpha \boldsymbol{\varepsilon} \pi_*(\mathbb{H}(\mathcal{S}_{iso.et}, (\mathbb{Q} \times \mathbb{Z}_\infty(B_\bullet GL_N)_\mathbb{Q})))$ for some N. Then the observation that $id_{GL_N} - N$ lies in the augmentation ideal of $R_\mathbb{Z}(GL_N)$ shows that $id_{GL_N} - N$ is λ-nilpotent (see [15, Proposition 8]

or [31]). Hence $\lambda^n(\alpha) = 0$ for $n \gg 0$; now a standard argument (see [31]) completes the proof in view of the definition of the λ-operations. $\quad\square$

Now there is a natural augmentation $\pi_*(\bar{K}(\mathcal{S}))_{\mathbb{Q}} \to \pi_*(\mathbb{H}(\mathcal{S}_{iso.et}, (\mathbb{Q} \times \mathbb{Z}_\infty(B_\bullet GL)_{\mathbb{Q}})))$. We take the inverse images of the components of the target in the above decomposition to define a decomposition of $\pi_*(\bar{K}(\mathcal{S}))_{\mathbb{Q}}$. Clearly this decomposition is contravariantly functorial on $\mathcal{S}_{iso.et}$ and is compatible with the Chern character into cohomology theories that are defined on the isovariant étale sites of algebraic stacks. For each n, we denote this decomposition as:

$$\pi_n(\bar{\mathbf{K}}(\ )_{\mathcal{S}})_{\mathbb{Q}} = \bigoplus_i \pi_n(\bar{\mathbf{K}}(\ )_{\mathcal{S}})_{\mathbb{Q}}(i). \tag{5.0.11}$$

5.0.12. Given the above decompositions of $\pi_*(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}}$ and $\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}})_{\mathbb{Q}}$ one may define an induced decomposition on $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ as follows.

Consider first the case when a coarse moduli space $\mathfrak{M}$ is assumed to exist for the given algebraic stack $\mathcal{S}$. For each $i \geqslant 0$, let $\pi_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}(i)$ be defined by the co-Cartesian square:

$$
\begin{array}{ccc}
\pi_*(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}}(i) & \longrightarrow & \pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}(i) \\
\uparrow & & \uparrow \\
\pi_*(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}} & \longrightarrow & \pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}
\end{array}
\tag{5.0.13}
$$

where $\pi_*(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}}(i)$ is the eigen-space with weight k^i for the action of ψ^k. Since $\pi_*(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}}(i)$ splits off $\pi_*(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}}$ and for each integer n,

$$\pi_n(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}} \cong \bigoplus_{i \geqslant 0} \pi_n(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}}(i),$$

it follows that $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}(i)$ splits off $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ and for each fixed integer n,

$$\pi_n(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \cong \bigoplus_{i \geqslant 0} \pi_n(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}(i).$$

Making use of (5.0.11), one obtains a similar decomposition of $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$.

Observe, as a result that $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ is a presheaf of *bi-graded* rings: the index denoting n in π_n will be called the *degree* while the index denoting i in the decomposition considered above will be called the *weight*.

At this point there are several alternate definitions of Bredon cohomology and homology each having its own advantages and defects. The following choice is more or less forced on us if our primary goals are to define theories that reduce to the usual theories when the stacks are schemes and to prove Riemann–Roch theorems for dg-stacks. See the discussion in 5.1.3 for possible alternate formulations.

Definition 5.4. Let S denote a given base scheme (or algebraic space) and let $\{\Gamma(r)|r\}$, $\{\Gamma^h(s)|s\}$ denote a collection of complexes of sheaves as in Definition 3.1 *on the category* (*alg.spaces*$/S$). Observe that the hyper-cohomology

$$\pi_*\big(\mathbb{H}(\,,Sp(\Gamma(\bullet)))\big)=\prod_{n,r}\pi_n\big(\mathbb{H}(\,,Sp(\Gamma(r))))\big)\quad\text{and}$$

$$\pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h(\bullet)))\big)=\prod_{n,r}\pi_n\big(\mathbb{H}(\,,Sp(\Gamma^h(r))))\big)$$

are also presheaves of bi-graded abelian groups. *The hyper-cohomology* $\mathbb{H}(\,,Sp(\Gamma(r)))$ *and* $\mathbb{H}(\,,Sp(\Gamma^h(r)))$ *denote hyper-cohomology computed on the isovariant étale site of the stack* $\mathcal{S}$. In this situation, we define

$$K\Gamma_{\mathcal{S}}^h(\bullet)=\mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\,))_{\mathcal{S}\mathbb{Q}}}\big(\pi_*(p_*\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}),\pi_*(\mathbb{H}(\,,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})\big)$$

where $\mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\,))_{\mathcal{S}\mathbb{Q}}}$ denotes the internal hom in the category of *presheaves on* $\mathcal{S}_{iso.et}$ *of modules over* $\pi_*(\bar{\mathbf{K}}(\,))_{\mathcal{S}\mathbb{Q}}$.

$$K\Gamma_{\mathcal{S}}(\bullet)=\pi_*(p_*\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}})\otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})}\pi_*\mathbb{H}(\,,Sp\Gamma(\bullet))_{\mathbb{Q}}$$

where the tensor product $\otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})}$ denotes the tensor product of *presheaves of modules over the presheaf of graded rings* $\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})$. (Recall from 2.1.3 that $p:\mathcal{S}_{smt}\to\mathcal{S}_{iso.et}$ is the obvious map of sites. Here π_* denotes the homotopy groups of the spectra considered above.)

5.0.14. Here we invoke the definition (9.0.8) to define $K\Gamma_{\mathcal{S}}^h(\bullet)$ with $\mathcal{A}=\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})$, $M=\pi_*(p_*(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}))$, $N=\pi_*(\mathbb{H}(\,,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})$, $m^*=\mathcal{H}om(\lambda_M,N)$ and $n_*=\mathcal{H}om(\mathcal{A}\otimes M,\lambda_N)$, $\lambda_M=$ the obvious map

$$\pi_*\big(p_*(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}})\big)\otimes\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})\to\pi_*\big(p_*(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}})\big)$$

given by the obvious module structure and $\lambda_N=$ the pairing

$$\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})\otimes\pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}\big)\to\pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}\big)$$

given by multiplication with $\pi_*(Ch)$. The presheaf $K\Gamma_{\mathcal{S}}(\bullet)$ is defined similarly making use of the definition (9.0.7). Now $K\Gamma_{\mathcal{S}}(\bullet)$ define presheaves of $\mathbb{Q}$-vector spaces on the site $\mathcal{S}_{iso.et}$. Observe that the presheaves $K\Gamma_{\mathcal{S}}(\bullet)$ ($K\Gamma_{\mathcal{S}}^h(\bullet)$) get an induced decomposition into bi-graded components (filtration indexed by degree and weight, respectively), induced from the decompositions in (5.0.13) above, the corresponding decomposition of the presheaves $\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})$ and the decomposition of the hyper-cohomology

$$\pi_*\big(\mathbb{H}(\,,Sp(\Gamma(\bullet)))\big)=\bigoplus_{n,t}\pi_n\mathbb{H}(\,,Sp(\Gamma(t))),$$

$$\pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h(\bullet)))\big)=\bigoplus_{n,t}\pi_n\mathbb{H}(\,,Sp(\Gamma^h(t))).$$

Definition 5.5 *(Bredon cohomology and homology for general algebraic stacks).* Assume the above situation. Let $\mathcal{S}$ denote an algebraic stack. We let *the total Bredon cohomology* be defined by

(i) $H_{Br}(\mathcal{S}, \Gamma(\bullet)) = \Gamma(\mathcal{S}; K\Gamma_{\mathcal{S}}(\bullet)) = \pi_*\mathbf{K}((\mathcal{S}, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \otimes_{\pi_*(\bar{\mathbf{K}}(\mathcal{S})_{\mathbb{Q}})} \pi_*(\mathbb{H}(\mathcal{S}, Sp(\Gamma(\bullet)))_{\mathbb{Q}})$

and *the total Bredon homology* be defined by

(ii) $H^{Br}(\mathcal{S}, \Gamma^h(\bullet)) = \Gamma(\mathcal{S}; K\Gamma_{\mathcal{S}}^h(\bullet)) = \Gamma(\mathcal{S}, \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}),$
  $\pi_*\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$.

We let

(iii) $H_{Br}^s(\mathcal{S}, \Gamma(t)) = Gr_{-s,t}(\mathbb{H}_{Br}(\mathcal{S}, \Gamma(\bullet))) = Gr_{-s,t}(\Gamma(\mathcal{S}, \pi_*(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \otimes_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathbb{Q}})}$
  $\pi_*\mathbb{H}(\ , Sp(\Gamma(\bullet))_{\mathbb{Q}})))$ and
(iv) $H_s^{Br}(\mathcal{S}, \Gamma^h(t)) = Gr_{s,t}(\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet))) = Gr_{s,t}(\Gamma(\mathcal{S}, \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}),$
  $\pi_*\mathbb{H}(\ , Sp(\Gamma^h(\bullet))))_{\mathbb{Q}}))$.

Here $Gr_{p,q}$ denotes the associated graded term with p denoting the degree and q denoting the weight. For cohomology the term $Gr_{-s,t}$ has contributions from $\pi_n(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}(m)$ and $\pi_{-s-n}\mathbb{H}(\ , Sp(\Gamma(\bullet)))_{\mathbb{Q}}(t - m)$. For homology, the term $Gr_{s,t}$ has contributions from

$$\pi_n(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}})(m) \quad \text{and} \quad \pi_{n+s}(\mathbb{H}(\ , Sp(\Gamma^h(\bullet))))_{\mathbb{Q}}(m + t).$$

The hyper-cohomology $\mathbb{H}(\mathcal{S}, Sp(\Gamma(\bullet)))$ and $\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))$ are computed on the isovariant étale site.

5.0.15. As observed in 9.0.9, the induced filtration

$$\{F_{p,q} | p, q\} \quad \text{on} \quad \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}), \pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$

has the property that the natural map

$$F_{p,q}\mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}), \pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$
$$\rightarrow \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}), \pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$

is a split monomorphism. Therefore, one may define a map $H^{Br}(\mathcal{S}, \Gamma^h(\bullet)) \rightarrow \prod_{s,t} H_s^{Br}(\mathcal{S}, \Gamma(t))$.

Example 5.6. As one of the simplest examples one may consider *toric stacks* over algebraically closed fields, i.e. X is a smooth projective toric variety defined over an algebraically closed field k and T is the dense torus. Now one may consider the associated quotient stack $\mathcal{S} = [X/T]$: such quotient stacks are what are often called toric stacks. Observe that except in trivial cases, the coarse moduli space does not exist.

For example, one may take $X = \mathbb{P}^1$ viewed as a toric variety for the torus $T = \mathbb{G}_m$. Let X_0 denote the open dense orbit. In this case one may verify readily that there are no isovariant étale maps to $[X_0/T]$ except finite disjoint copies of $[X_0/T]$ mapping to $[X_0/T]$ in the obvious way. To see that the same holds for $[\mathbb{A}^1/T]$ consider an isovariant étale surjective map $\phi : \mathcal{S}' \rightarrow [\mathbb{A}^1/T]$. As shown in Example 2.3, $\mathcal{S}' = [Y/T]$ for some algebraic space with a T-action so that the map $\phi : Y \rightarrow \mathbb{A}^1$ is equivariant and induces isomorphisms on the stabilizer groups. Now

$Y \times_{\mathbb{A}^1} X_0$ is a finite disjoint union of copies of X_0; the closures of these will be irreducible components of Y mapping étale surjectively onto $\mathbb{A}^1$. Therefore it is easy to see these irreducible components are in fact connected components, i.e. they do not intersect. Thus the only isovariant étale maps $S' = [Y/T] \to [\mathbb{A}^1/T]$ are finite disjoint unions of $[X_0/T]$, $[\mathbb{A}^1/T]$. From this one may show that all hyper-coverings of $[\mathbb{A}^1/T]$ in the isovariant étale site are dominated by the trivial hyper-covering which is $[\mathbb{A}^1/T]$ in each degree and with structure maps the identity. It will follow from this that $\mathbb{H}^*_{iso.et}([\mathbb{A}^1/T], \mathbb{Q}_l) = \mathbb{Q}_l$. Clearly $\mathbb{H}^*_{iso.et}([X_0/T], \mathbb{Q}_l) = \mathbb{Q}_l$ also. Therefore a Mayer–Vietoris sequence will show $\bar{\mathbb{H}}^*_{iso.et}([X/T], \mathbb{Q}_l) \cong \mathbb{Q}_l$ as well.

One may similarly conclude that $\bar{\mathbf{K}}([X_0/T]) \simeq K(Spec\,k)$ and $\bar{\mathbf{K}}([\mathbb{A}^1/T]) \simeq K(Spec\,k)$. *Assuming the existence of a Mayer–Vietoris sequence* for $\bar{\mathbf{K}}(\ )$, one may then conclude $\bar{\mathbf{K}}([X/T]) \simeq \bar{\mathbf{K}}(Spec\,k)$ as well. (See 5.1.3(3) to see how one may circumvent this issue.) Finally one readily computes $\pi_*(\mathbf{K}([X/T])) \cong \pi_* K(\mathbb{P}^1, T) \cong R(T) \otimes \pi_*(K(Spec\,k)) \otimes \pi_0(K(\mathbb{P}^1))$. Therefore (assuming the existence of a Mayer–Vietoris sequence for $\bar{\mathbf{K}}(\ )$), one obtains:

$$H^*_{Br}\big([X/T], \mathbb{Q}_l\big) \cong R(T) \otimes \pi_0\big(K\big(\mathbb{P}^1\big)\big) \otimes \mathbb{Q}_l \quad \text{while}$$
$$H^{Br}_*\big([X/T], \mathbb{Q}_l\big) \cong Hom\big(R(T) \otimes \pi_0\big(K\big(\mathbb{P}^1\big)\big), \mathbb{Q}_l\big).$$

One may want to *contrast* this with the computation of the Borel-style T-equivariant cohomology of $\mathbb{P}^1$: this is $H^*(ET \times_T \mathbb{P}^1, \mathbb{Q}_l) \cong H^*(BT, \mathbb{Q}_l) \otimes \mathbb{H}^*(\mathbb{P}^1, \mathbb{Q}_l)$. Thus the key difference between the two equivariant cohomology theories is the factor $R(T)$ in the place of $H^*(BT)$.

5.1. Cohomology–homology theories when a coarse moduli space exists

In this case we may adopt the following alternate formulation of cohomology and homology theories.

Definition 5.7 (*Bredon-style cohomology and homology for algebraic stacks when a coarse moduli space exists*). Let $\{\Gamma(r)|r\}$, $\{\Gamma^h(s)|s\}$ denote a collection of complexes of sheaves as in Definition 3.1 *on the category* (*alg.spaces/S*). If S is an algebraic stack with coarse moduli space $\mathfrak{M}$ belonging to the former category (with $p : S \to \mathfrak{M}$ denoting the obvious map), we define the *presheaves on* $\mathfrak{M}_{et}$

$$K\Gamma^h_S(\bullet) = \mathcal{H}om_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})}\big(\pi_*\big(p_*\big(\mathbf{K}(\ , \mathcal{A})_{S\mathbb{Q}}\big)\big), \pi_*\big(\mathbb{H}\big(\ , Sp\big(\Gamma^h(\bullet)\big)\big)_{\mathbb{Q}}\big)\big) \quad \text{and}$$
$$K\Gamma_S(\bullet) = \pi_*\big(p_*\big(\mathbf{K}(\ , \mathcal{A})_{S\mathbb{Q}}\big)\big) \otimes_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})} \pi_*\big(\mathbb{H}\big(\ , Sp\,\Gamma(\bullet)\big)_{\mathbb{Q}}\big) \qquad (5.1.1)$$

where the hyper-cohomology $\mathbb{H}(\ , Sp(\Gamma(\bullet)))$ and $\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))$ are *computed on the étale site of the coarse moduli space* $\mathfrak{M}$ *associated to the stack* S. (Once again we make use of the pairings as in 5.0.14 to define these presheaves.)

Now we define the *total Bredon cohomology* and *total Bredon homology* as follows:

(i) $H_{Br}(S, \Gamma(\bullet)) = \Gamma(\mathfrak{M}; K\Gamma_S(\bullet)) = \pi_*(\mathbf{K}(S, \mathcal{A})_{S\mathbb{Q}}) \otimes_{\pi_*(\mathbf{K}(\mathfrak{M})_{\mathfrak{M}\mathbb{Q}})} \pi_*(\mathbb{H}(\mathfrak{M}, Sp(\Gamma(\bullet))_{\mathbb{Q}}))$,
(ii) $H^{Br}(S, \Gamma^h_S(\bullet)) = \Gamma(\mathfrak{M}; K\Gamma^h_S(\bullet)) = \Gamma(\mathfrak{M}; \mathcal{H}om_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})}(\pi_*(p_*(\mathbf{K}(\ , \mathcal{A})_{S\mathbb{Q}})),$
     $\pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})))$.

Apart from this change the remaining definitions in Definition 5.5 carry over.

In this situation, one may define a map $\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet)) \to \prod_{s,t} \mathbb{H}_s^{Br}(\mathcal{S}, \Gamma(t))$ as before. We will show in Section 6, that this theory satisfies all the properties (i) through (vi) in Theorem 1.1. However, the remaining properties will be satisfied by homology only if $\mathfrak{M}$ is also *smooth* or only if one works at the level of Grothendieck groups: see [13, Theorem 18.2]. Therefore we will modify the definition of the presheaves $K\Gamma_{\mathcal{S}}^h(\bullet)$ as follows to handle situations where $\mathfrak{M}$ *need not* be smooth and when one wants Riemann–Roch for higher G-theory.

Definition 5.8. Assume the situation as in Definition 5.7. Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a fixed closed immersion into a smooth quasi-projective scheme. We define the *presheaf on* $\mathfrak{M}_{et}$

$$K\Gamma_{\mathcal{S}}^h(\bullet) = \mathcal{H}om_{i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*\big(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}\big)\big), \pi_*\big(\mathbb{H}\big(\ , Sp\big(\Gamma^h(\bullet)\big)\big)_{\mathbb{Q}}\big)\big) \quad \text{and}$$

$$K\Gamma_{\mathcal{S}}(\bullet) = \pi_*\big(p_*\big(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}\big)\big) \otimes_{i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})} \pi_*\big(\mathbb{H}\big(\ , Sp\Gamma(\bullet)\big)_{\mathbb{Q}}\big). \tag{5.1.2}$$

Observe that the action of the Adams' operations on $\pi_*(K(\ )_{\mathfrak{M}})_Q$ is compatible with the ones on $\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}$. Therefore we will define the total Bredon homology and cohomology as before and obtain a decomposition of these groups as before.

Examples 5.9. There are several examples we consider here. In all the examples we will consider quotient stacks associated to quasi-projective schemes over an algebraically closed field k. We will also assume that the dg-structure sheaf is the usual one, so that the dg-stack is simply an algebraic stack.

1. Let D denote a diagonalizable group scheme acting *trivially* on a quasi-projective scheme X all defined over k. Now the coarse moduli space X/D identifies with X. Moreover, the quotient stack $[X/D] \cong X \times [Spec\, k/D]$. Therefore, $\pi_*(K([X/D])) \cong R(D) \otimes \pi_*(K(X))$. It follows that for the versions of Bredon cohomology and homology considered in Definition 5.7 with $\mathcal{S} = [X/D]$, $H_{Br}^s([X/D], \Gamma(t)) \cong R(D) \otimes H_{et}^s(X, \Gamma(t))$ and $H_s^{Br}([X/D], \Gamma(t)) \cong Hom(R(D), H_s^{et}(X, \Gamma(t)) \otimes \mathbb{Q})$.

It is worth *contrasting* these with the Borel-style theories at least for the case of l-adic étale cohomology. One may readily see that $H^*(ED \times_D X, \mathbb{Q}_l) \cong H^*(BD, \mathbb{Q}_l) \otimes H_{et}^*(X, \mathbb{Q}_l)$. Thus the difference between the Borel-style and Bredon-style cohomology, at least (again) in this example, is that $H^*(BD, \mathbb{Q}_l)$ appears in the former in the place of $R(D)$.

2. Let G denote a smooth group scheme acting transitively on a quasi-projective scheme X with stabilizer H all over k. Therefore $X \cong G/H$. Now $[X/G] \cong [Spec\, k/H]$. Therefore $\pi_*(K([X/G])) \cong \pi_*(K([Spec\, k/H])) \cong \pi_*(K(Spec\, k, H)) = $ the H-equivariant K-theory of $Spec\, k$. The coarse moduli space, $\mathfrak{M}$, is clearly $Spec\, k$. Therefore

$$H_{Br}\big([X/G], \Gamma(\bullet)\big) \cong \pi_*\big(K(Spec\, k, H)\big) \otimes \mathbb{Q} \otimes_{\pi_*(K(Spec\, k)) \otimes \mathbb{Q}} H^*\big(Spec\, k, \Gamma(\bullet)\big) \otimes \mathbb{Q}.$$

In case H is diagonalizable as well, this identifies with $R(H) \otimes H^*(Spec\, k, \Gamma(\bullet)) \otimes \mathbb{Q}$. There is a similar description for Bredon homology where we may take $\tilde{\mathfrak{M}} = \mathfrak{M} = Spec\, k$.

3. Next assume G is a finite constant étale group scheme acting on a quasi-projective variety X again over k. In this case the coarse moduli space is the geometric quotient which is also quasi-projective. The total Bredon cohomology now is given by

$$H_{Br}\big([X/G], \Gamma(\bullet)\big) \cong \pi_* K(X, G) \otimes \mathbb{Q} \otimes_{\pi_*(K(X/G)) \otimes \mathbb{Q}} H_{et}^*\big(X/G, \Gamma(\bullet)\big) \otimes \mathbb{Q}.$$

Next suppose, in addition, that the G-action is *Galois*, i.e. the groupoid

$$X \times_{X/G} X \xrightarrow[pr_2]{\ pr_1\ } X$$

is isomorphic to the groupoid

$$G \times X \xrightarrow[pr_2]{\ \mu\ } X.$$

Then the category of G-equivariant vector bundles on X is equivalent by descent theory to the category of vector bundles on X/G. Therefore, in this case the Bredon cohomology identifies with $H^*_{et}(X/G, \Gamma(\bullet)) \otimes \mathbb{Q}$. There is a similar description for Bredon homology when we use Definition 5.7.

In the case of Deligne–Mumford stacks, we will also define the following variant. Given a Deligne–Mumford stack $\mathcal{S}$ defined over an algebraically closed field k with quasi-projective coarse moduli space $\mathfrak{M}$, we let $K_{et}(\mathcal{S}) = \mathbb{H}_{et}(\mathcal{S}, \mathbf{K}(\ )_{\mathcal{S},\mathbb{Q}})$ and the presheaf $K_{et}(\ )_{\mathcal{S}}$ be defined by $U \to K_{et}(U)$ with $U \in \mathcal{S}_{et}$. This will be useful in relating the Bredon homology theories with those of the inertia stacks (for Deligne–Mumford stacks) in the sense of [12,41,44]. Let $\mathcal{S}$ denote such an algebraic stack. Let $I_{\mathcal{S}}$ denote the associated inertia stack. Let $p_0 : I_{\mathcal{S}} \to \mathcal{S}$ denote the obvious map and let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a fixed closed immersion into a smooth quasi-projective scheme.

Definition 5.10 *(Relative étale form of cohomology and homology for inertia stacks, etc.).*

(i) $H_{Br\text{-}et}(I_{\mathcal{S}}/\mathcal{S}, \Gamma(\bullet)) = \Gamma(\mathfrak{M}, \pi_*(p_* p_{0*}(K_{et}(\ )_{I_{\mathcal{S}}\,\mathbb{Q}})) \otimes_{i^{-1}(\pi_*(K(\ ))_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}} \pi_*(\mathbb{H}(\ , Sp(\Gamma(\bullet)))_{\mathbb{Q}}));$

(ii) $H^{Br\text{-}et}(I_{\mathcal{S}}/\mathcal{S}, \Gamma(\bullet)) = \Gamma(\mathfrak{M}, \mathcal{H}om_{i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}(\pi_*(p_* p_{0*}(K_{et}(\ )_{I_{\mathcal{S}}\,\mathbb{Q}})),$
$\pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}));$

(iii) $H^{Br\text{-}et}(\mathcal{S}, \Gamma(\bullet)) = \Gamma(\mathfrak{M}, \mathcal{H}om_{i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}(\pi_*(p_*(K_{et}(\ )_{\mathcal{S}\,\mathbb{Q}})), \pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}));$

(iv) One now takes the decomposition of the K-theory presheaves as in (5.0.13). This induces a similar decomposition of $\pi_*(K_{et}(\ )_{\mathcal{S}})_{\mathbb{Q}}$. Making use of 9.0.9, one takes the associated graded terms as before to define the cohomology and homology groups $H^s_{Br\text{-}et}(I_{\mathcal{S}}/\mathcal{S}, \Gamma(t))$, $H^{Br\text{-}et}_s(I_{\mathcal{S}}/\mathcal{S}, \Gamma(t))$ and $H^{Br\text{-}et}_s(\mathcal{S}, \Gamma^h(t))$.

We will next consider Bredon-style *local cohomology* for algebraic stacks. We will always assume that the coarse moduli space is a quasi-projective scheme so that it admits a closed immersion into a regular scheme.

Definition 5.11 *(Bredon-style local cohomology).* Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a closed immersion of the moduli-space into a regular scheme. Recall that the complexes $\{\Gamma(r)|r\}$ are defined on (alg.spaces/S) and in particular on the étale site of $\tilde{\mathfrak{M}}$ as well. Therefore we may define the presheaf

$$\pi_*\big(i_*\big(p_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\,\mathbb{Q}})\big)\big) \otimes_{\pi_*(i_*(\mathbf{K}(\ )_{\mathfrak{M}\,\mathbb{Q}}))} \pi_*\big(\mathbb{H}(\ , i_* R i^! Sp(\Gamma(\bullet))_{\mathbb{Q}})\big)$$

on the Zariski site of $\tilde{\mathfrak{M}}$. (The presheaf $i_*\mathbf{K}(\ )_{\mathfrak{M}}$ is the presheaf of spectra defined on $\tilde{\mathfrak{M}}_{Zar}$ by extension by zero of the presheaf $\mathbf{K}(\ )_{\mathfrak{M}}$.) We will denote this presheaf by $i_*i^!(K\Gamma(\bullet))$ for convenience. Now we will let

$$\mathbb{H}^s_{Br,\mathcal{S}}\big(\tilde{\mathfrak{M}}, \Gamma(t)\big) = Gr_{-s,t}\big(\Gamma\big(\tilde{\mathfrak{M}}, i_*i^!K\Gamma(\bullet)\big)\big).$$

(The associated graded terms are defined as before.) Under the hypothesis in (iii)$'$ in Definition 3.1, one may show readily that this is independent of the chosen closed immersion.

5.1.3. Alternate definitions of Bredon-style cohomology and homology

There are several possible variations to the definitions we already provided. We proceed to consider some of these here rather briefly.

(1) One major variation is the following. Let $\mathcal{S}$ denote an algebraic stack with coarse moduli space $\mathfrak{M}$ which is quasi-projective and let $p : \mathcal{S} \to \mathfrak{M}$ denote the obvious map. Now we may define

$$K\Gamma^h_{\mathcal{S}}(\bullet) = \mathcal{R}\mathcal{H}om_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})}\big(\pi_*\big(p_*\big(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}\big)\big), \pi_*\big(\mathbb{H}(\ , Sp\big(\Gamma^h(\bullet)\big))_{\mathbb{Q}}\big)\big) \quad \text{and}$$

$$K\Gamma_{\mathcal{S}}(\bullet) = \pi_*\big(p_*\big(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}\big)\big) \otimes^L_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})} \pi_*\big(\mathbb{H}(\ , Sp\Gamma(\bullet))_{\mathbb{Q}}\big)$$

where the hyper-cohomology $\mathbb{H}(\ , Sp(\Gamma(\bullet)))$ and $\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))$ are *computed on the étale site of the coarse moduli space $\mathfrak{M}$ associated to the stack $\mathcal{S}$.* (Once again we make use of the pairings as in 5.0.14 to define these presheaves.) The derived functors are taken in the category of presheaves on $\mathfrak{M}_{et}$. (One may contrast this with the earlier definitions, where we do not take $\mathcal{R}\mathcal{H}om$, but only $\mathcal{H}om = $ an internal hom in the *category of presheaves on $\mathfrak{M}_{et}$ for defining $K\Gamma^h_{\mathcal{S}}(\bullet)$.* Similarly we do not take the left-derived functor of $\otimes_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})}$ in defining $K\Gamma_{\mathcal{S}}(\bullet)$.)

Now we may define the *Bredon cohomology* and *Bredon homology* spectra as follows:

(i) $\mathbb{H}_{Br}(\mathcal{S}, \Gamma(\bullet)) = R\Gamma(\mathfrak{M}; K\Gamma_{\mathcal{S}}(\bullet)) = R\Gamma(\mathfrak{M}; \pi_*(p_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}})) \otimes^L_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})}$
   $\pi_*(\mathbb{H}(\ , Sp(\Gamma(\bullet)))_{\mathbb{Q}}))$,

(ii) $\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h_{\mathcal{S}}(\bullet)) = R\Gamma(\mathfrak{M}; K\Gamma^h_{\mathcal{S}}(\bullet)) = \Gamma(\mathfrak{M}; \mathcal{R}\mathcal{H}om_{\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})}(\pi_*(p_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}})),$
   $\pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})))$.

Here the derived functors of Γ are taken on the étale site of $\mathfrak{M}$. Observe that the above objects are complexes of $\mathbb{Q}$-vector spaces and that their cohomology groups have a natural filtration induced by the decompositions of the K-theory presheaves and the cohomology presheaves. Therefore one may finally let

$$H^s_{Br}\big(\mathcal{S}, \Gamma(t)\big) = \bigoplus_{s=i-u} Gr_{u,t} H^i\big(\mathbb{H}_{Br}\big(\mathcal{S}, \Gamma(\bullet)\big)\big) \quad \text{and}$$

$$H^{Br}_s\big(\mathcal{S}, \Gamma^h(t)\big) = \prod_{s=-i+u} Gr_{u,t} H^i\big(\mathbb{H}^{Br}\big(\mathcal{S}, \Gamma^h(\bullet)\big)\big).$$

These definitions seem to be more in the spirit of the traditional Bredon-style cohomology and homology theories as we discussed in the introduction. However, there are

several *disadvantages*, the chief being that some of the properties considered in Theorem 1.1 will fail. For example, even if the stack $\mathcal{S}$ is a scheme, $K\Gamma_{\mathcal{S}}^h(\bullet)(K\Gamma_{\mathcal{S}}(\bullet))$ will not identify with $Sp(\Gamma^h(\bullet))$ ($Sp(\Gamma(\bullet))$, respectively) in general and therefore the resulting Bredon-style homology (cohomology) theories defined above will not reduce to the usual homology (cohomology) of schemes. Apart from this, these cohomology/homology theories satisfy the properties (i) through (iv) of Theorem 1.1.

(2) In this variant we may replace $\pi_*(p_*(\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}}))$ $(\pi_*(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}}))$ by $\pi_0(p_*(\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}}))$ $(\pi_0(\mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}})$, respectively) throughout the following discussion. Making use of Riemann–Roch at the level of Grothendieck groups for proper maps of quasi-projective schemes, the results of Section 8 will provide a Riemann–Roch theorem at the level of Grothendieck groups on dg-stacks.

(3) A possible variant of the definition of $K\Gamma(\bullet)$ and $K\Gamma^h(\bullet)$ as in Definition 5.4 is to use $\mathbb{H}_{iso.et}(\ ,\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}})$ $(\mathbb{H}_{iso.et}(\ ,\bar{\mathbf{K}}(\ )_{\mathcal{S}}))$ in the place of $\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}}$ $(\bar{\mathbf{K}}(\ )_{\mathcal{S}}$, respectively). This will be computationally preferable and will avoid the issue about the existence of a Mayer–Vietoris sequence in Example 5.6.

6. Proof of Theorem 1.1

We will adopt the following convention throughout.

Convention. If $\mathcal{S}$ is an algebraic stack ($f:\mathcal{S}' \to \mathcal{S}$ is a map of algebraic stacks), we may assume that it satisfies (both $\mathcal{S}$ and $\mathcal{S}'$ satisfy, respectively) the hypothesis that it has a coarse moduli space which exists as an algebraic space: in this case we will let $\{\Gamma^h(r)|r\}$ be any collection of complexes satisfying the general hypotheses in Definition 3.1. If the moduli spaces are not assumed to exist, we will need to assume the complexes $\{\Gamma^h(r)|r\}$ are defined as in Definition 4.5, i.e. they define continuous l-adic étale cohomology and homology. We will provide proofs of statements (i) through (vi) in detail only for the theories defined as in Definition 5.4. They readily extend to the variants in Definitions 5.7 and 5.8 and also to the variants considered in the remarks above.

Basic observations. When the moduli spaces are assumed to be quasi-projective, one may observe the following: if $f:\mathcal{S}' \to \mathcal{S}$ is a map of algebraic stacks and $\bar{f}:\mathfrak{M}' \to \mathfrak{M}$ is the corresponding map of the moduli spaces, one may find regular schemes $\tilde{\mathfrak{M}}'$ containing $\mathfrak{M}'$ as a closed sub-scheme ($\tilde{\mathfrak{M}}$ containing $\mathfrak{M}$ as a closed sub-scheme, respectively) and a map $\tilde{f}:\tilde{\mathfrak{M}}' \to \tilde{\mathfrak{M}}$ extending $\bar{f}$. The map $\tilde{f}$ may be chosen to be proper if the original map f is.

The hypothesis (iii) of Definition 3.1 which is assumed to hold on the big isovariant étale site shows that if $f:\mathcal{S}' \to \mathcal{S}$ is a map of algebraic stacks, there is an induced map $f^*:\pi_*\mathbb{H}(\ ,Sp(\Gamma_{\mathcal{S}}(r))) \to f_*\pi_*\mathbb{H}(\ ,Sp(\Gamma_{\mathcal{S}'}(r)))$ of presheaves for all r. Next observe that f also induces maps $f^*:K(U,\mathcal{A}) \to K(\mathcal{S}' \times_{\mathcal{S}} U,\mathcal{A}')$ (equivalently $f^*:\mathbf{K}_{\mathcal{S}}(\ ,\mathcal{A}) \to f_*\mathbf{K}_{\mathcal{S}'}(\ ,\mathcal{A}')$) and $f^*:\bar{K}(U) \to \bar{K}(\mathcal{S}' \times_{\mathcal{S}} U)$ (equivalently $f^*:\bar{\mathbf{K}}_{\mathcal{S}}(\ ) \to f_*\bar{\mathbf{K}}_{\mathcal{S}'}(\ ))$ of (symmetric) ring spectra, for $U \in \mathcal{S}_{iso.et}$. On taking the associated presheaves of homotopy groups, one obtains a map of the presheaves of graded rings.

To see that the induced maps on cohomology and homology preserve the weights as stated, one needs to observe first that if $f:(\mathcal{S}',\mathcal{A}') \to (\mathcal{S},\mathcal{A})$ is a map of dg-stacks, the induced map on K-theory presheaves $f^*:\pi_*\Gamma(U,\mathbf{K}_{\mathcal{S}}(\ ,\mathcal{A})) \to \pi_*\Gamma(\mathcal{S}' \times_{\mathcal{S}} U,\mathbf{K}_{\mathcal{S}'}(\ ,\mathcal{A}'))$ preserves weights, $U \in \mathcal{S}_{iso.et}$; this in turn follows from the observation that the induced map $f^*:\pi_*\Gamma(U,\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}}) \to \pi_*\Gamma(\mathcal{S}' \times_{\mathcal{S}} U,\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})$ (or in the presence of moduli-spaces, the in-

duced map $\bar{f}^*: \pi_* \Gamma(V, \mathbf{K}(\ )_{\mathfrak{M}\mathbb{Q}}) \to \pi_* \Gamma(\mathfrak{M}' \times_{\mathfrak{M}} V, \mathbf{K}(\ )_{\mathfrak{M}'\mathbb{Q}}), \ V \in \mathfrak{M}_{et})$ preserves weights. (See also (5.0.13).) In addition to this, one also needs to make use of the basic hypotheses on weights on the complexes $\{\Gamma(r)|r\}$ and $\{\Gamma^h(s)|s\}$ and how they behave as in Section 3. Consequently we observe that the map f induces the following map of presheaves of graded rings:

$$\pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) \otimes_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})} \pi_*\big(\mathbb{H}(\ , Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)$$
$$\to f_*\big(\pi_*\big(p'_*\mathbf{K}(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big)\big) \otimes_{f_*(\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}}))} f_*\pi_*\big(\mathbb{H}(\ , Sp(\Gamma_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)$$
$$\to f_*\big(\pi_*\big(p'_*\big(\mathbf{K}(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big)\big) \otimes_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})} \pi_*\big(\mathbb{H}(\ , Sp(\Gamma_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big).$$

These observations suffice to prove the contravariance and the ring structure on Bredon-style cohomology and these extend to local cohomology readily.

Now we consider the covariance property for Bredon homology. It suffices to show that, if $f : \mathcal{S}' \to \mathcal{S}$ is a proper map of algebraic stacks, one obtains an induced map of presheaves

$$f_* \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}\big(\pi_*\big(p'_*K(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big)$$
$$\xrightarrow{f_*} \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}\big(\pi_*\big(K(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)\big).$$

By (9.0.11) in Appendix A, this is adjoint to a map

$$f_* \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}\big(\pi_*\big(p'_*K(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big) \otimes_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}$$
$$\big(\pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big)\big) \to \pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big).$$

This map may be obtained as follows. One first observes there are natural maps $\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}})_{\mathbb{Q}} \to f_*(\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'})_{\mathbb{Q}})$ and $p_*(\pi_*(\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}) \to f_*p'_*(\pi_*(\mathbf{K}(\ ,\mathcal{A}')_{\mathcal{S}'})_{\mathbb{Q}})$ of presheaves of graded rings. Therefore, we obtain the following sequence of maps:

$$f_* \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}\big(\pi_*\big(p_*K(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big) \otimes_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})} \pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big)$$
$$\to f_* \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}\big(\pi_*\big(p'_*K(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big)$$
$$\otimes_{f_*\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})} f_*\pi_*\big(p'_*\mathbf{K}(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big)$$
$$\to f_*\big(\mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}\big(\pi_*\big(p'_*K(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big)$$
$$\otimes_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})} \pi_*\big(p'_*\mathbf{K}(\ ,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big)\big).$$

f_* composed with the obvious evaluation map defines a map from the last term to

$$f_*\pi_*\big(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big).$$

Finally, the hypothesis in definition 3.0.3(iii) shows there exists a natural map from the last term to $\pi_*(\mathbb{H}(\ , Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$. (Such a map exists in general for all Artin stacks, only for continuous

étale cohomology, while it exists for all the cohomology–homology theories we consider provided a coarse moduli space exists.) The composition of the above maps provides the required covariant functoriality of Bredon homology.

It will be important for later applications to observe that the composition of the above maps also factors as the following composition:

$$
\begin{aligned}
f_* \, \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}'\mathbb{Q}})} & \big(\pi_*\big(p'_*\mathbf{K}(\,,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big) \\
& \rightarrow \mathcal{H}om_{f_*(\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}'\mathbb{Q}}))}\big(f_*\pi_*\big(p'_*(\mathbf{K}(\,,\mathcal{A}')_{\mathcal{S}'\mathbb{Q}})\big), f_*\big(\pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}\big)\big)\big) \\
& \rightarrow \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})}\big(\pi_*\big(p_*\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)\big)
\end{aligned}
\tag{6.0.4}
$$

where the last map is defined by its adjoint as above.

The compatibility of the direct image maps in Bredon homology with the map

$$
\pi_*\big(\mathbb{H}^{Br}\big(\mathcal{S}, \Gamma^h(\bullet)\big)\big)_{\mathbb{Q}} \rightarrow \prod_t \mathbb{H}^{Br}_*\big(\mathcal{S}, \Gamma(t)\big)_{\mathbb{Q}}
$$

as in 5.0.15 follows from the basic observations above. (For example: the inverse image maps on K-theory preserve the weight filtrations considered above.)

Next we consider the third property. By (9.0.11) in Appendix A, it suffices to show that there exists a map:

$$
\begin{aligned}
\pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) & \otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})} \pi_*\big(\mathbb{H}(\,,Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big) \\
& \otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})} \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})}\big(\pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)\big) \\
& \otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})} \pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) \rightarrow \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big).
\end{aligned}
$$

We obtain such a map by composing the pairing

$$
\pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) \otimes \pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) \rightarrow \pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big)
$$

with the evaluation map

$$
\begin{aligned}
\pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) & \otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})} \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})}\big(\pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)\big) \\
& \rightarrow \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)
\end{aligned}
$$

which provides the map:

$$
\begin{aligned}
\pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) & \otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})} \pi_*\big(\mathbb{H}(\,,Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big) \\
& \otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})} \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})}\big(\pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)\big) \\
& \otimes_{\pi_*(\bar{\mathbf{K}}(\,)_{\mathcal{S}\mathbb{Q}})} \pi_*\big(\mathbf{K}(\,,\mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) \\
& \rightarrow \pi_*\big(\mathbb{H}(\,,Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big) \otimes \pi_*\big(\mathbb{H}(\,,Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big).
\end{aligned}
$$

Now we compose with the pairing:

$$Sp\big(\Gamma_{\mathcal{S}}(\bullet)\big) \otimes Sp\big(\Gamma_{\mathcal{S}}^{h}(\bullet)\big) \to Sp\big(\Gamma_{\mathcal{S}}^{h}(\bullet)\big)$$

to complete the required pairing. (One may readily verify the required associativity of the pairing.) This is the pairing for the theory defined using the presheaves in Definition 5.4 and an entirely similar argument works for the theories defined in Definitions 5.7 and 5.8. The pairing between local cohomology and homology is defined similarly.

The projection formula in (iv) may be derived as follows. Let $\mathcal{E} \in \pi_*(K(\mathcal{S}))_{\mathbb{Q}}$, $\alpha \in \pi_*(\mathbb{H}(\mathcal{S}_{iso.et}, Sp(\Gamma(\bullet))))$ and let

$$\phi \in \mathcal{H}om_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}'\mathbb{Q}})}\big(\pi_*\big(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}'\mathbb{Q}}\big), \pi_*\big(\mathbb{H}(\ , Sp(\Gamma_{\mathcal{S}'}^{h}(\bullet)))_{\mathbb{Q}}\big)\big).$$

Now

$$(\mathcal{E} \otimes \alpha) \circ f_*(\phi) = \bar{f}_*\big(\phi\big(f^*(\mathcal{E}) \circ (\ )\big)\big) \circ \alpha \in H_*^{Br}\big(\mathcal{S}, \Gamma(\bullet)\big)$$

denotes the composition of the maps in the left column and bottom row of the corresponding square applied to $\mathcal{E} \otimes \alpha \otimes \phi$. ($\circ$ denotes the appropriate pairings and $\bar{f}_*$ denotes the induced map $f_*\pi_*(\mathbb{H}_{iso.et}(\ , Sp(\Gamma^{h}(\bullet)))_{\mathcal{S}'})_{\mathbb{Q}} \to \pi_*(\mathbb{H}_{iso.et}(\ , Sp(\Gamma^{h}(\bullet)))_{\mathcal{S}})_{\mathbb{Q}}$.) The composition of the top row and right column applied to the same class, defines the class $\bar{f}_*(\phi(f^*(\mathcal{E}) \circ (\ )) \circ \bar{f}^*(\alpha)$. This identifies with the former class by the usual projection formula on $\bar{f}$.

Now we consider (v). Observe that in this case the presheaves $\bar{\mathbf{K}}(\ )_{\mathcal{S}}$, $\mathbf{K}(\ )_{\mathfrak{M}}$ and $\mathbf{K}(\ )_{\mathcal{S}} = \mathbf{K}(\ , \mathcal{A})_{\mathcal{S}}$ are identical, so that

$$\pi_*\big(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}\mathbb{Q}}\big) \otimes_{\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}\mathbb{Q}})} \pi_*\big(\mathbb{H}(\ , Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big) \simeq \pi_*\big(\mathbb{H}(\ , Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}\big)$$

thereby proving the assertion in (v) for cohomology. The reasoning for homology is similar when the definitions in Definitions 5.4 or 5.7 are used. This completes the proof of property (v).

Remark 6.1. Other possible alternate approaches to defining Bredon cohomology and homology as in 5.1.3 (except for 5.1.3(2)) will, in general, *fail* to satisfy this property.

The statement in (ix) follows from the homotopy property for K-theory for smooth objects. Observe that in this case the K-theory identifies with G-theory: see Remarks 2.10(3) and the hypothesis in Definition 3.1(v). In more detail: the presheaves $p_*(\mathbf{K}(\ ))$ and $i^{-1}(K(\ )_{\tilde{\mathfrak{M}}})$ on $\mathfrak{M}_{Zar}$ have the homotopy property where $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ is a closed immersion into a smooth scheme and the given stack is smooth. Now the presheaf $\pi_*(p_*(\mathbf{K}(\ ))) \otimes_{i^{-1}(\pi_*(\bar{\mathbf{K}}(\ )_{\tilde{\mathfrak{M}}}))} \pi_*\mathbb{H}(\ , Sp(\Gamma(\bullet)))$ also inherits the homotopy property. This proves the homotopy property for cohomology and the case of local cohomology is similar.

Now we consider (x). Let $\tilde{\mathcal{E}}$ denote a vector bundle on $\tilde{\mathfrak{M}}$ and let $\mathcal{E}$ denote its pull-back to the stack $\mathcal{S}$. Assume that $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are of rank $= n$. Let $\phi : \mathbb{P}(\mathcal{E}) \to \mathcal{S}$ and $\bar{\phi} : \mathbb{P}(\tilde{\mathcal{E}}_{|\mathfrak{M}}) \to \mathfrak{M}$, $p : \mathcal{S} \to \mathfrak{M}$ and $p_0 : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\tilde{\mathcal{E}}_{|\mathfrak{M}})$ denote the obvious maps. Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote the given closed immersion and $i_P : \mathbb{P}(\tilde{\mathcal{E}}_{|\mathfrak{M}}) \to \mathbb{P}(\tilde{\mathcal{E}})$ denote the induced closed immersion. Now the hypothesis

in Definition 3.1(vi) shows that there exists a canonical class $c_1(\tilde{\mathcal{E}}) \in H^d_{et}(\mathbb{P}(\tilde{\mathcal{E}}_{|\mathfrak{M}}); Sp\Gamma(1))$ and that the map $\sum_{i=0}^n \pi^*( ) \cap c_1(\tilde{\mathcal{E}})^i$ induces a quasi-isomorphism:

$$R\bar{\phi}_*\big(Sp\Gamma^h(\bullet)\big) \simeq \bigsqcup_{i=0}^n Sp\big(\Gamma^h(\bullet)\big) \tag{6.0.5}$$

where the derived functors are computed on the appropriate étale sites. One also obtains a quasi-isomorphism $\mathbb{H}_{et}(\mathbb{P}(\tilde{\mathcal{E}}_{|\mathfrak{M}}); Sp\Gamma^h(\bullet)) \simeq \bigsqcup_{i=0}^n \mathbb{H}_{et}(\mathfrak{M}; Sp\Gamma^h(\bullet))$ and similarly for $\Gamma(\bullet)$ in the place of $\Gamma^h(\bullet)$.

We compute the K-theory of a projective space bundle over a dg-stack in [26, Section 3]. There it is shown that the usual formula holds, i.e. the following result holds.

Proposition 6.2 *(K-theory of projective space bundles over dg-stacks). The maps $\phi^*( ) \otimes \mathcal{O}_{\mathbb{P}}(-i) : \mathbf{K}(\mathcal{S}, \mathcal{A}) \to K(\mathbb{P}(\mathcal{E}), \phi^*(\mathcal{A}))$ induce a weak-equivalence*:

$$\bigsqcup_{i=0}^{r-1} \mathbf{K}(\mathcal{S}, \mathcal{A}) \to \mathbf{K}\big(\mathbb{P}(\mathcal{E}), \phi^*(\mathcal{A})\big).$$

Clearly one also has the weak-equivalence:

$$\mathbf{K}\big(\mathbb{P}(\tilde{\mathcal{E}})\big) \simeq \bigsqcup_{i=0}^{r-1} \mathbf{K}(\tilde{\mathfrak{M}}).\big[\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-i)\big].$$

Moreover, the corresponding assertion holds when $\tilde{\mathfrak{M}}$ is replaced by an object $\tilde{U} \in \tilde{\mathfrak{M}}_{et}$ and when the stack $\mathcal{S}$ is replaced by the pull-back $\mathcal{S} \times_{\mathfrak{M}} U$, $U = \tilde{U} \times_{\tilde{\mathfrak{M}}} \mathfrak{M}$. Therefore, one obtains the isomorphism:

$$\pi_*\big(K\big(\mathbb{P}(\mathcal{E}), \phi^*(\mathcal{A})\big)\big) \cong \pi_*\big(K\big(\mathbb{P}(\tilde{\mathcal{E}})\big)\big) \otimes_{\pi_*(K(\tilde{\mathfrak{M}}))} \pi_*\big(K(\mathcal{S}, \mathcal{A})\big)$$

and the isomorphism of presheaves on $\mathfrak{M}_{Zar}$:

$$p_*\phi_*\big(\pi_*\big(\mathbf{K}(, \phi^*(\mathcal{A}))\big)_{\mathbb{P}(\mathcal{E})}\big) = \bar{\phi}_* p_{0*}\big(\pi_*\big(\mathbf{K}(, \phi^*(\mathcal{A}))\big)_{\mathbb{P}(\mathcal{E})}\big)$$

$$\simeq i^{-1}\big(\bar{\phi}_*\big(\pi_*\big(\mathbf{K}( )_{\mathbb{P}(\tilde{\mathcal{E}})}\big)\big)\big) \otimes_{i^{-1}\pi_*(\mathbf{K}( )_{\tilde{\mathfrak{M}}})} \pi_*\big(p_*\mathbf{K}(, \mathcal{A})_{\mathcal{S}}\big)$$

and similarly $\bar{\phi}_*\pi_*(\mathbb{H}_{et}(, Sp\Gamma^h(\bullet))) \simeq \bigoplus_{i=0}^n \pi_*(\mathbb{H}_{et}(, Sp\Gamma^h(\bullet)))$. Therefore,

$$\mathcal{H}om_{\bar{\phi}_* i_P^{-1}(\pi_*(\mathbf{K}( )_{\mathbb{P}(\tilde{\mathcal{E}})}))}\big(\bar{\phi}_* p_{0*}\big(\pi_*\big(\mathbf{K}(, \pi^*(\mathcal{A}))\big)_{\mathbb{P}(\mathcal{E})}\big)\big), \bar{\phi}_*\big(\pi_*\big(\mathbb{H}_{et}(; Sp\Gamma^h(\bullet))\big)\big)\big)$$

$$\simeq \mathcal{H}om_{i^{-1}(\pi_*(\mathbf{K}( )_{\tilde{\mathfrak{M}}}))}\big(\pi_*\big(p_*\mathbf{K}(, \mathcal{A})_{\mathcal{S}}\big), \bar{\phi}_*\big(\pi_*\big(\mathbb{H}_{et}( ); Sp\Gamma^h(\bullet)\big)\big)\big)$$

$$\simeq \bigoplus_{i=0}^n \mathcal{H}om_{i^{-1}\pi_*(\mathbf{K}( )_{\tilde{\mathfrak{M}}})}\big(\pi_*\big(p_*\mathbf{K}(, \mathcal{A})_{\mathcal{S}}\big), \pi_*\big(\mathbb{H}_{et}(; Sp\Gamma^h(\bullet))\big)\big).$$

Now the property (x) for Bredon cohomology follows immediately from its definition. The corresponding assertion there on homology now follows from the observation in (6.0.4) and the definition of Bredon homology. (Take p' (p, f) in (6.0.4) to be p_0 (p, $\bar{\phi}$, respectively).)

The rest of the discussion will be devoted to defining the Chern character map, the Riemann–Roch transformations and in establishing their properties.

We begin by *defining the Chern character*. In general, we let $Ch : \pi_*(\mathbf{K}(\mathcal{S})) \to H^*_{Br}(\mathcal{S}, \Gamma(\bullet))$ be defined by the (natural) map

$$\pi_*\big(K(\mathcal{S}, \mathcal{A})\big) = \pi_*\big(K(\mathcal{S}, \mathcal{A})\big) \otimes_{\pi_*(\bar{\mathbf{K}}_{\mathcal{S}}(\,))_{\mathbb{Q}}} \pi_*\big(\bar{\mathbf{K}}_{\mathcal{S}}(\,)\big)_{\mathbb{Q}}$$

$$\xrightarrow{id \otimes ch} \Gamma\big(\mathcal{S}, K\Gamma(\bullet)\big) = \Gamma\big(\mathcal{S}, \pi_*(\mathbf{K}_{\mathcal{S}}(\,, \mathcal{A}))_{\mathbb{Q}} \otimes_{\pi_*(\bar{\mathbf{K}}_{\mathcal{S}}(\,))_{\mathbb{Q}}} \pi_*\big(\mathbb{H}_{iso.et}(\,, \Gamma(\bullet))_{\mathbb{Q}}\big)\big) \quad (6.0.6)$$

where ch denotes the Chern character on $\pi_*(\bar{\mathbf{K}}_{\mathcal{S}}(\,))_{\mathbb{Q}}$. One may also define *a local Chern character* as follows in case a moduli-space exists as a quasi-projective scheme. We assume the situation of Definition 5.11. Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote the closed immersion into a smooth scheme. Recall

$$i_* i^! K\Gamma(\bullet) = i_*\big(\pi_*\big(p_*(\mathbf{K}(\,, \mathcal{A})_{\mathcal{S}})\big)_{\mathbb{Q}}\big) \otimes_{i_* \pi_*(\mathbf{K}(\,)_{\mathfrak{M}})_{\mathbb{Q}}} \pi_*\big(\mathbb{H}(\,, i_* Ri^! Sp(\Gamma(\bullet)))_{\mathbb{Q}}\big)$$

where $i_* \mathbf{K}(\,)_{\mathfrak{M}}$ is the presheaf defined on the site $\tilde{\mathfrak{M}}_{et}$ by $V \to \bar{\mathbf{K}}(V \times_{\tilde{\mathfrak{M}}} \mathfrak{M})$. Therefore, there is a map

$$Ch_{\mathcal{S}|\tilde{\mathfrak{M}}} : \pi_*\big(\mathbf{K}(\mathcal{S}, \mathcal{A})\big) \xrightarrow{id \otimes ch} \Gamma\big(\tilde{\mathfrak{M}}, i_* i^! K\Gamma(\bullet)\big). \quad (6.0.7)$$

Replacing $\tilde{\mathfrak{M}}$ everywhere by $\mathfrak{M}$ defines similarly a (natural) map

$$Ch : \pi_*\big(\mathbf{K}(\mathcal{S}, \mathcal{A})\big) \xrightarrow{id \otimes ch} \Gamma\big(\mathfrak{M}, K\Gamma(\bullet)\big). \quad (6.0.8)$$

Definition 6.3 *(Chern character and local Chern character).* We define the Chern character to be given by the map in (6.0.6) in general and by (6.0.8) when a coarse moduli space exists satisfying our hypotheses. The local Chern character with respect to a closed immersion $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ of the moduli-space into a smooth scheme is defined to be the map in (6.0.7).

Remarks 6.4. (1) Observe that, by replacing $\pi_*(\bar{\mathbf{K}}_{\mathcal{S}}(\,))_{\mathbb{Q}}$ by $\pi_*(K(\,)_{\mathfrak{M}})$ (by $i^{-1}(\pi_*(K(\,)_{\tilde{\mathfrak{M}}}))$ for a fixed closed immersion $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ into a smooth scheme) defines a Chern character for the cohomology theory defined using the presheaves in Definition 5.7 (Definition 5.8, respectively).

(2) To understand these Chern characters, one needs to observe that they define operations on the Bredon-homology groups. In this sense they are *operational Chern classes* (in the same spirit as the Chern classes considered in [13]). When viewed as operations on Bredon-homology we see from the properties below that they have all the expected properties. One may also see that, when the stack reduces to a scheme or algebraic space, it identifies with the usual Chern character.

Next we proceed to define the Riemann–Roch transformations as specific maps from the G-theory of a stack to its Bredon-homology. It will have the advantage that the Riemann–Roch transformation is defined also for singular stacks and makes intrinsic use of the Riemann–Roch

transformation at the level of the moduli spaces. Throughout the rest of this section we will assume that a moduli-space exists as a quasi-projective scheme and that the projection $p : \mathcal{S} \to \mathfrak{M}$ is proper and of finite cohomological dimension.

Proposition 6.5. *Let $\mathfrak{M}$ denote quasi-projective scheme and let $\{\Gamma(\bullet)\}$, $\{\Gamma^h(\bullet)\}$ denote a duality theory in the sense of Definition* 3.1 *that is defined on the category* (alg.spaces/S). *Then the Riemann–Roch transformation $\tau_{\mathfrak{M}} : \pi_* G(\mathfrak{M})_{\mathbb{Q}} \to \pi_*(\mathbb{H}_{et}(\mathfrak{M}, Sp(\Gamma^h(\bullet))))$ extends to a map of presheaves*:

$$\tau_{\mathfrak{M}} : \pi_*\big(G(\ )_{\mathfrak{M}}\big)_{\mathbb{Q}} \to \pi_*\big(\mathbb{H}_{et}\big(\ , Sp\big(\Gamma^h(\bullet)\big)\big)\big)_{\mathbb{Q}}$$

on $\mathfrak{M}_{et}$.

Proof. We will fix a closed immersion $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ with the latter *smooth* for the rest of the discussion. Given any presheaf P on $\mathfrak{M}_{et}$ we will consider its extension by zero, $i_* P$, on $\tilde{\mathfrak{M}}_{et}$. Moreover, given a presheaf P on $\tilde{\mathfrak{M}}_{et}$, we define $i^! P$ to be the homotopy fiber of the map $P \to j_*(P)$, where j is the complementary open immersion, complimentary to i. It follows from [25, Theorem 3.22], that for each U in $\mathfrak{M}_{et}$, there exists a (smooth) algebraic space $\tilde{U}$ in $\tilde{\mathfrak{M}}_{et}$ so that $\mathfrak{M} \times_{\tilde{\mathfrak{M}}} \tilde{U} \cong U$. For each U in $\mathfrak{M}_{et}$ and $\tilde{U}$ in $\tilde{\mathfrak{M}}_{et}$ with U closed in $\tilde{U}$, we may write the map $\tau_U = \tau_{\tilde{U}|U} : \pi_* G(U) \to \pi_*(\mathbb{H}_{et}(U, Sp(\Gamma^h(\bullet))))$ as the composition of the following maps:

$$\pi_* G(U)_{\mathbb{Q}} \simeq \pi_*\big(\text{homotopy fiber } \big(K(\tilde{U}) \to K(\tilde{U} - U)\big)\big)_{\mathbb{Q}} \xrightarrow{ch} \pi_*\big(\mathbb{H}_{et}\big(\tilde{U}, i_* Ri^! Sp\big(\Gamma^h(\bullet)\big)\big)\big)_{\mathbb{Q}}$$

$$\xrightarrow{\bigcap i_* i^*(Td_{\tilde{U}}) \circ \bigcap [\tilde{U}]} \pi_*\big(\mathbb{H}_{et}\big(U, Sp\big(\Gamma^h(\bullet)\big)\big)\big)_{\mathbb{Q}} \tag{6.0.9}$$

where *ch* denotes the local Chern character. Observe that *ch*, $Td_{\tilde{\mathfrak{M}}}$ (which denotes the Todd-class of $\tilde{\mathfrak{M}}$) and $[\tilde{\mathfrak{M}}]$ (which denotes the fundamental class of $\tilde{\mathfrak{M}}$) all localize on $\tilde{\mathfrak{M}}_{et}$ to denote the corresponding objects over $\tilde{U}$. Let $V \to U$ denote a map in $\mathfrak{M}_{et}$ and let $\tilde{V} \to \tilde{U}$ denote a map in $\tilde{\mathfrak{M}}_{et}$ so that $V = \tilde{V} \times_{\tilde{\mathfrak{M}}} \mathfrak{M}$ and $U = \tilde{U} \times_{\tilde{\mathfrak{M}}} \mathfrak{M}$. Now the following diagram commutes:

$$
\begin{array}{ccccc}
\pi_*(G(U)) & \xrightarrow{ch} & \pi_*(\mathbb{H}_{et,U}(\tilde{U}, Sp(\Gamma(\bullet)))) & \xrightarrow{\bigcap Td_{\tilde{U}|U} \circ \bigcap [\tilde{U}]} & \pi_*(\mathbb{H}_{et}(U, Sp(\Gamma^h(\bullet)))) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_*(G(V)) & \xrightarrow{ch} & \pi_*(\mathbb{H}_{et,V}(\tilde{V}, Sp(\Gamma(\bullet)))) & \xrightarrow{\bigcap Td_{\tilde{V}|V} \circ \bigcap [\tilde{V}]} & \pi_*(\mathbb{H}_{et}(V, Sp(\Gamma^h(\bullet)))).
\end{array}
$$

Next let $V = U$ as above and let $\tilde{V}$ denote another object in $\tilde{\mathfrak{M}}_{et}$, so that $\tilde{V}$ dominates $\tilde{U}$ in $\tilde{\mathfrak{M}}_{et}$ and contains U as a closed sub-scheme. Then the middle column is an isomorphism by excision: see Definition 3.1(ix). This shows that the map in the top row of the above diagram depends only on U. $\quad\square$

Remark 6.6. When we define $\tau_{\mathfrak{M}}$ with respect to a closed immersion of $\mathfrak{M}$ in $\tilde{\mathfrak{M}}$, we will often denote $\tau_{\mathfrak{M}}$ by $\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}$.

6.0.10. Behavior of $\tau_{\mathfrak{M}}$ with respect to the Chern character

Next we observe that the map $\tau_{\mathfrak{M}}$ in (6.0.9) is a map of modules over $i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}}))$ if $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ is a closed immersion into a smooth scheme: see [3]. In particular, it will be a map of modules over $\pi_*(K(\ )_{\mathfrak{M}})$ if $\mathfrak{M}$ itself is smooth or if we restrict to Grothendieck groups only: the latter follows from [13, Theorem 18.2].

Now we fix a closed immersion $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ into a smooth quasi-projective scheme. We then obtain the following key identification:

$$
\begin{aligned}
\mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})} & \big(\pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*K(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \\
& \pi_*\big(\mathbb{H}(\ ,Sp(\Gamma^h(\bullet)))\big)_{\mathbb{Q}}\big)\big) \\
\simeq \mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})} & \big(\pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})} \pi_*\big(p_*K(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \\
& \pi_*\big(\mathbb{H}(\ ,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}\big)\big).
\end{aligned}
\tag{6.0.11}
$$

The term $\pi_*(p_*G(\ ,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \otimes_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})} \pi_*(K(\ ,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ makes use of the module-structure of $\pi_*(p_*G(\ ,\mathcal{A})_{\mathcal{S}})$ and $\pi_*(p_*K(\ ,\mathcal{A})_{\mathcal{S}})$ over $i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}}))$. This enables us to define one step of the Riemann–Roch transformation as the following map.

Definition 6.7. We define the map of (complexes of) presheaves on $\mathfrak{M}_{et}$:

$$
\tau'_{\mathcal{S},\mathfrak{M}} : \pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} \to \mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \pi_*\big(\mathbb{H}(\ ,Sp(\Gamma^h(\bullet))_{\mathbb{Q}})\big)\big)
$$

as the map corresponding under the adjunction in (6.0.11) to the following map:

$$
\begin{aligned}
\pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} & \otimes_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})} \pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} \\
& \to \pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} \xrightarrow{p_*} \pi_*\big((G(\ )_{\mathfrak{M}})\big)_{\mathbb{Q}} \xrightarrow{\tau_{\mathfrak{M}}} \pi_*\big(\mathbb{H}(\ ,Sp(\Gamma^h(\bullet))_{\mathbb{Q}})\big)
\end{aligned}
$$

where the first map is given by the module structure of $G(\ ,\mathcal{A})_{\mathcal{S}}$ over $K(\ ,\mathcal{A})_{\mathcal{S}}$, the second is the push-forward by p and the third map is the Riemann–Roch transformation $\tau_{\mathfrak{M}}$ defined on $\mathfrak{M}$. (Recall from our hypotheses that p is of finite cohomological dimension.)

Remarks 6.8. (1) Observe that the map $\tau'_{\mathcal{S},\mathfrak{M}}$ admits the following alternate description. First the composition of the maps

$$
\begin{aligned}
\pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} & \otimes_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})} \pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} \xrightarrow{p_*} \pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} \\
& \xrightarrow{p_*} \pi_*\big((G(\ )_{\mathfrak{M}})\big)_{\mathbb{Q}}
\end{aligned}
$$

defines a map

$$
\pi_*\big(p_*G(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}} \to \mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \pi_*\big(G(\ )_{\mathfrak{M}}\big)_{\mathbb{Q}}\big).
\tag{6.0.12}
$$

Next we compose this with $\mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}(id, \tau_{\mathfrak{M}})$.

(2) First the projection formula implies the map p_* is a map of module-spectra over $K(\ )_{\mathfrak{M}}$. Now the observation in 6.0.10 shows the map $\tau_{\mathfrak{M}}$ is a map of $i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}}))_{\mathbb{Q}}$-modules so that the adjunction as in (6.0.11) applies (see (9.0.11) in Appendix A).

Definition 6.9 *(The Riemann–Roch transformations).* Now we pre-compose $\tau'_{\mathcal{S},\mathfrak{M}}$ with the obvious augmentation $\pi_*(G(\mathcal{S},\mathcal{A})) \to \Gamma(\mathfrak{M}, \pi_*(p_*G(\ ,\mathcal{A})_{\mathcal{S}}))_{\mathbb{Q}}$ to define the Riemann–Roch transformation

$$\tau_{\mathcal{S}} : \pi_*\big(G(\mathcal{S},\mathcal{A})\big) \to H_*^{Br}\big(\mathcal{S}, \Gamma^h(\bullet)\big).$$

Remark 6.10. Now the proof of the compatibility of the Chern character and the Riemann–Roch transformations as in Theorem 1.1(vii) follows immediately from the definitions.

Proposition 6.11. *Let* $\bar{i}: \tilde{\mathfrak{M}} \to \hat{\mathfrak{M}}$ *denote a closed immersion of smooth algebraic spaces containing* $\mathfrak{M}$ *as a closed sub-scheme. Let* $i: \mathfrak{M} \to \tilde{\mathfrak{M}}$ *and* $\hat{i} = \bar{i} \circ i$. *Then one has a natural map*

$$\mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \pi_*\big(\mathbb{H}_{et}(\ , Sp\big(\Gamma^h(\bullet)\big)\big)_{\mathbb{Q}}\big)\big)$$

$$\to \mathcal{H}om_{\hat{i}^{-1}(\pi_*(K(\ )_{\hat{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \pi_*\big(\mathbb{H}_{et}(\ , Sp\big(\Gamma^h(\bullet)\big)\big)_{\mathbb{Q}}\big)\big).$$

Given any class $\mathcal{F} \in \pi_*(G(\mathcal{S}))$, *the class* $\tau_{\mathcal{S}}(\mathcal{F})$ *in the first group maps to the corresponding class in the second, i.e. the Riemann–Roch transformations defined using the imbeddings of* $\mathfrak{M}$ *into* $\tilde{\mathfrak{M}}$ *and* $\hat{\mathfrak{M}}$ *are compatible.*

Proof. Here we make use of the factorization of $\tau'_{\mathcal{S},\mathfrak{M}}$ as in Remarks 6.8. Since the existence of the natural map in the first statement is clear, it suffices to show that the map in (6.0.9) is independent of the imbedding of $\mathfrak{M}$ in $\tilde{\mathfrak{M}}$. for this it suffices to show the squares

$$
\begin{array}{ccccccc}
\pi_*(G(\mathfrak{M})) & \longrightarrow & \pi_*(K_{\mathfrak{M}}(\tilde{\mathfrak{M}})) & \xrightarrow{ch((\ ))\circ Td(N)^{-1}} & \pi_*(\mathbb{H}_{et,\mathfrak{M}}(\tilde{\mathfrak{M}}, Sp(\Gamma(\bullet)))) & \xrightarrow{\cap Td_{\tilde{\mathfrak{M}}|\mathfrak{M}}\circ\cap[\tilde{\mathfrak{M}}]} & \pi_*(\mathbb{H}_{et}(\mathfrak{M}, Sp(\Gamma^h(\bullet)))) \\
\downarrow{id} & & \downarrow{\bar{i}_*} & & \downarrow{\bar{i}_*} & & \downarrow{id} \\
\pi_*(G(\mathfrak{M})) & \xrightarrow{} & \pi_*(K_{\mathfrak{M}}(\hat{\mathfrak{M}})) & \xrightarrow{ch} & \pi_*(\mathbb{H}_{et,\mathfrak{M}}(\hat{\mathfrak{M}}, Sp(\Gamma(\bullet)))) & \xrightarrow{\cap Td_{\hat{\mathfrak{M}}|\mathfrak{M}}\circ\cap[\hat{\mathfrak{M}}]} & \pi_*(\mathbb{H}_{et}(\mathfrak{M}, Sp(\Gamma^h(\bullet))))
\end{array}
$$

commute. This may be proved by deformation to the normal cone of $\tilde{\mathfrak{M}}$ in $\hat{\mathfrak{M}}$: see [3] for more details. Now $ch(\ ).\bar{i}_* = \bar{i}_*(ch(\ ) \circ Td(N)^{-1})$ where N is the normal bundle associated to the closed immersion $\bar{i}$ and where ch denotes the local Chern character. The $\bar{i}_*$ on the right is the Gysin map in local cohomology which is given by cup-product with the Koszul–Thom class, T, of the normal bundle N associated to the closed immersion $\bar{i}$. The $\bar{i}_*$ on the left is the Gysin map $\pi_*(K_{\mathfrak{M}}(\tilde{\mathfrak{M}})) \to \pi_*(K_{\mathfrak{M}}(\hat{\mathfrak{M}}))$ and is again given by cup product with an appropriate Koszul–Thom class. Finally observe that $Td_{\hat{\mathfrak{M}}} = Td(N)^{-1}.Td_{\hat{\mathfrak{M}}|\tilde{\mathfrak{M}}}$ (where the last term is the restriction of $Td_{\hat{\mathfrak{M}}}$ to $\tilde{\mathfrak{M}}$) and $[T] \cap [\hat{M}] = [\tilde{M}]$. These prove the commutativity of the last square and completes the proof of the proposition. $\quad\square$

In view of the last proposition, one may make the following definition of Bredon homology, which will show it is independent of the imbedding of $\mathfrak{M}$ into a smooth quasi-projective

scheme $\tilde{\mathfrak{M}}$. Consider the direct system of closed immersions: $\{\mathbb{P}^n \to \mathbb{P}^{n+1} \to \cdots\}$. Clearly the set of all open sub-schemes of each fixed $\mathbb{P}^n$ containing the given $\mathfrak{M}$ as a closed sub-scheme is a directed set, ordered by inclusion. Therefore, one may take the iterated colimit:

Definition 6.12 *(Intrinsic Bredon homology).*

$$\lim_{n \to \infty} \operatorname*{colim}_{\mathfrak{M} \subset \tilde{\mathfrak{M}} \subset \mathbb{P}^n} \mathcal{H}om_{i^{-1}(\pi_*(K( )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*\mathbf{K}(, \mathcal{A})_{\mathcal{S}}\big)_{\mathbb{Q}}, \pi_*\big(\mathbb{H}_{et}\big(, Sp\big(\Gamma^h(\bullet)\big)\big)_{\mathbb{Q}}\big)\big) \quad (6.0.13)$$

to obtain a definition of Bredon homology that is *intrinsic*. (Observe however, that if one restricts to Grothendieck groups of coherent sheaves, then the last statement in 6.0.10 shows this construction is not needed.)

We conclude this section by defining a Riemann–Roch transformation for the relative form of homology involving inertia stacks as defined in Definition 5.10.

Definition 6.13. Assume the algebraic stack $\mathcal{S}$ is (i) Deligne–Mumford and defined over an algebraically closed field, (ii) is smooth, and (iii) is separated so that the diagonal $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ and hence the obvious induced projection $p_0 : I_{\mathcal{S}} \to \mathcal{S}$ are proper. (iv) Moreover, we will assume that p_0 has finite cohomological dimension. We will fix a closed immersion $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ into a smooth scheme. We recall that in this setting one has an isomorphism of presheaves on $\mathcal{S}_{iso.et}$:

$$\phi_{\mathcal{S}} : \pi_*\big(p_{0*}K( )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \to \pi_*\big(p_{0*}K_{et}( )_{I_{\mathcal{S}}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty),$$

see [41], [43] and [25, Theorem 1.3]. Since the stack is smooth, so is $I_{\mathcal{S}}$ and therefore,

$$p_*K( )_{\mathcal{S}} \simeq p_*G( )_{\mathcal{S}}, \qquad p_*p_{0*}K_{et}( )_{I_{\mathcal{S}}} \simeq p_*p_{0*}G_{et}( )_{I_{\mathcal{S}}}.$$

We will let the map

$$\pi_*G_{et}(I_{\mathcal{S}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \to \pi_*G(S)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$$

denote the inverse of the isomorphism $\phi_{\mathcal{S}}$.

We let $\tau_{I_{\mathcal{S}}/\mathcal{S}} : \pi_*G_{et}(I_{\mathcal{S}}) \to H_*^{Br\text{-}et}(I_{\mathcal{S}}/\mathcal{S}, \Gamma^h(\bullet))$ be defined by its adjoint: this is the composition of the map $\pi_*(G_{et}(I_{\mathcal{S}})) \to \Gamma(\mathfrak{M}, \pi_*(p_*p_{0*}\mathbf{G}_{et}( )_{I_{\mathcal{S}}})_{\mathbb{Q}})$ with the adjoint to the map of presheaves on $\mathfrak{M}_{et}$ induced by the following (see Definition 5.10):

$$\pi_*\big(p_*p_{0*}\mathbf{G}_{et}( )_{I_{\mathcal{S}}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \otimes_{i^{-1}(\pi_*(K( )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)} \pi_*\big(p_*p_{0*}\mathbf{K}_{et}( )_{I_{\mathcal{S}}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$$

$$\to \pi_*\big(p_*p_{0*}\mathbf{G}_{et}( )_{I_{\mathcal{S}}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \xrightarrow{\phi_{\mathcal{S}}^{-1}} \pi_*\big(p_*G( )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$$

$$\xrightarrow{\pi_*(p_*)} \pi_*G( )_{\mathfrak{M}_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \xrightarrow{\tau_{\mathfrak{M}}} \pi_*\big(H\big(, Sp\big(\Gamma^h(\bullet)\big)\big)_{\mathbb{Q}}\big) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty).$$

Next we proceed to define *fundamental classes*. For simplicity we will restrict to the situation where the integer d as in Definition 3.1(viii) in Section 3.1 is 2. Observe from the Proposition 6.14 below that $H_*^{Br}(\mathcal{S}, \Gamma(m)) = \pi_*(\mathbb{H}^{Br}(\mathcal{S}, Sp(\Gamma^h(m)))) = 0$ for all $m > n$ if $\mathcal{S}$ is an algebraic stack for which a coarse moduli space of dimension n exists. Therefore, in general we define the fundamental class to be *the nonzero term in* $\tau_{\mathcal{S}}(\mathcal{A})$ *of the highest weight* k and

degree $= 2k$, where $\mathcal{A}$ is the given dg-structure sheaf of the dg-stack $\mathcal{S}$. (More detailed definition when the dg-structure sheaf $\mathcal{A}$ is obtained from a perfect obstruction theory is considered in [27]. There it is shown that the integer k coincides with the *virtual dimension* of the dg-stack $(\mathcal{S}, \mathcal{A})$.)

Next we will consider the case when the dg-structure sheaf $\mathcal{A}$ is just the usual structure sheaf $\mathcal{O}_{\mathcal{S}}$. Now it suffices to show that $H_*^{Br}(\mathcal{S}, \Gamma(n)) \neq 0$ where n is the dimension of the moduli space of the stack $\mathcal{S}$. Assuming this we let

$$[\mathcal{S}] = \text{the term of degree } 2n \text{ and weight } n \text{ in } \tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}}). \tag{6.0.14}$$

Next we show that $H_*^{Br}(\mathcal{S}, \Gamma(n)) \neq 0$ under the assumption that the map $p : \mathcal{S} \to \mathfrak{M}$ is *finite*. In view of the existence of an obvious restriction $H_*^{Br}(\mathcal{S}, \Gamma(n)) \to H_*^{Br}(\mathcal{S}_U, \Gamma(n))$ for each $U \in \mathfrak{M}_{Zar}$, with $\mathcal{S}_U = \mathcal{S} \times_{\mathfrak{M}} U$, it suffices to do this generically on the moduli space. Therefore, since the base scheme is a field, we may assume the moduli space $\mathfrak{M}$ is in fact smooth. Next observe that $H_{2n}(\mathfrak{M}, \Gamma(n)) = \mathbb{H}_{et}^{-2n}(\mathfrak{M}, \Gamma^h(n))$ has a *fundamental class*, $[\mathfrak{M}]$ by our hypothesis: see Definition 3.1(viii). Moreover, by the relationship between the Riemann–Roch transformation for the moduli space $\mathfrak{M}$ and its fundamental class $\tau_{\mathfrak{M}}(\mathcal{O}_{\mathfrak{M}}) = [\mathfrak{M}] +$ lower dimensional terms. (See [13, Theorem 18.3, (5)].) From the definition of the Riemann–Roch transformation for the stack $(\mathcal{S}, \mathcal{O})$ (see Definitions 6.7 and 6.9 above), observe that $\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}})$ identifies with the morphism $\mathcal{E} \mapsto \tau_{\mathfrak{M}}(p_*(\mathcal{E}))$, $\mathcal{E} \in \pi_k(K(\mathcal{S}))$, $k \geqslant 0$. The map p is finite by assumption, and it induces a map $p_* : H_*^{Br}(\mathcal{S}, \Gamma(\bullet)) \to H_*^{Br}(\mathfrak{M}, \Gamma(\bullet)) = \mathbb{H}_{et}^*(\mathfrak{M}; \Gamma^h(\bullet))$. One may verify that $p_*(\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}}))$ identifies with the map $\tilde{\mathcal{E}} \mapsto \tau_{\mathfrak{M}}(p_* p^*(\tilde{\mathcal{E}})) = ch(\tilde{\mathcal{E}}) . \tau_{\mathfrak{M}}(p_* p^*(\mathcal{O}_{\mathfrak{M}}))$, $\tilde{\mathcal{E}} \in \pi_k(K(\mathfrak{M}))$, $k \geqslant 0$. Therefore, $p_*(\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}})_{2n}(n)) \in \mathbb{H}_{et}^*(\mathfrak{M}; \Gamma^h(\bullet))$ is nothing other than a multiple of $\tau_{\mathfrak{M}}(\mathcal{O}_{\mathfrak{M}})_{2n}(n)$ by the degree of the projection map $p : \mathcal{S} \to \mathfrak{M}$. This shows $H_{2n}^{Br}(\mathcal{S}, \Gamma(n)) \neq 0$.

The compatibility of the Chern character and the Riemann–Roch transformation follows readily in view of the pairing established between Bredon cohomology and Bredon homology.

This completes the proof of Theorem 1.1.

Proposition 6.14. *Let $\mathcal{S}$ denote an algebraic stack for which a coarse moduli space of dimension n exists. Then $H_*^{Br}(\mathcal{S}; \Gamma(m)) = 0$ for $m > n$.*

Proof. Recall

$$H_*^{Br}\big(\mathcal{S}; \Gamma^h(m)\big)$$
$$= Gr_m\Big(\Gamma\big(\mathfrak{M}; \mathcal{H}om_{i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}\big(\pi_*\big(p_*\mathbf{K}(\ , \mathcal{A})_{\mathcal{S},\mathbb{Q}}\big), \pi_*\big(\mathbb{H}_{et}\big(\ ; Sp\big(\Gamma_{\mathfrak{M}}^h(\bullet)\big)\big)_{\mathbb{Q}}\big)\big)\Big)\Big)$$

where $\mathfrak{M} \to \tilde{\mathfrak{M}}$ is a closed immersion into a smooth scheme and one decomposes $i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}$ using Adams operations and $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ is decomposed correspondingly as in (5.0.13). Finally one takes the pieces of total weight m, coming from the graded terms of weight k in $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ and weight $m + k$ in $\Gamma^h(\bullet)$. By our hypothesis, the moduli space of $\mathcal{S}$ has dimension n and therefore, $\Gamma_{\mathfrak{M}}^h(m) = 0$ for all $m > n$: see Definition 3.1(i). Since there are no terms of *negative* weight in $\pi_*(\mathbf{K}(\ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$, it follows that the highest weighted terms appearing in $H_{Br}^*(\mathcal{S}; \Gamma^h(\bullet))$ are with weight $m = n$. $\square$

Theorem 6.15.

(i) *Assume that the Bredon homology and cohomology theories are defined as in any of the Definitions 5.4, 5.7 or 5.8. Let $x : X \to \mathcal{S}$ denote an atlas for the stack and let $B_x \mathcal{S} = cosk_0^{\mathcal{S}}(X)$. Then there exists a map $\sigma_* : H_*^{Br}(\mathcal{S}, \Gamma(\bullet)) \to \mathbb{H}_{et}^*(B_x \mathcal{S}^+, \bar{x}^* \Gamma^h(\bullet))_{\mathbb{Q}}$ where $B_x \mathcal{S}^+$ denotes the semi-simplicial space as in Appendix B and for any of the complexes $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ considered in Section 4. ($\bar{x}^* = \{x_n^*|n\}$.) For separated Deligne–Mumford stacks (when the complexes $\Gamma(\bullet)$ are defined on the site $\mathcal{S}_{lis\text{-}et}$) the target identifies with $\mathbb{H}_{et}^*(\mathcal{S}, \Gamma^h(\bullet))_{\mathbb{Q}}$ ($\mathbb{H}_{lis\text{-}et}^*(\mathcal{S}, \Gamma^h(\bullet))_{\mathbb{Q}}$, respectively). This map is compatible with push-forward maps associated to closed immersions of algebraic stacks. When the Bredon homology is defined as in Definition 5.8, this provides a fundamental class in $H_*^{et}(B_x \mathcal{S}^+, \bar{x}^* \Gamma(\bullet))$ for algebraic stacks with coarse moduli spaces that are quasi-projective over a field.*

(ii) *Assume the dg-structure sheaf is the usual structure sheaf $\mathcal{O}_{\mathcal{S}}$ and that Bredon cohomology is defined as in Definitions 5.7 or 5.8. Then there exists a map $\sigma^* : H_{Br}^*(\mathcal{S}, \Gamma(\bullet)) \to H_{et}^*(B_x \mathcal{S}^+, \bar{x}^* \Gamma(\bullet))_{\mathbb{Q}}$. For separated Deligne–Mumford stacks (when the complexes $\Gamma(\bullet)$ are defined on the site $\mathcal{S}_{lis\text{-}et}$) the target identifies with $\mathbb{H}_{et}^*(\mathcal{S}, \Gamma^h(\bullet))_{\mathbb{Q}}$ ($\mathbb{H}_{lis\text{-}et}^*(\mathcal{S}, \Gamma^h(\bullet))_{\mathbb{Q}}$, respectively). Moreover, $\sigma^* \circ ch = Ch$ where ch (Ch) denotes the Chern character in Bredon cohomology (étale or lisse-étale cohomology, respectively).*

(iii) *Let $\mathcal{S}$ denote a Deligne–Mumford stack over an algebraically closed field k with quasi-projective coarse moduli space. Assume that the dg-structure sheaf $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$. Then the finer variant of Bredon homology $H_*^{Br,et}(I_{\mathcal{S}}/\mathcal{S}, \Gamma^h(\bullet)) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$ defined in Definition 5.10 maps to $H_*^{Br}(\mathcal{S}, \Gamma^h(\bullet)) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$. This map is compatible with respect to proper push-forwards. (This map will be denoted $\phi_{\mathcal{S}}^*$ henceforth.) Moreover, the latter map is an isomorphism when the stack $\mathcal{S}$ is smooth and the orders of the stabilizer groups of the stack at every point are prime to the characteristic of k.*

Remarks 6.16. (1) If the complexes $\Gamma^h(\bullet)$ and $\Gamma(\bullet)$ are the ones associated to l-adic cohomology as in Section 4, they are defined on the site $\mathcal{S}_{lis\text{-}et}$, for all algebraic stacks $\mathcal{S}$ satisfying the hypotheses of Definition 4.5. If the stack $\mathcal{S}$ is smooth, then the motivic complexes considered in Section 4 are also defined on $\mathcal{S}_{lis\text{-}et}$.

(2) Suppose $\mathcal{S}$ is a separated Deligne–Mumford stack. Then the maps considered in (i) provide maps $H_*^{Br}(I_{\mathcal{S}}, \Gamma(\bullet)) \to \mathbb{H}_{et}^*(I_{\mathcal{S}}, \Gamma^h(\bullet))$ and similarly $H_{Br}^*(I_{\mathcal{S}}, \Gamma(\bullet)) \to \mathbb{H}_{et}^*(I_{\mathcal{S}}, \Gamma(\bullet))$. Recall the targets of these maps identify with the (finer) homology and cohomology of the stack $\mathcal{S}$ as defined in [41] or [12].

Proof of Theorem 6.15. (i) The map from Bredon homology to the homology computed on the étale site $\mathcal{B}_x \mathcal{S}^+$ may be obtained as follows. For the proof we will consider explicitly only the theory defined in Definition 5.4. Sending a vector bundle that is locally trivial on the isovariant étale site of an algebraic stack $\mathcal{S}$ to the same vector bundle, but now viewed as a vector bundle on the stack and then pulled back to a perfect complex of $\mathcal{A}_{\mathcal{S}}$-modules (i.e. tensored with $\mathcal{A}_{\mathcal{S}}$), defines a natural map of presheaves of spectra $\bar{\mathbf{K}}(\ )_{\mathcal{S}} \to \mathbf{K}(\ , \mathcal{A})_{\mathcal{S}}$. Moreover, the natural map of simplicial objects $x_\bullet : B_x \mathcal{S}^+ \to \mathcal{S}$ induces a map of sites $x : B_x \mathcal{S}_{et}^+ \to \mathcal{S}_{iso.et}$ and hence a map of presheaves on $\mathcal{S}_{iso.et}$, sending $\mathcal{S}' \to \mathcal{S}$ to

$$\mathbb{H}_{iso.et}\big(\mathcal{S}', \Gamma^h(\bullet)\big)_{\mathbb{Q}} \to \mathbb{H}_{et}\big(B_{x'} \mathcal{S}'^+, \bar{x}'^*\big(\Gamma^h(\bullet)\big)\big)_{\mathbb{Q}}.$$

We will denote this map of presheaves by ϕ. Here $x' = x \times_{\mathcal{S}} \mathcal{S}'$ and $\bar{x}'^* = \{x_n'^* | n\}$. Therefore, one obtains a map

$$\sigma_* : H_{Br}^*\big(\mathcal{S}, \Gamma(\bullet)\big) \to \mathbb{H}_{et}^*\big(B_x \mathcal{S}^+, \bar{x}^* \Gamma^h(\bullet)\big) \otimes \mathbb{Q},$$

sending a map $\pi_*(\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}}) \to \pi_*(\mathbb{H}_{iso.et}(\ , \Gamma^h(\bullet))_{\mathbb{Q}})$ of presheaves of $\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}})_{\mathbb{Q}}$-modules to the map obtained by pre-composing with the map $\pi_*(\bar{\mathbf{K}}(\ )_{\mathcal{S}})_{\mathbb{Q}} \to \pi_*(\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}\mathbb{Q}})$ and composing at the end with the map ϕ. When a coarse moduli space exists one may replace $\bar{\mathbf{K}}(\ )_{\mathcal{S}}$ ($\mathbb{H}_{iso.et}(\ , \Gamma(\bullet))$) with $i^{-1}(\mathbf{K}(\ )_{\mathfrak{M}})$ ($\mathbb{H}_{et}(\ , \Gamma^h(\bullet))$) computed on the étale site of the coarse moduli space, respectively). The stated identification of $\mathbb{H}_{et}(B_x \mathcal{S}^+, \bar{x}^* \Gamma^h(\bullet))$ with $\mathbb{H}_{et}(\mathcal{S}, \Gamma^h(\bullet))$ ($\mathbb{H}_{lis\text{-}et}(\mathcal{S}, \Gamma^h(\bullet))$) in the case of Deligne–Mumford stacks (when the complexes $\Gamma(\bullet)$ are defined on the site $\mathcal{S}_{lis\text{-}et}$, respectively) is clear.

The compatibility of the map above with push-forward for closed immersions follows by observing that an obvious a base change formula for push-forward by closed immersions (and then a pull-back) holds. (More precisely, let $i : \mathcal{S}' \to \mathcal{S}$ denote a closed immersion of algebraic stacks and let $Bi_\bullet : B_{x'}\mathcal{S}'_\bullet \to B_x \mathcal{S}_\bullet$ denote the induced map, where $x' = x \times_{\mathcal{S}} \mathcal{S}'$. Now one may observe readily that the natural (base-change) map $\bar{x}_n^* i_*(F) \to Bi_{n*}\bar{x}_n'^*(F)$ for all $n \geqslant 0$ is an isomorphism on all abelian sheaves F.) These prove the statements in (i) of Theorem 6.15.

(ii) The map from Bredon cohomology to the étale cohomology of the semi-simplicial classifying space (denoted σ^*) may be obtained as follows. Recall the dg-structure sheaf is assumed to be the usual structure sheaf $\mathcal{O}_{\mathcal{S}}$. Therefore,

$$H_{Br}^s\big(\mathcal{S}, \Gamma(t)\big) = Gr_{s,t}\big(\pi_*\big(K(\mathcal{S})\big)_{\mathbb{Q}} \otimes_{\pi_* K(\mathfrak{M})_{\mathbb{Q}}} \mathbb{H}_{et}^*\big(\mathfrak{M}, \Gamma(\bullet)\big)_{\mathbb{Q}}\big).$$

Clearly this maps to

$$Gr_{s,t}\big(\pi_*\big(K(\mathcal{S})\big)_{\mathbb{Q}} \otimes_{\pi_* K(\mathcal{S})_{\mathbb{Q}}} \mathbb{H}_{et}^*\big(B_x \mathcal{S}^+, \bar{x}^* \Gamma(\bullet)\big)_{\mathbb{Q}}\big) \cong \mathbb{H}_{et}^{-s}\big(B_x \mathcal{S}^+, \bar{x}^* \Gamma(t)\big)_{\mathbb{Q}}.$$

This defines σ^* for the variant in Definition 5.7 and the proof for the variant in Definition 5.8 is similar. These prove all but the last statement in (ii). To see this, observe that the module structure of $H_{et}^*(B_x \mathcal{S}^+, \bar{x}^*(\Gamma(\bullet)))_{\mathbb{Q}}$ over $\pi_*(K(\mathcal{S}))_{\mathbb{Q}}$ is given by the Chern character. Clearly this Chern character is compatible with the Chern character on the moduli space under pull-back by the map $\bar{x}^*$. This proves the last statement in (ii).

Assume the hypotheses of Theorem 6.15(iii). Clearly there is an obvious morphism

$$\pi_*\big(K(\ )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \to \pi_*\big(K(\ )_{I_{\mathcal{S}}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \to \pi_*\big(K_{et}(\ )_{I_{\mathcal{S}}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty).$$

Since the action of the inertia stack on vector bundles on $\mathcal{S}$ is diagonalizable, one may break up the last term into a sum of terms indexed by the characters of the inertia stack. This way one obtains a map from the last term into $\pi_*(K_{et}(\ )_{I_{\mathcal{S}}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$. That this composite map is an isomorphism was shown in [41] and [43]. Therefore, the second statement in Theorem 6.15(iii) is an immediate consequence of the following isomorphisms (which are obtained using the observation that the algebra $\mathbb{Q}(\mu_\infty)$ is flat over $\mathbb{Q}$ and that therefore pull-back from presheaves of $\mathbb{Q}$-vector spaces to presheaves of $\mathbb{Q}(\mu_\infty)$-modules is an exact functor):

$$\Gamma\big(\mathfrak{M}, \pi_*\big(p_* p_{0*}\big(\big(K_{et}(\ )_{I_{\mathcal{S}}\mathbb{Q}}\big)\big)\big)\big) \otimes_{i^{-1}(\pi_*(\bar{\mathbf{K}}(\ )_{\mathfrak{M}})_{\mathbb{Q}})} \pi_*\big(\mathbb{H}\big(\ , Sp\big(\Gamma(\bullet)\big)_{\mathbb{Q}}\big)\big) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$$

$$\simeq \Gamma\big(\mathfrak{M}, p_*p_{0*}\big(\pi_*\big(K_{et}(\,)_{I_S}\big)_\mathbb{Q}\big) \otimes_\mathbb{Q} \mathbb{Q}(\mu_\infty)$$

$$\otimes_{i^{-1}(\pi_*(\bar{\mathbf{K}}(\,)_\mathfrak{M})_\mathbb{Q})\otimes_\mathbb{Q}\mathbb{Q}(\mu_\infty)} \pi_*\big(\mathbb{H}\big(\,, Sp\big(\Gamma(\bullet)\big)\big)_\mathbb{Q}\big) \otimes_\mathbb{Q} \mathbb{Q}(\mu_\infty)\big) \quad \text{and}$$

$$\Gamma\big(\mathfrak{M}, \mathcal{H}om_{i^{-1}(\pi_*(\bar{\mathbf{K}}(\,)_\mathfrak{M})_\mathbb{Q})}\big(\pi_*\big(p_*p_{0*}\big(K_{et}(\,)_{I_S\,\mathbb{Q}}\big)\big), \pi_*\big(\mathbb{H}\big(\,, Sp\big(\Gamma^h(\bullet)\big)\big)_\mathbb{Q}\big)\big) \otimes_\mathbb{Q} \mathbb{Q}(\mu_\infty)\big)$$

$$\simeq \Gamma\big(\mathfrak{M}, \mathcal{H}om_{i^{-1}(\pi_*(\bar{\mathbf{K}}(\,)_\mathfrak{M})_\mathbb{Q})}\big(\pi_*\big(p_*p_{0*}\big(K_{et}(\,)_{I_S\,\mathbb{Q}}\big)\big), \pi_*\big(\mathbb{H}\big(\,, Sp\big(\Gamma^h(\bullet)\big)\big)_\mathbb{Q}\big)\big) \otimes_\mathbb{Q} \mathbb{Q}(\mu_\infty)\big)$$

$$\simeq \Gamma\big(\mathfrak{M}, \mathcal{H}om_{i^{-1}(\pi_*(\bar{\mathbf{K}}(\,)_\mathfrak{M})_\mathbb{Q})\otimes_\mathbb{Q}\mathbb{Q}(\mu_\infty)}\big(\pi_*\big(p_*p_{0*}\big(K_{et}(\,)_{I_S}\big)_\mathbb{Q}\big)\big) \otimes_\mathbb{Q} \mathbb{Q}(\mu_\infty),$$

$$\pi_*\big(\mathbb{H}\big(\,, Sp\big(\Gamma^h(\bullet)\big)\big)_\mathbb{Q}\big) \otimes_\mathbb{Q} \mathbb{Q}(\mu_\infty)\big).$$

This concludes the proof of Theorem 6.15. □

7. Applications to virtual fundamental classes

7.0.15. Proof of Theorem 1.5

First observe in view of our hypotheses that if $\mathcal{S}$ is a Deligne–Mumford stack of finite type, for each fixed weight r, $H^i_{et}(\mathcal{S}, \Gamma(r)) \otimes \mathbb{Q} = 0$ for all but finitely many i. Therefore, the Chern classes for any perfect complex on the stack $\mathcal{S}$ with values in this cohomology that lie in positive degrees are all nilpotent; now the (usual) formula for the Todd class of any perfect complex with values in this cohomology shows the Todd class of any perfect complex is invertible. Next observe that if we let $\mathcal{F} = \mathcal{O}_\mathcal{S}^{virt}$, then its Chern character $ch^{Br}(\mathcal{O}_\mathcal{S}^{virt}) = 1$ in $H^*_{Br}(\mathcal{S}, \Gamma(\bullet)) \otimes \mathbb{Q}$: see the remark below. Therefore (by the definition of τ^{et}), $\tau^{et}(\mathcal{O}_\mathcal{S}^{virt}) = \sigma_*([\mathcal{S}]^{virt}_{Br}) \cap Td(T\mathcal{S}^{virt}) = [\mathcal{S}]^{virt} \cap Td(T\mathcal{S}^{virt})$. Since the Todd class, $Td(T\mathcal{S}^{virt})$ is invertible, one multiplies by its inverse to obtain the required identification. This completes the proof of Theorem 1.5.

Remarks 7.1. (1) It is worthwhile pointing out that, in the proof of the last theorem, it is important to consider the K-theory of the dg-stack, and not the stack with its usual structure sheaf $\mathcal{O}_\mathcal{S}$. It is only because we used the K-theory of the dg-stack both for the source of the Todd-homomorphism τ^{et} and also in the definition of Bredon-style homology that we are able to obtain $ch(\mathcal{O}_\mathcal{S}^{virt}) = 1$ in $H^*_{Br}(\mathcal{S}, \Gamma(\bullet))$: see Proposition 2.9(vi).

(2) In [29, p. 9], Kontsevich conjectures that the usual formula expressing the fundamental class of a smooth algebraic variety in terms of the Riemann–Roch transformation applied to the structure sheaf and the Todd class of the tangent bundle extends to the virtual setting. A similar statement is also conjectured in [5, Remark 5.4] where they remark that if one had a good enough Riemann–Roch transformation, one could express the virtual fundamental class in terms of the virtual Todd class and the Riemann–Roch transformation applied to the virtual structures sheaf. A full form of this conjecture very likely involves the virtual setting where one works with the derived moduli stack of stable curves. However, the framework of derived moduli stacks is not yet sufficiently developed (except for work that is appearing currently and work still in preparation) that it would take us a major effort to work out the corresponding formula in this setting; we think such an effort would also not serve the interests of the present paper well. Therefore, we restrict ourselves to the situation above, where the virtual objects are defined by an obstruction theory. Theorem 1.5 shows that, at least when the virtual objects are defined using an obstruction theory, the conjectured formula expressing the virtual fundamental class in terms of a Riemann–Roch transformation and the virtual Todd class of the obstruction theory holds.

(3) The sequel, [27] is devoted entirely to applications to virtual structure sheaves and fundamental classes. There we show that most formulae for virtual fundamental classes may be first derived at the level of virtual structure sheaves; then by invoking the Riemann–Roch theorems proved in the next section these extend readily to Bredon-style homology theories as discussed here. Finally making use of the relationship of Bredon-style theories to other more traditional theories as discussed in Theorem 6.15, one obtains various expected formulae (some of them new) for the virtual fundamental classes.

8. Riemann–Roch theorems

In this section we will let $(\mathcal{S}', \mathcal{A}')$ and $(\mathcal{S}, \mathcal{A})$ denote dg-stacks with $p' : \mathcal{S}' \to \mathfrak{M}'$ and $p : \mathcal{S} \to \mathfrak{M}$ the obvious proper map to their moduli-spaces. We will assume throughout that p and p' are of finite cohomological dimension so that proper push-forward maps $p_* : G(\mathcal{S}, \mathcal{A}) \to G(\mathfrak{M})$, $p'_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathfrak{M}')$ are defined and that both $\mathfrak{M}'$ and $\mathfrak{M}$ are quasi-projective over the base-scheme S which is assumed to be Noetherian and smooth. We will let $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a proper of map of dg-stacks. Recall from 2.1.1 that a map $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ of dg-stacks is *proper* if the underlying morphism of algebraic stacks is proper.

8.1. For the rest of the discussion in this section, we will fix closed immersions $i' : \mathfrak{M}' \to \tilde{\mathfrak{M}}'$ and $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ with $\tilde{\mathfrak{M}}'$ and $\tilde{\mathfrak{M}}$ *smooth* along with an induced proper map $\tilde{f} : \tilde{\mathfrak{M}}' \to \tilde{\mathfrak{M}}$ extending the induced proper map $\bar{f} : \mathfrak{M}' \to \mathfrak{M}$.

The first step in the proof of the Riemann–Roch is to reduce to the case where the dg-structure sheaf $\mathcal{A}' = f^*(\mathcal{A})$. This is achieved in the following proposition.

Proposition 8.1. *Let* $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ *denote a proper map of dg-stacks.*

(i) *Then the map sending an $\mathcal{A}'$-module M to M viewed as an $f^*(\mathcal{A})$-module, induces a direct-image map* $f_{\mathcal{S}',*} : \mathbf{G}(\mathcal{S}', \mathcal{A}') \to \mathbf{G}(\mathcal{S}', f^*(\mathcal{A}))$. *There is also an induced inverse-image map* $f_{\mathcal{S}'}^* : \mathbf{K}(\mathcal{S}', f^*(\mathcal{A})) \to \mathbf{K}(\mathcal{S}', \mathcal{A}')$. *Moreover, f_* induces a direct image map* $f_* : \mathbf{G}(\mathcal{S}', f^*(\mathcal{A})) \to \mathbf{G}(\mathcal{S}, \mathcal{A})$ *provided* $f : (\mathcal{S}', \mathcal{O}_{\mathcal{S}'}) \to (\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ *is of finite cohomological dimension.*

(ii) *One obtains a commutative square of presheaves on* $\mathfrak{M}_{et}$:

$$
\begin{array}{ccc}
\pi_*(\mathbf{G}(\,,\mathcal{A}')_{\mathcal{S}'})_{\mathbb{Q}} & \xrightarrow{\tau'} & \mathcal{H}om_{i'^{-1}(\pi_*(K(\,)_{\tilde{\mathfrak{M}}'})_{\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\,,\mathcal{A}')_{\mathcal{S}'})_{\mathbb{Q}}, \pi_*(\mathbb{H}_{et}(\,;Sp(\Gamma^h(\bullet))))_{\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\pi_*(\mathbf{G}(\,,f^*(\mathcal{A}))_{\mathcal{S}'})_{\mathbb{Q}} & \xrightarrow{\tau} & \mathcal{H}om_{i'^{-1}(\pi_*(K(\,)_{\tilde{\mathfrak{M}}'})_{\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\,,f^*(\mathcal{A}))_{\mathcal{S}'})_{\mathbb{Q}}, \pi_*(\mathbb{H}_{et}(\,;Sp(\Gamma^h(\bullet))))_{\mathbb{Q}})
\end{array}
$$

where τ' (τ) denotes the Riemann–Roch transformation as in Definition 6.9 for the dg-stack $(\mathcal{S}', \mathcal{A}')$ ($(\mathcal{S}', f^*(\mathcal{A}))$, respectively).

Proof. The assertion that the map sending an $\mathcal{A}'$-module M to M, viewed as an $f^*(\mathcal{A})$-module induces a map $\mathbf{G}(\mathcal{S}', \mathcal{A}') \to \mathbf{G}(\mathcal{S}', f^*(\mathcal{A}))$ follows readily from Proposition 2.13(i). (See also Proposition 2.9.) The key observation here is that an $\mathcal{A}'$-module ($f^*(\mathcal{A})$-module) N is coherent

as an $\mathcal{A}'$-module ($f^*(\mathcal{A})$-module, respectively) if and only it is a pseudo-coherent complex of $\mathcal{O}_{\mathcal{S}'}$-modules with coherent cohomology sheaves. The assertion about the inverse-image map follows readily from Proposition 2.9(iv). Moreover, since $f : (\mathcal{S}', \mathcal{O}_{\mathcal{S}'}) \to (\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ is proper and of finite cohomological dimension, it sends a pseudo-coherent complex with bounded coherent cohomology sheaves to a complex of sheaves with the same property. Therefore, Proposition 2.13(i) shows f_* induces a direct image map $f_* : \mathbf{G}(\mathcal{S}', f^*(\mathcal{A})) \to \mathbf{G}(\mathcal{S}, \mathcal{A})$. These prove the first assertion.

In view of Remarks 6.8(1), it suffices to prove the commutativity of the following square in the place of the one in (ii):

$$
\begin{array}{ccc}
\pi_* \mathbf{G}(\ , \mathcal{A}')_{\mathcal{S}' \mathbb{Q}} & \longrightarrow & \mathcal{H}om_{i'^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}'})_{\mathbb{Q}})}(\pi_*(p_* \mathbf{K}(\ , \mathcal{A}')_{\mathcal{S}'})_{\mathbb{Q}}, \pi_*(G(\ )_{\mathfrak{M}})_{\mathbb{Q}}) \\
\Big\downarrow {\scriptstyle f_{\mathcal{S}',*}} & & \Big\downarrow {\scriptstyle \mathcal{H}om_{id}(f_{\mathcal{S}'}^*, id)} \\
\pi_* \mathbf{G}(\ , f^*(\mathcal{A}))_{\mathcal{S}' \mathbb{Q}} & \longrightarrow & \mathcal{H}om_{i'^{-1}((\pi_*(K(\ )_{\tilde{\mathfrak{M}}'}))_{\mathbb{Q}})}(\pi_*(p_* \mathbf{K}(\ , f^*(\mathcal{A}))_{\mathcal{S}'})_{\mathbb{Q}}, \pi_*(G(\ )_{\mathfrak{M}})_{\mathbb{Q}}).
\end{array}
$$

The commutativity of this square follows from the commutativity of the diagram:

$$
\begin{array}{ccccc}
\pi_*(\mathbf{G}(\ , \mathcal{A}')_{\mathcal{S}'}) \otimes_R S & \xrightarrow{id \otimes f_{\mathcal{S}'}^*} & \pi_*(\mathbf{G}(\ , \mathcal{A}')_{\mathcal{S}'}) \otimes_{i'^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}'}))} \pi_*(\mathbf{K}(\ , \mathcal{A}')_{\mathcal{S}'}) & \longrightarrow & \pi_*(\mathbf{G}(\ , \mathcal{A}')_{\mathcal{S}'}) \\
\Big\downarrow {\scriptstyle f_{\mathcal{S}',*} \otimes id} & & & & \Big\downarrow {\scriptstyle f_{\mathcal{S}',*}} \\
\pi_*(\mathbf{G}(\ , f^*(\mathcal{A}))_{\mathcal{S}'}) \otimes_R S & & \longrightarrow & & \pi_*(\mathbf{G}(\ , f^*(\mathcal{A}))_{\mathcal{S}'})
\end{array}
$$

where $R = i'^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}'}))$ and $S = \pi_*(\mathbf{K}(\ , f^*(\mathcal{A}))_{\mathcal{S}'})$. This is clear from the projection formula (in fact, in the above case, this reduces to a standard identity for derived tensor products) and completes the proof of the proposition. $\quad\square$

The next step in the proof of the Riemann–Roch theorem is to be able to factor the proper map $f : (\mathcal{S}', f^*(\mathcal{A})) \to (\mathcal{S}, \mathcal{A})$ into factors that are manageable. We begin with the following definition.

Definition 8.2. Let $f : \mathcal{S}' \to \mathcal{S}$ denote a map between algebraic stacks. We say f is *purely non-representable* if the induced map $\bar{f} : \mathfrak{M}' \to \mathfrak{M}$ of the corresponding coarse moduli-spaces is a purely inseparable (i.e. radical and bijective) map. A map $f : \mathcal{S}' \to \mathcal{S}$ is *purely representable*, if $\mathcal{S}' = \mathcal{S} \times_{\mathfrak{M}} \mathfrak{M}'$ and f is induced by a map $\bar{f} : \mathfrak{M}' \to \mathfrak{M}$ of the corresponding coarse moduli-spaces.

Proposition 8.3. *Let $f : \mathcal{S}' \to \mathcal{S}$ denote a map between two algebraic stacks. Now one has a canonical factorization of f as the composition*

$$
\mathcal{S}' \xrightarrow{n} \mathcal{S}'' \xrightarrow{r} \mathcal{S}
$$

where n is purely non-representable and r is purely representable. In case f is proper (finite) so are n and r. Moreover, if the stacks $\mathcal{S}'$ and $\mathcal{S}$ are provided with the action of a smooth group

scheme G and the map f is G-equivariant, the group scheme G acts on the stack $\mathcal{S}''$ and the resulting maps n and r are also G-equivariant.

Proof. We will consider first the case when the stack $\mathcal{S}$ is an algebraic space. In this case, the stack $\mathcal{S}''$ will be defined as the coarse moduli space for $\mathcal{S}'$ which has the universal property for maps from $\mathcal{S}'$ to algebraic spaces and is therefore unique. In general, the stack $\mathcal{S}''$ will be defined as a relative moduli-space for maps from $\mathcal{S}'$ to $\mathcal{S}$. Let $\mathfrak{M}'$ ($\mathfrak{M}$) denote the coarse moduli space for the stack $\mathcal{S}'$ ($\mathcal{S}$, respectively) and let $\bar{f} : \mathfrak{M}' \to \mathfrak{M}$ denote the induced map. Now we let $\mathcal{S}'' = \mathfrak{M}' \times_{\mathfrak{M}} \mathcal{S}$, with $n : \mathcal{S}' \to \mathcal{S}''$ and $r : \mathcal{S}'' \to \mathcal{S}$ the obvious induced maps. (In the equivariant case, the induced map $\bar{f} : \mathfrak{M}' \to \mathfrak{M}$ is equivariant so that so is the induced map $\mathcal{S}' \to \mathcal{S}''$.) If $\mathfrak{M}''$ denotes the coarse moduli space for the stack $\mathcal{S}''$, one may observe that there is a radicial bijective map $\mathfrak{M}' \to \mathfrak{M}''$. (In the equivariant case, one may show this is equivariant as well.) This shows n (r) is purely non-representable (representable, respectively). Since going from an algebraic stack to its coarse moduli space is canonical, one can see that the above factorization of f is in fact canonical. Moreover, both the maps n and r are equivariant in the equivariant case.

Observe that the map r, being obtained by base-change from a map between the moduli-spaces (which are separated by our hypotheses), is also separated. Therefore, one may show readily, using the valuative criterion for properness (see [32, Théorème (7.3)]) that the map n is also proper, if f is proper. Now it follows readily that if f is finite, so is n. Since the maps $p' : \mathcal{S}' \to \mathfrak{M}'$ and $p : \mathcal{S} \to \mathfrak{M}$ are proper, one may easily see that the properness (finiteness) of f implies that of r as well. $\square$

8.1.1. Assume the above situation. Now we may further factor the map r as the composition of the following two maps π and i which are defined as follows. Let $\bar{r} : \mathfrak{M}' \to \mathfrak{M}$ denote the induced proper map of the moduli spaces. Now recall the moduli spaces are quasi-projective. Therefore, one may factor the map $\bar{r}$ as $\bar{\pi} \circ \bar{i}$, where $\bar{i} : \mathfrak{M}' \to \mathfrak{M} \times \mathbb{P}^n$ is a closed immersion for some large enough integer n and $\bar{\pi} : \mathfrak{M} \times \mathbb{P}^n \to \mathfrak{M}$ is the obvious projection. Now recall $r : \mathcal{S}'' = \mathfrak{M}' \times_{\mathfrak{M}} \mathcal{S} \to \mathcal{S}$ is the map induced by $\bar{r}$. Therefore we let $\pi : \mathcal{S} \times \mathbb{P}^n \cong (\mathfrak{M} \times \mathbb{P}^n) \times_{\mathfrak{M}} \mathcal{S} \to \mathcal{S}$ and $i : \mathcal{S}' \cong \mathfrak{M}' \times_{\mathfrak{M} \times \mathbb{P}^n} (\mathcal{S} \times \mathbb{P}^n) \to \mathcal{S} \times \mathbb{P}^n$. Moreover, the above factorization shows that the map $\bar{f} = \bar{r} : \mathfrak{M}' \to \mathfrak{M}$, and hence r, is of finite cohomological dimension. Therefore, f is of finite cohomological dimension if n is; but the converse need not be true. Therefore, we make the following definition.

Definition 8.4. Assume the above situation. Then the map $f : \mathcal{S}' \to \mathcal{S}$ is *strongly of finite cohomological dimension* if the induced map $n : \mathcal{S}' \to \mathcal{S}''$ is of finite cohomological dimension.

Theorem 8.5 (Riemann–Roch: first form). *Let $f : \mathcal{S}' \to \mathcal{S}$ denote a proper map strongly of finite cohomological dimension between dg-stacks. Assume that a coarse moduli space $\mathfrak{M}'$ ($\mathfrak{M}$) exists for the stack $\mathcal{S}$ ($\mathcal{S}'$, respectively) in the sense of* 1.0.3. *Moreover, we assume that the obvious projections $p' : \mathcal{S}' \to \mathfrak{M}'$ and $p : \mathcal{S} \to \mathfrak{M}$ are of finite cohomological dimension.*

If the moduli-spaces are quasi-projective schemes, one obtains the commutative square:

$$
\begin{array}{ccc}
\pi_* G(\mathcal{S}', \mathcal{A}') & \xrightarrow{\tau_{\mathcal{S}'}} & H_*^{Br\text{-}G}(\mathcal{S}', \Gamma^h(*)) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\pi_* G(\mathcal{S}, \mathcal{A}) & \xrightarrow{\tau_{\mathcal{S}}} & H_*^{Br\text{-}G}(\mathcal{S}, \Gamma^h(*)).
\end{array}
$$

Proof. In view of Proposition 8.1, we may assume without loss of generality that the dg-structure sheaf $\mathcal{A}' = f^*(\mathcal{A})$. We will adopt the following terminology throughout the proof. It suffices to consider separately the three cases when $f = n$ is a purely non-representable morphism, $f = i$ is a closed immersion induced by a closed immersion $\mathfrak{M} \to \tilde{\mathfrak{M}}$ of the associated moduli spaces and $f = \pi$ is the projection $\mathcal{S} \times \mathbb{P}^n \to \mathcal{S}$ induced by the corresponding projection on the moduli spaces. (This follows from the factorization of f as in the last proposition. Observe that if $\mathcal{S}'' = \mathfrak{M}' \times_{\mathfrak{M}} \mathcal{S}$ and its moduli space is $\mathfrak{M}''$, the obvious projection from $\mathcal{S}''$ to $\mathfrak{M}'$ factors uniquely through $\mathfrak{M}''$. On the other hand, the map $n : \mathcal{S}' \to \mathcal{S}''$ induces a unique map $\mathfrak{M}' \to \mathfrak{M}''$ as well. It follows both the maps $\mathfrak{M}' \to \mathfrak{M}''$ and $\mathfrak{M}'' \to \mathfrak{M}'$ are purely inseparable.)

We consider the first case where the morphism f itself is purely non-representable. The proof now reduces to checking the commutativity of the following squares of presheaves on $\mathfrak{M}_{et}$:

$$
\begin{array}{ccc}
\bar{f}_*\pi_*(G(\ ,f^*(\mathcal{A}))_{\mathcal{S}'})_{\mathbb{Q}} & \xrightarrow{\tau_{\mathcal{S}'}} & \bar{f}_*\mathcal{H}om_{i'^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}'}})_{\mathbb{Q}})}(\pi_*(p'_*\mathbf{K}(\ ,f^*(\mathcal{A}))_{\mathcal{S}'})_{\mathbb{Q}}, \pi_*(\mathbb{H}_{et}(\ ,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\pi_*(G(\ ,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} & \xrightarrow{\tau_{\mathcal{S}}} & \mathcal{H}om_{i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\ ,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}, \pi_*(\mathbb{H}_{et}(\ ,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})).
\end{array}
$$

$$(8.1.2)$$

Observe that the induced maps $\bar{f}_* : \pi_* G(U \times_{\mathfrak{M}} \mathfrak{M}')_{\mathbb{Q}} \to \pi_* G(U)_{\mathbb{Q}}$, $\bar{f}_* : H_*(U \times_{\mathfrak{M}} \mathfrak{M}', \Gamma(*))_{\mathbb{Q}} \to H_*(U)_{\mathbb{Q}}$ are isomorphisms for all $U \in \mathfrak{M}_{et}$ since the map f is purely non-representable and hence the induced map $\bar{f} : \mathfrak{M}' \to \mathfrak{M}$ is *purely inseparable*. The definition of the Riemann–Roch transformations $\tau_{\mathcal{S}'}$ and $\tau_{\mathcal{S}}$ shows the commutativity of the above square reduces to the commutativity of the square:

$$
\begin{array}{ccc}
\pi_*(\mathbf{G}(\ ,f^*(\mathcal{A}))_{\mathcal{S}'}) \otimes_R S & \xrightarrow{id \otimes f^*} \pi_*(\mathbf{G}(\mathcal{S}',f^*(\mathcal{A}))) \otimes_{i'^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}'}}))} \pi_*(\mathbf{K}(\mathcal{S}',f^*(\mathcal{A}))) \longrightarrow \pi_*(\mathbf{G}(\mathcal{S}',f^*(\mathcal{A}))) \\
\downarrow{\scriptstyle f_* \otimes id} & & \downarrow{\scriptstyle f_*} \\
\pi_*(\mathbf{G}(\mathcal{S},\mathcal{A})) \otimes_R S & \xrightarrow{\hspace{8cm}} & \pi_*(\mathbf{G}(\mathcal{S},\mathcal{A}))
\end{array}
$$

$$(8.1.3)$$

where $R = i^{-1}(\pi_*(\mathbf{K}(\ )_{\tilde{\mathfrak{M}}}))$ and $S = \pi_*(\mathbf{K}(\mathcal{S},\mathcal{A}))$. This reduces to the projection formula. The remaining two cases are handled by explicit computations in the following two propositions: observe that the proofs essentially reduce to the proofs of the corresponding Riemann–Roch at the level of the moduli-spaces. $\quad\square$

Proposition 8.6 (*Riemann–Roch for a purely representable closed immersion*). *Assume in addition to the hypothesis of Theorem* 8.5 *that* $\bar{f} : \mathfrak{M}' \to \mathfrak{M}$ *is a closed immersion and that* $\mathcal{S}' \cong \mathfrak{M}' \times_{\mathfrak{M}} \mathcal{S}$ *with* f *the corresponding induced map. Now the square in Theorem* 8.5 *commutes.*

Proof. Since, as we showed above, we may assume the dg-structure sheaf $\mathcal{A}' = f^*(\mathcal{A})$, we will omit it altogether from the following discussion. Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ be a closed immersion into

a smooth scheme and let $i' : \mathfrak{M}' \to \tilde{\mathfrak{M}}$ be the composite closed immersion of the closed subscheme $\mathfrak{M}' \to \mathfrak{M} \to \tilde{\mathfrak{M}}$. In view of the interpretation of the Riemann–Roch transformation as in Remark 6.8 (see (6.0.12)), it suffices to prove the commutativity of the two squares:

$$
\begin{array}{ccc}
\bar{f}_* \pi_* (p'_* \mathbf{G}(\ )_{\mathcal{S}'})_{\mathbb{Q}} & \longrightarrow & \mathcal{H}om_{\bar{f}_* i'^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}'}})_{\mathbb{Q}}} (\bar{f}_* \pi_* (p'_* \mathbf{K}(\ )_{\mathcal{S}'})_{\mathbb{Q}}, \bar{f}_* \pi_* (G(\ )_{\mathfrak{M}'})_{\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\pi_* (p_* \mathbf{G}(\ )_{\mathcal{S}})_{\mathbb{Q}} & \longrightarrow & \mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}} (\pi_* (p_* \mathbf{K}(\ )_{\mathcal{S}})_{\mathbb{Q}}, \pi_* (G(\ )_{\mathfrak{M}})_{\mathbb{Q}}),
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{H}om_{\bar{f}_* i'^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}'}})_{\mathbb{Q}}}(X, \bar{f}_* \pi_* (G(\ )_{\mathfrak{M}'})_{\mathbb{Q}}) & \longrightarrow & \mathcal{H}om_{\bar{f}_* i'^{-1}(\pi_*(K(\,)_{\tilde{\mathfrak{M}'}})_{\mathbb{Q}}}(X, \bar{f}_* \pi_* (\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathfrak{M}'})_{\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}}(Y, \pi_* (G(\ )_{\mathfrak{M}})_{\mathbb{Q}}) & \longrightarrow & \mathcal{H}om_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}}(Y, \pi_* (\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathfrak{M}})_{\mathbb{Q}})
\end{array}
$$

where the top horizontal map (bottom horizontal map) in the last diagram are induced by the Riemann–Roch transformation $\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}'}$ ($\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}$, respectively). Here $X = \bar{i}_* \pi_* (p'_* \mathbf{K}(\ )_{\mathcal{S}'})_{\mathbb{Q}}$ and $Y = \pi_* (p_* \mathbf{K}(\ )_{\mathcal{S}})_{\mathbb{Q}}$. The commutativity of the first square reduces to that of a square as in (8.1.3). This is clear by a projection formula once again. Therefore, now, it suffices to prove the commutativity of the square:

$$
\begin{array}{ccc}
i_* \bar{f}_* (\pi_* G(\ )_{\mathfrak{M}'}) & \xrightarrow{\ \tau_{\tilde{\mathfrak{M}}|\mathfrak{M}'}\ } & i_* \bar{f}_* (\pi_* \mathbb{H}_{et}(\ , Sp(\Gamma^h(*))))_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
i_* \pi_* G(\ )_{\mathfrak{M}} & \xrightarrow{\ \tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}\ } & i_* \pi_* \mathbb{H}_{et}(\ , Sp(\Gamma^h(*)))_{\mathbb{Q}}.
\end{array}
$$

Recall $\bar{f}_* G(\ )_{\mathfrak{M}'} = i'_* G(\ )_{\mathfrak{M}'} \simeq K_{\mathfrak{M}'}(\ )_{\tilde{\mathfrak{M}}}$ and $i_* G(\ )_{\mathfrak{M}} \simeq K_{\mathfrak{M}}(\ )_{\tilde{\mathfrak{M}}}$. Therefore we may identify $i_* \bar{i}_* G(\ )_{\mathfrak{M}'}$ with the (canonical) homotopy fiber of the map $K_{\mathfrak{M}}(\ )_{\tilde{\mathfrak{M}}} \to K_{\mathfrak{M}-\mathfrak{M}'}(\ )_{\tilde{\mathfrak{M}}-\mathfrak{M}'}$. Let $j : \mathfrak{M} - \mathfrak{M}' \to \tilde{\mathfrak{M}} - \mathfrak{M}'$ denote the obvious locally closed immersion. The commutativity of the last square follows from that of the square of maps obtained by taking the canonical homotopy fibers of the vertical maps:

$$
\begin{array}{ccc}
K_{\mathfrak{M}}(\ )_{\tilde{\mathfrak{M}}} & \xrightarrow{\ \tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}\ } & i_* \mathbb{H}_{et}(\ , Sp(\Gamma^h(*)))_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
K_{\mathfrak{M}-\mathfrak{M}'}(\ )_{\tilde{\mathfrak{M}}-\mathfrak{M}'} & \xrightarrow{\ \tau_{\tilde{\mathfrak{M}}-\mathfrak{M}'|\mathfrak{M}-\mathfrak{M}'}\ } & i_* \mathbb{H}_{et}(\ , Rj_* Rj^! Sp(\Gamma^h(*)))_{\mathbb{Q}}.
\end{array}
$$

This square homotopy commutes, since the higher Chern classes, the Todd class of the ambient space $\tilde{\mathfrak{M}}$ and the fundamental class of the ambient space $\tilde{\mathfrak{M}}$ localize. $\quad\square$

Proposition 8.7 (*Riemann–Roch for a purely representable projection*). *Assume in addition to the hypothesis of Theorem* 8.5 *that* $\bar{\pi} : \mathfrak{M}' = \mathfrak{M} \times \mathbb{P}^n \to \mathfrak{M}$ *is the obvious projection. Let* $\mathcal{S}' = \mathcal{S} \times_{\mathfrak{M}} \mathfrak{M}'$. *Now the square in Theorem* 8.5 *commutes.*

Proof. Once again we may assume the structure sheaf $\mathcal{A}' = \pi^*(\mathcal{A})$ and therefore we will omit it altogether from the discussion. Let $\bar{\mathcal{E}}$ denote the trivial vector bundle of rank $= n$ on the coarse moduli space $\mathfrak{M}$ and let $\mathcal{E}$ denote its pull-back to the stack $\mathcal{S}$ so that $\mathbb{P}(\mathcal{E}) = \mathfrak{M} \times \mathbb{P}^n$. Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{S}$ and $\bar{\pi} : \mathbb{P}(\bar{\mathcal{E}}) \to \mathfrak{M}$, $p : \mathcal{S} \to \mathfrak{M}$ and $p_0 : \mathbb{P}(\mathcal{E}) = \mathcal{S} \times \mathbb{P}^n \to \mathbb{P}(\bar{\mathcal{E}}) = \mathfrak{M} \times \mathbb{P}^n$ denote the obvious maps. Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a fixed closed immersion into a smooth quasi-projective scheme. Let $\tilde{U} \in \tilde{\mathfrak{M}}_{et}$, $U = \tilde{\mathfrak{M}} \times_{\tilde{\mathfrak{M}}} U$, $\mathcal{S}_U = \mathcal{S} \times_{\mathfrak{M}} U$ and $\mathcal{S}'_U = \mathcal{S}' \times_{\mathfrak{M} \times \mathbb{P}^n} U \times \mathbb{P}^n = \mathcal{S}_U \times \mathbb{P}^n$. We will denote the maps between these objects corresponding to the ones above with the subscript U.

The composition of the top row and the right column in Theorem 8.5 will correspond to sending the class of $F \in \pi_*(G(\mathcal{S}'))$ to the map $\pi_*(\mathbf{K}(\ )_{\mathcal{S}})_{\mathbb{Q}} \to \pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathcal{S}})_{\mathbb{Q}}$ of presheaves that sends $\mathcal{F} \in \pi_*(K(\mathcal{S}_U))_{\mathbb{Q}}$ to

$$\bar{\pi}_{U*}\big(\tau_{\tilde{U} \times \mathbb{P}^n | U \times \mathbb{P}^n}\big(p_{0U*}\big(F_{|U} \circ \pi_U^*(\mathcal{F})\big)\big)\big) \in \pi_*\big(\mathbb{H}_{et}\big(U, Sp\big(\Gamma^h(\bullet)\big)\big)\big).$$

(Here $\circ$ denotes the given pairing.) By the usual Riemann–Roch theorem at the level of the moduli-spaces (see, for example, [13]), this identifies with $\tau_{\tilde{U}|U}\bar{\pi}_{U*}((p_{0U*}(F_{|U} \circ \pi_U^*(\mathcal{F}))))$. Now $\bar{\pi}_{U*}p_{0U*} = p_{U*}\pi_{U*}$. Therefore, the latter identifies with

$$\tau_{\tilde{U}|U}\, p_{U*}\big(\big(\pi_{U*}\big(F_{|U} \circ \pi_U^*(\mathcal{F})\big)\big)\big) = \tau_{\tilde{U}|U}\, p_{U*}\big(\pi_{U*}(F_U) \circ \mathcal{F}\big).$$

The last isomorphism is by the projection formula. One may readily see that the composition of the maps in the left column and the bottom row is given by sending F to the map $\pi_*(\mathbf{K}(\ )_{\mathcal{S}})_{\mathbb{Q}} \to \pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})$ of presheaves, where

$$\mathcal{F} \in \pi_* K(\mathcal{S}_U)_{\mathbb{Q}} \mapsto \tau_{\tilde{U}|U}\, p_{U*}\big(\pi_{U*}(F_{|U}) \circ \mathcal{F}\big). \qquad \square$$

Example 8.8. As an example of our Riemann–Roch theorem, one may consider the map $f = p : \mathcal{S} \to \mathfrak{M}$, i.e. the obvious projection from the stack to its coarse moduli space. Assume that this is of finite cohomological dimension. Under the identification of $H_*^{Br}(\mathfrak{M}, \Gamma(*)) \cong H_*^{et}(\mathfrak{M}, \Gamma(*))$, one may show that the Riemann–Roch square commutes as follows. Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a fixed closed immersion into a smooth quasi-projective scheme. We will adopt the terminology in the last proposition: i.e. $\tilde{U} \in \tilde{\mathfrak{M}}_{Zar}$, $U = \tilde{U} \times_{\tilde{\mathfrak{M}}} \mathfrak{M}$ and $\mathcal{S}_U = \mathcal{S} \times_{\mathfrak{M}} U$. Observe that for $K \in \pi_*(G(\mathcal{S}))$, $p_* \circ \tau_{\mathcal{S}}(K)$ identifies with the map $\pi_*(\mathbf{K}(\ )_{\mathfrak{M}})_{\mathbb{Q}} \to \pi_*(\mathbb{H}(\ , Sp(\Gamma^h(\bullet)))_{\mathfrak{M}})_{\mathbb{Q}}$ of presheaves that sends $\bar{F} \in \pi_*(K(U))_{\mathbb{Q}}$ to $\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}(p_*(p^*(\bar{F}) \circ K)) = \tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}(\bar{F} \circ p_*(K))$. One may see readily that $\tau_{\mathfrak{M}}(p_*(K))$ is the same map of presheaves, thereby verifying the Riemann–Roch theorem for the map p. Observe that, in case $\mathfrak{M}$ is itself smooth one may let $\tilde{\mathfrak{M}} = \mathfrak{M}$ (or if we only consider Grothendieck groups: see 6.0.10), one may identify $\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}(\bar{F} \circ p_*(K))$ with $ch(\bar{F}) \circ \tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}(p_*(K))$.

We will conclude with the following form of Riemann–Roch for the map relating the inertia stack $I_{\mathcal{S}}$ with the original stack $\mathcal{S}$.

Theorem 8.9 *(Riemann–Roch for inertia stacks). Assume the situation of Definition* 6.13. *Now the square*

$$
\begin{array}{ccc}
\pi_*(G_{et}(I_{\mathcal{S}}))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) & \xrightarrow{\tau_{I_{\mathcal{S}}/\mathcal{S}}} & H_*^{Br,et}(I_{\mathcal{S}}/\mathcal{S}; \Gamma^h(*)) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \\
\downarrow{\phi_{\mathcal{S}}^{-1}} & & \downarrow{\phi_{\mathcal{S}}^*} \\
\pi_*(G(\mathcal{S}))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) & \xrightarrow{\tau_{\mathcal{S}}} & H_*^{Br}(\mathcal{S}; \Gamma^h(*)) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)
\end{array}
$$

commutes. (The right vertical map is the one induced by $\phi_{\mathcal{S}}: \pi_*(K(\ ,G)_{\mathcal{S}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \to \pi_*(K_{et}(\ ,G)_{I_{\mathcal{S}}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty).)$

Proof. The proof of the statement follows by considering the adjoint to the Riemann–Roch transformations as in Definition 6.13. We will let $i: \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a fixed closed immersion into a smooth quasi-projective scheme. Observe that the top row corresponds by adjunction to the composite map

$$
\pi_*(G_{et}(\ )_{I_{\mathcal{S}}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \otimes_{i^{-1}(\pi_*(K(\ )_{\tilde{\mathfrak{M}}})_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)} \pi_*(K_{et}(\ )_{I_{\mathcal{S}}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)
$$

$$
\to \pi_*(G_{et}(\ )_{I_{\mathcal{S}}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \xrightarrow{\phi_{\mathcal{S}}^{-1}} \pi_*(G(\ )_{\mathcal{S}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \xrightarrow{p_*} \pi_*(G(\ )_{\mathfrak{M}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)
$$

$$
\xrightarrow{\tau} \pi_*(\mathbb{H}_{et}(\ ; Sp(\Gamma^h(\bullet)))_{\mathfrak{M}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty).
$$

Therefore, the composition of the top row and the right vertical map corresponds to the map that sends the class of $\mathcal{F} \in \pi_* G_{et}((I_{\mathcal{S}}, G))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$ to the map

$$
\pi_*\big(K(\ )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \to \pi_*\big(\mathbb{H}_{et}(\ , Sp(\Gamma^h(\bullet)))_{\mathfrak{M}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty),
$$

$$
\mathcal{E}' \mapsto \tau_{\mathfrak{M}}\big(p_* \phi_{\mathcal{S}}^{-1}\big(\mathcal{F} \circ \phi_{\mathcal{S}}(\mathcal{E}')\big)\big).
$$

By the multiplicative property of the isomorphism $\phi_{\mathcal{S}}$ and hence that of $\phi_{\mathcal{S}}^{-1}$, the latter identifies with the map $\mathcal{E}' \mapsto \tau_{\mathfrak{M}}(p_*(\phi_{\mathcal{S}}^{-1}(\mathcal{F}) \circ \mathcal{E}'))$.

The bottom row corresponds under the adjunction to the composite map

$$
\pi_*\big(G(\ )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \otimes_{i^{-1}(\pi_*(K(\ )_{\mathfrak{M}}))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)} \pi_*\big(K(\ )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)
$$

$$
\to \pi_*\big(G(\ )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \xrightarrow{p_*} \pi_*\big(G(\ )_{\mathfrak{M}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)
$$

$$
\xrightarrow{\tau} \pi_*\big(\mathbb{H}_{et}(\ ; Sp(\Gamma^h(\bullet)))_{\mathfrak{M}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty).
$$

One may see readily that this map sends $\mathcal{F} \in \pi_* G_{et}((I_{\mathcal{S}}))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$ to same map as above. This proves the commutativity of the Riemann–Roch square. □

8.2. Incorporating the inertia stack and the second form of Riemann–Roch

For the rest of this section, we will restrict to smooth Deligne–Mumford stacks defined over an algebraically closed field k. We assume the dg-structure sheaf is the usual structure sheaf.

Theorem 8.10 (*Riemann–Roch: second form*). *Let $\mathcal{S}$ and $\mathcal{S}'$ denote smooth Deligne–Mumford stacks and let $f : \mathcal{S}' \to \mathcal{S}$ be a proper map. Let $i_c : I_{\mathcal{S}} \to \mathcal{S}$ and $i'_c : I_{\mathcal{S}'} \to \mathcal{S}'$ denote the associated local imbeddings. Now the following diagram commutes*:

$$
\begin{array}{ccc}
\pi_*(G(\mathcal{S}')) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) & \xrightarrow{\phi_{\mathcal{S}'}} & \pi_*(G_{et}(I_{\mathcal{S}'})) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) \\
\end{array}
$$

The Bredon homology is defined with respect to a chosen duality theory $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ as in Section 3: we have omitted the coefficients $\Gamma^h(\bullet)$ for notational simplicity. The map $\tau_I = \tau_{I^{\mathcal{S}'}/\mathcal{S}'}$.

Proof. The commutativity of the left-most square follows by the Riemann–Roch theorem: first form, discussed in the last section. In view of the description of the middle and bottom horizontal maps as given above, the commutativity of the (bottom) right square follows from the observation that the map

$$
\phi^*_{\mathcal{S}} : H^{Br}_{et,*}(I_{\mathcal{S}}/\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) \to H^{Br}_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty)
$$

is covariantly functorial in $\mathcal{S}$ for proper maps. This follows readily from the definition of the map $\phi^*_{\mathcal{S}}$ (see Theorem 6.15(iii)) and the observation that the isomorphism

$$
\phi_{\mathcal{S}} : \pi_*\big(K(\ )_{\mathcal{S}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \to \pi_*\big(K_{et}(\ )_{I_{\mathcal{S}}}\big)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)
$$

defined in [41] is contravariantly functorial in $\mathcal{S}$.

Again in view of the definition of the maps $\phi_{\mathcal{S}'}$ and $\phi_{\mathcal{S}'}^*$ considered above, the commutativity of the top square reduces to the commutativity of the square:

$$
\begin{array}{ccc}
\pi_*(G(\mathcal{S}')) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) & \xleftarrow{\ \phi_{\mathcal{S}'}^{-1}\ } & \pi_*(G_{et}(I_{\mathcal{S}'})) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) \\
\Big\downarrow{\scriptstyle \tau_{\mathcal{S}'}} & & \Big\downarrow{\scriptstyle \tau_I} \\
H_*^{Br}(\mathcal{S}') \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) & \xleftarrow[\ \phi_{\mathcal{S}'}^*\ ]{} & H_{et,*}^{Br}(I_{\mathcal{S}'}/\mathcal{S}') \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty).
\end{array}
$$

The commutativity of the above square follows from the Riemann–Roch for inertia stacks. Observe that the map $\phi_{\mathcal{S}'}$ of the theorem corresponds to the inverse of the isomorphism in the top row in the last diagram. $\quad\square$

Remark 8.11. Let X denote a quasi-projective scheme. We will view X as an algebraic stack in the obvious manner. Then one obtains the isomorphisms:

$$
H_{Br}^*\big(X; \varGamma(\bullet)\big) \cong \mathbb{H}_{et}^*\big(X; \varGamma(\bullet)\big) \otimes \mathbb{Q} \quad \text{and}
$$
$$
H_*^{Br}\big(X; \varGamma(\bullet)\big) \cong \mathbb{H}_{et}^*\big(X, \varGamma^h(\bullet)\big) \otimes \mathbb{Q} = H_*^{et}\big(X, \varGamma(\bullet)\big).
$$

In this case the inertia stack also identifies with X.

Corollary 8.12. *Let $\mathcal{S}'$ denote a smooth Deligne–Mumford stack provided a proper map $f : \mathcal{S}' \to X$ where X is a quasi-projective scheme. Assume that all of the above are defined over an algebraically closed field k. Let F denote a coherent sheaf on the stack $\mathcal{S}'$. Now we obtain the equality in $H_*(X, \varGamma(\bullet)) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty)$:*

$$
\big(\tau_X\big(Rf_*(F)\big)\big) = f_*\big(\tau_{\mathcal{S}'}(F)\big) = f_*^I\big(\tau_I\big(\phi_{\mathcal{S}'}(F)\big)\big) = f_*\phi_{\mathcal{S}'}^*\big(\tau_I\big(\phi_{\mathcal{S}'}(F)\big)\big). \qquad (8.2.1)
$$

Proof. The statement is clear from the Riemann–Roch theorem considered above. Observe that since X $(= \mathcal{S}$, in the last theorem$)$ is an algebraic space, the map ϕ_X^* is the identity. $\quad\square$

Acknowledgments

Appendix A

9.0.2. Throughout this section $\mathcal{S}$ will denote a site satisfying the following hypotheses.

9.0.3. In the language of [38, Exposé IV], there exists a conservative family of points on $\mathcal{S}$. Recall this means the following. Let (sets) denote the category of sets. Now there exists a set $\bar{\mathcal{S}}$ with a map $p : (\text{sets})^{\bar{\mathcal{S}}} \to \mathcal{S}$ so that the map $F \to p_* \circ U \circ a \circ p^*(F)$ is injective for all abelian sheaves F on $\mathcal{S}$. (Equivalently, if $i_{\bar{s}} : (\text{sets}) \to \mathcal{S}$ denotes the map of sites corresponding to a point $\bar{s}$ of $\bar{\mathcal{S}}$, an abelian sheaf F on $\mathcal{S}$ is trivial if and only if $i_{\bar{s}}^* F = 0$ for all $\bar{s} \in \bar{\mathcal{S}}$.) Here $(\text{sets})^{\bar{\mathcal{S}}}$ denotes the product of the category (sets) indexed by $\bar{\mathcal{S}}$. a is the functor sending a presheaf to the associated sheaf and U is the forgetful functor sending a sheaf to the underlying presheaf. We will also assume that the corresponding functor $p^{-1} : \mathcal{S} \to (\text{sets})^{\bar{\mathcal{S}}}$ commutes with fibered products. Given a presheaf $P \in Mod(\underline{S})$, we let $G^\bullet P : P \dots G P \dots G^2 P \dots G^n P \dots$ denote the obvious cosimplicial object in $Mod(\underline{S})$, where $G = p_* \circ U \circ a \circ p^*$. Now we let

$$\mathcal{G} P = \underset{\Delta}{holim}\{G^n P | n\} \qquad (9.0.4)$$

where $holim_\Delta\{G^n P | n\}$ denotes the homotopy inverse limit: see [22, Section 6] and [9].

We will further assume that $\mathcal{S}$ is *essentially small* and for every object U in $\mathcal{S}$ the category of coverings of U in $\mathcal{S}$ is also *essentially small*.

9.0.5. If X is an object in the site $\mathcal{S}$, we will let $\mathcal{S}/X$ denote the category whose objects are maps $u : U \to X$ in $\mathcal{S}$ and where a morphism $\alpha : u \to v$ (with $v : V \to X$ in $\mathcal{S}$) is a commutative triangle

$$
\begin{array}{ccc}
U & \longrightarrow & V \\
& u \searrow \quad \swarrow v & \\
& X. &
\end{array}
$$
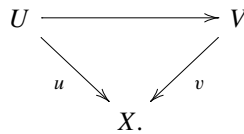
We will further assume that the site $\mathcal{S}$ has a terminal object which will be denoted X (i.e. $\mathcal{S}/X = \mathcal{S}$) and that the category $\mathcal{S}$ is closed under finite inverse limits.

9.0.6. $Prsh(\mathcal{S})$ will denote the category of presheaves of abelian groups. An algebra in $Prsh(\mathcal{S})$ will mean an object which has the additional structure of a presheaf of bi-graded (commutative) algebras. Given such an algebra $\mathcal{A}$ in $Prsh(\mathcal{S})$, $Mod(\mathcal{S}, \mathcal{A})$ will denote the sub-category of presheaves that are presheaves of modules over $\mathcal{A}$. Observe that $Prsh(\mathcal{S})$ has a tensor structure defined by the tensor product of two presheaves. It also has an internal *Hom* which we denote by $\mathcal{H}om$. Given an algebra $\mathcal{A}$ in $Prsh(\mathcal{S})$, $M \in Mod(\mathcal{S}, \mathcal{A})$ and $N \in Mod(\mathcal{S}, \mathcal{A})$, $M \otimes_{\mathcal{A}} N$ is defined as the co-equalizer:

$$Coeq\left(M \otimes \mathcal{A} \otimes N \underset{n}{\overset{m}{\rightrightarrows}} M \otimes N \right) \qquad (9.0.7)$$

where $m : M \otimes \mathcal{A} \otimes N \to M \otimes N$ ($n : M \otimes \mathcal{A} \otimes N \to M \otimes N$) is the map $m = \lambda_M \otimes id_N$, with $\lambda_M : M \otimes \mathcal{A} \to M$ the module structure on M ($n = id_M \otimes \lambda_N$, with $\lambda_N : \mathcal{A} \otimes N \to N$ the module structure on N, respectively). Let $\mathcal{H}om$ denote the internal hom in the category

$Prsh(\mathcal{S})$: this exists as a right adjoint to $\otimes$ since the category $Prsh(\mathcal{S})$ has a small generating set. If $M \in Mod(\mathcal{S}, \mathcal{A})$ and $N \in Mod(\mathcal{S}, \mathcal{A})$, we also define:

$$\mathcal{H}om_{\mathcal{A}, l}(M, N) = Equalizer\left(\mathcal{H}om(\mathcal{A}, N) \underset{n_*}{\overset{m*}{\rightrightarrows}} \mathcal{H}om(\mathcal{A} \otimes M, N)\right) \qquad (9.0.8)$$

where $m^* = \mathcal{H}om(\lambda_M, N)$ and $n_* = \mathcal{H}om(\mathcal{A} \otimes M, \lambda_N)$.

In case M and N in $Mod(\mathcal{S}, \mathcal{A})$ are also bi-graded, so that the module structures are compatible with the grading (i.e. $\mathcal{A}_{i,j} \otimes M_{i',j'}$ maps to $M_{i+i', j+j'}$ and similarly for N), one may observe readily that $M \otimes_{\mathcal{A}} N$ has an induced bi-grading.

9.0.9. One may filter M (N) by $F_k M = \bigoplus_{i \leqslant k} M(i)$ and $F_k N = \bigoplus_{i \leqslant k} N(i)$ so that the above definitions apply to define a filtration on $\mathcal{H}om_{\mathcal{A}}(M, N)$. One may define this filtration explicitly by $F_k \mathcal{H}om_{\mathcal{A}}(M, N) = \{f : M \to N \mid f(F_i M) \subseteq F_{i+k} N\}$. By projecting onto the summands in N, one may see readily that the natural maps $F_k \mathcal{H}om_{\mathcal{A}}(M, N) \to F_{k+1} \mathcal{H}om_{\mathcal{A}}(M, N)$ and $F_k \mathcal{H}om_{\mathcal{A}}(M, N) \to \mathcal{H}om_{\mathcal{A}}(M, N)$ are split mono-morphisms.

9.0.10. In case $\mathcal{A}$ and $\mathcal{B}$ are algebras in $Prsh(\mathcal{S})$ and $M \in Mod(\mathcal{S}, \mathcal{A})$, $N \in Mod(\mathcal{S}, \mathcal{B})$ and P is a presheaf of $(\mathcal{A}, \mathcal{B})$-bi-modules, then one obtains the usual adjunction:

$$\mathcal{H}om_{\mathcal{A}}\big(M, \mathcal{H}om_{\mathcal{B}}(P, N)\big) \cong \mathcal{H}om_{\mathcal{B}}(M \otimes_{\mathcal{A}} P, N). \qquad (9.0.11)$$

The category $Mod(\mathcal{S}, \mathcal{A})$ has enough injectives which enables us to define $\mathcal{R}\mathcal{H}om_{\mathcal{A}}(M, N)$ if $M, N \in Mod(\mathcal{S}, \mathcal{A})$. Then the above conclusions on $\mathcal{H}om_{\mathcal{A}}(M, N)$ extend to $\mathcal{R}\mathcal{H}om_{\mathcal{A}}(M, N)$.

One may also easily define functorial *flat* resolutions of any $M \in Mod(\mathcal{S}, \mathcal{A})$ making use of the hypothesis that our site $\mathcal{S}$ is essentially small: see details in Appendix B. Moreover, if M is a bi-graded object one may find a resolution by presheaves of bigraded flat modules over $\mathcal{A}$. This shows one may define $M \otimes_{\mathcal{A}}^{L} N$ in the obvious manner and that it gets an induced bi-grading if M and N are presheaves of bigraded $\mathcal{A}$-modules. Then the adjunction in (9.0.11) extends to an adjunction between the derived functors $\mathcal{R}\mathcal{H}om_{\mathcal{A}}$ and $\otimes_{\mathcal{A}}^{L}$.

9.1. From co-chain complexes to symmetric spectra

We begin by recalling the functor

$$Sp : \text{(abelian groups)} \to \text{(symmetric spectra)} \qquad (9.1.1)$$

from [19, Example 1.2.5]. Let S^1 denote the simplicial 1-sphere which is obtained by identifying the boundary of $\Delta[1]$ to a point. We let $S^n = \wedge^n S^1 =$ the n-sphere. If A is an abelian group we let $Sp'(A) = \{Sp'(A)_n \mid n\}$ denote the spectrum defined by $Sp'(A)_n = A \otimes (S^n) =$ the simplicial group given in degree k by the sum of A indexed by the non-degenerate k-simplices of S_k^n and with the base-point identified to the zero element. The symmetric group Σ_n acts on $Sp'(A)_n$ in the obvious way by permuting the n-factors of S^1. If $A^\bullet$ is a co-chain complex (trivial in negative degrees), we let $DN(A^\bullet)$ denote the cosimplicial abelian group obtained in the usual manner. Now we apply the functor Sp' to $DN(A^\bullet)$ to obtain a cosimplicial object of symmetric spectra.

The $holim_\Delta$ of the resulting object will define a symmetric spectrum we denote by $Sp'(A^\bullet)$. So defined Sp' now extends to a functor

$$Sp' : \text{(co-chain complexes trivial in neg. degrees)} \to \text{(symmetric spectra)}. \qquad (9.1.2)$$

This functor sends short exact sequences of co-chain complexes to fibration sequences and quasi-isomorphisms to weak-equivalences.

Next assume that $A^\bullet$ is a co-chain complex that is trivial in degrees lower than N. Now we let $Sp'_N(A^\bullet) = (S^N) \wedge Sp'(A^\bullet[-N])$. One may verify that this extends the functor Sp' to all co-chain complexes that are trivial in degrees lower than N and having similar properties. Finally we skip the verification that there exists a natural weak-equivalence $Sp'_N(A^\bullet[-N]) \to S^k \wedge Sp'(A^\bullet[-N-k]) = Sp'_{N+k}(A^\bullet[-N-k])$. We let $Sp(A^\bullet) = lim_{N \to \infty} Sp'_N(A^\bullet[-N])$. It follows in straightforward manner that this defines a functor

$$Sp : \text{(co-chain complexes)} \to \text{(symmetric spectra)} \qquad (9.1.3)$$

and that this functor sends short exact sequences (quasi-isomorphisms) of complexes to fibration sequences (weak-equivalences) of spectra.

Lemma 9.1.

(i) *If A is an abelian sheaf and $A[-k]$ denotes the co-chain complex of abelian sheaves concentrated in degree k, $\pi_i(Sp(A[-k])) = 0$ unless $i = -k$ and $\pi_{-k}(Sp(A[-k])) \cong A$.*

(ii) *If $K^\bullet$ is a co-chain complex of abelian sheaves bounded below, then there exists an integer $N \gg 0$ so that $\pi_i(Sp(K^\bullet)) = 0$ if $i > N$.*

(iii) *If $K^\bullet$ is a co-chain complex of abelian sheaves, then $H^s(K^\bullet) \cong \pi_{-s}(Sp(K^\bullet))$.*

Proof. Assume the situation in (ii). Now there exists a spectral sequence:

$$E_2^{s,t} = H^s\big(\big\{\pi_t\big(Sp\big(K^k\big)\big)|k\big\}\big) = H^s\big(\big\{\pi_t\big(Sp\big(K^k[0]\big)\big)|k\big\}\big) \Rightarrow \pi_{-s+t}\big(Sp\big(K^\bullet\big)\big).$$

Now let $K^\bullet = A[-k]$. In this case, the spectral sequence degenerates since $E_2^{s,t} = 0$ unless $t = 0$ and $s = k$. Therefore one computes $\pi_i(Sp(A[-k])) \cong A$ if $i = -k$ and trivial otherwise. This proves (i).

We may assume without loss of generality that $K^i = 0$ if $i < 0$. Now (i) shows that, for each fixed k, $\pi_i(Sp(K^k[0])) = 0$ unless $i = 0$. Therefore the $E_2^{s,t} = 0$ unless $t = 0$ and $H^s(\{\pi_0(Sp(K^k[0]))|k\}) = \pi_{-s}(Sp(K^\bullet))$. Since this is trivial for $s < 0$, it follows that $\pi_i(Sp(K^\bullet)) = 0$ unless $i \leqslant 0$. This proves (ii). The last statement follows similarly by the degeneration of the same spectral sequence. $\quad\square$

Lemma 9.2.

(i) *In case $\Gamma(\bullet) = \prod_r \Gamma(r)$ is a differential graded algebra, $Sp(\Gamma(\bullet))$ is a ring object in the category of symmetric spectra.*

(ii) *If $\Gamma^h(\bullet) = \prod_r \Gamma^h(r)$ is a left (right, bi) differential graded module over $\Gamma(\bullet)$, then $Sp(\Gamma^h(\bullet))$ is a left (right, bi) module spectrum over the ring spectrum $Sp(\Gamma(\bullet))$.*

(iii) *Let $\mathcal{S}$ denote a site with enough points. Then $\mathbb{H}(X, Sp(K^\bullet)) \cong Sp(\mathbb{H}(X, K^\bullet))$ where X denotes an object in the site $\mathcal{S}$ and $K^\bullet$ is a chain complex of abelian sheaves. Moreover, $\pi_{-n}\mathbb{H}(X, Sp(K^\bullet)) \cong \mathbb{H}^n(X, K^\bullet)$.*

Proof. Ring objects in the category of co-chain complexes of abelian groups may be identified with differential graded algebras. Now it suffices to show the functor Sp sends ring objects to ring objects, which may be checked readily. This proves the first assertion and the second one may be checked similarly. Now we consider the last property. Since the site has enough points, one may use Godement resolution to compute the hyper-cohomology. Now first statement in (iii) follows from the fact the homotopy inverse limits involved in the definition of hyper-cohomology commute with the homotopy inverse limit involved in the definition of the functor Sp. The second statement in (iii) follows from this and Lemma 9.1(iii) applied to the complex $\mathbb{H}(X, K^\bullet)$.   □

Appendix B. Replacement for the smooth site and inverse image functors

The discussion in the first part of this appendix is to address the issues with the smooth site (or more precisely the *lisse-étale site* of an algebraic stack in [32] that have come to light recently. We will essentially invoke the detailed paper of Martin Olsson (see [35]) where these issues are dealt with at length and consider only those results that are relevant for the K-theory and G-theory of algebraic stacks. After-wards, we discuss functorial flat resolutions that come in handy at several places in the paper.

Let $\mathcal{S}$ denote a Noetherian algebraic stack defined over a Noetherian base scheme S, let $x : X \to \mathcal{S}$ denote a presentation and let $B_x\mathcal{S} = cosk_0^{\mathcal{S}}(X)$ denote the corresponding classifying simplicial algebraic space. If we assume that X is affine and that the stack is separated, one verifies readily that each $B_x\mathcal{S}_n$ is an affine scheme. In general one may find an étale hyper-covering $U_{\bullet,\bullet} \to B_x\mathcal{S}_\bullet$ as in 10.0.4 with each $U_{i,j}$ an affine Noetherian scheme. Let $\Delta U_{\bullet,\bullet}$ denote the diagonal of $U_{\bullet,\bullet}$. Following [35] we will adopt the following terminology: given a simplicial object $V_\bullet$, $V_\bullet^+$ will denote the associated *semi-simplicial object* obtained by forgetting the degeneracies. When $V_\bullet$ is a simplicial scheme or simplicial algebraic space, the étale site and the lisse-étale of $V_\bullet^+$ are defined as in the case of $V_\bullet$ except that there are no degeneracies as structure maps of $V_\bullet^+$. A sheaf F on $V_{\bullet,et}^+$ (on $V_{\bullet,lis-et}^+$) consists of a collection of sheaves $\{F_n | n\}$, with F_n a sheaf on the étale site (the lisse-étale site) of V_n along with a compatible collection of morphisms $\{\alpha^*(F_n) \to F_m\}$ for each structure map $\alpha : V_m^+ \to V_n^+$. We say that a sheaf F on $V_{\bullet,et}$ or $V_{\bullet,et}^+$ has *descent* (i.e. is Cartesian as in [35]) if all the above structure maps $\alpha^*(F_n) \to F_m$ are isomorphisms. A sheaf F on $V_{\bullet,lis-et}^+$ is *Cartesian* if each F_n is Cartesian on the lisse-étale site of V_n (see [32, Definition 12.3]) and in addition it has descent. Clearly there is a restriction functor $res : Sh(V_{\bullet,et}) \to Sh(V_{\bullet,et}^+)$ where Sh denotes the category of sheaves.

Let $Mod(\mathcal{S}, \mathcal{O})$ $(Qcoh(\mathcal{S}, \mathcal{O}))$ denote the category of all $\mathcal{O}_\mathcal{S}$-modules (all quasi-coherent $\mathcal{O}_\mathcal{S}$-modules, respectively). (Recall that we have defined quasi-coherent $\mathcal{O}_\mathcal{S}$-modules to be those $\mathcal{O}_\mathcal{S}$-modules whose restriction to the étale sites of all atlases for $\mathcal{S}$ are quasi-coherent: see Definition 2.7. We do not require these to be Cartesian.) Similarly, for a simplicial scheme $V_\bullet$, let $Mod(V_{\bullet,et}, \mathcal{O})$ and $Mod(V_{\bullet,et}^+, \mathcal{O})$ denote the category of all $\mathcal{O}$-modules on $V_{\bullet,et}$ ($V_{\bullet,et}^+$, respectively); let $Qcoh(V_{\bullet,et}, \mathcal{O})$ and $Qcoh(V_{\bullet,et}^+, \mathcal{O})$ denote the corresponding categories of quasi-coherent $\mathcal{O}$-modules. If $\mathbf{A}$ denotes any of the abelian categories above, we will let $D^b(\mathbf{A})$ denote the corresponding bounded derived category. $D_{cart}^b(Qcoh(\mathcal{S}, \mathcal{O}))$ $(D_{qcoh,cart}^b(Mod(\mathcal{S}, \mathcal{O})))$ will denote the full sub-category of $D^b(Qcoh(\mathcal{S}, \mathcal{O}_\mathcal{S}))$ $(D^b(Mod(\mathcal{S}, \mathcal{O}_\mathcal{S})))$ consisting of complexes

whose cohomology sheaves are Cartesian (Cartesian and quasi-coherent, respectively). Replacing $\mathcal{S}$ by $V^+_{\bullet,lis\text{-}et}$, $V^+_{\bullet,et}$ and $V_{\bullet,et}$ defines the corresponding categories on these sites.

Theorem 10.1. (See [24].) *The obvious inclusion functor*

$$D^b_{cart}\big(Qcoh(\mathcal{S},\mathcal{O})\big) \to D^b_{qcoh,cart}\big(Mod(\mathcal{S},\mathcal{O})\big)$$

is an equivalence of categories.

Proof. Assume that either $B_x\mathcal{S}_n$ is affine for each n or that $u : U_{\bullet,\bullet} \to B_x\mathcal{S}$ is a fixed étale hypercovering as above with each $U_{i,j}$ affine. To handle both situations, we will denote $B_x\mathcal{S}_\bullet$ in the first case and $\Delta U_{\bullet,\bullet}$ in the second case by $V_\bullet$. Our hypotheses show that we may assume each V_n is also Noetherian and affine.

Let $\bar{v}^* : Mod(\mathcal{S}_{lis\text{-}et}, \mathcal{O}) \to Mod(V^+_{\bullet,lis\text{-}et}, \mathcal{O})$ denote the obvious inverse image functor $M \mapsto \{v_n^*(M)|n\}$. Here $v_n : V_n \to \mathcal{S}$ denotes the map induced by $v_\bullet$. Let $\bar{v}_* : Mod_{cart}(V^+_{\bullet,lis\text{-}et}, \mathcal{O}) \to Mod_{cart}(\mathcal{S}_{lis\text{-}et}, \mathcal{O})$ denote the functor sending $F = \{F_n|n\}$ to $ker(\delta^0 - \delta^1 : v_{0*}(F_0) \to v_{1*}(F_1))$. One observes that the composition $R\bar{v}_* \circ \bar{v}^*$ is naturally quasi-isomorphic to the identity. This shows the functors

$$\bar{v}^* : D^b_{qcoh,cart}\big(Mod(\mathcal{S}_{lis\text{-}et}, \mathcal{O})\big) \to D^b_{qcoh,cart}\big(Mod(V^+_{\bullet,lis\text{-}et}, \mathcal{O})\big) \quad \text{and}$$

$$\bar{v}^* : D^b_{cart}\big(Qcoh(\mathcal{S}_{lis\text{-}et}, \mathcal{O})\big) \to D^b_{cart}\big(Qcoh(V^+_{\bullet,lis\text{-}et}, \mathcal{O})\big)$$

are fully-faithful. Moreover they induce an equivalence of the hearts of the corresponding categories: therefore, they are in fact equivalences. (See [4, Lemma 1.4].) The obvious map of sites $\eta : V^+_{\bullet,lis\text{-}et} \to V^+_{\bullet,et}$ induces functors

$$R\eta_* : D^b_{cart}\big(Qcoh(V^+_{\bullet,lis\text{-}et}, \mathcal{O})\big) \to D^b_{des}\big(Qcoh(V^+_{\bullet,et}, \mathcal{O})\big),$$

$$R\eta_* = \eta_* : D^b_{qcoh,cart}\big(Mod(V^+_{\bullet,lis\text{-}et}, \mathcal{O})\big) \to D^b_{qcoh,des}\big(Mod(V^+_{\bullet,et}, \mathcal{O})\big)$$

which are known to be equivalence of categories: see [35, Section 4].

Now one obtains a commutative diagram of derived categories:

$$
\begin{array}{ccccc}
D^b_{qcoh,cart}(Mod(\mathcal{S},\mathcal{O})) & \xrightarrow{\eta_* \circ \bar{v}^*} & D^b_{des,qcoh}(Mod(V^+_{\bullet,et}, \mathcal{O})) & \xleftarrow{res} & D^b_{des,qcoh}(Mod(V_{\bullet,et}, \mathcal{O})) \\
\uparrow & & \uparrow & & \uparrow \\
D^b_{cart}(Qcoh(\mathcal{S},\mathcal{O})) & \xrightarrow{\eta_* \circ \bar{v}^*} & D^b_{des}(Qcoh(V^+_{\bullet,et}, \mathcal{O})) & \xleftarrow{res} & D^b_{des}(Qcoh(V_{\bullet,et}\mathcal{O})).
\end{array}
$$

In view of the above observations, the maps in the top and bottom rows are all equivalences of categories. Therefore it suffices to show that the right vertical map is an equivalence. This follows using the quasi-coherator defined in [40] and adapted to the étale site in 10.0.4 below. The discussion there along with our hypothesis that each V_n is a Noetherian affine scheme, shows that the quasi-coherator Q is right adjoint to the inclusion $i : D^b_{des}(Qcoh(V_{\bullet,et}, \mathcal{O})) \to D^b_{des,qcoh}(Mod(V_{\bullet,et}, \mathcal{O}))$ and that the compositions $Q \circ i$ and $i \circ Q$ are the appropriate identity maps proving that the last vertical map in the diagram above is an equivalence. $\quad\square$

Proposition 10.2. *Assume that $U_{\bullet,\bullet} \to S$ is a fixed hyper-covering of S and that $V_\bullet = \Delta(U_{\bullet,\bullet})$. Then the functors*

$$\eta_* \bar{v}^* : D_{cart}^b\big(Mod(S, \mathcal{O})\big) \to D_{des}^b\big(Mod(V_{\bullet,et}^+, \mathcal{O})\big) \quad and$$

$$res : D_{des}^b\big(Mod(V_{\bullet,et}, \mathcal{O})\big) \to D_{des}^b\big(Mod(V_{\bullet,et}^+, \mathcal{O})\big)$$

are equivalences of categories.

Proof. This follows along the same lines as in the proof of the last theorem. The details are therefore skipped. $\square$

Let $(S, \mathcal{A})$ denote a dg-stack as in Section 2 and let $v : V_\bullet \to S$ denote the same simplicial scheme as above. Then $\mathcal{A}$ defines, by pull-back a sheaf of dgas on $V_{\bullet,et}^+$ and on $V_{\bullet,et}$. *Recall from 2.1.2 that for all sheaves of $\mathcal{A}$-modules on $S_{lis\text{-}et}$ we consider, the cohomology sheaves are all assumed to be Cartesian.* A sheaf of $\mathcal{A}$-modules M is *coherent* if the cohomology sheaves $\mathcal{H}^*(M)$ are bounded and are sheaves of finitely generated $\mathcal{H}^*(\mathcal{A})$-modules. The category of all coherent $\mathcal{A}$-modules on S ($V_{\bullet,et}$, $V_{\bullet,et}^+$) will be denoted $Coh(S, \mathcal{A})$ ($Coh(V_{\bullet,et}, \mathcal{A})$, $Coh(V_{\bullet,et}^+, \mathcal{A})$, respectively). Similarly one defines the category of *perfect complexes* on S ($V_{\bullet,et}$, $V_{\bullet,et}^+$): see Section 2. These will be denoted $Perf(S, \mathcal{A})$, $Perf(V_{\bullet,et}^+, \mathcal{A})$ and $Perf(V_{\bullet,et}, \mathcal{A})$, respectively. Observe that these are all Waldhausen categories with fibrations and weak-equivalences where the fibrations are degree-wise surjections and the weak-equivalences are maps of $\mathcal{A}$-modules that are quasi-isomorphisms.

Proposition 10.3. *Let $(S, \mathcal{A})$ denote a dg-stack as in Section 2. Then the following hold:*

(i) *The obvious functors $Coh(S, \mathcal{A}) \to Coh_{des}(V_{\bullet,et}^+, \mathcal{A}) \leftarrow Coh_{des}(V_{\bullet,et}, \mathcal{A})$ induce weak-equivalences of Waldhausen K-theories where the Waldhausen structure is as above.* (*The subscript des denotes the full-subcategory of complexes whose cohomology sheaves have descent.*)

(ii) *The obvious functors $Perf(S, \mathcal{A}) \to Perf_{des}(V_{\bullet,et}^+, \mathcal{A}) \leftarrow Perf_{des}(V_{\bullet,et}, \mathcal{A})$ induce weak-equivalences of Waldhausen K-theories.* (*Again the subscript des denotes the full-subcategory of complexes whose cohomology sheaves have descent.*)

Proof. Observe first that all the functors $\bar{v}^*$, $R\bar{v}_*$ and η_* (considered above) preserve the structure of being $\mathcal{A}$-modules. The functor $\bar{v}^*$ clearly preserves the structure of Waldhausen categories with fibrations and weak-equivalences. Moreover, the observations above show $\bar{v}^*(f)$ is a quasi-isomorphism if and only if f is. Therefore the weak-equivalence of the K-theory spectra of $Coh(S, \mathcal{A})$ and $Coh_{des}(V_{\bullet,lis\text{-}et}^+, \mathcal{A})$ follows from the Waldhausen approximation theorem: see [40]. In view of the equivalence of derived categories associated to $Coh_{des}(V_{\bullet,lis\text{-}et}^+, \mathcal{A})$ and $Coh_{des}(V_{\bullet,et}^+, \mathcal{A})$, another application of the Waldhausen approximation theorem shows the K-theory of the last two Waldhausen categories are weakly-equivalent. One observes that the restriction functor $res : Coh_{des}(V_{\bullet,et}, \mathcal{A}) \to Coh_{des}(V_{\bullet,et}^+, \mathcal{A})$ is fully-faithful at the level of the associated derived categories. Since the functor $\eta_* \bar{v}^* : Coh(S, \mathcal{A}) \to Coh_{des}(V_{\bullet,et}^+, \mathcal{A})$ factors through $Coh_{des}(V_{\bullet,et}, \mathcal{A})$ (observe that the degeneracies are sections to the face maps) it follows that for each object $K^+ \in D(Coh_{des}(V_{\bullet,et}^+, \mathcal{A}))$ there exists a unique object $K \in D(Coh_{des}(V_{\bullet,et}, \mathcal{A}))$ so that $res(K) \cong K^+$. These prove the functor $res : Coh_{des}(V_{\bullet,et}, \mathcal{A}) \to Coh_{des}(V_{\bullet,et}^+, \mathcal{A})$ induces an

equivalence of the associated derived categories proving it induces a weak-equivalence of the corresponding Waldhausen K-theory spectra.

To see (ii) it suffices now to observe that all the functors $\bar{v}^*$, $R\bar{v}_*$ and η_* preserve the property of being a perfect $\mathcal{A}$-module. In view of the definition of this in Section 2, we reduce to proving the above functors preserve the property of being a perfect $\mathcal{O}$-module. This is clear for $\bar{v}^*$. Next suppose $P \in Perf(V^+_{\bullet,lis\text{-}et}, \mathcal{O})$ and has cohomology sheaves with descent. Then $P = \bar{v}^*(Q)$ for some $Q \in D^b(Mod(\mathcal{S}_{lis\text{-}et}, \mathcal{O}))$. Since P is perfect, Q is perfect as a complex of $\mathcal{O}$-modules on $\mathcal{S}_{lis\text{-}et}$. Now observe that $R\bar{v}_*(P) = Q$. Hence $R\bar{v}_*(P)$ is perfect. (We skip the proof that η_* also preserves perfection.) $\quad\square$

10.0.4. Quasi-coherator on the étale site of affine schemes

Given a Noetherian affine scheme X one has the obvious map $\epsilon : X_{et} \to X_{Zar}$ of sites. Given a sheaf of $\mathcal{O}_X$-modules F on X_{et}, we define the associated quasi-coherent sheaf on X_{et} as follows. First one takes the $\Gamma(X, \mathcal{O}_X)$-module $\Gamma(X, \epsilon_* F)$, and then produces the quasi-coherent sheaf $\Gamma(X, \epsilon_* F)\tilde{\,}$ on the Zariski site of X. Next one takes the pull-back of this to X_{et} by ϵ^*. Since this map of sites is natural in X, it follows that the above construction defines a quasi-coherent sheaf on $B_x \mathcal{S}_{et}$.

The following properties are proved in [32, Chapter 13]. Let X denote any Noetherian scheme. Then $\epsilon_* \circ \epsilon^* = id$, $R^i \epsilon_* \epsilon^* = 0$ for $i > 0$ and for a quasi-coherent sheaf F on X_{et}, $\epsilon^* \epsilon_*(F) \cong F$. It follows from these properties that if X is affine the right adjoint to the inclusion functor $\phi : Qcoh(X_{et}, \mathcal{O}) \to Mod(X_{et}, \mathcal{O})$ is defined by the composite functor $M \mapsto \epsilon^*(\Gamma(X, \epsilon_*(K))\tilde{\,})$, $M \in Mod(X_{et}, \mathcal{O})$. We will denote this functor by Q. One may define

$$RQ(M) = \epsilon^*\big(\Gamma\big(X, \epsilon_*(\mathcal{G}M)\big)\tilde{\,}\big), M \in D^b\big(Mod(X_{et}, \mathcal{O})\big)$$

where $\mathcal{G}M$ denotes the functorial Godement resolution. This will be right adjoint to the inclusion $\phi : D^b(QCoh(X_{et}, \mathcal{O})) \to D^b(Mod(X_{et}, \mathcal{O}))$. It is now straight-forward to verify that if $K \in D^b(Mod(X_{et}, \mathcal{O}))$ with quasi-coherent cohomology sheaves, the natural map $\phi(RQ(K)) \to K$ is a quasi-isomorphism. (This follows from the degeneration of the spectral sequence for the composite derived functor.)

10.1. Inverse image functors

One may see readily from the discussion above that, an inverse image functor $f^*: Perf(\mathcal{S}, \mathcal{A}) \to Perf(\mathcal{S}', \mathcal{A}')$ may be defined where $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ is a map of dg-stacks.

Next let $f : \mathcal{S}' \to \mathcal{S}$ denote a representable map of Noetherian Artin stacks so that it *has finite tor dimension*. In view of the issues with the smooth site, one needs to define the functor $Lf^* : D^b_{cart}(Mod(\mathcal{S}, \mathcal{O})) \to D^b_{cart}(Mod(\mathcal{S}', \mathcal{O}'))$ in the following manner. One first considers the induced map $Bf : B_{x'}\mathcal{S} \to B_x\mathcal{S}$ of the associated classifying simplicial algebraic spaces; here $x : X \to \mathcal{S}$ is an atlas and $x' : X' \to \mathcal{S}'$ is the corresponding induced atlas. In view of the equivalence of categories in the proof of Proposition 10.2, the functor $Lf^* : D^b_{des}(Mod(B_x\mathcal{S}_{et}, \mathcal{O})) \to D^b_{des}(Mod(B_{x'}\mathcal{S}'_{et}, \mathcal{O}'))$ will induce the required functor $Lf^* : D^b_{cart}(Mod(\mathcal{S}, \mathcal{O})) \to D^b_{cart}(Mod(\mathcal{S}', \mathcal{O}'))$.

10.2. Functorial flat resolutions

Assume the above situation. In this case we will consider the following *functorial flat resolutions* which are often convenient. In view of the functoriality of the resolution considered below, it respects the simplicial structure on the classifying spaces of algebraic stacks; therefore, we may assume the stack $\mathcal{S}'$ ($\mathcal{S}$) is in fact a Noetherian scheme X' (X, respectively) and that $f : X' \to X$ is a map of finite tor dimension.

Since X is Noetherian, one may assume its étale site X_{et} is small. Let $Mod(X, \mathcal{O}_X)$ ($FMod(X, \mathcal{O}_X)$) denote the category of sheaves of $\mathcal{O}_X$-modules (the full sub-category of sheaves of $\mathcal{O}_X$-modules that are also flat, respectively). One may define a functor $\mathfrak{F} : Mod(X, \mathcal{O}_X) \to FMod(X, \mathcal{O}_X)$ as follows. Given any $M \in Mod(X, \mathcal{O}_X)$ and $j_U : U \to X$ in the site X_{et}, let $S(M)(U) = Hom_{\mathcal{O}_X}(j_{U!}j_U^*(\mathcal{O}_X), M)$. $M \mapsto S(M)$ is a functor $Mod(X, \mathcal{O}_X) \to$ (sheaves of sets on X_{et}). This has a left adjoint defined by $F(T) = \bigoplus_{U \in X_{et}} \bigoplus_{t \in T(U)} j_{U!}j_U^*(\mathcal{O}_X)$. Now one may let $\mathfrak{F} = F \circ S$. This is a functor $Mod(X, \mathcal{O}_X) \to FMod(X, \mathcal{O}_X)$ and explicitly $\mathfrak{F}(M) = \bigoplus_{U \in X_{et}, \phi \in S(M)(U)} j_{U!}j_U^*(\mathcal{O}_X)$. There is a canonical surjective map $\epsilon : \mathfrak{F}(M) \to M$ obtained by sending the summand indexed by (U, ϕ) to M by the morphism ϕ. Given $M, N \in Mod(X, \mathcal{O}_X)$ and $\alpha : j_{U!}j_U^*(\mathcal{O}_X) \to M$, $\beta : j_{V!}j_V^*(\mathcal{O}_X) \to N$, one observes first the natural isomorphism

$$j_{U \times_F V!}j_{U \times_F V}^*(\mathcal{O}_X) \xrightarrow{\cong} j_{U!}j_U^*(\mathcal{O}_X) \otimes_{\mathcal{O}_X} j_{V!}j_V^*(\mathcal{O}_X).$$

Therefore one obtains the map

$$j_{U \times_F V!}j_{U \times_F V}^*(\mathcal{O}_X) \xrightarrow{\cong} j_{U!}j_U^*(\mathcal{O}_X) \otimes_{\mathcal{O}_X} j_{V!}j_V^*(\mathcal{O}_X) \xrightarrow{\alpha \otimes \beta} M \otimes_{\mathcal{O}_X} N.$$

This defines a pairing $\mathfrak{F}(M) \otimes_{\mathcal{O}_X} \mathfrak{F}(N) \to \mathfrak{F}(M \otimes_{\mathcal{O}_X} N)$. One may now verify readily that the functor $\mathfrak{F}$ is compatible with the symmetric monoidal structure on $Mod(X, \mathcal{O}_X)$ and $FMod(X, \mathcal{O}_X)$ provided by the tensor product. It follows that if A is a sheaf of commutative algebras in $Mod(X, \mathcal{O}_X)$, $\mathfrak{F}(A)$ is also a sheaf of commutative algebras.

Let $Simp(Mod(X, \mathcal{O}_X))$ ($Simp(FMod(X, \mathcal{O}_X))$) denote the category of simplicial objects in $Mod(X, \mathcal{O}_X)$ ($FMod(X, \mathcal{O}_X)$, respectively). One may now readily verify that if $\mathcal{A}$ is a commutative monoid in $Simp(Mod(X, \mathcal{O}_X))$ (i.e. what we may call a commutative simplicial algebra in $Mod(X, \mathcal{O}_X)$), $\mathfrak{F}(\mathcal{A}) \in Simp(FMod(X, \mathcal{O}_X))$ is also a commutative monoid. Moreover, the functor F along with S provides a triple, which provides a *functorial flat* resolution of any object $M \in Mod(X, \mathcal{O}_X)$. Such a resolution will be denoted $\mathfrak{F}_\bullet(M)$. In view of the above observations, it follows that if A is a commutative monoid in $Simp(Mod(X, \mathcal{O}_X))$, then $\Delta\mathfrak{F}_\bullet(A) \to A$ will be a quasi-isomorphism and $\Delta\mathfrak{F}_\bullet(A) \in Simp(Mod(X, \mathcal{O}_X))$ will be a commutative monoid, i.e. a commutative simplicial algebra, which in each degree is flat.

Let $C_0(Mod(X, \mathcal{O}_X))$ ($C_0(FMod(X, \mathcal{O}_X))$, respectively) denote the category of chain complexes with differentials of degree $+1$ and trivial in positive degrees. There is a functor $N : Simp(Mod(X, \mathcal{O}_X)) \to C_0(Mod(X, \mathcal{O}_X))$ that is an equivalence of categories and sends $Simp(FMod(X, \mathcal{O}_X))$ to $C_0(FMod(X, \mathcal{O}_X))$. This functor is compatible with the tensor structures. Moreover, for $A, B \in Simp(Mod(X, \mathcal{O}_X))$, the canonical map $N(A) \otimes N(B) \to N(A \otimes B)$ is provided by shuffle maps which commute strictly with the obvious action of the symmetric group interchanging the two factors. Passing to a classifying simplicial space associated to the given stack, we may therefore conclude the following from the above discussion:

10.2.1. (a) There exists a functorial flat resolution:

$$\Delta\mathfrak{F}_\bullet : C_0\big(Mod(\mathcal{S}, \mathcal{O}_\mathcal{S})\big) \to C_0\big(FMod(\mathcal{S}, \mathcal{O}_\mathcal{S})\big).$$

(b) This is compatible with tensor structures, so that if A is a commutative dga in $C_0(Mod(\mathcal{S}, \mathcal{O}_\mathcal{S}))$, then $\Delta\mathfrak{F}_\bullet(A) \in C_0(FMod(\mathcal{S}, \mathcal{O}_\mathcal{S}))$ is also a commutative dga.

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