GENERALIZED t-STRUCTURES: t-STRUCTURES FOR SHEAVES OF DG-MODULES OVER A SHEAF OF DG-ALGEBRAS AND DIAGONAL t-STRUCTURES

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Abstract. t-structures, in the abstract, apply to any triangulated category. However, for the most part, they have been studied so far only in the context of sheaves of modules over sites provided with sheaves of rings. In this paper we define and study t-structures for categories of modules over sites provided with sheaves of dgas and E_∞-dgas. A close variant, as we show, are the diagonal t-structures that come up in the context of crystalline cohomology (as in the work of Ekedahl). All of this is carried out in the unified frame-work of *aisles*. We conclude with several examples: ℓ-adic equivariant derived categories of toric varieties, the diagonal t-structures in crystalline derived categories as well as t-structures on motivic derived categories that are compatible with étale realization.

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1. Introduction.

t-structures were originally introduced in [BBD] to define and study perverse sheaves and soon afterwords in [Ek] to study crystalline cohomology problems. Since then, they have appeared in other contexts: for example, they appear prominently in certain conjectures on algebraic cycles where a formalism similar to ℓ-adic derived categories for algebraic cycles is formulated.

In this paper we utilize the techniques of *pre-aisles and aisles* to provide a painless way to define and study (generalized) t-structures for many of the above contexts. As applications of our work, we also show how to apply our results to several of the above situations.

We had prepared a draft of this paper as early as 2007: since then this preprint remained on the author's personal website, while he was working on more applications of the results of this paper. (See [J-3], for example.) During this period, the existence of this preprint was apparently discovered by several colleagues who were presumably searching for such results and who contacted the author. Clearly the author is very grateful to all such colleagues who found interesting applications way beyond those anticipated by the author.

The following theorem is typical of our results in the paper. (The notion of *pre-aisles* and *aisles* are discussed below.) For the purposes of this introduction we may assume the site $\mathcal{S}$ is the small Zariski, Nisnevich or étale sites associated to a given Noetherian scheme X, but it could also be any of the corresponding big sites.

Theorem 1.1. *Let $(\mathcal{S}, \mathcal{R})$ denote a ringed site as in 2.1 so that the underlying site $\mathcal{S}$ is essentially small and has enough points. Let $C(\mathcal{S}, \mathcal{R})$ denote the category of all unbounded complexes of sheaves of $\mathcal{R}$-modules on $\mathcal{S}$. Moreover we assume the following:*

(i) for each object U in the site $\mathcal{S}$, there exists a large enough integer N (depending on U) so that $H^i(U, F) = 0$ for all $i > N$ and all sheaves of $\mathcal{R}$-modules F on the site $\mathcal{S}$ and

(ii) for all filtered direct systems $\{F_\alpha|\alpha\}$ of sheaves of $\mathcal{R}$-modules and every object U in the site $\mathcal{S}$, $\underset{\alpha}{colim} H^*(U, F_\alpha) \cong H^*(U, \underset{\alpha}{colim} F_\alpha)$.

Let $\mathcal{A}$ be a sheaf of E_∞-dgas or dgas on the ringed site $(\mathcal{S}, \mathcal{R})$. Let $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ denote the pre-aisle (see Definition 1.6 below) in $DMod(\mathcal{S}, \mathcal{A})$ generated by $j_{U!}j_U^*(\mathcal{A}[n])$, $n \geq 0$, U in the site $\mathcal{S}$.

Then (i) $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ is an aisle in $DMod(\mathcal{S}, \mathcal{A})$, i.e. defines a t-structure on $DMod(\mathcal{S}, \mathcal{A})$.

(ii) Assume next the hypotheses of 2.3 hold. i.e. We will assume that $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \rightarrow \mathcal{R}$ which is assumed to be a map of sheaves of E_∞-dgas and that $\mathcal{A}$ is connected, i.e. $\mathcal{A}^i = 0$ for $i < 0$ and $\mathcal{A}^0 = \mathcal{R}$. We will also assume that if V and W are objects in the site $\mathcal{S}$ and X is the terminal object of $\mathcal{S}$, then the fibered product $V \underset{X}{\times} W$ exists in the site $\mathcal{S}$.

Then (a) $j_{U!}(\mathcal{A}_{|U}) \epsilon DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ = the heart of the above t-structure, where $j_U : U \rightarrow \mathcal{S}$ is the structure map of the object U and $\mathcal{A}_{|U} = j_U^*(\mathcal{A})$. In particular, $\mathcal{A} = \mathcal{A}_{|X}$ belongs to the heart of the above t-structure. (b) Moreover, every object M in $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ satisfies the property that the natural map $\tau_{\leq 0}(\mathcal{R} \overset{L}{\underset{\mathcal{A}}{\otimes}} M) \rightarrow \mathcal{R} \overset{L}{\underset{\mathcal{A}}{\otimes}} M$ is a quasi-isomorphism in $DMod(\mathcal{S}, \mathcal{R})$. In other words, the functor $\mathcal{R} \overset{L}{\underset{\mathcal{A}}{\otimes}} (\quad) : DMod(\mathcal{S}, \mathcal{A}) \rightarrow DMod(\mathcal{S}, \mathcal{R})$ sends $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ to $DMod(\mathcal{S}, \mathcal{R})^{\leq 0}$.

We discuss several applications in this paper: one of these is to the diagonal t-structures appearing in the context of crystalline cohomology and the second is to the construction of a t-structure for certain dg-categories associated to equivariant (ℓ-adic) derived categories. We discuss the last only in the case of toric varieties in positive characteristics in this paper. A third application is to the construction of t-structures on motivic derived categories that are preserved by étale realization functors.

Example 1.2. The equivariant derived categories of sheaves of $\mathbb{Q}$-vector spaces on symmetric varieties, equivariant for the action of the corresponding complex reductive group (or its compact form) is shown to be equivalent to the derived category of sheaves over a sheaf of dgas in [Guil] and a related equivalence is conjectured to hold under fairly general hypotheses: see [Soe]. We obtain an extension of this to positive characteristics, for projective toric varieties, making use of the iso-variant étale site and ℓ-adic coefficients. This is summarized as the following theorem.

Theorem 1.3. *Let X denote a projective toric variety for the action of a torus T over an algebraically closed field of characteristic $p \geq 0$. Let ℓ denote a fixed prime different from p. Let $\pi : [X/T]_{sm} \rightarrow [X/T]_{iso.et}$ denote the map of sites associated to the quotient stack $[X/T]$. (Here the subscript sm (iso.et) denotes the smooth site (the isovariant étale site: see* [J-2]*). Let $\mathcal{A} = R\pi_*(\underline{\mathbb{Q}}_\ell)$ denote the sheaf of E_∞-dgas on $[X/T]_{iso.et}$. Then the following hold:*

(i) The points of the site $[X/T]_{iso.et}$ correspond to T-orbits of the geometric points on X and the stalk of $\mathcal{A}$ at such a point $\bar{x} = T\bar{x}$, $\bar{x} \in \bar{X}$, is given by $\mathcal{A}_{\bar{x}} = \mathbb{H}^*(BT_{\bar{x}}, \mathbb{Q}_l)$ where $T_{\bar{x}}$ denotes the stabilizer at $\bar{x}$.

(ii) $\mathcal{A}$ is formal in the sense that $\mathcal{A} \simeq \mathcal{H}^*(\mathcal{A})$ and

(iii) one obtains an equivalence of derived categories (of bounded below complexes):

$$D^T_{b,c}(X, \mathbb{Q}_\ell) \simeq D_c([X/T]_{iso.et}, \mathcal{A}).$$

The category on the left is the usual derived category of complexes of $\mathbb{Q}_l$-sheaves that are bounded and with constructible T-equivariant cohomology sheaves while the category on the right is defined in (7.0.17)*. The t-structure on the right-hand-side is defined as in the last theorem and corresponds to the usual t-structure on the left-hand-side, under the equivalence of categories provided by the theorem.*

We briefly consider the following application to motivic derived categories and étale realization. This is put in more as a sample of what is possible in this direction, than as the most definitive result in this direction. See section 8 and for more details. One may also consult the forthcoming paper [J-3] for more definitive results in the framework of the more commonly adopted triangulated categories of motives.

Let X denote a smooth scheme of finite type over a field k. Let X_{Nis} (X_{Et}) denote the corresponding big Nisnevich site (the big étale site, respectively) consisting of smooth schemes over k, whose structure map to k factor through X and where morphisms between two objects are compatible with the given structure maps to X. Let $\epsilon : X_{Et} \rightarrow X_{Nis}$ denote the obvious maps of sites. We will fix a prime ℓ different from the characteristic of k and let $\nu \geq 0$ be also a fixed integer. Then we let $real_\nu$ denote the functor $K \mapsto \epsilon^*(K \otimes Z/\ell^\nu) = \oplus_r \epsilon^*(K(r) \otimes Z/\ell^\nu)$ sending the (graded) $\mathcal{A}$-module K to the pull-back to the étale site of the corresponding mod-ℓ^ν-reduction. We let $\mathcal{A}$ denote one of the following sheaves of E_∞-dgas restricted to X_{Nis}.

(i) $\mathbb{Z} = \underset{r}{\oplus}\mathbb{Z}(r)$ will denote the integral *motivic complex* on the sites $(\mathrm{Sm_k})_{Nis}$ and $(\mathrm{Sm_k})_{Et}$.

(ii) Let $\mathbb{Z}/\ell = \underset{r}{\oplus}\mathbb{Z}/\ell(r)$ denote the corresponding $mod - \ell$ motivic complex.

Making use of [J-1, Theorem 1.1] (see also [BJ, section 5]), these are in fact sheaves of E_∞-differential graded algebras. (If one prefers, one can also work with the corresponding motivic Eilenberg-Maclane spectrum, which is the framework adopted in [J-3].)[1] We define the *realization functor real* : $D(X_{Nis}, \mathcal{A}) \rightarrow D(X_{Et}, \mu_{\ell^\nu}(0)) \simeq D(X_{Et}, Z/\ell^\nu)$ to be the composition of the functor sending $K = \oplus_r K(r) \mapsto \oplus_r real_\nu(K(r))$ with the one taking the graded piece in degree 0 of the graded module $\oplus_r real_\nu(K(r))$. (The derived categories above are the derived categories of complexes of sheaves of dg-modules over the corresponding sheaf of dgas and where morphisms are defined only upto $\mathbb{A}^1$-equivalence.)

Theorem 1.4. *Assume that the base field has a primitive l^ν-th root of unity, for example, that it is a perfect field and that it has finite ℓ-étale cohomological dimension. Then the realization functor real : $D(X_{Nis}, \mathcal{A}) \rightarrow D(X_{Et}, \mu_{\ell^\nu}(0)) \simeq D(X_{Et}, Z/\ell^\nu)$ is compatible with the above t-structures where the t-structure on $D(X_{Et}, Z/\ell^\nu)$ is the usual one. i.e. There exist natural transformations*

$$real \circ \sigma_{\leq 0} \rightarrow \sigma_{\leq 0} \circ real \text{ and } real \circ \sigma_{\geq 1} \rightarrow \sigma_{\geq 1} \circ real.$$

We conclude this introduction by recalling the notion of pre-aisles and aisles (see [KV] and [TLSS]).

1.0.1. Let $\mathcal{T}$ be a triangulated category whose translation functor is denoted by $(-)[1]$ and its iterates by $(-)[n]$, with $n \in \mathbb{Z}$. A *t-structure* on $\mathcal{T}$ in the sense of ([BBD, Définition 1.3.1]) is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that, denoting $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$, the following conditions hold:

(t1) For $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$, $\mathcal{H}om_{\mathcal{T}}(X, Y) = 0$.

[1]The E_∞-structure on the motivic complexes discussed in [J-1] and [BJ], is quite explicit, is over the Barratt-Eccles operad and therefore has several nice features. For the purposes of this paper though, all we require is an E_∞-structure on the motivic complex compatible with the given pairing on it. Such a structure is often assumed in the literature and therefore, one may also assume its existence.

(t2) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 0} \supset \mathcal{T}^{\geq 1}$.
(t3) For each $X \in \mathcal{T}$ there is a distinguished triangle $A \to X \to B \to A[1]$ with$A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 1}$.

The subcategory $\mathcal{T}^{\leq 0}$ is called the *aisle* of the t-structure, and $\mathcal{T}^{\geq 0}$ is called the *co-aisle*. For a subcategory $\mathcal{C} \subset \mathcal{T}$, we denote the associated orthogonal subcategories as $\mathcal{C}^{\perp} = \{Y \in \mathcal{T} \mid \mathcal{H}om_{\mathcal{T}}(Z, Y) = 0,\ \forall Z \in \mathcal{C}\}$ and ${}^{\perp}\mathcal{C} = \{Z \in \mathcal{T} \mid \mathcal{H}om_{\mathcal{T}}(Z, Y) = 0,\ \forall Y \in \mathcal{C}\}$. The following are immediate formal consequences of the definition.

Proposition 1.5. *Let $\mathcal{T}$ be a triangulated category, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ a t-structure in $\mathcal{T}$, and $n \in \mathbb{Z}$. Then*

1. *$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ is a pair of* orthogonal *subcategories of $\mathcal{T}$, i.e. $\mathcal{T}^{\geq 1} = \mathcal{T}^{\leq 0 \perp}$ and $\mathcal{T}^{\leq 0} = {}^{\perp}\mathcal{T}^{\geq 1}$.*
2. *The subcategories $\mathcal{T}^{\leq n}$ are stable for positive translations and the subcategories $\mathcal{T}^{\geq n}$ are stable for negative translations.*
3. *The canonical inclusion $\mathcal{T}^{\leq n} \to \mathcal{T}$ has a right adjoint denoted $\tau^{\leq n}$, and $\mathcal{T}^{\geq n} \to \mathcal{T}$ a left adjoint denoted $\tau^{\geq n}$. Moreover, $X \in \mathcal{T}^{\leq n}$ if, and only if, $\tau^{\geq n+1}(X) = 0$. ($X \in \mathcal{T}^{\geq n}$ if, and only if, $\tau^{\leq n-1}(X) = 0$.)*
4. *For X in $\mathcal{T}$ there is a distinguished triangle $\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \to \tau^{\leq 0}X[1]$.*
5. *The subcategories $\mathcal{T}^{\leq n}$ and $\mathcal{T}^{\geq n}$ are stable under extensions, i.e. given a distinguished triangle $X \to Y \to Z \xrightarrow{+}$, if X and Z belong to one of these categories, so does Y.*

The subcategories $\mathcal{T}^{\leq n}$ and $\mathcal{T}^{\geq n}$, in general, are not triangulated subcategories but they come close. In fact, each subcategory $\mathcal{T}^{\leq n}$ has the structure of a *suspended category* in the sense of Keller and Vossieck [KV]. Let us recall this definition.

An additive category $\mathcal{U}$ is suspended if and only if it is graded by an additive translation functor T (sometimes called *shifting*) and there is given a class of diagrams of the form $X \to Y \to Z \to TX$ called *distinguished triangles* such that the following axioms, analogous to those for triangulated categories in Verdier's exposition [V, p. 266] hold:

(SP1) Every triangle isomorphic to a distinguished one is distinguished. For $X \in \mathcal{U}$, $0 \to X \xrightarrow{id} X \to 0$ is a distinguished triangle. Every morphism $u : X \to Y$ can be completed to a distinguished triangle $X \xrightarrow{u} Y \to Z \to TX$
(SP2) If $X \xrightarrow{u} Y \to Z \to TX$ is a distinguished triangle in $\mathcal{U}$ then so is $Y \to Z \to TX \xrightarrow{Tu} TY$.
(SP3) = (TR3) in Verdier's *loc. cit.*
(SP4) = (TR4) in Verdier's *loc. cit.*

The main difference with triangulated categories is that the translation functor in a suspended category may not have an inverse and therefore, some objects cannot be shifted back. The formulation of axioms (SP1) and (SP2) reflects this fact. If $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure on a triangulated category T, the aisle $\mathcal{T}^{\leq 0}$ is a suspended subcategory of $\mathcal{T}$ whose distinguished triangles are diagrams in $\mathcal{T}^{\leq 0}$ that are distinguished triangles in $\mathcal{T}$ (Proposition 1.5). Moreover, the aisle $\mathcal{T}^{\leq 0}$ determines the t-structure because the co-aisle $\mathcal{T}^{\geq 0}$ is recovered as $(\mathcal{T}^{\leq 0})^{\perp}[1]$. The terminology "aisle" and "co-aisle" comes from [KV].

Definition 1.6. A *pre-aisle* is a suspended full subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$, where the triangulation in $\mathcal{U}$ is given by the triangles which are distinguished in $\mathcal{T}$ and the shift functor is induced by the one in $\mathcal{T}$.

Observe that, a full subcategory $\mathcal{U}$ of $\mathcal{T}$ is a pre-aisle, if it satisfies the following:

- For any X in $\mathcal{U}$, $X[1]$ is also in $\mathcal{U}$.
- Given a distinguished triangle $X \to Y \to Z \to X[1]$, if X and Z belong to $\mathcal{U}$, then so does Y.

Once these two facts hold for $\mathcal{U}$, the verification of axioms (SP1) through (SP4) is immediate.

The following are the key techniques we use to construct t-structures in this paper.

Theorem 1.7. *(*[KV, Section 1]*) A suspended subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$ is an aisle (i.e. $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a t-structure on $\mathcal{T}$) if and only if the canonical inclusion functor $\mathcal{U} \to \mathcal{T}$ has a right adjoint.*

Definition 1.8. Let $\mathcal{T}$ be a triangulated category. An object E of $\mathcal{T}$ is called *compact* if the functor $\mathcal{H}om_{\mathcal{T}}(E, -)$ commutes with arbitrary (small) co-products. Another way of expressing this condition is that a map from E to a co-product factors through a finite subcoproduct.

Theorem 1.9. *(*[TLSS, Theorem A.1]*) Let $\mathcal{S} = \{E_{\alpha} \mid \alpha \in A\}$ denote a* set *of compact objects in a triangulated category $\mathcal{T}$. Let $\mathcal{U}$ the smallest co-complete (i.e. closed under all small sums) pre-aisle of $\mathcal{T}$ which contains the family $\mathcal{S}$. Then $\mathcal{U}$ is an aisle in $\mathcal{T}$.*

2. The Basic Contexts

In this section we discuss two of the different contexts we consider in this paper. These are the following:

(i) *Derived categories of modules over sites provided with sheaves of dgas and E_∞-dgas.* and
(ii) *Presheaves of E_∞-module spectra and CW-cell modules*

A third context, namely that of crystalline derived categories is discussed separately in section 5: we will show that this is closely related to the set-up in (i).

2.1. **The common frame-work.** Let $\mathcal{S}$ denote a *site* with the following properties: (i) it is essentially small (ii) has enough points. We will denote the points of the site $\mathcal{S}$ by $\bar{\mathcal{S}}$. In addition we will assume the following:

(i) that every object U in the site $\mathcal{S}$ is quasi-compact,
(ii) that the objects of the site form a subcategory of the category of all schemes of finite type over a fixed Noetherian base scheme B and
(iii) the site $\mathcal{S}$ is provided with a Grothendieck topology specified by giving a family of coverings of each object in the site satisfying the usual axioms of coverings.

The coverings of each object U in the site will be denoted $Cov(U)$. We will also assume the site has a terminal object X.

We will need to consider both small and big sites, especially in the context of applications of the main results of this paper. Therefore, we will presently set up the framework for considering both types of sites. *The big site* associated to $\mathcal{S}$ will simply be the given site $\mathcal{S}$. The objects of *the corresponding associated small site* will be objects of the following form: maps $u : U \to X$ in $\mathcal{S}$ so that u factors as the composition $U \to U_0 \to X$ so that $U \to U_0$ is a covering of U_0 in the given topology and $U_0 \to X$ is some Zariski open immersion. The coverings in the small site will be the same as the coverings in the big site.

Recall the notion of *coherence* from [SGA4, Exposé VI] (2.3). Recall that an object U in $\mathcal{S}$ is *quasi-separated* if for any two maps $V \to U$ and $W \to U$ in the site $\mathcal{S}$ (with V and W quasi-compact), the fibered product $V \underset{U}{\times} W$ is quasi-compact. An object U is *coherent* if it is both quasi-compact and quasi-separated. A site with a terminal object X is *coherent* if every object U quasi-separated in $\mathcal{S}$ is quasi-separated over the terminal object X (i.e. the map $U \to X$ is quasi-separated) and the terminal object X is coherent. Given the site $\mathcal{S}$ and an object X in $\mathcal{S}$, the site $\mathcal{S}/X$ will denote the site whose objects are morphisms $u : U \to X$ and morphisms are morphisms in $\mathcal{S}$ over X. We say a site $\mathcal{S}$ is *locally coherent* if it has a covering $\{U_i|i\}$ so that each of the sites $\mathcal{S}/U_i$ is coherent.

The main observation we make now is the following. Assume that for each object $U \,\epsilon\, \mathcal{S}$, the small site associated to $\mathcal{S}/U$ has the property that it is coherent in the above sense. Then

$$\underset{\alpha}{\mathrm{colim}} H^n(U, F_\alpha) \cong H^n(U, \underset{\alpha}{\mathrm{colim}} F_\alpha) \tag{2.1.1}$$

for each U in the site $\mathcal{S}$ and for each filtered direct system $\{F_\alpha|\alpha\}$ of abelian sheaves on $\mathcal{S}$ and for each n. (One may consult [SGA4, Exposé VI] or [St, 21.16: Cohomology and colimits] for proofs.)

2.1.2. We will let $\mathcal{R}$ denote either one of the following: (i) a sheaf of commutative Noetherian rings (or graded commutative Noetherian rings) with unit on the site $\mathcal{S}$ or (ii) the constant sphere spectrum Σ^0. (For simplicity, we will just consider the usual S^1-sphere spectrum, though we could equally well consider a $\mathbb{P}^1$-sphere spectrum.) We will let $C(\mathcal{S}, \mathcal{R})$ denote the category of all unbounded complexes of $\mathcal{R}$-modules (with differentials of degree +1) in the first case and the category of all sheaves of spectra on the site $\mathcal{S}$ in the second case.

In case $\mathcal{R}$ is graded, we will assume that $\mathcal{R} = \underset{i \,\epsilon\, \mathbb{Z}}{\oplus} \mathcal{R}_i$ and that $C(\mathcal{S}, \mathcal{R})$ will denote the category of complexes of sheaves of graded modules over $\mathcal{R}$: a sheaf of graded modules $M = \underset{i \,\epsilon\, \mathbb{Z}}{\oplus} M_i$. For a sheaf of graded $\mathcal{R}$-modules M, $M(t)$ will denote the object with a shift of grading given by: $M(t)_i = M_{t+i}$. Moreover, when $\mathcal{R}$ denotes a sheaf of graded rings, $\mathcal{R}[s](t)$ will denote the complex concentrated in degree s and given by the sheaf $\mathcal{R}(t)$ there. In the second case when $\mathcal{R}$ denotes the constant sheaf of sphere spectra Σ^0, $\mathcal{R}[s](t)$ will denote the s-fold suspension Σ^s, for all t.

We will let $\mathcal{R} \otimes \Delta[1]$ denote the following object in $C(\mathcal{S}, \mathcal{R})$: if $\mathcal{R}$ is a sheaf of rings, then this is the obvious chain complex associated to the simplicial object defined by $n \mapsto \underset{\alpha \,\epsilon\, \Delta[1]_n}{\oplus} \mathcal{R}$ (and with the obvious structure maps). If $\mathcal{R}$ denotes Σ^0, then this is the suspension spectrum associated to the pointed simplicial set $\Delta[1]_+$. Observe that we have canonical morphisms $d_i : \mathcal{R} \cong \mathcal{R} \otimes \Delta[0] \to \mathcal{R} \otimes \Delta[1]$, $i = 0$, $i = 1$. If F is a complex of abelian sheaves on $\mathcal{S}$, $\mathcal{H}^n(F)$ will denote the cohomology sheaf in degree n of the complex F; in case F is a sheaf of spectra, this will denote $\pi_{-n}(F)$ = the sheaf of $-n$-th homotopy groups of F. Any map $f : F' \to F$ that induces an isomorphism on $\mathcal{H}^n$ will be called a *quasi-isomorphism* (following the terminology when $\mathcal{R}$ is a sheaf of graded rings).

Definition 2.1. For each $U \,\epsilon\, S$, $\mathbb{H}(U, \mathcal{G}F) = \Gamma(U, \mathcal{G}F)$ where $\mathcal{G}F$ denotes a the cosimplicial object defined by the Godement resolution. If F is chain complex (spectrum), this is also a chain complex (spectrum, respectively).

Remark 2.2. Instead of the Godement resolution, one may make use of suitable fibrant replacements. For example, in the case of chain complexes of sheaves of modules over a sheaf of rings, one may use injective resolutions. In the case of $\mathbb{P}^1$-spectra, one will in fact need to use fibrant replacements in the $\mathbb{A}^1$-local structure.

Proposition 2.3. *Assume the following hypothesis: with $\mathcal{R}$ as above (i.e. denoting either a sheaf of commutative Noetherian rings on the site $\mathcal{S}$ or the sphere spectrum), and for each U in the site $\mathcal{S}$, there exists an integer $N > 0$ so that $H^n(U, F) = 0$ for all $n > N$ and all sheaves F of $\mathcal{R}$-modules. Assume also that each of the small sites $\mathcal{S}/U$ is coherent.*

Let $\{F_\alpha|\alpha\}$ denote a filtered direct system of objects in $C(\mathcal{S}, \mathcal{R})$ where $C(\mathcal{S}, \mathcal{R})$ denotes the category as above.

Then one obtains the quasi-isomorphism:

$$colim\mathbb{H}^n(U, F_\alpha) \cong \mathbb{H}^n(U, colimF_\alpha) \tag{2.1.3}$$

for each n and each U in the site $\mathcal{S}$.

Proof. This follows by comparing the spectral sequences

$$E_2^{s,t} = \underset{\alpha}{\mathrm{colim}} H^s(U, \mathcal{H}^t(F_\alpha)) \Rightarrow \underset{\alpha}{\mathrm{colim}} \mathbb{H}^{s+t}(U, F_\alpha) \text{ and}$$

$$E_2^{s,t} = H^s(U, \mathcal{H}^t(\underset{\alpha}{\mathrm{colim}} F_\alpha)) \Rightarrow \mathbb{H}^{s+t}(U, \underset{\alpha}{\mathrm{colim}} F_\alpha).$$

Since both spectral sequences converge strongly under the above hypotheses, and one obtains an isomorphism at the E_2-terms by (2.1.1), the required isomorphism of the abutments follows. ☐

Definitions 2.4. (i) Let $\mathcal{R}$ denote a sheaf of commutative Noetherian rings with 1 on the site $\mathcal{S}$. A sheaf of dgas will mean an unbounded complex $\mathcal{A}$ in $C(\mathcal{S}, \mathcal{R})$ which has the structure of a sheaf $\mathcal{A}$ of differential graded algebras on the site $\mathcal{S}$. A sheaf of E_∞ dgas will similarly mean an unbounded complex $\mathcal{A}$ in $C(\mathcal{S}, \mathcal{R})$ which is a sheaf of algebras over an E_∞-operad.

(ii) In addition to these situations, we will also consider cases where $\mathcal{A}$ is a sheaf of E_∞-ring spectra on the site $\mathcal{S}$. (An E_∞-ring spectrum will mean an object in the category of spectra that is also an algebra over an E_∞-operad.) Denoting the constant sheaf of sphere spectra by $\mathcal{R}$, such sheaves of E_∞-ring spectra may be viewed as E_∞-ring objects of $C(\mathcal{S}, \mathcal{R})$.

2.1.4. *Basic conventions.* Henceforth $\mathcal{A}$ will denote a sheaf of E_∞-algebras or a sheaf of E_∞-ring spectra. $Mod(\mathcal{S}, \mathcal{A})$ will denote the category of sheaves of E_∞-modules over $\mathcal{A}$. The obvious pairing $Mod(\mathcal{S}, \mathcal{A}) \times Mod(\mathcal{S}, \mathcal{A}) \to C(\mathcal{S}, \mathcal{R})$ will be denoted $\otimes$. (Here $\mathcal{R}$ will denote a sheaf of commutative Noetherian rings with 1 in case $\mathcal{A}$ is a sheaf of E_∞-algebras, and will denote the constant sheaf of sphere spectra Σ^0 in case $\mathcal{A}$ is a sheaf of E_∞-ring spectra.) We will refer to E_∞-ring spectra (sheaves of E_∞-ring spectra) as E_∞-dgas (sheaves of E_∞-dgas, respectively). In case $\mathcal{R} = \oplus_i \mathcal{R}_i$ is a sheaf of graded rings and that $\mathcal{A} \,\epsilon\, C(\mathcal{S}, \mathcal{R})$ is an E_∞-ring object, then $A[s](t) = \mathcal{A} \underset{\mathcal{R}}{\otimes} \mathcal{R}[s](t)$.

Next we consider the homotopy category associated to $Mod(\mathcal{S}, \mathcal{A})$: this will have the same objects as $Mod(\mathcal{S}, \mathcal{A})$, but morphisms will be homotopy classes of morphisms, where a homotopy H between two morphisms $f, g : K \to L$ is a morphism $K \otimes \Delta[1] \to L$ so that $f = H \circ d_0$ and $g = H \circ d_1$. We define a morphism $f : K \to L$ to be a *quasi-isomorphism* if $f : K \to L$ induces an isomorphism on the cohomology sheaves. A diagram $K' \xrightarrow{f} K \xrightarrow{g} K''$ is a *distinguished triangle* if there is a map from the mapping cone, $Cone(f)$ to K'' that is a quasi-isomorphism. (Observe that $Cone(f) \,\epsilon\, Mod(\mathcal{S}, \mathcal{A})$.) The *derived category* $DMod(\mathcal{S}, \mathcal{A})$ is the category obtained by inverting these quasi-isomorphisms. We will let $D_+(Mod(\mathcal{S}, \mathcal{A}))$ $(D_-(Mod(\mathcal{S}, \mathcal{A})))$ denote the full sub-category of $DMod(\mathcal{S}, \mathcal{A})$ consisting of complexes that are bounded below (above, respectively).

Proposition 2.5. *$DMod(\mathcal{S}, \mathcal{A})$ is a triangulated category with the above structure.*

Proof. This is skipped and left as an exercise. ☐

For each U in the site $\mathcal{S}$, let $j_U^*: C(\mathcal{S},\mathcal{R}) \to C(\mathcal{S}/U,\mathcal{R})$ denote the obvious restriction functor; let $j_{U!}$ (Rj_{U*}) denote the left-adjoint (right-adjoint) to j_U^*.

2.2. **The standard t-structure on $D(\mathcal{S},\mathcal{R})$.** One defines the standard t-structure on $D(\mathcal{S},\mathcal{R})$ to be given by the two full subcategories:

$$D(\mathcal{S},\mathcal{R})^{\leq 0} = \{K \,\epsilon\, D(\mathcal{S},\mathcal{R}) | \mathcal{H}^i(K) = 0, i > 0\}, D(\mathcal{S},\mathcal{R})^{\geq 0} = \{K \,\epsilon\, D(\mathcal{S},\mathcal{R}) | \mathcal{H}^i(K) = 0, i < 0\}. \tag{2.2.1}$$

One may readily observe that $D(\mathcal{S},\mathcal{R})^{\leq 0}$ is generated by $\{j_{U!}(\mathcal{R}_{|U})[n] | n \geq 0\}$. The (obvious) inclusion functor $D(\mathcal{S},\mathcal{R})^{\leq 0} \to D(\mathcal{S},\mathcal{R})$ has a right adjoint which is denoted $\tau_{\leq 0}$.

Proposition 2.6. *Assume the hypotheses as in Proposition 2.3 with $C(\mathcal{S},\mathcal{R})$ denoting any one of the categories considered there.*

Let $\mathcal{A}$ denote a sheaf of E_∞-dgas in $C(\mathcal{S},\mathcal{R})$. Then, for each U in the site $\mathcal{S}$, and each integer n, $j_{U!}j_U^*(\mathcal{A})[n]$ is a compact object in $D(Mod(\mathcal{S},\mathcal{A}))$.

Proof. Let $M \,\epsilon\, DMod(\mathcal{S},\mathcal{A})$. Let $RHom_{\mathcal{A}}$ ($\mathcal{RH}om_{\mathcal{A}}$) denote the external (internal) Hom in the derived category $DMod(\mathcal{S},\mathcal{A})$. Then

$$RHom_{\mathcal{A}}(j_{U!}j_U^*(\mathcal{A})[n], M) = R\Gamma(\mathcal{S}, \mathcal{RH}om_{\mathcal{A}}(j_{U!}j_U^*(\mathcal{A})[n], M)) = R\Gamma(\mathcal{S}_{|U}, \mathcal{RH}om_{j_U^*(\mathcal{A})}(j_U^*(\mathcal{A}), M[-n]))$$

$$= R\Gamma(\mathcal{S}_{|U}, \mathcal{RH}om_{\mathcal{R}_{|U}}(\mathcal{R}_{|U}, M[-n])) = R\Gamma(\mathcal{S}_{|U}, M[-n]).$$

Next let $\{M_\alpha|\alpha\}$ denote a direct system of objects in $DMod(\mathcal{S},\mathcal{A})$. In view of the identifications in the last paragraph, one observes, using Proposition 2.3 that $\underset{\alpha}{\operatorname{colim}} RHom_{\mathcal{A}}(j_{U!}j_U^*(\mathcal{A})[n], M_\alpha) \cong RHom_{\mathcal{A}}(j_{U!}j_U^*(\mathcal{A})[n], \underset{\alpha}{\operatorname{colim}} M_\alpha)$. This proves the proposition. □

Examples 2.7. One may consider the following as typical examples where the last proposition applies:

(1) R = the constant sheaf $\mathbb{Q}$ and $\mathcal{A}$ is any sheaf of dgas in $C(\mathcal{S},\mathbb{Q})$ with $\mathcal{S}$ the big Zariski, étale, Nisnevich or cdh sites associated to schemes of finite types over a Noetherian base scheme of finite Krull dimension.

(2) R = the constant sheaf $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}/p$ for some prime p and $\mathcal{A}$ is any sheaf of dgas or E_∞-dgas in $C(\mathcal{S},R)$ with $\mathcal{S}$ the big Zariski, Nisnevich or cdh sites associated to schemes of finite type over a Noetherian base scheme of finite Krull dimension.

(3) $R = Z/\ell$, where ℓ is a prime different from the characteristic of the base scheme, which we assume is a field k. We will also assume that k has finite ℓ-étale cohomological dimension, for example, it is either an algebraically closed or a finite field. $\mathcal{S}$ will denote the big étale site of k or the small étale site of a scheme of finite type over k, and $\mathcal{A}$ will denote a sheaf of E_∞-dgas in $C(\mathcal{S},R)$.

(4) In addition to the above, one may let $\mathcal{S}$ denote any site so that for every object U in $\mathcal{S}$, the small site $\mathcal{S}/U$ is coherent and has finite cohomological dimension with respect to all abelian sheaves, for example, the transcendental site associated to the complex points of any complex algebraic variety. Now $\mathcal{R}$ may denote any sheaf of of commutative rings, or the constant sheaf of sphere spectra and $\mathcal{A}$ any E_∞-ring object in $C(\mathcal{S},\mathcal{R})$.

2.3. **The connectedness assumption on the E_∞-dga.** In this framework, we will make the following basic hypotheses throughout:

$\mathcal{A}$ is a sheaf of E_∞-dgas or dgas on the ringed site $(\mathcal{S},\mathcal{R})$. We will assume that $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \to \mathcal{R}$ which is assumed to be a map of sheaves of E_∞-dgas and that $\mathcal{A}$ is *weakly connected*, i.e. $\mathcal{A}^i = 0$ for $i < 0$. We will say $\mathcal{A}$ is *strongly connected* if it is weakly connected and in addition, $\mathcal{A}^0 = \mathcal{R}$. It suffices to assume that $\mathcal{A}$ is weakly-connected for the rest of paper.

Here are two particularly interesting cases of such E_∞-dgas.

(i) Let $f: \mathcal{X} \to \mathcal{Y}$ denote a map of sites and let $\mathcal{R}$ denote a sheaf of rings on $\mathcal{X}$. Then $Rf_*(\mathcal{R})$ is a sheaf of E_∞-rings on the site $\mathcal{Y}$. This follows from the discussion in 9.1. (We also discuss an ℓ-adic variant there.) The connectedness is clear since abelian sheaf cohomology is trivial in negative degrees.

(ii) Another particularly interesting case is when $\mathcal{A}$ denotes the rational motivic dga (constructed in [J-1]): in this case the above condition that $\mathcal{A}$ is weakly connected (strongly connected) is equivalent to the weak form (strong form, respectively) of the Beilinson-Soulé vanishing conjecture. More generally one may assume $\mathcal{A}$ is a sheaf of bi-graded dgas, $\mathcal{A} = \oplus_{r,s}\mathcal{A}(s)^r$ with r denoting the degree of the chain complex and s another index we call the weight, so that $\mathcal{A}^\bullet = \oplus \mathcal{A}^\bullet(s)$, i.e. each $\mathcal{A}^\bullet(s)$ is a sub-complex of $\mathcal{A}^\bullet$. Then, if $\mathcal{A}^\bullet(0) = \mathcal{R}[0]$ one clearly obtains an augmentation in the above sense. Observe that these conditions are met by the (integral) motivic E_∞-dga. The weak-connectedness (strong-connectedness) is then equivalent to an appropriate integral form of the Beilinson-Soulé vanishing conjecture.

3. Construction of (standard) t-structures on sites provided with E_∞ sheaves of dgas

In this section we will provide each of the basic situations considered in the last section with standard t-structures.

Proposition 3.1. *Assume the hypotheses as in Proposition (2.6) and that $\mathcal{A} \epsilon C(\mathcal{S}, \mathcal{R})$ is a sheaf of E_∞ dgas. Let $\mathbf{C} = \{j_{U!}(\mathcal{A}_{|U})[n] | U \epsilon \mathcal{S}, n \geq 0\}$.*

(i) Then $\mathbf{C}$ is a set of compact objects in the triangulated category $DMod(\mathcal{S}, \mathcal{A})$.

Let $Coh(\mathbf{C})^{\leq 0}$ denote the smallest sub-category of $Mod(\mathcal{S}, \mathcal{A})$ containing all of $\mathbf{C}$ and closed under the following operations: (i) finite sums, (ii) mapping cones, (iii) translations [1] and (iv) extensions (i.e. if $F' \xrightarrow{f} F \to Cone(f) \to F'[1]$ is a distinguished triangle with F', $Cone(f)$ and $F'[1] \epsilon Coh(\mathcal{S})^{\leq 0}$, then $F \epsilon Coh(\mathbf{C})^{\leq 0}$.)

(ii) Let $QCoh(\mathbf{C})^{\leq 0}$ denote the full sub-category of $Mod(\mathcal{S}, \mathcal{A})$ consisting of filtered colimits $\underset{\alpha}{colim} F_\alpha$, each $F_\alpha \epsilon Coh(\mathbf{C})^{\leq 0}$ and with the indexing set for the filtered colimit being small. Then the smallest co-complete pre-aisle of $DMod(\mathcal{S}, \mathcal{A})$ containing all of $\mathbf{C}$ identifies with the image of $QCoh(\mathbf{C})^{\leq 0}$ in $DMod(\mathcal{S}, \mathcal{A})$.

(iii) Let $\mathrm{Comp}(\mathbf{C})$ *denote the smallest full sub-category of $QCoh(\mathbf{C})^{\leq 0}$ containing $Coh(\mathbf{C})^{\leq 0}$ and closed under summands. Now* $\mathrm{Comp}(\mathbf{C})$ *identifies with the full sub-category of compact objects in $QCoh(\mathbf{C})^{\leq 0}$.*

Proof. The first assertion follows from Proposition 2.6. Clearly $QCoh(\mathbf{C})^{\leq 0}$ is closed under all small sums: any such sum may be written as a filtered colimit of finite sums. Next observe that the sub-category $Coh(\mathbf{C})^{\leq 0}$ consists of compact objects. Now it suffices to show that $QCoh(\mathbf{C})^{\leq 0}$ is closed under mapping cones, translations [1] and extensions. Let $F' = \underset{i}{\mathrm{colim}}\{F'_i | i \epsilon I\} \to F = \underset{j}{\mathrm{colim}}\{F_i | i \epsilon J\}$ denote a map of objects in $QCoh(\mathbf{C})^{\leq 0}$ with each F'_i, $F_i \epsilon Coh(\mathbf{C})^{\leq 0}$. Since each F'_i is a compact object, one observes that for each $i \epsilon I$, there exists an index $j_i \epsilon J$ so that the map $F'_i \to F' \to \underset{j}{\mathrm{colim}}\{F_j | j\}$ factors through F_{j_i}. Therefore, after re-indexing $F = \{F_j | j \epsilon J\}$ one may assume that both F' and F are indexed by the same indexing set I and the map f is given by a map $\{f_i : F'_i \to F_i | i \epsilon I\}$. Clearly the mapping cone $Cone(f_i) \epsilon Coh(\mathbf{C})^{\leq 0}$ and therefore $Cone(f) \cong \underset{\to i}{\lim} Cone(f_i) \epsilon QCoh(\mathbf{C})^{\leq 0}$. Similarly one may show that $QCoh(\mathbf{C})^{\leq 0}$ is closed under the translations [1]. Next consider an extension: $F' \xrightarrow{f} F \xrightarrow{g} F'' \xrightarrow{h} F'[1]$ with F', $F'' \epsilon QCoh(\mathbf{C})^{\leq 0}$. Now F identifies with $Cone(h)[-1]$. Clearly the argument above shows that one may write the map h as $\underset{i}{\mathrm{colim}} h_i : F''_i \to F'_i[1]$; therefore one has a diagram of extensions $F'_i \to Cone(h_i)[-1] \to G_i \to F'_i[1]$. Therefore $Cone(h_i)[-1] \epsilon Coh(\mathbf{C})^{\leq 0}$; since $F \cong \underset{i}{\mathrm{colim}} Cone(h_i)[-1]$, it follows that $F \epsilon QCoh(\mathbf{C})^{\leq 0}$. Therefore $QCoh(\mathbf{C})^{\leq 0}$ is a co-complete pre-aisle containing all of $\mathbf{C}$. This proves (ii).

(iii). Let $F = \underset{\to i}{\lim} F_i$ denote an object in $QCoh(\mathbf{C})^{\leq 0}$ which is a compact object. Then the identity map of F must factor through some F_j, so that F is a split summand of F_j which clearly belongs to $Coh(\mathbf{C})^{\leq 0}$. This proves that the full subcategory of compact objects in $QCoh(\mathbf{C})^{\leq 0}$ contains $Coh(\mathbf{C})^{\leq 0}$, and the objects in the former are in general summands of objects in $Coh(\mathbf{C})^{\leq 0}$. Therefore, the full subcategory of compact objects in $QCoh(\mathbf{C})^{\leq 0}$ is a subcategory of $\mathrm{Comp}(\mathbf{C})$. One may also show readily that the full subcategory of compact objects in $QCoh(\mathbf{C})^{\leq 0}$ contains $Coh(\mathbf{C})^{\leq 0}$ and is closed under taking summands. Therefore, $\mathrm{Comp}(\mathbf{C})$ is a subcategory of the full subcategory of compact objects in $QCoh(\mathbf{C})^{\leq 0}$. This proves the last assertion. □

One of the main results we prove in this paper is the following:

Theorem 3.2. *Assume the hypotheses as in Proposition (2.6) and that $\mathcal{A} \epsilon C(\mathcal{S}, \mathcal{R})$ is a sheaf of E_∞ dgas. Let $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ denote the pre-aisle in $DMod(\mathcal{S}, \mathcal{A})$ generated by $j_{U!}j_U^*(\mathcal{A}[n])$, $n \geq 0$, U in the site $\mathcal{S}$.*

Then (i) $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ is an aisle in $DMod(\mathcal{S}, \mathcal{A})$, i.e. defines a t-structure on $DMod(\mathcal{S}, \mathcal{A})$.

(ii) Assume next that $\mathcal{A}$ is weakly-connected as in the hypotheses of 2.3. Assume also that if X denotes the terminal object of the site $\mathcal{S}$, and $U \to X$ and $W \to X$ are maps in the site, then the fibered product $U \underset{X}{\times} W$ exists in the site $\mathcal{S}$. Let V denote an object in the site $\mathcal{S}$ and let $j_V : V \to X$ denote the structure map. Then

(a) $j_{V!}(\mathcal{A}_{|V}) \, \epsilon \, DMod(\mathcal{S},\mathcal{A})^{\leq 0} \cap DMod(\mathcal{S},\mathcal{A})^{\geq 0}$ *= the heart of the above t-structure. In particular,* $\mathcal{A} = \mathcal{A}_{|X}$ *belongs to the heart of the above t-structure.*

(b) Moreover, every object M in $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ satisfies the property that the natural map $\tau_{\leq 0}(\mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}M) \to \mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}M$ is a quasi-isomorphism in $DMod(\mathcal{S},\mathcal{R})$. In other words, the functor

$$\mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}(\quad) : DMod(\mathcal{S},\mathcal{A}) \to DMod(\mathcal{S},\mathcal{R})$$

sends $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ to $DMod(\mathcal{S},\mathcal{R})^{\leq 0}$.

Proof. The first statement is clear from the last proposition in view of Theorem 1.9. We will now prove the remaining statements. To prove (ii)(a), observe that $RHom_{\mathcal{A}}(j_{U!}(\mathcal{A}_{|U})[n], j_{V!}(\mathcal{A}_{|V})) \simeq R\Gamma(U, j_!\mathcal{A}_{U\underset{X}{\times}V}[-n])$, where $j: U\underset{X}{\times}V \to U$ is the obvious map so that

$$Hom_{DMod(\mathcal{S},\mathcal{A})}(j_{U!}(\mathcal{A}_{|U})[n], j_{V!}(\mathcal{A}_{|V})) = H^0(RHom_{\mathcal{A}}(j_{U!}(\mathcal{A}_{|U})[n], j_{V!}(\mathcal{A}_{|V})) = H^0(R\Gamma(U, j_!\mathcal{A}_{U\underset{X}{\times}V}[-n])) = 0$$

for any $n \geq 1$. The last equality follows from the hypothesis that $\mathcal{A}^i = 0$ for all $i < 0$ and some basic properties of the functor $j_!$ including, in particular, that it is exact in this case. (See [Mi, p. 78] or [St, Sites: localization] for a proof that the functor $j_!$ is exact in this case and that $j_{U!}j_U^*(j_{V!}(\mathcal{A}_{|V}))$ identifies with $j_!\mathcal{A}_{U\underset{X}{\times}V}$.) Next one shows that the collection of objects K in $D(Mod(\mathcal{S},\mathcal{A}))$ for which $Hom_{DMod(\mathcal{S},\mathcal{A})}(K, j_{V!}(\mathcal{A}_{|V})) = 0$ is closed under finite sums, translations $[n]$, $n \geq 0$, extensions and mapping cones. The assertion for mapping cones follows by considering the exact sequence

$$Hom_{DMod(\mathcal{S},\mathcal{A})}(K'[1], j_{V!}(\mathcal{A}_{|V})) \to Hom_{DMod(\mathcal{S},\mathcal{A})}(Cone(\alpha), j_{V!}(\mathcal{A}_{|V})) \to Hom_{DMod(\mathcal{S},\mathcal{A})}(K, j_{V!}(\mathcal{A}_{|V}))$$

associated to the distinguished triangle: $K' \overset{\alpha}{\to} K \to Cone(\alpha) \to K'[1]$. This shows that

$$Hom_{DMod(\mathcal{S},\mathcal{A})}(K, j_{V!}(\mathcal{A}_{|V})) = 0$$

for all objects $K \, \epsilon \, Coh(\mathbf{C})^{\leq -1}$. Next let $M \, \epsilon \, QCoh(\mathbf{C})^{\leq -1}$. Then M is the filtered colimit of a diagram forming a direct system of objects in $Coh(\mathbf{C})^{\leq -1}$). First observe that if the direct system is a (co)-tower $\{K_n | n \geq 0\}$ of objects in $QCoh(\mathbf{C})^{\leq -1}$, then one obtains the short-exact sequence (where Hom denotes $Hom_{DMod(\mathcal{S},\mathcal{A})}$):

$$0 \to \lim_{\infty \leftarrow n}{}^1 Hom(K_n, j_{V!}(\mathcal{A}_{|V}[-1]) \to Hom(\lim_{n\to\infty} K_n, j_{V!}(\mathcal{A}_{|V}) \to \lim_{\infty\leftarrow n} Hom(K_n, j_{V!}(\mathcal{A}_{|V}) \to 0 \tag{3.0.1}$$

By what is shown above, the two end terms are zero, thereby showing the middle term is also zero.

In general, M can only be realized as the filtered colimit of some small diagram of objects in $Coh(\mathbf{C})^{\leq -1}$. In this case, one may apply a simplicial replacement to this diagram as in [BK, Chapter XII, section 5], so that M is identified with the homotopy colimit of a simplicial object $S_\bullet$ in $QCoh(\mathbf{C})^{\leq -1}$, with each S_n a sum of objects in $Coh(\mathbf{C})^{\leq -1}$. On applying $RHom(\quad, j_{V!}(\mathcal{A}_{|V})$, the above homotopy colimit comes out as a homotopy inverse limit. In fact this homotopy inverse limit may be identified with the homotopy inverse limit of a tower, by truncating the simplicial replacement $S_\bullet$ at finite degrees. The required conclusion that $Hom_{DMod(\mathcal{S},\mathcal{A})}(K, j_{V!}(\mathcal{A}_{|V})) = 0$ for all objects $M \, \epsilon \, QCoh(\mathbf{C})^{\leq -1}$ now follows readily in view of (3.0.1). This proves

$$j_{V!}(\mathcal{A}_V) \, \epsilon \, (DMod(\mathcal{S},\mathcal{A})^{\leq -1})^{\perp} = DMod(\mathcal{S},\mathcal{A})^{\geq 0}.$$

Since $j_{V!}(\mathcal{A}_V) \, \epsilon \, DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ by definition, the assertion (a) in (ii) is proved.

By the definition of $D(Mod(\mathcal{S},\mathcal{A})^{\leq 0}$ above (and Proposition 3.1 above), there exist a sequence $\{M_i | i \, \epsilon \, I\}$ in $Coh(\mathbf{C})^{\leq 0}$ so that $M \cong \underset{i}{\operatorname{colim}}\{M_i|i\}$.

One of our key observations now is that each $M_i\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} \, \epsilon \, Coh(\mathcal{S},\mathcal{R})^{\leq 0}$. This is clear if $M_i = j_{U!}(\mathcal{A}_{|U}[n])$ for some $U \, \epsilon \, \mathcal{S}$ and $n \geq 0$: in this case $M\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} \cong j_{U!}(\mathcal{R}_{|U}[n])$. In general, recall that M is obtained by finitely many operations from the set $\{j_{U!}(\mathcal{A}_{|U})[n] | U \, \epsilon \, \mathcal{S}, n \geq 0\}$ where the allowed operations are finite sums, mapping cones, translations [1] and extensions. Since $Coh(\mathcal{S},\mathcal{R})^{\leq 0}$ is closed under these operations, one may show readily that each $M_i\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} \, \epsilon \, Coh(\mathcal{S},\mathcal{R})^{\leq 0}$. Next recall that $QCoh(\mathcal{S},\mathcal{R})^{\leq 0}$ is the set of objects obtained as filtered colimits of objects in $Coh(\mathcal{S},\mathcal{R})^{\leq 0}$. Therefore $M\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} = (\underset{i}{\mathrm{colim}} M_i)\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} \cong \underset{i}{\mathrm{colim}}(M_i\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R}) \; \epsilon \; QCoh(\mathcal{S},\mathcal{R})^{\leq 0}$. This completes the proof of the second statement and hence that of the theorem. □

Remark 3.3. It does not seem possible to say, in general, that $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ is the full sub-category of $DMod(\mathcal{S},\mathcal{A})$ that the functor $\mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}(\quad)$ sends to $DMod(\mathcal{S},\mathcal{R})^{\leq 0}$. Nevertheless, the above theorem shows that one has meaningful, non-trivial t-structures defined on the category $DMod(\mathcal{S},\mathcal{A})$.

Definition 3.4. (The truncation functors for the standard t-structure) Assume the above situation. We will define

$$\sigma_{\leq n} : DMod(\mathcal{S},\mathcal{A}) \to DMod(\mathcal{S},\mathcal{A})^{\leq n}$$

as right-adjoint to the obvious imbedding $DMod(\mathcal{S},\mathcal{A})^{\leq n} \subseteq DMod(\mathcal{S},\mathcal{A})$. Let $K \;\epsilon\; DMod(\mathcal{S},\mathcal{A})$. We will define $\sigma_{\geq n+1}(K)$ by requiring

$$\sigma_{\leq n}(K) \to K \to \sigma_{\geq n+1}(K) \to \sigma_{\leq n}K[1]$$

to be a distinguished triangle. Then $\sigma_{\geq n+1}$ will be left-adjoint to the obvious imbedding $DMod(\mathcal{S},\mathcal{A})^{\geq n+1} \to DMod(\mathcal{S},\mathcal{A})$.

In the next section, we proceed to define and study the notion of *constructibility* in the category $D(Mod(\mathcal{S},\mathcal{A}))$, in particular how the t-structures defined above relate to the full sub-category of constructible objects.

3.1. **Constructibility and t-structures.**

Definition 3.5. In general, we will say that a sheaf of $\mathcal{A}$-modules M is *of finite type* if *is a compact object* in $DMod(\mathcal{S},\mathcal{A})$. In case $\mathcal{R}$ is the constant sheaf of rings associated to a commutative ring R, we will say that a sheaf of $\mathcal{A}$-modules M is *a constructible $\mathcal{A}$-module*, if it is of finite type. The full sub-category of compact objects in $DMod(\mathcal{S},\mathcal{A})$ will be denoted $D_c(Mod(\mathcal{S},\mathcal{A}))$.

Proposition 3.6. *The truncation functors $\sigma_{\leq n}$ and $\sigma_{\geq n+1}$ as in Definition 3.4 preserve compactness and hence the property of being of finite type as well as being constructible.*

Proof. Let $K' \to K \to K'' \to K'[1]$ denote a distinguished triangle in $DMod(\mathcal{S},\mathcal{A})$. Then if two of K', K and K'' are compact, so is the third. This follows by comparing the distinguished triangle obtained by applying $\underset{i}{\mathrm{colim}}\mathcal{RH}om_{\mathcal{A}}(\quad, L_i)$ and $\mathcal{RH}om_{\mathcal{A}}(\quad, \underset{i}{\mathrm{colim}} L_i)$ to the distinguished triangle $K' \to K \to K'' \to K'[1]$ where $\{L_i|i\}$ is a filtered direct system of objects in $DMod(\mathcal{S},\mathcal{A})$. Therefore, in view of the fact that $\sigma_{\leq n}(K) \to K \to \sigma_{\geq n+1}(K) \to \sigma_{\leq n}(K)[1]$ is a distinguished triangle for any $K \;\epsilon\; DMod(\mathcal{S},\mathcal{A})$, it suffices to prove this for the functor $\sigma_{\geq 0}$.

Next we will show that the filtered colimit of a collection $\{K_i | i \;\epsilon\; I\}$ of objects $K_i \;\epsilon\; DMod(\mathcal{S},\mathcal{A})^{\geq 0}$ also belongs to $DMod(\mathcal{S},\mathcal{A})^{\geq 0}$. To see this, consider

$$Hom_{DMod(\mathcal{S},\mathcal{A})}(j_{U!}(\mathcal{A}_{|U})[n], \lim_{\to i} K_i) = H^0(RHom_{\mathcal{A}}(j_{U!}(\mathcal{A}_{|U})[n], \lim_{\to i} K_i))$$

$$\cong H^0(R\Gamma(U, \lim_{\to i} j_U^*(K_i)[-n])) \cong \lim_{\to i} H^0(R\Gamma(U, j_U^*(K_i)[-n]) = \lim_{\to i} H^0(RHom_{\mathcal{A}}(j_{U!}\mathcal{A}_{|U})[n], K_i)) = 0$$

for all $n \geq 1$. The last equality is from the hypothesis that each $K_i \;\epsilon\; DMod(\mathcal{S},\mathcal{A})^{\geq 0}$ and $n \geq 1$. The isomorphism prior to that follows from our hypotheses on the site as in 2.1: see Proposition 2.3.

Let $\phi : DMod(\mathcal{S},\mathcal{A})^{\geq 0} \to DMod(\mathcal{S},\mathcal{A})$ denote the obvious inclusion functor. Clearly ϕ commutes with filtered colimits since the former category is a full sub-category of the latter and the former category is itself closed under the formation of filtered colimits as we just showed.

Next let $M \;\epsilon\; DMod(\mathcal{S},\mathcal{A})$ denote a compact object and let $\{K_i|i \;\epsilon\; I\}$ be a collection of objects $K_i \;\epsilon\; DMod(\mathcal{S},\mathcal{A})^{\geq 0}$. Then:

$$RHom_{\mathcal{A}}(\sigma_{\geq 0}(M), \lim_{\to i} K_i) \cong RHom_{\mathcal{A}}(M, \phi(\lim_{\to i} K_i)) \cong RHom_{\mathcal{A}}(M, \lim_{\to i} \phi(K_i))$$

$$\cong \lim_{\to i} RHom_{\mathcal{A}}(M, \phi(K_i)) \cong \lim_{\to i} RHom_{\mathcal{A}}(\sigma_{\geq 0}(M), K_i).$$

The last and first isomorphisms use the fact that $\sigma_{\leq 0}$ is left adjoint to ϕ. These prove that $\sigma_{\geq 0}$ preserves compactness. □

Corollary 3.7. *In case $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \to \mathcal{R}$, then the functor $M \mapsto M \overset{L}{\underset{\mathcal{A}}{\otimes}} \mathcal{R}$, $DMod(\mathcal{S}, \mathcal{A}) \to DMod(\mathcal{S}, \mathcal{R})$ induces a functor $D_c(Mod(\mathcal{S}, \mathcal{A}))^{\leq 0} \to D_c(Mod(\mathcal{S}, \mathcal{R}))^{\leq 0}$. Every object in $D_c(Mod(\mathcal{S}, \mathcal{R}))^{\leq 0}$ is in the image of this functor.*

Proof. In view of Theorem 3.2 (ii)(b), it suffices to show that the above functor preserves compactness for objects in $QCoh(\mathbf{C})^{\leq 0}$ as in Proposition 3.1. As shown there, the compact objects in $QCoh(\mathcal{S}, \mathcal{A})^{\leq 0}$ identify with split summands of objects in $Coh(\mathcal{S}, \mathcal{A})^{\leq 0}$. Clearly the functor $M \mapsto M \overset{L}{\underset{\mathcal{A}}{\otimes}} \mathcal{R}$ sends split summands of objects in $Coh(\mathcal{S}, \mathcal{A})^{\leq 0}$ to split summands of objects in $Coh(\mathcal{S}, \mathcal{R})^{\leq 0}$. The last statement follows since the composition of the functor $K \mapsto K \overset{L}{\underset{\mathcal{R}}{\otimes}} \mathcal{A}$, $K \,\epsilon\, Coh(\mathcal{S}, \mathcal{R})$ with the functor $M \mapsto M \overset{L}{\underset{\mathcal{A}}{\otimes}} \mathcal{R}$, $M \,\epsilon\, Coh(\mathcal{S}, \mathcal{A})$ is the identity. $\square$

Let $\mathcal{C} = DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ denote the heart of the above t-structure on $DMod(\mathcal{S}, \mathcal{A})$. Observe that this is an abelian category. Let $\mathcal{C}_c = \mathcal{C} \cap D_c(Mod(\mathcal{S}, \mathcal{A}))$, i.e. the full sub-category of all objects in the heart that are also compact.

Theorem 3.8. *$\mathcal{C}_c$ is an additive sub-category of $\mathcal{C}$ closed under extensions.*

Proof. Observe that any short-exact sequence $M' \to M \to M''$ in $\mathcal{C}$ corresponds to a distinguished triangle in $DMod(\mathcal{S}, \mathcal{A})$ with each M', M and M'' in $\mathcal{C}$. (i.e. If $i : M' \to M$ is a monomorphism in $\mathcal{C}$, the mapping cone of i identifies with M'' which is the cokernel of i in $\mathcal{C}$.) Moreover, in such a short-exact sequence $M' \to M \to M''$, M is compact if both M' and M'' are. $\square$

Remark 3.9. One may also prove the following identity straight from the definition:

$$\sigma_{\leq n-1}(K[1]) \simeq (\sigma_{\leq n} K)[1], K \,\epsilon\, DMod(\mathcal{S}, \mathcal{A}) \tag{3.1.1}$$

3.1.2. **The induced t-structures on bounded derived categories.**

Proposition 3.10. *(i) The t-structure defined above induces a t-structure on $D_+Mod(\mathcal{S}, \mathcal{A})$, which is the full subcategory of bounded below complexes in $DMod(\mathcal{S}, \mathcal{A})$, i.e. complexes $K \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})$ whose cohomology sheaves $\mathcal{H}^i(K) = 0$ for all $i << 0$.*

(ii) In case $\mathcal{A}$ is bounded above, i.e. $\mathcal{A}^i = 0$ for $i >> 0$, then the above t-structure induces a t-structure on $D_-Mod(\mathcal{S}, \mathcal{A})$ which is the full subcategory of $DMod(\mathcal{S}, \mathcal{A})$ consisting of complexes that are bounded above i.e. complexes $K \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})$ whose cohomology sheaves $\mathcal{H}^i(K) = 0$ for all $i >> 0$. In this case it also induces a t-structure on the bounded derived category $D_bMod(\mathcal{S}, \mathcal{A})$.

Proof. We will first show that every object in $DMod(\mathcal{S}, \mathcal{A})^{\geq 1}$ is bounded below. Let $M \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})^{\geq 1}$. Now observe that for each $i < 0$,

$$H^i R\Gamma(U, M) = RHom_{\mathcal{A}}(j_{U!}(\mathcal{A}_{|U}[-i]), M) = 0.$$

The last equality comes from the observation that $j_{U!}(\mathcal{A}_{|U}[-i]) \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$, (since $i < 0$) and the assumption that $M \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})^{\geq 1}$. Varying U in the site, it follows that $\mathcal{H}^i(M) = 0$ for all $i < 0$. In particular M is bounded below. Therefore, for any $K \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})$, $\sigma_{\geq 1}(K)$ belongs to $DMod(\mathcal{S}, \mathcal{A})^{\geq 1}$ and therefore is bounded below. Now it follows from the distinguished triangle

$$\sigma_{\leq 0} K \to K \to \sigma_{\geq 1} K \to \sigma_{\leq 0} K[1]$$

that, if $K \,\epsilon\, D_+Mod(\mathcal{S}, \mathcal{A})$ also, then so does $\sigma_{\leq 0} K$. This proves that the functor $\sigma_{\leq 0}$ preserves $D_+Mod(\mathcal{S}, \mathcal{A})$. Next observe that $D_+Mod(\mathcal{S}, \mathcal{A})$ is stable by finite applications of both the positive shift [1] and the negative shift $[-1]$. Making use of (3.1.1), (i) follows from these observations.

Next we will consider (ii). Since $\mathcal{A}$ is bounded above, there exists an integer N so that $\mathcal{H}^i(\mathcal{A}) = 0$ for all $i > N$. Now the aisle $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ is generated by the objects $j_{U!}(\mathcal{A}_{|U})[n]$, $n \geq 0$, which all have cohomology sheaves trivial in degrees $> N$. Moreover, one may see readily that any object generated by the above objects by taking positive shifts, extensions, mapping cones, finite sums and small filtered colimits all have cohomology sheaves trivial above degree N. For any $K \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})$, $\sigma_{\leq 0}(K) \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ and therefore has cohomology sheaves trivial above degree N. Now the distinguished triangle

$$\sigma_{\leq 0} K \to K \to \sigma_{\geq 1} K \to \sigma_{\leq 0} K[1]$$

shows that, if K is also bounded above, then so is $\sigma_{\geq 1} K$. This proves the first statement in (ii). The second statement now follows from (i) and the first statement in (ii). $\square$

4. Non-standard t-structures, generalized perverse sheaves and perverse extensions

In this section we will show briefly how to define generalized perverse sheaves and perverse extensions of generalized perverse sheaves.

Assume one is given a stratified site i.e. one is provided with a decomposition of the terminal object X of the site $\mathcal{S}$ into a disjoint union of finitely many *locally closed sub-objects*. By taking the unions of the strata one defines a finite increasing filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$ of the object X as well as a filtration of sites $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_n = \mathcal{S}$, where each inclusion is an open immersion (i.e. the corresponding functor of the associated topoi of sheaves of sets is an open immersion.) For the most part, the site $\mathcal{S}$ will denote either the étale site of a scheme or the Nisnevich site of a smooth scheme, in which the case the stratification of X will be by locally closed smooth subschemes, though the example considered in section 7 shows that one may also consider quotient stacks provided with suitable topologies and stratifications. It is shown in [J-3] that one may also consider possibly singular schemes over fields of characteristic 0 and provided with the cdh-topology.

Let a (perversity) function $p : \{S_{i+1} - S_i | i\} \rightarrow (integers)$ be given. We will assume that p is non-decreasing and $p(\mathcal{S}_0) = 0$. Let $\mathcal{A}$ denote sheaf of E_∞ dgas on $\mathcal{S}$. Now $DMod(\mathcal{S}, \mathcal{A})$ and $DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}})$ will denote the obvious derived categories of $\mathcal{A}$-modules. Now one may glue together standard t-structures on each stratum, shifted suitably by the perversity $p(\mathcal{S}_i - \mathcal{S}_{i-1})$ to define a non-standard t-structure on $DMod(\mathcal{S}, \mathcal{A})$ exactly as in [BBD, Chapter 1].

In fact we will follow the terminology in [BBD, Chapter 1, 1.4] rather closely. (The only difference will be that we take the t-structure on the open stratum to be given by the usual t-structure, whereas in [BBD], they shift this.) Accordingly if $j_i : \mathcal{S}_i - \mathcal{S}_{i-1} \rightarrow \mathcal{S}$ is the obvious inclusion, then $\mathcal{A}_{S_i - S_{i-1}}$ will denote $j_i^*(\mathcal{A})$ which is simply the restriction of $\mathcal{A}$ to the stratum $\mathcal{S}_i - \mathcal{S}_{i-1}$. For such a stratified object, $j_i^!$ will denote the right-adjoint to $j_{i!} : DMod(\mathcal{S}_i - S_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}}) \rightarrow DMod(\mathcal{S}, \mathcal{A})$ whose existence can be shown fairly easily. However, we need to assume that the following distinguished triangle (i.e. *localization sequence*) holds, whenever $i : Y \rightarrow X$ is a closed immersion with open complement $j : U \rightarrow X$:

$$j_! j^*(M) \rightarrow M \rightarrow i_* i^*(M) \rightarrow j_! j^*(M)[1], M \ \epsilon \ DMod(\mathcal{S}, \mathcal{A}). \tag{4.0.3}$$

We will also require that the two compositions

$$j^* i_* \text{ and } i^* j_! \text{ are both trivial.} \tag{4.0.4}$$

One may observe that taking adjoints in (4.0.3) provides a second localization sequence: $i_* Ri^!(M) \rightarrow M \rightarrow Rj_* j^*(M) \rightarrow i_* Ri^!(M)[1]$, for any $M \ \epsilon \ DMod(\mathcal{S}, \mathcal{A})$ and that taking adjoints of the first relation in (4.0.4) implies, $Ri^! Rj_*$ is also trivial. We will call the combination of (4.0.3) and (4.0.4) the *gluing property*. Depending on the situation, one may only need to assume this when all the schemes are smooth. (This is what often happens in the motivic case, since the schemes are all assumed to be smooth.) In [J-3], we also consider the motivic case for non-smooth schemes over a field of characteristic 0 using the cdh-topology. Accordingly we will define

$$\begin{aligned} DMod(\mathcal{S}, \mathcal{A})^{\leq 0} &= \{K \ \epsilon \ DMod(\mathcal{S}, \mathcal{A}) | j_i^*(K) \ \epsilon \ DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}})^{\leq p(\mathcal{S}_i - \mathcal{S}_{i-1})} \text{ for all } i\} \\ DMod(\mathcal{S}, \mathcal{A})^{\geq 0} &= \{K \ \epsilon \ DMod(\mathcal{S}, \mathcal{A}) | j_i^!(K) \ \epsilon \ DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}})^{\geq p(\mathcal{S}_i - \mathcal{S}_{i-1})} \text{ for all } i\}. \end{aligned} \tag{4.0.5}$$

Definition 4.1. (Generalized Perverse Sheaves) Since we prove in the theorem below that the above objects define a t-structure on $DMod(\mathcal{S}, \mathcal{A})$, we call objects in $DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ *Generalized Perverse Sheaves.*

Definition 4.2. The functors $\sigma_{\leq 0}^{\mathcal{S}}$ and $\sigma_{\geq 1}^{\mathcal{S}}$. We define the functor $\sigma_{\leq 0}^{\mathcal{S}} : DMod(\mathcal{S}, \mathcal{A}) \rightarrow DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ as follows. Let $K \ \epsilon \ DMod(\mathcal{S}, \mathcal{A})$ be given. We let $L =$ the canonical homotopy fiber of the map $K \rightarrow Rj_*(\sigma_{\geq p(S_0)+1} j^*(K))$. Then we define $\sigma_{\leq 0}^{\mathcal{S}}(K) =$ the canonical homotopy fiber of the map $L \rightarrow i_*(\sigma_{\geq p(S_1 - S_0)+1} i^*(L))$. One checks that, so defined $\sigma_{\leq 0}^{\mathcal{S}}(K) \ \epsilon \ DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$. $\sigma_{\geq 1}^{\mathcal{S}}(K)$ is defined to be the mapping cone of the obvious map $\sigma_{\leq 0}^{\mathcal{S}}(K) \rightarrow K$.

Theorem 4.3. *(i) The above structures define a t-structure on $DMod(\mathcal{S}, \mathcal{A})$ with the aisle $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ and co-aisle $DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$.*

(ii) Given a generalized perverse sheaf $P_0 \ \epsilon \ DMod(\mathcal{S}_0, \mathcal{A}_{\mathcal{S}_0})^{\leq p(\mathcal{S}_0)} \cap DMod(\mathcal{S}_0, \mathcal{A}_{\mathcal{S}_0})^{\geq p(\mathcal{S}_0)}$, there exist extensions

$$P \ \epsilon \ DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$$

of P_0, i.e. $j_0^*(P) \simeq P_0$.

(iii) Given a generalized perverse sheaf $P_0 \, \epsilon \, DMod(\mathcal{S}_0, \mathcal{A}_{\mathcal{S}_0})^{\leq p(\mathcal{S}_0)} \cap DMod(\mathcal{S}_0, \mathcal{A}_{\mathcal{S}_0})^{\geq p(\mathcal{S}_0)}$*, the extension*

$$P \, \epsilon \, DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0} \text{ of } P_0$$

is unique if $j_i^*(P) \, \epsilon \, DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}})^{\leq p(\mathcal{S}_i - \mathcal{S}_{i-1})-1}$ *and* $j_i^!(P) \, \epsilon \, DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}})^{\geq p(\mathcal{S}_i - \mathcal{S}_{i-1})+1}$.

Proof. This is essentially the argument in [BBD, Theorem 1.4.10]: we provide some details mainly for the sake of completeness. We will restrict to the case where there are only two strata, i.e. $\mathcal{S}_1 = \mathcal{S}$. Let $K \, \epsilon \, DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ and $L \, \epsilon \, DMod(\mathcal{S}, \mathcal{A})^{\geq 1}$. To show $Hom(K, L) = 0$, one may argue as follows. First one observes the existence of the distinguished triangle:

$$j_!(j^*(K)) \to K \to i_* i^*(K) \to j_! j^*(K)[1]$$

where $j : \mathcal{S}_0 \to \mathcal{S}$ and $i : \mathcal{S} - \mathcal{S}_0 \to \mathcal{S}$ are the obvious maps. This provides us with the long-exact-sequence:

$$\cdots \to H^0(RHom_{\mathcal{A}}(i_* i^*(K), L)) \to H^0(RHom_{\mathcal{A}}(K, L)) \to H^0(RHom_{\mathcal{A}}(j_! j^*(K), L)) \to H^1(RHom_{\mathcal{A}}(i_* i^*(K), L)) \cdots .$$

Now

$$H^0(RHom_{\mathcal{A}}(i_* i^*(K), L)) \cong H^0(RHom_{\mathcal{A}}(i^*(K), i^!(L)) \cong 0 \text{ and}$$
$$H^0(RHom_{\mathcal{A}}(j_! j^*(K), L)) \cong H^0(RHom_{\mathcal{A}}(j^*(K), j^*(L))) \cong 0$$

by our hypotheses. Therefore $H^0(RHom_{\mathcal{A}}(K, L)) = 0$ as well. It is clear from the definitions that $DMod(\mathcal{S}, \mathcal{A})^{\leq n} \subseteq DMod(\mathcal{S}, \mathcal{A})^{\leq n+1}$ and similarly $DMod(\mathcal{S}, \mathcal{A})^{\geq n+1} \subseteq DMod(\mathcal{S}, \mathcal{A})^{\geq n}$. These prove (i).

Given P_0 as in (ii), we will discuss one extension. First let $Y = \mathcal{S} - \mathcal{S}_0$ denote the closed stratum. Then for a $K \, \epsilon \, DMod(\mathcal{S}, \mathcal{A})$, we define

$$\sigma^Y_{\leq n} K = \text{homotopy fiber of } (K \to i_* \sigma_{\geq p(Y)+n+1} i^*(K)). \tag{4.0.6}$$

Now we let

$$P = \sigma^Y_{\leq 0}(Rj_*(P_0)). \tag{4.0.7}$$

To see this is an extension of P_0 we proceed as follows. Clearly $j^* \circ i_*$ is trivial, so that $j^* i_* \sigma_{\geq p(Y)+1} i^*(Rj_*(P_0)) \simeq *$ and therefore, $j^*(\sigma^Y_{\leq 0}(Rj_*(P_0)) \simeq j^*(Rj_*(P_0)) \simeq P_0$.

To show P is in fact a generalized perverse sheaf, now it suffices to show the following:

$$i^*(\sigma^Y_{\leq 0}(Rj_*(P_0)) \, \epsilon \, DMod(Y, \mathcal{A}_Y)^{\leq p(Y)} \text{ and } Ri^!(\sigma^Y_{\leq 0}(Rj_*(P_0)) \, \epsilon \, DMod(Y, \mathcal{A}_Y)^{\geq p(Y)}.$$

Applying i^* to to the fiber sequence in (4.0.6) with $K = Rj_*(P_0)$, shows that

$$i^*(\sigma^Y_{\leq 0} Rj_*(P_0) \simeq \sigma_{\leq p(Y)} i^*(Rj_*(P_0)).$$

Since $Ri^! Rj_*$ is trivial, applying $Ri^!$ to the fiber sequence in (4.0.6) also shows that

$$Ri^! \sigma^Y_{\leq 0}(Rj_*(P_0)) \simeq (Ri^! i_* \sigma_{\geq p(Y)+1} i^*(Rj_*(P_0)))[-1] = \sigma_{\geq p(Y)+2} i^* Rj_*(P_0).$$

These complete the verification that P defined in (4.0.7) is indeed a generalized perverse sheaf and completes the proof of (ii).

(iii) Let P denote a generalized perverse sheaf extending the perverse sheaf

$$P_0 \, \epsilon \, DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\leq p(\mathcal{S}_0)} \cap DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\geq p(\mathcal{S}_0)}$$

satisfying the hypotheses in (iii). The key diagram is:

$$\begin{tikzcd}
 & & i_* i^*(P) \arrow[dr] & & \\
 & P \arrow[ur] \arrow[dr] & & i_*(Rj_*/j_!) j^*(P) \arrow[dr] & \\
j_!(j^*(P)) \arrow[ur] & & Rj_* j^*(P) \arrow[rr] & & i_* Ri^!(P)[1]
\end{tikzcd}$$

where

$$\begin{aligned} &j_!(j^*(P)) \to P \to i_* i^*(P) \to j_!(j^*(P))[1], i_* i^*(P) \to i_*(Rj_*/j_!) j^*(P) \to i_* Ri^!(P)[1] \to i_* i^*(P)[1] \text{ and} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P \to Rj_* j^*(P) \to i_* Ri^!(P)[1] \to P[1] \end{aligned} \tag{4.0.8}$$

are distinguished triangles. Now the hypotheses imply that $i^*(P) \,\epsilon\, DMod(\mathcal{S}-\mathcal{S}_0, \mathcal{A}_{\mathcal{S}-\mathcal{S}_0})^{\le p(\mathcal{S}-\mathcal{S}_0)-1}$ and that $Ri^!(P) \,\epsilon\, DMod(\mathcal{S}-\mathcal{S}_0, \mathcal{A}_{\mathcal{S}-\mathcal{S}_0})^{\ge p(\mathcal{S}-\mathcal{S}_0)+1}$. Therefore, the following lemma with $A = i^*(P)$ applied to the distinguished triangle $i_*i^*(P) \to i_*(Rj_*/j_!)j^*(P) \to i_*Ri^!(P)[1] \to i_*i^*(P)[1]$ as well as to the distinguished triangle $i^*P \to i^*Rj_*j^*(P) \to i^*i_*Ri^!(P)[1] \cong Ri^!(P)[1]$ shows that

$$i_*Ri^!(P)[1] \simeq i_*(\sigma_{\ge p(\mathcal{S}-\mathcal{S}_0)}(Rj_*/j_!)j^*(P)) \simeq i_*(\sigma_{\ge p(\mathcal{S}-\mathcal{S}_0)} i^*Rj_*j^*(P)).$$

This implies that P identifies with the canonical homotopy fiber of the map $Rj_*(P_0) \to i_*\sigma_{\ge p(\mathcal{S}-\mathcal{S}_0)} i^*(Rj_*(P_0))$. Therefore it is unique. This completes the proof of the theorem. $\square$

Lemma 4.4. *Let $A \to B \to C \to A[1]$ denote a distinguished triangle in $DMod(\mathcal{S}, \mathcal{A})$ and let n be an integer so that the natural map $\sigma_{\le n-1}A \to A$ is a quasi-isomorphism. Then the natural map $\sigma_{\ge n}(B) \to \sigma_{\ge n}(C)$ is a quasi-isomorphism.*

Proof. Let $K \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})^{\ge n}$. Since $\sigma_{\le n-1}A \to A$ is a quasi-isomorphism, it follows that $Hom_{DMod(\mathcal{S},\mathcal{A})}(A, K) = 0 = Hom_{DMod(\mathcal{S},\mathcal{A})}(A[1], K)$. Therefore the map $B \to C$ induces an isomorphism

$$Hom_{DMod(\mathcal{S},\mathcal{A})}(C, K) \overset{\cong}{\to} Hom_{DMod(\mathcal{S},\mathcal{A})}(B, K).$$

Now the definition of the functor $\sigma_{\ge n}$ as left-adjoint to the inclusion $DMod(\mathcal{S}, \mathcal{A})^{\ge n} \to DMod(\mathcal{S}, \mathcal{A})$ shows that the induced map $\sigma_{\ge n}(B) \to \sigma_{\ge n}(C)$ is also a quasi-isomorphism. $\square$

5. A counter-example to the existence of non-trivial t-structures: modules over -1-connected dgas

The main result of this section is Theorem 5.8 which shows that under the hypothesis that the sheaf of E_∞-dgas $\mathcal{A}$ is -1-*connected* (as in Definitions 5.1 below), the functor $\sigma_{\le n}$ identifies with the functor that kills cohomology in degrees above n. This puts strong restrictions on what can be in the heart of the corresponding t-structure on $DMod(\mathcal{S}, \mathcal{A})$.

Definitions 5.1. (i) We will assume henceforth, *but only in this section*, that $\mathcal{R} = \oplus_i \mathcal{R}(i)$ is a sheaf of graded rings and that the sheaf of E_∞-dgas $\mathcal{A}$ is -1-*connected*, i.e. $\mathcal{H}^s(\mathcal{A}) = 0$ for all $s \ge 1$. This terminology is derived from the case where $\mathcal{A}^i = \mathcal{B}_{-i}$ for a chain-complex $\mathcal{B}$ (i.e. one whose differentials are of degree -1.) In this case, the theory developed below is entirely similar to the homotopy theory of CW-complexes. We say a sheaf of $\mathcal{A}$-modules M is $-n$-*connected* if $\mathcal{H}^i(M) = 0$ for all $i \ge n$. Since $\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, M) \cong \mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M) \cong M$, this is equivalent to $\mathcal{H}^i(\mathcal{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, M)) \cong \mathcal{H}^i(\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M)) = 0$ for all $i \ge n$. We say M is *connected if it is $-n$-connected for some* $n > 0$.

(ii) A map $f : M' \to M$ in $DMod(\mathcal{S}, \mathcal{A})$ is a k-*equivalence* if the induced map $\mathcal{H}^i(f) : \mathcal{H}^i(M') \to \mathcal{H}^i(M)$ is an isomorphism for all $i > k$ and an epimorphism for $i = k$.

Definition 5.2. (i) A *free* $\mathcal{R}$-module is an object $M \,\epsilon\, C(\mathcal{S}, \mathcal{R})$ so that M is given by a sum $\underset{s_U, t_U \,\epsilon\, \mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U)$, where U ranges over the objects of the site $\mathcal{S}$. A *free* $\mathcal{A}$-module is an object $M \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})$ so that M is given by a sum $\underset{s_U, t_U \,\epsilon\, \mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{A})[s_U](t_U)$, where U ranges over the objects of the site $\mathcal{S}$. We call $-s_U$ $(-t_U)$ the *dimension (weight)* of the free module $j_{U!}j_U^*(\mathcal{A})[s_U](t_U)$.

(ii)An $\mathcal{R}$-module M is a *cone* $\mathcal{R}$-module if $M = Cone(id : \underset{s_U, t_U \,\epsilon\, \mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U) \to \underset{s_U, t_U \,\epsilon\, \mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U))$ for some free $\mathcal{R}$-module $\underset{s_U, t_U \,\epsilon\, \mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U)$. A *cone* $\mathcal{A}$-module is defined similarly. A *cell-module* $M \,\epsilon\, C(\mathcal{S}, \mathcal{R})$ is an object $M \,\epsilon\, C(\mathcal{S}, \mathcal{R})$ provided with a decreasing filtration $\{F_iM | i \le 0\}$ by sub-objects in $C(\mathcal{S}, \mathcal{R})$ so that $F_0(M)$ is a free $\mathcal{R}$-module and each successive quotient $F_iM/F_{i+1}M$ is also a free $\mathcal{R}$-module, for all $i \le 0$. Moreover F_iM is the mapping cone of a map $f_i : \mathcal{F}_i \to F_{i+1}M$ of a map in $C(\mathcal{S}, \mathcal{R})$) with $\mathcal{F}_i$ a free $\mathcal{R}$-module. (Observe that this mapping cone may be realized as a quotient of $F_{i+1}M \oplus Cone(\mathcal{F}_i)$.) In this case we say that F_iM is obtained from $F_{i+1}M$ by *attaching free $\mathcal{R}$-cell modules* for each summand in $\mathcal{F}_i$. One defines *cell $\mathcal{A}$-modules* $M \,\epsilon\, D(Mod(\mathcal{S}, \mathcal{A}))$ similarly.

(iii) A *CW* $\mathcal{R}$-module is a cell $\mathcal{R}$-module $M \,\epsilon\, C(\mathcal{S}, \mathcal{R})$ so that the dimension of each of the summands

$$j_{U!}j_U^*(\mathcal{R})[s_U](t_U) \text{ in } F_iM/F_{i+1}M$$

are *strictly smaller* than the dimension of each of the summands $j_{U!}j_U^*(\mathcal{R})[s_U'](t_U')$ in F_{i-1}/F_iM. One defines *CW* $\mathcal{A}$-modules similarly.

Remark 5.3. To see the intuition behind the last definition, consider the case where $F_iM/F_{i+1}M$ is a wedge of terms of the form $\mathcal{A}_U[i]$ for each $i \le 0$. In particular, $F_{i-1}M/F_iM$ is a wedge of terms of the form $\mathcal{A}_V[i-1]$. Now the dimension of $\mathcal{A}_U[i] = -i$ and the dimension of $\mathcal{A}_V[i-1] = -i+1$ and $-i < -i+1$.

Throughout this section we will assume the basic hypotheses as in 5.1, i.e. $\mathcal{A}$ is -1-connected or equivalently $\mathcal{H}^s(\mathcal{A}) = 0$ for all $s > 0$. We will assume that $\mathcal{R}$ is a constant sheaf. In this section we develop the basic theory of cell and CW cell-modules over a sheaf of E_∞-dgas. Now the following basic results show that the theory of $CW-\mathcal{A}$-modules is indeed similar to the homotopy theory of CW-complexes: see, for example, [Gray, Chapter 16].

5.0.8. *Convention.* : Henceforth, we will denote $j_{U!}j_U^*(\mathcal{R})$ ($j_{U!}j_U^*(\mathcal{A})$) by $\mathcal{R}_U$ ($\mathcal{A}_U$, respectively).

Proposition 5.4. *Throughout let P, K be $\mathcal{A}$-modules. (i) Let $g : \mathcal{A}_U[n-1] \to P$, $f : P \to K$ be $\mathcal{A}$-maps and assume $\mathcal{H}^i(K) = 0$ for $i = -n+1$. Then there exists a covering $\{V_\alpha \to U|\alpha\}$ of U so that each $f_{|V_\alpha}$ extends to a map $Cone(g_{|V_\alpha}) \to K_{|V_\alpha}$ where $Cone(g_{|V_\alpha}))$ denotes the mapping cone of $g_{|V_\alpha}$. (In this case we say that f extends locally to a map from $Cone(g)$ to K.)*

(ii) Let S denote a finite set of integers and (P, Q) a relative $CW-\mathcal{A}$-module, i.e. Q is obtained from P by attaching finitely many free $\mathcal{A}$-modules $\mathcal{A}_U[n_{\alpha,U}]$ with $n_{\alpha,U} \,\epsilon\, S$. Suppose K is an $\mathcal{A}$-module so that $\mathcal{H}^{-i}(K) = 0$ for all $i \,\epsilon\, S$. Then any map $f : P \to K$ of $\mathcal{A}$-modules admits a local extension $\tilde{f} : Q \to K$, i.e. there exists a covering $\{V_\alpha \to \mathcal{S}|\alpha\}$ so that each restriction $f_{|V_\alpha} : P_{|V_\alpha} \to K_{|V_\alpha}$ extends to a map $\tilde{f}_{|V_\alpha} : Q_{|V_\alpha} \to R_{|V_\alpha}$.

(iii) Suppose that there exists a covering $\{V_\alpha \to \mathcal{S}\}$ so that $(P_{|V_\alpha}, Q_{|V_\alpha})$ is a relative $CW-\mathcal{A}$-module in the above sense so that $Q_{|V_\alpha}$ is obtained from $P_{|V_\alpha}$ by attaching free-$\mathcal{A}$-cells in dimensions $\le -n$. Then $\mathcal{H}^i(Q/P) = 0$ for all $i > -n$.

(iv) If Q is a CW-$\mathcal{A}$-module obtained by attaching free $\mathcal{A}$-cells in dimensions $\le -n$, then $\mathcal{H}^i(Q) = 0$ for all $i > -n$.

Proof. For each point p of the site $\mathcal{S}$, the only obstruction to extending f_p to $Cone(g_p)$ is that the composition $f_p \circ g_p$ be null-homotopic: this is clear since $H^0(RHom_{\mathcal{A}_p}(\mathcal{A}_p[n-1], Q_p)) = H^{-n+1}(Q_p) = 0$ by the hypothesis. This proves (i). To prove (ii) one uses (i) as a starting point to handle the case when Q_p is obtained from P_p by attaching a single $\mathcal{A}_p$-cell. In general one uses ascending induction on the cardinality of the set S.

(iii) It is enough to assume that Q is obtained by attaching finitely many free $\mathcal{A}$-cells to P. In this case one uses an ascending induction on the number of these cells and the exact sequence $\mathcal{H}^i(Q'/P) \to \mathcal{H}^i(Q/P) \to \mathcal{H}^i(Q/Q')$ where Q' is obtained from P by attaching one *less* free $\mathcal{A}$-cell. Observe that key use is made of the hypothesis that $\mathcal{H}^i(\mathcal{A}) = 0$ for all $i > 0$: in fact, the last assertion is false if this is not the case. Clearly (iii) implies (iv). ☐

Theorem 5.5. *Let $\mathcal{M} \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})$ so that $\mathcal{M}$ is $-n$-connected in the above sense for some n.*

(i) Then there exists a CW-cell $\mathcal{A}$-module $P(M) \,\epsilon\, DMod(\mathcal{S}, \mathcal{A})$ with a map $P(M) \to M$ which is a quasi-isomorphism. If $\mathcal{H}^i(M) = 0$ for all $i > N$, then $P(M)$ can be constructed with $\mathcal{A}$-cells of dimension $\le N$ (We say $P(M) \to M$ is a CW-$\mathcal{A}$-resolution.)

(ii) Moreover, if $M' \to M$ is a map between two such objects in $DMod(\mathcal{S}, \mathcal{A})$, there exist CW-$\mathcal{A}$-resolutions $P(M') \to M'$, $P(M) \to M$ and a map $P(M') \to P(M)$ preserving the given filtrations so that one obtains a commutative square

$$\begin{array}{ccc} P(M') & \longrightarrow & P(M) \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \end{array}$$

Proof. Assume that $\mathcal{H}^i(M) = 0$ and $\mathcal{H}^i(M') = 0$ for all $i > N_0$. For each class $[\alpha_{N_0}] \,\epsilon\, \mathcal{H}^{N_0}(M)$, let α_{N_0} : $\mathcal{A}_U[-N_0] \to j_{U!}j_U^*(M) \to M$ denote a map representing $[\alpha_{N_0}]$. Now let $P_{N_0}(M) = \underset{[\alpha_{N_0}] \,\epsilon\, \mathcal{H}^{N_0}(M)}{\oplus} \mathcal{A}_U[-N_0]$: we will map this to M by mapping the summand indexed by $[\alpha_{N_0}]$ by the corresponding map α_{N_0} to M. We will denote this map $P_{N_0}(M) \to M$ by $p_{N_0}(M)$.

Consider the cone $\mathcal{A}$-module $Cone(P_{N_0}(M))$ and also the mapping cone $Cone(p_{N_0}(M))$. Observe that one has the distinguished triangle: $P_{N_0}(M) \overset{p_{N_0}(M)}{\to} M \to Cone(p_{N_0}(M)) \to P_{N_0}(M)[1]$ which results in the long-exact sequence:

$$\cdots \to \mathcal{H}^i(P_{N_0}(M)) \to \mathcal{H}^i(M) \to \mathcal{H}^i(Cone(p_{N_0}(M))) \to \mathcal{H}^{i+1}(P_{N_0}(M)) \to \cdots .$$

Since $\mathcal{H}^{N_0+k}(P_{N_0}(M)) = 0$ for all $k > 0$ and $\mathcal{H}^{N_0}(P_{N_0}(M)) \to \mathcal{H}^{N_0}(M)$ is a surjection by our choice of $P_{N_0}(M)$, it follows that

$$\mathcal{H}^i(Cone(p_{N_0}(M)) = 0, i \geq N_0 \tag{5.0.9}$$

i.e. the map $p_{N_0} : P_{N_0}(M) \to M$ *is an* N_0*-equivalence.*

5.0.10.

5.0.11. We will construct a sequence of complexes $P_k(M)$, $k \leq N_0$, which are $\mathcal{A}$-modules, which in each degree consist of terms of the form $\oplus_\alpha \mathcal{A}_{U_\alpha}[n_\alpha]$ and are provided with compatible maps $p_k : P_k(M) \to M$ which are k-equivalences, i.e. induce an isomorphism on $\mathcal{H}^i$ for $i > k$ and an epimorphism on $\mathcal{H}^k$. In order to construct these inductively, we will assume that N is an integer for which such a $P_N(M)$ has been already constructed. To start the induction, we may let $N = N_0$ and let $P_N(M)$ denote the complex constructed above. Observe that $\mathcal{H}^i(Cone(p_N)) = 0$ for all $i \geq N$. Therefore, we will now replace M by $Cone(p_N)$ and for each class $[\alpha_{-N+1}] \,\epsilon\, \mathcal{H}^{N-1}(Cone(p_N)$, let

$$\mathcal{A}_{U_{\alpha_{-N+1}}}[-N+1] \to Cone(p_N) = Cyl(p_N)/P_N(M)$$

denote a representative. This provides us a map

$$\alpha_{-N+1} : \oplus_{\alpha_{-N+1}} \mathcal{A}_{U_{\alpha_{-N+1}}}[-N+1] \to Cone(p_N) = Cyl(p_N)/P_N(M) \to P_N(M)[1]$$

i.e. a map

$$q_{-N+1} = \alpha_{-N+1}[-1] : \oplus_{\alpha_{-N+1}} \mathcal{A}_{U_{\alpha_{-N+1}}}[-N] \to P_N(M).$$

We let $P_{N-1}(M) = Cone(q_{-N+1})$. We now observe that the induced map $q_{-N+1} : \oplus_{\alpha_{-N+1}} \mathcal{A}_{U_{\alpha_{-N+1}}}[-N] \to P_N(M)$ also factors through $Cone(p_N)[-1]$, which is the homotopy fiber of the obvious map $p_N : P_N(M) \to M$. This shows that the composition $p_N \circ q_{-N+1}$ is chain homotopically trivial. Therefore, one obtains an induced map $p_{N-1} : P_{N-1}(M) = Cone(q_{-N+1}) \to M$ making the triangle

$$\begin{tikzcd}
P_N(M) \arrow[r, "p_N"] \arrow[d] & M \\
P_{N-1}(M) = Cone(q_{-N+1}) \arrow[ur, "p_{N-1}"] &
\end{tikzcd}$$

commute. Observe also that the induced map

$$\mathcal{H}^{N-1}(\oplus_{\alpha_{-N+1}} \mathcal{A}_{U_{\alpha_{-N+1}}}[-N+1]) \to \mathcal{H}^{N-1}(Cone(p_N)) \tag{5.0.12}$$

is an epimorphism by our assumptions.

Since $\oplus_{\alpha_{-N+1}} \mathcal{A}_{U_{\alpha_{-N+1}}}[-N])$ is the homotopy fiber of the map $P_N(M) \to P_{N-1}(M)$, a comparison of the long exact sequences in cohomology associated to the distinguished triangles $P_N(M) \overset{p_N}{\to} M \to Cone(p_N) \to P_N(M)[1]$ and $P_{N-1}(M) = Cone(q_{N-1}) \overset{p_{N-1}}{\to} M \to Cone(p_{N-1}) \to P_{N-1}(M)[1]$ shows that the homotopy fiber of the induced map $Cone(p_N) \to Cone(p_{N-1})$ identifies with $\oplus_{\alpha_{-N+1}} \mathcal{A}_{U_{\alpha_{-N+1}}}[-N+1])$. In view of (5.0.12) and the observation that $H^i(\oplus_{\alpha_{-N+1}} \mathcal{A}_{U_{\alpha_{-N+1}}}[-N+1]) = 0$ for all $i > N-1$, it follows that $\mathcal{H}^{N-1}(Cone(p_{N-1})) = 0$. Therefore, $\mathcal{H}^i(p_{N-1})$ is an epimorphism for $i = N-1$. By construction, one may readily see that $\mathcal{H}^i(p_{N-1})$ is an isomorphism for $i \geq N$. Therefore, p_{N-1} is an $N-1$-equivalence.

We may therefore, continue the inductive construction and define $P_k(M)$ as an $\mathcal{A}$-module, consisting of free cell $\mathcal{A}$-modules in each degree and provided with a map $p_k(M) : P_k(M) \to M$, $k \leq N_0$ which is a k-equivalence, i.e. where $\mathcal{H}^i(p_k(M))$ is an isomorphism for $i > k$ and an epimorphism for $i = k$. Finally one lets $P(M) = \underset{k \to -\infty}{\operatorname{colim}} P_k(M)$ along with the map $p(M) : P(M) \to M$ defined as $\underset{k \to -\infty}{\operatorname{colim}} p_k(M)$. One verifies immediately that $p(M)$ is a quasi-isomorphism: clearly $P(M)$ is a CW $\mathcal{A}$-module. This proves the first statement in the theorem. The construction also shows that $P(M)$ was built with $\mathcal{A}$-cells of dimension no greater than N_0.

To make the construction of CW-$\mathcal{A}$-resolutions functorial, we will need to make the following modifications to the arguments above. Instead of choosing representative cohomology classes as in (5.0.9), we choose all possible maps $j_{U!}j_U^*(\mathcal{A})[-N] \to M$, for all U in the site $\mathcal{S}$. (Since our site $\mathcal{S}$ is assumed to be essentially small this causes no difficulties.) We will then need to repeat the same construction in 5.0.11 and at every stage of the inductive process. Next observe that since both M' and M are assumed to be connected, we may choose a large enough N so that $\mathcal{H}^i(M) = \mathcal{H}^i(M') = 0$ for all $i > N$. Now the construction of the resolution $P(M)$ is made functorial in M. Moreover the construction of the CW-$\mathcal{A}$ resolution by descending induction shows that the induced map $P(f) : P(M') \to P(M)$ preserves the CW-filtration. This proves the second statement in the theorem. ☐

Definition 5.6. Let $Mod^{cw}(\mathcal{S}, \mathcal{A})$ denote the category whose objects are all $CW - \mathcal{A}$-modules and morphisms are morphisms that preserve the given cell-filtrations. A morphism between two $CW - \mathcal{A}$-modules will be called a quasi-isomorphism if it is a quasi-isomorphism in the underlying category $DMod(\mathcal{S}, \mathcal{A})$. The corresponding derived category obtained by inverting these quasi-isomorphisms will be denoted $DMod^{cw}(\mathcal{S}, \mathcal{A})$.

Corollary 5.7. *The obvious functor $DMod^{cw}(\mathcal{S}, \mathcal{A}) \to DMod(\mathcal{S}, \mathcal{A})$ is an equivalence of categories.*

Proof. This follows from the last Theorem. ☐

The following is a key result of this section.

Theorem 5.8. *Let $M \,\epsilon\, D(Mod(\mathcal{S}, \mathcal{A}))$ denote an $\mathcal{A}$-module. Then $\mathcal{H}^i(\sigma_{\le n}(M)) = 0$ for all $i > n$ and $\cong \mathcal{H}^i(M)$ if $i \le n$, i.e. the functor $\sigma_{\le n}$ in Definition 3.4 identifies with the functor that kills the cohomology in degrees above n.*

Proof. This is a direct consequence of the last theorem and Proposition 5.4(iv). The definition of the t-structure on $DMod(\mathcal{S}, \mathcal{A})$ shows that $DMod(\mathcal{S}, \mathcal{A})^{\le n}$ is generated using the operations of small sums, mapping cones, translations [1], extensions and by filtered colimits by the cells $j_{U!}(\mathcal{A}_U)[s_U](t_U)$ with $s_U \ge -n$. Since all such complexes have cohomology sheaves that are trivial in degrees $> n$, it follows that $DMod(\mathcal{S}, \mathcal{A})^{\le n}$ identifies with the full sub-category of $DMod(\mathcal{S}, \mathcal{A})$ consisting of complexes M whose cohomology is trivial in degrees larger than n. Therefore the right-adjoint to the imbedding $DMod(\mathcal{S}, \mathcal{A})^{\le n} \to DMod(\mathcal{S}, \mathcal{A})$ is in fact the functor that kills the cohomology in degrees larger than n. i.e. $\sigma_{\le n}$ is the functor killing the cohomology in degrees larger than n. ☐

Corollary 5.9. *The dga $\mathcal{A}_V$ cannot belong to the heart of this t-structure unless $\mathcal{A}$ is concentrated in degree 0, i.e. $\mathcal{A} = \mathcal{R}$.*

Remark 5.10. This shows that, in general, there are no meaningful t-structures for sites provided with presheaves of spectra or even -1-connected spectra. From this point of view, it seems also preferable to view the motivic complex as an E_∞-dga rather than as an E_∞-ring (or symmetric ring) spectrum.

Corollary 5.11. *The t-structure obtained above on $DMod(\mathcal{S}, \mathcal{R})$ coincides with the usual one, i.e. if $K \,\epsilon\, DMod(\mathcal{S}, \mathcal{R})$, $\sigma_{\le 0}K$ identifies with the functor killing cohomology sheaves above degree 0.*

Proof. This is clear in view of the above discussion. ☐

6. Example I: the diagonal t-structures in crystalline cohomology

We will begin by recalling the basic framework from [Ek] and [Ill]. Let k denote a perfect field of characteristic $p > 0$ and let $W = W(k) =$ the ring of Witt vectors of k. The Raynaud ring, R, is the graded W-ring, and generated by F, V in degree 0 and d in degree 1 with the following relations:

$$FV = VF = p, Fa = a^\sigma F, V = Va^\sigma, \\ da = ad, FdV = d, d^2 = 0, a \,\epsilon\, W \tag{6.0.13}$$

Here $(-)^\sigma$ is the Frobenius endomorphism of W. Given a scheme X defined over k, one may adapt the definition above to define a sheaf of rings $\mathcal{R}$ on the Zariski site of X, called the sheaf of Raynaud rings on X. One may then extend the discussion below to the Zariski site of the given scheme X. However, for the sake of simplicity, we will keep $X = Spec\, k$ throughout the following discussion.

We let R^i, $i = 0, 1$, denote the piece of R in degree i. The above relations enable us to view the graded ring R as a (non-commutative) dga, so that the methods developed in the earlier sections of this paper apply, at least on a heuristic level. In fact that is the reason for discussing the following as an application.

In view of the above relations, one observes that a left R-module M is the same as a complex of R^0-modules and where the differential $d : M^n \to M^{n+1}$ satisfies $FdV = d$. Moreover any left R^0-module can be viewed as a left R-module concentrated in degree 0. Henceforth an R-module will mean a left R-module. Given an R-module M, $M(n)$ will denote the R-module defined by $M(n)^i = M^{n+i}$ and the differential d given by $(-1)^n d$.

A complex M of R-modules can be viewed as a double complex $M^{\bullet\bullet}$ where the first degree (called the horizontal direction) corresponds to the R-grading. The second degree will be called the vertical direction. Thus $M^{\bullet,n}$ denotes the n-th row of $M^{\bullet\bullet}$ and this is an R-module. Observe that one may take the cohomology of the double complex with respect to the vertical differential : these cohomology objects will be all R-modules. Thus $H^n_v(M^{\bullet\bullet})$ denotes the n-th (vertical) cohomology. We define the derived category of cohomologically bounded R-modules, $D(R)^b$ to be the category of all complexes of R-modules M so that $H^n_v(M) = 0$ for all but finitely many n and where we invert maps that induce isomorphisms on H^*_v. $D(R)$ will denote the corresponding unbounded derived category.

For a complex M of R-modules, we define $(M(n)[m])$ as the complex of R-modules defined by $(M(n)[m])^{i,j} = M(n+i)^{m+j}$.

Next recall the *diagonal t-structures* of [Ek]. (See also [Ill, 6.4] for a particularly clear discussion.) First, for each R-module M and an integer n, one defines the R-module $\tilde{\tau}_{\le n}M$ by

$$\tilde{\tau}_{\le n}M = (\cdots \to M^{n-1} \xrightarrow{d} M^n \to F^\infty B^{n+1} \to 0) \tag{6.0.13}$$

where $B^{n+1} = Im(d^n : M^n \to M^{n+1})$ and $F^\infty B^{n+1} = \underset{i\ge 0}{\cup} F^i B^{n+1}$. Clearly this is a sub-$R$-module of M. Next $\tilde{\tau}_{\ge n+1}M$ is defined to be the quotient $M/\tilde{\tau}_{\le n}M$. A complex of R-modules $M^{\bullet\bullet} \,\epsilon\, D(R)^{\le 0}$ if for each n, the natural map $\tilde{\tau}_{\le n}H^{-n}_v(M^{\bullet\bullet}) \to H^{-n}_v(M^{\bullet\bullet})$ is an isomorphism of R-modules. We say a complex of R-modules $M \,\epsilon\, D(R)^{\ge 1}$ if the natural maps $H^{-n}_v(M^{\bullet\bullet}) \to \tilde{\tau}_{\ge n+1}(H^{-n}_v(M^{\bullet\bullet}))$ is an isomorphism.

It was shown in [Ek] that this defines a t-structure on $D(R)^b$, *the diagonal t-structure* with the *heart of the t-structure* given by $D(R)^{b,\le 0} \cap D(R)^{b,\ge 0}$. We proceed to show that the derived category $D(R)$ is compactly generated and that the above t-structure is defined by a family of compact objects as in Theorem 1.9.

For each pair of integers i and j we define the complex $R(-j)[i]$ of R-modules which is the following (double) complex: we put the R-module $R(-j)$ (viewed as a complex) as the $-i$-th row and put zeros elsewhere. Observe that $H^n_v(R(-j)[i]) = R(-j)$ if $n = -i$ and 0 otherwise. Therefore, if $j \le i$, $R(-j)[i] \,\epsilon\, D(R)^{b,\le 0}$. We also consider the complex of R-modules, $R_0(-i)[i]$ which is the following (double) complex: we put the R-module $R_0(-i)$ as the $-i$-th row and put zeros elsewhere.

The following is the main result of this section.

Theorem 6.1. *(i) Let $\bar{D}(R)^{\le 0}$ denote the smallest pre-aisle generated by the R-modules $R(-j)[i]$, $j \le i$. These R-modules are compact objects in $D(R)$ and hence the above pre-aisle is an aisle which defines a t-structure on $D(R)$.*

(ii) Let $D(R)^b$ denote the full subcategory of complexes in $D(R)$ that are bounded. Then $\bar{D}(R)^{b,\le 0} = D(R)^b \cap \bar{D}(R)^{\le 0}$ and $\bar{D}(R)^{b,\ge 0} = D(R)^b \cap \bar{D}(R)^{\ge 0}$ defines a t-structure on $D(R)^b$. This t-structure agrees with the t-structure defined on $D(R)^b$ in [Ek] *making using the truncation functors* (6.0.13).

Proof. A key observation is the following identification:

$$Hom_{D(R)}(R(-j)[i], M^{\bullet\bullet}) = H^{-i}_v(M^{j,\bullet}). \tag{6.0.14}$$

Therefore, $Hom_{D(R)}(R(-j)[i], M^{\bullet\bullet})$ commutes with arbitrary small sums in the argument $M^{\bullet\bullet}$ and therefore, each $R(-j)[i]$ is a compact object in $D(R)$. One may now invoke Theorem 1.0.1 to obtain (i). Observe that there is a surjection $R \to R^0$ defined by the identity in degree 0 and the trivial map in degree 1. It is often convenient to add $R_0(-i)[i]$ to the collection of generators: but the above observation shows that it suffices to consider just $\{R(-j)[i] | j \le i\}$ as a set of generators.

Next we consider (ii). Since R is bounded above, one may show as in the proof of Proposition 3.10 that the above t-structure induces a t-structure on $D(R)^b$. In view of the definition (6.0.13), one next observes that the conditions

$$\tilde{\tau}_{\le i-1}H^{-i}_v(M^{\bullet\bullet}) = 0 \text{ and } H^{-i}_v(M^{j,\bullet}) = 0, \text{ for all } j < i \tag{6.0.15}$$

are equivalent.

Let $\bar{D}(R)^{b,\geq 0} = D(R)^b \cap \bar{D}(R)^{\geq 0}$ and let $D(R)^{b,\leq 0}$, $D(R)^{b,\geq 0}$ denote the t-structure defined on $D(R)$ by [Ek] making use of the truncation functors (6.0.13). Then it follows from the identification (6.0.14) and the above observation that a bounded complex $M^{\bullet\bullet}$ of R-modules belongs to $\bar{D}(R)^{b,\geq 0}$ if and only if $M^{\bullet\bullet}$ satisfies the equivalent conditions in (6.0.15). But the first of these conditions characterizes $M^{\bullet\bullet}$ belonging to $D(R)^{b,\geq 0}$ and the second characterizes $M^{\bullet\bullet}$ belonging to $\bar{D}(R)^{b,\geq 0}$ in view of (6.0.14). Therefore, we have shown $D(R)^{b,\geq 0} = \bar{D}(R)^{b,\geq 0}$. By [Ek], the truncation functors $\tilde{\tau}_{\leq n}$ define a t-structure on $D(R)^b$ with the co-aisle given by $D(R)^{b,\geq 0}$. Therefore, $D(R)^{b,\leq 0}$ is determined as the aisle corresponding to the co-aisle $D(R)^{b,\geq 0}$. Similarly $\bar{D}(R)^{b,\leq 0}$ is the aisle corresponding to the co-aisle $\bar{D}(R)^{b,\geq 0}$. Therefore, $\bar{D}(R)^{b,\leq 0} = D(R)^{b,\leq 0}$, which completes the proof of the theorem. □

7. **Example II: Equivariant Derived Categories**

We will fix an algebraically closed field k of characteristic $p \geq 0$. Let G denote a smooth group scheme acting on a scheme X of finite type over k. Now $[X/G]$ will denote the associated *quotient stack*. One associates several sites to the stack $[X/G]$: $[X/G]_{sm}$ denotes the site whose objects are smooth maps $s : S \to [X/G]$ with S an algebraic stack of finite type over k and where the coverings of a given object $s : S \to [X/G]$ are smooth coverings. In [J-2], we introduced the *iso-variant étale* site, $[X/G]_{iso.et}$ as follows: the objects are G-isovariant etale maps $Y \to X$ of schemes. These correspond to iso-variant étale maps $[Y/G] \to [X/G]$ of the associated stacks, or to G-equivariant maps $Y \to X$ that induce isomorphism on the stabilizer groups. One may verify readily that this site is closed under fibered products. Therefore, sending an iso-variant étale map to the same map viewed simply as an étale map defines a morphism of sites: $\pi : [X/G]_{sm} \to [X/G]_{iso.et}$. It is shown in [J-2] that the latter site has enough points and that the points correspond to G-orbits of geometric points of X. (One may consult [J-2] for more details on the isovariant étale site.)

Proposition 7.1. *Let X denote a scheme of finite type over k provided with the action of an algebraic group G where G acts with finitely many orbits. Then objects in $[X/G]_{iso.et}$ consist of schemes Y of finite type over k provided with a G-action making the given map $Y \to X$ G-isovariant and étale (i.e. an étale map which is G-equivariant and inducing an isomorphism on the stabilizer groups.) Therefore any such Y also has only finitely many G-orbits. If X is an imbedding of G, i.e. there is an open orbit where G acts freely, then the same is true for Y.*

Proof. These are clear from the definition of isovariant maps. □

Next assume that the group G is a torus T and X is a toric variety associated to T. Observe that each object in the site $[X/T]_{iso.et}$ is a toric variety for T provided with a T-isovariant étale map to X. Now T acts with finitely many orbits on X so that there are only finitely many points on the site $[X/T]_{iso.et}$, each corresponding to the T-orbits of geometric points of X. In characteristic zero it is possible to define a topological space corresponding to the site $[X/T]_{iso.et}$. However, this approach clearly fails in positive characteristic and necessitates the use of the site $[X/T]_{iso.et}$.

Next given a geometric point $\bar{x}$ of X, one observes from (7.0.16) below that it has a T-stable neighborhood of the form $T\bar{x} \times V_{\bar{x}}$ where $V_{\bar{x}}$ denotes an affine toric variety for $T_{\bar{x}}$ which contracts $T_{\bar{x}}$-equivariantly to $\bar{x}$. (Here $T_{\bar{x}}$ denotes the stabilizer at $\bar{x}$.) Therefore one may readily compute $R\pi_*(\underline{\mathbb{Q}}_l)$ (where $\pi : [X/G]_{sm} \to [X/G]_{iso.et}$ is the map of sites) and show that one has the isomorphism:

$R^n\pi_*(\mathbb{Q}_l)_{T\bar{x}} \simeq H^n(BT_{\bar{x}}; \mathbb{Q}_l)$, for each $n \geq 0$.

However, we need to consider $R\pi_*(\mathbb{Q}_l)$ as a sheaf of E_∞-dgas. This makes it necessary to first develop certain background material, which is done in the next section.

Proposition 7.2. *$R\pi_*(\mathbb{Q}_l)$ is a sheaf of E_∞-dgas on $[X/T]_{iso.et}$.*

Proof. This follows from Example 9.5 in the last section. □

Observe that if $K \epsilon D_+([X/T]_{sm}, \mathbb{Q}_l)$, then $R\pi_*(K)$ has the structure of a sheaf of E_∞-dg modules over $Rp_*(\mathbb{Q}_l)$. If $L \epsilon D_+([X/T]_{iso.et}, Rp_*(\mathbb{Q}_l))$, $Lp^*(L) = \mathbb{Q}_l \underset{Rp_*(\mathbb{Q}_l)}{\overset{L}{\otimes}} L \epsilon D_+([X/T]_{sm}, \mathbb{Q}_l)$.

Next one observes that for toric varieties, the stabilizers $T_{\bar{x}}$ are all connected. Therefore, every T-equivariant ℓ-adic local system on $ET\underset{T}{\times}\mathcal{O}$ is constant, for any T-orbit $\mathcal{O}$ on X. Therefore, the generators of the derived category $D_+([X/T]_{lis.et}, \mathbb{Q}_l)$ are $j_{\mathcal{O}!}(\mathbb{Q}_l)$, where $j_{\mathcal{O}} : ET\underset{T}{\times}\mathcal{O} \rightarrow ET\underset{T}{\times}X$ is the obvious locally closed immersion.

Observe also that each orbit $\mathcal{O}$ has an open neighborhood $V_{\mathcal{O}}$ which is stable by T so that

$$V_{\mathcal{O}} \cong \mathcal{O} \times \mathcal{S}_{\mathcal{O}} \tag{7.0.16}$$

where $\mathcal{S}_{\mathcal{O}}$ is a toric variety for $T_{\bar{x}}$, $x \epsilon \mathcal{O}$. Moreover, $\mathcal{S}_{\mathcal{O}}$ is an attractive slice (as in [BJ-1, (0.3)]) for the action of a 1-parameter subgroup of $T_{\bar{x}}$ and $\mathcal{O}$ is the only closed T-orbit in this T-stable neighborhood. Therefore, T-equivariant ℓ-adic local systems on $V_{\mathcal{O}}$ and $\mathcal{O}$ correspond and they are just the constant systems. Let $j_{V_{\mathcal{O}}} : V_{\mathcal{O}} \rightarrow X$ denote the open immersion and let $D_{+,c}([X/T]_{sm}, \mathbb{Q}_l)$ denote the full subcategory of bounded below complexes of ℓ-adic sheaves on $[X/T]_{sm}$ with constructible cohomology sheaves. Then it follows that, therefore, $j_{V_{\mathcal{O}}!}(\mathbb{Q}_l)$ as $\mathcal{O}$ varies among the T-orbits, form a set of generators for $D_{+,c}([X/T]_{sm}, \mathbb{Q}_l)$. We let
(7.0.17)
$D_c([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l))$ denote the full subcategory of $D_+([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l))$ generated by $R\pi_*(j_{V_{\mathcal{O}},!}\mathbb{Q}_l)$
as $\mathcal{O}$ varies among the T-orbits. (Since each $Rp_*(j_{V_{\mathcal{O}},!}\mathbb{Q}_l)$ will be shown to be a compact object in $D_+([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l))$ below, the above category is a full subcategory of the category of compact objects in $D_+([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l))$.)

Proposition 7.3. *Assume the above situation. Then one obtains the quasi-isomorphisms:*

$$R\pi_*(j_{V_{\mathcal{O}}!}(\mathbb{Q}_l)) \simeq j_{V_{\mathcal{O}}!}(R\pi_{V_{\mathcal{O}}*}(\mathbb{Q}_l)) \text{ and} \tag{7.0.18}$$

$$L\pi^*(j_{V_{\mathcal{O}}!}(R\pi_{V_{\mathcal{O}}*}(\mathbb{Q}_l))) \simeq j_{V_{\mathcal{O}}!}(\mathbb{Q}_l) \tag{7.0.19}$$

where $\pi_{V_{\mathcal{O}}} : [V_{\mathcal{O}}/T]_{sm} \rightarrow [V_{\mathcal{O}}/T]_{iso.et}$ is the obvious map of sites.

Proof. First observe that $L\pi^*$ and $j_{V_{\mathcal{O}}!}$ commute. (To see this, observe that their right adjoints are $R\pi_*$ and $j^*_{V_{\mathcal{O}}}$ which evidently commute since $j_{V_{\mathcal{O}}}$ is an open immersion.) Therefore the second quasi-isomorphism follows from the observation that $L\pi^*_{V_{\mathcal{O}}}(R\pi_{V_{\mathcal{O}}*}(\mathbb{Q}_l) \simeq \mathbb{Q}_l$.

Now we consider the proof of the first quasi-isomorphism. Let $\mathcal{O}'$ denote a T-orbit so that the closure of $\mathcal{O}$ contains $\mathcal{O}'$. Now either $\mathcal{O}' = \mathcal{O}$ or $\mathcal{O}' \neq \mathcal{O}$. In the first case, as observed above, $j^*_{V_{\mathcal{O}}}$ and $R\pi_{V_{\mathcal{O}}*}$ commute so that the stalk of $R\pi_*(j_{V_{\mathcal{O}}!}\mathbb{Q}_l)$ at $\mathcal{O}$ identifies with $R\pi_{\mathcal{O}*}(\mathbb{Q}_l)$. Therefore, it suffices to consider the case when $\mathcal{O}' \neq \mathcal{O}$. Let $\tilde{X} \rightarrow X$ denote any object in $[X/T]_{iso.et}$ that is an isovariant étale neighborhood of the orbit $\mathcal{O}'$. Then $\tilde{X}$ is also a toric variety for T with orbits $\tilde{\mathcal{O}}$ ($\tilde{\mathcal{O}}'$) lying above $\mathcal{O}$ ($\mathcal{O}'$, respectively). Observe that the same slice structure holds for $\tilde{X}$. Let $\tilde{V}_{\tilde{\mathcal{O}}}$ ($\tilde{V}_{\tilde{\mathcal{O}}'}$) denote the corresponding Zariski neighborhood of the orbit $\tilde{\mathcal{O}}$ ($\tilde{\mathcal{O}}'$, respectively).

Letting $\tilde{j} : \tilde{V}_{\tilde{\mathcal{O}}'}\underset{\tilde{X}}{\times}\tilde{V}_{\tilde{\mathcal{O}}} \rightarrow \tilde{V}_{\tilde{\mathcal{O}}'}$, one obtains the identification:

$$H^n_T(\tilde{V}_{\tilde{\mathcal{O}}'}, \tilde{j}_!(\mathbb{Q}_l)) \cong H^n_T(\tilde{V}_{\tilde{\mathcal{O}}'}, \tilde{V}_{\tilde{\mathcal{O}}'} - \tilde{V}_{\tilde{\mathcal{O}}'}\underset{\tilde{X}}{\times}\tilde{V}_{\tilde{\mathcal{O}}}, \mathbb{Q}_l). \tag{7.0.20}$$

Moreover, in view of Proposition 7.1, it follows that the stalk $R^n\pi_*(j_{V_{\mathcal{O}}!}\mathbb{Q}_l)_{\mathcal{O}'}$ may be obtained by taking the colimit of groups of the form $H^n_T(\tilde{V}_{\tilde{\mathcal{O}}'}, \tilde{j}_!(\mathbb{Q}_l))$, as one runs over T-toric varieties that are iso-variant étale neighborhoods of the orbit $\mathcal{O}'$. Therefore, it suffices to show that the groups in (7.0.20) are 0. Now $\tilde{V}_{\tilde{\mathcal{O}}'} - \tilde{V}_{\tilde{\mathcal{O}}'}\underset{\tilde{X}}{\times}\tilde{V}_{\tilde{\mathcal{O}}} \cong \tilde{\mathcal{O}}' \times (\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} - \tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} \cap \tilde{V}_{\tilde{\mathcal{O}}})$. Since both the slice $\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'}$ and $\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} - (\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} \cap \tilde{V}_{\tilde{\mathcal{O}}})$ contract to x (where x is a any chosen fixed point of $\tilde{\mathcal{O}}'$) under the attractive action of a 1-parameter subgroup of T_x, one obtains the isomorphism

$$H^n_{T_x}(\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} - (\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} \cap \tilde{V}_{\tilde{\mathcal{O}}}), \mathbb{Q}_l) \overset{\cong}{\rightarrow} H^n_{T_x}(\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'}, \mathbb{Q}_l).$$

In view of the long -exact sequence

$$H^n_{T_x}(\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} - (\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} \cap \tilde{V}_{\tilde{\mathcal{O}}}), \mathbb{Q}_l) \rightarrow H^n_{T_x}(\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'}, \mathbb{Q}_l) \rightarrow H^n_{T_x}(\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'}, \tilde{\mathcal{S}}_{\mathcal{O}'} - (\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} \cap \tilde{V}_{\tilde{\mathcal{O}}}), \mathbb{Q}_l),$$

this implies that

$$H^n_T(\tilde{V}_{\tilde{\mathcal{O}}'}, \tilde{V}_{\tilde{\mathcal{O}}'} - \tilde{V}_{\tilde{\mathcal{O}}'}\underset{\tilde{X}}{\times}\tilde{V}_{\tilde{\mathcal{O}}}, \mathbb{Q}_l) \cong H^n_{T_x}(\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'}, \tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} - (\tilde{\mathcal{S}}_{\tilde{\mathcal{O}}'} \cap \tilde{V}_{\tilde{\mathcal{O}}}), \mathbb{Q}_l) \cong 0, \text{ for all n}.$$

This proves that $H^n_T(\tilde{V}_{\tilde{\mathcal{O}}'}, \tilde{V}_{\tilde{\mathcal{O}}'} - \tilde{V}_{\tilde{\mathcal{O}}'}\underset{\tilde{X}}{\times}\tilde{V}_{\tilde{\mathcal{O}}}, \mathbb{Q}_l) = 0$ for all n thereby completing the proof of the proposition. □

Theorem 7.4. *Assume that X is a toric variety for the action of the torus T.*

(i) Then the functor

$$R\pi_* : D_{b,c}([X/T]_{sm}, \mathbb{Q}_l) \rightarrow D_c([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l)) \text{ is fully faithful and}$$

$$L\pi^* : D_c([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l)) \to D_{b,c}([X/T]_{sm}, \mathbb{Q}_l)$$

is a left-inverse.

(ii) These functors induce an equivalence of categories:

$$D_c([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l)) \simeq D_{b,c}([X/T]_{sm}, \mathbb{Q}_l) \simeq D^T_{b,c}(Et(ET\underset{T}{\times}X), \mathbb{Q}_l) = D^T_{b,c}(X, \mathbb{Q}_l).$$

Here $ET\underset{T}{\times}X$ *denotes the simplicial scheme defined by the presentation of the stack* $X \to [X/T]$ *and* $Et(ET\underset{T}{\times}X)$ *denotes the étale site of the simplicial scheme* $ET\underset{T}{\times}X$. *Under the above equivalence, the standard* t*-structure on the left defined as in section 3, corresponds to the usual* t*-structure on the right while the non-standard* t*-structure on the left defined as in Theorem 4.3 where the strata are the* T*-orbits corresponds to the* t*-structure on the right obtained by gluing.*

Proof. In view of the last proposition, it follows readily that the two compositions $L\pi^* \circ R\pi_*$ and $R\pi_* \circ L\pi^*$ are naturally equivalent to the corresponding identity functors. Therefore, the equivalence of the above derived categories and the fully faithfullness of Rp_* follows. The category $D_{+,c}([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l))$ inherits a t-structure from $D_+([X/T]_{iso.et}, R\pi_*(\mathbb{Q}_l))$. The definition of the t-structures shows that $R\pi_*$ preserves these. This is clear for the standard t-structures and for the non-standard t-structures, it suffices to consider the case where there are only two strata, which are both T-orbits. Now the observation that the stabilizers are all connected shows that the T-equivariant local systems on the orbits are constant. This enables one to show readily that the t-structures are preserved by $R\pi_*$. The last equality is because one defines $D^T_{b,c}(X, \mathbb{Q}_l)$ to be $D^T_{b,c}(Et(ET\underset{T}{\times}X), \mathbb{Q}_l)$. This completes the proof of the theorem. $\square$

Remarks 7.5. (i) The formality of the E_∞-dg algebra $R\pi_*(\mathbb{Q}_l)$ may be shown by first observing that the stalks $R\pi_*(\mathbb{Q}_l)_{\bar{x}} \cong R\Gamma(BT_{\bar{x}}, \mathbb{Q}_l)$. Since $BT_{\bar{x}}$ is the product of r-copies of $\mathbb{P}^\infty$, if r is the rank of $T_{\bar{x}}$, one may show readily that $R\Gamma(BT_{\bar{x}}, \mathbb{Q}_l)$ breaks up into the sum $\Sigma_i R^n\Gamma(BT_{\bar{x}}, \mathbb{Q}_l)$. Finally the local structure of the toric variety considered above shows such a decomposition holds locally on the site $[X/T]_{iso.et}$.

(ii) In case the toric variety is defined over the complex numbers, making use of the transcendental topology, it is easy to define a replacement for the isovariant étale site. However, in positive characteristics using ℓ-adic coefficients the use of the isovariant étale site seems unavoidable. With the above theorem in place, it would be straight-forward to provide a proof of the conjecture of Soergel (see [Soe]) for toric varieties in positive characteristics: we will return to this elsewhere.

8. **Example III: Motivic Derived Categories**

In this section we will provide certain motivic derived categories with t-structures so that this t-structure is compatible with a t-structure on the corresponding mod-ℓ^ν étale derived categories. This is put in more as a sample of what is possible in this direction, than as the most definitive result in this direction. More definitive results, in particular how they relate to the motivic t-structures conjectured in [VV92], and also in the setting of Voevdosky's motivic derived categories are discussed in the forthcoming paper [J-3].

We will fix a ground field k, of arbitrary characteristic p throughout the paper and will only consider smooth schemes of finite type over k. This category will be denoted $(\mathrm{Sm_k})$. When provided with the big Zariski (Nisnevich, étale) topologies, we obtain the big-sites $(\mathrm{Sm_k})_{Zar}$, $((\mathrm{Sm_k})_{Nis}$, $(\mathrm{Sm_k})_{Et}$, respectively). $\mathbb{Z} = \underset{r}{\oplus}\mathbb{Z}(r)$ will denote the integral *motivic complex* on the sites $(\mathrm{Sm_k})_{Zar}$ and $(\mathrm{Sm_k})_{Nis}$. ℓ will denote a prime different from p, $\nu > 0$ an integer and $\mathbb{Z}/\ell^\nu = \underset{r}{\oplus}\mathbb{Z}/\ell^\nu(r)$ will denote the corresponding *mod*$-\ell^\nu$ motivic complex with $\mathbb{Z}/\ell^\nu_{et}$ the corresponding complex on the big-étale site $(\mathrm{Sm_k})_{Et}$. Making use of [J-1, Theorem 1.1], these are in fact sheaves of E_∞-differential graded algebras. $\mathbb{Q} = \underset{r}{\oplus}\mathbb{Q}(r) = \underset{r}{\oplus}\mathbb{Z}(r) \otimes Q$. The E_∞-structure on the motivic complexes discussed in [J-1] and [BJ, section 5], is quite explicit, is over the Barratt-Eccles operad and therefore has several nice features. For the purposes of this paper though, all we require is an E_∞-structure on the motivic complex compatible with the given pairing on it. Such a structure is often assumed in the literature and therefore, one may also assume its existence.

In general, we will fix a commutative Noetherian ring R and consider $\mathbb{Z} \otimes \mathcal{R}$, where $\mathcal{R}$ denotes the constant sheaf associated to R: this is a sheaf of E_∞-dgas over the ring R and we will denote this by $\mathcal{A}$ (with its weight r-part denoted $\mathcal{A}(r)$) throughout. Observe that now one has augmentations $\mathcal{R} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{R}$ the composition of which is the identity. For the rest of the discussion we will take $R = \mathbb{Z}/\ell^\nu$ for some prime $\ell \neq char(k)$ and ν a positive integer. Observe that, in this case the weak-form of the Beilinson-Soulé vanishing condition holds as a consequence of the Bloch-Kato conjecture, now a theorem: see [VV, Introduction] and also [A].

8.1. Given a scheme $X \epsilon (\mathrm{Sm_k})$, we let $(\mathrm{Sm_k}/X)$ denote the sub-category of $(\mathrm{Sm_k})$ that are of finite type over X with morphisms $Y' \to Y$ being morphisms of smooth schemes compatible with the given maps to X. The site $(\mathrm{Sm_k}/X)_{Zar}$ $((\mathrm{Sm_k}/X)_{Nis}$, $(\mathrm{Sm_k}/X)_{et})$ is the corresponding *big* site and will be often denoted X_{Zar} (X_{Nis}, X_{Et}, respectively).

Similarly $Sh(X_{Nis}, \mathcal{R})$ will denote the category of all sheaves of $\mathcal{R}$-modules on the site X_{Nis} for a given scheme X. $C(Sh((\mathrm{Sm_k})_{Nis}, \mathcal{R}))$ ($C(Sh(X_{Nis}, \mathcal{R}))$) will denote the category of all (unbounded) complexes of objects in $Sh((\mathrm{Sm_k})_{Nis}, \mathcal{R})$ ($Sh(X_{Nis}, \mathcal{R})$, respectively). We first define $Mod((\mathrm{Sm_k})_{Nis}, \mathcal{A})$ to consist of all complexes of sheaves K on $(\mathrm{Sm_k})_{Nis}$ with the following properties:

(i) $K = \underset{r}{\oplus} K(r)$ has homotopy invariant cohomology sheaves and
(ii) K has the structure of a complex of sheaves of E_∞-modules over the sheaf of E_∞-dgas $\mathcal{A}$.

A morphism $f : K' \to K$ between two such objects will be a map that preserves the last two structures. The objects of the derived category $D((\mathrm{Sm_k})_{Nis}, \mathcal{A})$ are the same as those of $Mod((\mathrm{Sm_k})_{Nis}, \mathcal{A})$, but where morphisms are defined up to $\mathbb{A}^1$-equivalences. $D(X_{Nis}, \mathcal{A})$ will denote the corresponding derived category where morphisms are defined up to $\mathbb{A}^1$-equivalences. Since $\mathcal{A}$ is an E_∞-dga, (see [J-1, Theorem 1.1]), we may make use of Theorem 1.1 to define a t-structure on $D(X_{Nis}, \mathcal{A})$.

For each map $f : X \to Y$ of smooth schemes over k, we obtain a map of sheaves of E_∞-dgas: $\mathcal{A}_{|Y} \to Rf_*(\mathcal{A}_{|X})$ as well as $f^{-1}(\mathcal{A}_{|Y}) \to \mathcal{A}_{|X}$. These induce derived functors $Rf_* : D(X_{Nis}, \mathcal{A}) \to D(X_{Nis}, \mathcal{A})$ and $Lf^* : D(Y_{Nis}, \mathcal{A}) \to D(X_{Nis}, \mathcal{A})$.

Let $\epsilon : X_{Et} \to X_{Nis}$ denote the obvious maps of sites. We will assume for the rest of this discussion that the field k has finite ℓ-étale cohomological dimension so that the hypotheses of Theorem 1.1 apply with $\mathcal{R} = \mathbb{Z}/\ell^\nu$ (i.e. the integers mod ℓ) to complexes of sheaves of $\mathcal{R}$-modules on the étale site of X. Then we let $real_\nu$ denote the functor $K \mapsto \epsilon^*(K \otimes \mathbb{Z}/\ell^\nu) = \oplus_r \epsilon^*(K(r) \otimes \mathbb{Z}/\ell^\nu)$ sending the (graded) $\mathcal{A}$-module K to the pull-back to the étale site of the corresponding mod-ℓ^ν-reduction. We let $D(X_{Et}, real_\nu(\mathcal{A}))$ denote the derived category of complexes of sheaves of modules over $real_\nu(\mathcal{A})$ where the morphisms are defined again up to $\mathbb{A}^1$-equivalence. (Observe that $\mathbb{A}^1$ is acyclic in the étale topology only with respect to locally constant sheaves of $\mathbb{Z}/\ell^\nu$ -modules, with ℓ different from $char(k)$. Therefore, the $\mathbb{A}^1$-localization is needed in general to be the target of any functor from $D(X_{Nis}, \mathcal{A})$.) Since $real_\nu(\mathcal{A}) = \oplus_r real_\nu(A(r))$ is an E_∞-dga on the big étale site of X we may make use of Theorem 1.1 to define a t-structure on $D(X_{Et}, real_\nu(\mathcal{A}))$. Since $real_\nu(\mathcal{A}) = \oplus_r \mu_\ell(r)$, we obtain the equivalence of derived categories: $D(X_{Et}, real_\nu(\mathcal{A})) \simeq D(X_{Et}, \oplus_r \mu_{\ell^\nu}(0))$. Moreover, the t-structure on $D(X_{Et}, real_\nu(\mathcal{A}))$ provided by Theorem 1.1 identifies with the usual t-structure on $D(X_{Et}, \oplus_r \mu_{\ell^\nu})$. We also let $D(X_{Et}, \mu_{\ell^\nu}(0)) \simeq D(X_{Et}, \mathbb{Z}/\ell^\nu)$ denote the corresponding $\mathbb{A}^1$-localized derived categories.

Definition 8.1. Assume that the base field has a primitive ℓ^ν-th root of unity, for example, that it is a perfect field and has finite ℓ-étale cohomological dimension. We define the *realization functor real* : $D(X_{Nis}, \mathcal{A}) \to D(X_{Et}, \mu_{\ell^\nu}(0)) \simeq D(X_{Et}, \mathbb{Z}/\ell^\nu)$ to be the composition of the functor sending $K = \oplus_r K(r) \mapsto \oplus_r real_\nu(K(r))$ with the one taking the graded piece in degree 0 of the graded module $\oplus_r real_\nu(K(r))$.

Theorem 8.2. *Assume that the base field has a primitive ℓ^ν-th root of unity and that it is a perfect field of finite ℓ-étale cohomological dimension. Then the realization functor real* : $D(X_{Nis}, \mathcal{A}) \to D(X_{Et}, \mu_{\ell^\nu}(0)) \simeq D(X_{Et}, \mathbb{Z}/\ell^\nu)$ *is compatible with the above t-structures where the t-structure on $D(X_{Et}, \mathbb{Z}/\ell^\nu)$ is the usual one. i.e. There exist natural transformations*

$$real \circ \sigma_{\leq 0} \to \sigma_{\leq 0} \circ real \text{ and } real \circ \sigma_{\geq 1} \to \sigma_{\geq 1} \circ real.$$

Proof. A key observation is that the realization functor $real_\nu$ commutes with the extension by zero functors. (One way to see this is to observe that the right adjoint of ϵ^* is ϵ_* while the right adjoint of $j_{U!}$ is j_U^*. One may readily show that ϵ_* and j_U^* commute. Therefore, their left-adjoints also commute.) Therefore, $real_\nu(j_{U!}(\mathcal{A}_{|U})[n]) \simeq j_{U!}(real_\nu(\mathcal{A}_{|U}[n])) = \oplus_r j_{U!}(\mu_{\ell^\nu |U}(r)[n])$. Now the definition of t-structures as in Theorem 1.1 shows first that the realization functor $real_\nu$ preserves the aisles. i.e. The following conclusions hold.

Let D_{Nis} ($D_{Nis}^{\leq 0}$) denote the category $D(X_{Nis}, \mathcal{A})$ ($D(X_{Nis}, \mathcal{A})^{\leq 0}$, respectively) and let D_{Et} ($D_{Et}^{\leq 0}$) denote the category $D(X_{Et}, real_\nu(\mathcal{A}))$ ($D(X_{Et}, real_\nu(\mathcal{A}))^{\leq 0}$, respectively). If $i_{Nis} : D_{Nis}^{\leq 0} \to D_{Nis}$ and $i_{Et} : D_{Et}^{\leq 0} \to D_{Et}$ are the obvious inclusions, then $real_\nu$ sends $D_{Nis}^{\leq 0}$ to $D_{Et}^{\leq 0}$ and moreover $real_\nu \circ i_{Nis} = i_{Et} \circ real_\nu$.

Therefore, we obtain the following maps for any $K \epsilon D^{\leq 0}_{Nis}$ and $L \epsilon D_{Nis}$:

$$Hom_{D^{\leq 0}_{Nis}}(K, \sigma_{\leq 0}L) \stackrel{\cong}{\to} Hom_{D_{Nis}}(i_{Nis}(K), L) \to Hom_{D_{Et}}(real_\nu(i_{Nis}(K)), real_\nu(L))$$
$$\stackrel{\cong}{\to} Hom_{D_{Et}}(i_{Et}(real_\nu((K))), real_\nu(L)) \stackrel{\cong}{\to} Hom_{D^{\leq 0}_{Et}}(real_\nu((K)), \sigma_{\leq 0}real_\nu(L)) \tag{8.1.1}$$

Therefore, taking $K = \sigma_{\leq 0}L$, the identity map $\sigma_{\leq 0}L \to \sigma_{\leq 0}L$ induces a map $real_\nu(\sigma_{\leq 0}L) \to \sigma_{\leq 0}(real_\nu(L))$. This proves the realization functor, $real_\nu$, is compatible with the truncation functor $\sigma_{\leq 0}$. Then, since the realization functor sends distinguished triangles to distinguished triangles, it follows that it also preserves the co-aisles. i.e. One also obtains a natural transformation

$$real_\nu \circ \sigma_{\geq 1} \to \sigma_{\geq 1} \circ real_\nu.$$

Therefore, the realization is clearly compatible with the t-structures. Finally, one may see that the functor sending the graded module $\oplus_r real_\nu(K(r))$ to $real_\nu(K(0))$ is induced by pull-back along the map $\mu_{\ell^\nu}(0) \to \oplus_r \mu_{\ell^\nu}(r)$ of sheaves of dgas and that therefore it also is compatible with the passage from the t-structure on $D(X_{Nis}, \mathcal{A})$ to the t-structure on $D(X_{Et}, \mathbb{Z}/\ell^\nu)$. ☐

Remark 8.3. In view of the validity of the Beilinson-Soulé vanishing condition with mod-ℓ^ν coefficients (at least when X is smooth), the heart of the t-structure on $D(X_{Nis}, \mathcal{A})$ contains interesting objects and the notion of motivic perverse sheaves makes sense using the non-standard t-structures obtained with respect to a stratification of X, i.e. provided the gluing property (i.e. (4.0.3) and (4.0.4)) holds. Therefore, the functor $real_\nu$ would send motivic perverse sheaves (defined as in 4) to perverse sheaves of $\mathbb{Z}/l^\nu$-modules on the étale site of X.

One could replace the derived categories considered above with the derived categories of complexes of sheaves with transfers both on the Nisnevich and étale sites. This will lead to similar results as above.

9. The *adic*-formalism and *adic* dg-algebras

Since we would like the following discussion to be useful in rather general contexts, we start by considering an an arbitrary site $\mathcal{C}$, whose objects are schemes of finite type over a given base scheme S. We will also assume that R is a commutative Noetherian ring with 1: by providing $\mathcal{C}$ with the corresponding constant sheaf R, we obtain the ringed site $(\mathcal{C}, R)$. We will let $Sh(\mathcal{C}, R)$ denote the category of sheaves of R-modules on $\mathcal{C}$. We will further assume that $\mathcal{C}$ has a conservative family of points, i.e. a sequence $0 \to F' \to F \to F'' \to 0$ of sheaves in $Sh(\mathcal{C}, R)$ is exact if and only if the corresponding sequence $0 \to F'_p \to F_p \to F''_p \to 0$ of stalks is exact, for every point p. In this case one may show readily that $Sh(\mathcal{C}, R)$ is a Grothendieck category so that it has enough injectives.

Next let $\Lambda_\bullet = \{\Lambda_n | n \epsilon \mathbb{Z}, n \geq 0\}$ denote an inverse system of rings and let $\Lambda = \lim_{\infty \leftarrow n} \Lambda_n$. For example, let R denote a local ring of dimension 1 with maximal ideal m so that the residue field R/m is of characteristic ℓ which is prime to the residue characteristics of the base scheme S and R is complete in the m-adic topology. For example, $R = \mathbb{Z}_l$ and $m = l(\mathbb{Z})_l$ or E is a finite extension of $\mathbb{Q}_l$ and R is the integral closure of $\mathbb{Z}_l$ in E. We then let $\Lambda_n = R/m^{n+1}$. We let

$$Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}} = \lim_{\infty \leftarrow n} Sh(\mathcal{C}, \Lambda_n) \tag{9.0.2}$$

which is the topos of inverse systems of sheaves $F_n \epsilon Sh(\mathcal{C}, \Lambda_n)$, i.e. the map $F_m \to F_n$ is compatible with the map $\Lambda_m \to \Lambda_n$, for all $m \geq n$. In this set-up, the functor sending $F \epsilon Sh(\mathcal{C}, \Lambda)$ to $\{F_n = \Lambda_n \underset{\Lambda}{\otimes} F | n\}$ defines a map of ringed topoi:

$$\pi : Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}} \to Sh(\mathcal{C}, \Lambda)$$

so that $\pi_*(\{F_n|n\}) = \lim_{\infty \leftarrow n} F_n$. Recall that for any $F = \{F_n|n\} \in Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}}$, the sheaf $R^n\pi_*F$ is the sheaf associated to the presheaf $U \mapsto H^n(U, F)$. Then one obtains the following exact sequence [Ek85, 0.4.6]

$$0 \to \varprojlim{}^1 H^{n-1}(U, F_n) \to H^n(U, F) \to \varprojlim H^n(U, F_n) \to 0. \tag{9.0.3}$$

Recall (cf. [SGA5, exp. V]) that a projective system $M_n, n \geq 0$ in an additive category is *AR-null* if there exists an integer r such that for every n the composite $M_{n+r} \to M_n$ is zero. Let $U \epsilon \mathcal{C}$ and let $\mathcal{C}_U$ denote the subcategory of all objects in $\mathcal{C}$ defined over U with morphisms between two such objects being morphisms in $\mathcal{C}$ over U.

Let $j_{U!} : \mathcal{C}_U \to \mathcal{C}$ denote the functor sending an object $V \to U$ to $V \epsilon \mathcal{C}$. Then one may show that the corresponding functor $j_{U!} : Sh(\mathcal{C}_U, R)^{\mathbb{N}} \to Sh(\mathcal{C}, R)^{\mathbb{N}}$ is exact. Therefore, its right adjoint j^* preserves injectives and therefore, it follows that

$j_U^* R\pi_*(F) \simeq R\pi_*(j_U^* F)$.

Definition 9.1. A complex M of objects in $Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}}$ is

- *AR-null* if all the $\mathcal{H}^i(M)$ are AR-null and
- is *almost zero* if for any $U \,\epsilon\, \mathcal{C}$, the restriction of $\mathcal{H}^i(M)$ to $\mathcal{C}_U$ is AR-null.

Definition 9.2. We say that

- a system $M = (M_n)_n$ of $Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}}$ is *adic* if all morphisms
$$R_n \otimes_{R_{n+1}} M_{n+1} \to M_n$$
are isomorphisms; it is called *almost adic* if for every $U \in \mathcal{C}$ there is a morphism $N_U \to M_U$ with almost zero kernel and cokernel with N_U adic in $Sh(\mathcal{C}_U, \Lambda_\bullet)^{\mathbb{N}}$.
- a complex $M = (M_n)_n$ of objects in $Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}}$ is called a λ-*complex* if all the cohomology modules $\mathcal{H}^i(M)$ are almost adic. Let $\mathrm{C}_\lambda(\mathcal{C}, \Lambda) \subset \mathrm{C}(Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}})$ denote the full subcategory whose objects are λ–complexes. We let $\mathrm{C}_\lambda(\mathcal{C}, \Lambda) \otimes \mathbb{Q}$ denote the quotient of the category $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$ by the full sub-category of torsion sheaves.
- We let $\mathcal{D}_{\lambda_0}(\mathcal{C}, \Lambda) \subseteq \mathcal{D}(Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}})$ denote the full subcategory whose objects belong to $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$. The full subcategory of $\mathcal{D}_\lambda(Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}})$ of complexes concentrated in degree 0 is called the category of λ-*modules*.
- The category $\mathcal{D}_\lambda(\mathcal{C}, \Lambda)$ (sometimes written just $\mathcal{D}(\mathcal{C})$ if the reference to Λ is clear) is the quotient of the category $\mathcal{D}_{\lambda_0}(\mathcal{C}, \Lambda)$ by the full subcategory of almost zero complexes. The category $\mathcal{D}_\lambda(\mathcal{C}, \Lambda) \otimes \mathbb{Q}$ will denote the quotient of the category $\mathcal{D}_\lambda(\mathcal{C}, \Lambda)$ by the full subcategory of all torsion sheaves.
- Assume that the categories of complexes and the associated derived categories above are defined using Λ_n being either $\mathbb{Z}/l^{n+1}$ or R/m^{n+1} where R is the ring of integers in a finite field extension E of $\mathbb{Q}_l$. We then let $C_\lambda(\mathcal{C}, E) = C_\lambda(\mathcal{C}, \Lambda) \otimes \mathbb{Q}$, $\mathcal{D}_\lambda(\mathcal{C}, E) = \mathcal{D}_\lambda(\mathcal{C}, \Lambda) \otimes \mathbb{Q}$ and let $C_\lambda(\mathcal{C}, \bar{\mathbb{Q}}_l) = \underset{\underset{E}{\rightarrow}}{\lim} C_\lambda(\mathcal{C}, E)$. $\mathcal{D}_\lambda(\mathcal{C}, \bar{\mathbb{Q}}_l)$ will denote the corresponding derived category.

9.1. **Coherently homotopy associative and commutative Dg-algebras.** A dg-algebra is a λ-complex $\mathcal{A} \,\epsilon\, \mathrm{C}_\lambda(\mathcal{C}, \Lambda)$ which is also an algebra in $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$. Observe that the last condition means, there exists a coherently associative pairing $\mu : \mathcal{A} \otimes_\Lambda \mathcal{A} \to \mathcal{A}$ compatible with a unit map $i : \Lambda \to \mathcal{A}$. Moreover, the last condition means $\mathcal{A} = \{A_n | n\}$, with $A_n \,\epsilon\, Sh(\mathcal{C}, \Lambda_n)$ being an associative algebra. Such a dg-algebra is commutative, if each of the dg-algebras A_n is a commutative dg-algebra in $Sh(\mathcal{C}, \Lambda_n)$. For the rest of the discussion, we will assume that $\mathcal{A}$ is a commutative dg-algebra.

A homotopy associative and commutative dg-algebra is a λ-complex $\mathcal{A} \,\epsilon\, \mathrm{C}_\lambda(\mathcal{C}, \Lambda)$ where the pairing μ is only homotopy associative and commutative and the required identities involving composition with the unit map i holds only up to homotopy. Much more useful are *coherently homotopy associative and commutative dg-algebras or E^∞-algebras* which may be briefly (or informally) defined as a λ-complex $A \,\epsilon\, \mathrm{C}_\lambda(\mathcal{C}, \Lambda)$ with prescribed higher order homotopies for all iterated compositions involving μ and i. The above informal definition may be formalized as follows making use of the Eilenberg-Zilber operad as in [H-Sch]. (While this notion seems familiar to topologists, it does not seem to be familiar to algebraists (at least as much) and therefore the following paragraphs are justified.)

9.1.1. *The (classical) Eilenberg-Zilber operad in $Sh(\mathcal{C}, \Lambda)$.* Consider the functor $\mathcal{Z}$ defined by $n \mapsto C_*(\Delta[n], \Lambda)$, where $\Delta[n]$ denotes the simplicial set $\{Hom_\Delta([k], [n]) | \quad [k] \,\epsilon\, \Delta\}$. This will be denoted $\mathcal{Z} = C_*(\Delta, \Lambda)$. Clearly $\mathcal{Z}$ is a cosimplicial object in $Sh(\mathcal{C}, \Lambda)$. We will denote the category of cosimplicial objects in $Sh(\mathcal{C}, \Lambda)$ by $Sh(\mathcal{C}, \Lambda)^\Delta$.

The *(classical) Eilenberg-Zilber operad* in $Sh(\mathcal{C}, \Lambda)$ is defined by the sequence $\mathcal{O}_{EZ}(n) = Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(\mathcal{Z}, \mathcal{Z}^{\otimes^n})$ where $\mathcal{Z}^{\otimes^n}(m) = \mathcal{Z}(m)^{\otimes^n}$. The operad structure is defined by the compositions:

$$\gamma_n : Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(\mathcal{Z}, \mathcal{Z}^{\otimes^n}) \otimes Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(\mathcal{Z}, \mathcal{Z}^{\otimes^{k_1}}) \otimes \cdots \otimes Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(\mathcal{Z}, \mathcal{Z}^{\otimes^{k_n}}) \to Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(\mathcal{Z}, \mathcal{Z}^{\otimes^{\Sigma_i k_i}})$$

(See [H-Sch] for more details.) An *algebra* over the above operad is a complex K in $Sh(\mathcal{C}, \Lambda)$ provided with pairings $\mu_k : \mathcal{O}_{EZ}(k) \otimes K^{\otimes^n} \to K$ which are compatible with the pairings $\{\gamma_k | k\}$ and with the action of the symmetric group Σ_n (which acts on the left as follows: if $\sigma \,\epsilon\, \Sigma_n$, σ acts on $\mathcal{O}_{EZ}(n)$ by permuting the n-factors $\mathcal{Z}^{\otimes^n}$ and it acts by permuting the n-factors of $K^{\otimes^n}$ using σ^{-1}.) One may readily show that the complexes $O_{EZ}(n)$ are all acyclic. The pairings μ_k encode the higher order homotopies.

It is shown in [H-Sch, (2.4.1) Proposition] that if A is a coismplicial object in $Sh(\mathcal{C}, \Lambda)$, then it normalization is the co-chain complex in $Sh(\mathcal{C}, \Lambda)$ defined by $Norm(A) = Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(Z, A)$. Then a main result in [H-Sch] is the following:

Theorem 9.3. *(Hinich and Schectmann: see* [H-Sch]*.) If A is a cosimplicial algebra in $Sh(\mathcal{C}, \Lambda)$, then its normalization $Norm(A)$ has the structure of an algebra over the classical Eilenberg-Zilber operad.*

Since we will need to make use of their proof, we will provide the following explanation of it. The required algebra structure on $Norm(A)$ over the Eilenberg-Zilber operad is provided by the following sequence of maps:

$$O_{EZ}(n) \otimes Norm(A)^{\otimes^n} = Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(Z, Z^{\otimes^n}) \otimes Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(Z, A)^{\otimes^n}$$

$$\to Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(Z, Z^{\otimes^n}) \otimes Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(Z^{\otimes^n}, A^{\otimes^n}) \to Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(Z, A^{\otimes^n})$$

$$\to Hom_{Sh(\mathcal{C},\Lambda)^\Delta}(Z, A) = Norm(A)$$

where the last map is given by the structure of a cosimplicial algebra on A and the one before that is the obvious map obtained by composition.

Corollary 9.4. *If A is a cosimplicial algebra in $Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}}$, then its normalization $Norm(A)$ has the structure of an algebra over the classical Eilenberg-Zilber operad.*

Proof. The proof of the last theorem discussed above shows that the algebra structures on $Norm(A_n)$, for $A_n \,\epsilon\, C(Sh(\mathcal{C}, \Lambda_n))$ over the operad $\{O_{E.Z}(m)|m \geq 0\}$ are compatible as n varies proving the corollary. $\square$

9.1.2. **Modules over an E_∞-dga.** Given such an E_∞-dg-algebra $\mathcal{A}$, we let $Mod(\mathcal{C}, \mathcal{A})$ denote the sub-category of $Sh(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}}$ consisting of objects $M = \{M_n|n\}$ which are λ-complexes, with the following extra structure: one is given pairings $\lambda_n : \mathcal{O}_{EZ}(n) \otimes \mathcal{A}^{\otimes^{n-1}} \otimes M \to M$ which satisfy certain obvious compatibility conditions involving the pairings $\{\mu_m|m \geq 0\}$: see [H-Sch]. Morphisms in this category between two E_∞-dg-modules M and N will be a map $M \to N$ in $\mathrm{C}_\lambda(\mathcal{C}, \Lambda_\bullet)^{\mathbb{N}}$ compatible with the above structures.

Example 9.5. A basic example of such an E_∞-dg algebra may be obtained as follows. Let $\phi : (\mathcal{C}', \Lambda') \to (\mathcal{C}, \Lambda)$ denote a map of ringed sites both of which have enough points and where $\Lambda' = \lim_{\infty \leftarrow n} \Lambda'_n$ and $\Lambda = \lim_{\infty \leftarrow n} \Lambda_n$. Then $R\phi_*(\Lambda'_\bullet) = \{R\phi_*(\Lambda'_n)|n\}$ is an E_∞-dg-algebra in $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$, where $R\phi_*$ is defined using the Godement resolution (which is clearly functorial).

9.1.3. **Conversion from E^∞-dg algebras to dg-algebras.** It is well-known that any E^∞ dg-algebra may be converted functorially to a quasi-isomorphic commutative dg-algebra after tensoring with $\mathbb{Q}$. We apply this functor to any of the E_∞-dg algebras in $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$ to obtain quasi-isomorphic commutative dg-algebras in $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$. i.e. Given an E_∞-dg algebra $\mathcal{A}$ in $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$, $W(\mathcal{A} \otimes Q)$ denotes a quasi-isomorphic commutative dg-algebra in $\mathrm{C}_\lambda(\mathcal{C}, \Lambda) \otimes \mathbb{Q}$. (The latter is the quotient of the category $\mathrm{C}_\lambda(\mathcal{C}, \Lambda)$ by the full subcategory of torsion sheaves.)

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