HIGHER INTERSECTION THEORY ON ALGEBRAIC STACKS:II

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Abstract. This is the second part of our work on the intersection theory of algebraic stacks. *The main results* here are the following. We provide an intersection pairing for all smooth Artin stacks (locally of finite type over a field) which we show reduces to the known intersection pairing on the Chow groups of smooth Deligne-Mumford stacks of finite type over a field as well as on the Chow groups of quotient stacks associated to actions of linear algebraic groups on smooth quasi-projective schemes modulo torsion. The former involves also showing the existence of Adams operations on the rational étale K-theory of all smooth Deligne-Mumford stacks of finite type over a field. In addition, we show that our definition of the higher Chow groups is *intrinsic to the stack* for all smooth stacks and also stacks of finite type over the given field. Next we establish the existence of Chern classes and Chern character for Artin stacks with values in our Chow groups and extend these to higher Chern classes and a higher Chern character for *perfect complexes* on an algebraic stack, taking values in cohomology theories of algebraic stacks that are defined with respect to complexes of sheaves on a big smooth site. As a by-product of our techniques we also provide an extension of higher intersection theory to all schemes locally of finite type over a field. As the higher cycle complex, by itself, is a bit difficult to handle, the stronger results like contravariance for arbitrary maps between smooth stacks and the intersection pairing for smooth stacks are established by comparison with motivic cohomology.

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0. Introduction

In this paper we continue our work on the higher intersection theory for all Artin stacks locally of finite type over a field. We establish Chern classes and Chern character along with an (integer valued) intersection pairing for all *smooth Artin stacks.* We show that this intersection theory reduces modulo torsion to the known intersection theory (at the level of the usual Chow groups) for all smooth Deligne-Mumford stacks of finite type over a field and also for quotient stacks associated to linear actions of linear algebraic groups on smooth quasi-projective schemes. The higher Chow groups of a stack are defined as in the first part of the paper, [J-1] section 4; we will also adopt the basic terminology from that paper.

We summarize the main results here. Let (alg.stacks) denote the category of algebraic stacks in the sense of Artin and locally of finite type over a given field k.

Let $CH_*(\quad ,.)$:(alg.stacks) $\rightarrow$(bi-graded abelian groups)

denote the integral higher Chow groups defined in the first part of our work i.e. [J-1](4.5.1). In general, this depends on the choice of an atlas for the stack; but for Deligne-Mumford stacks, smooth Artin stacks and Artin stacks of finite type, this is independent of the choice of the atlas. (The last assertion is proved in section 3. See (1.2) for an explanation of the notation.)

Theorem 1. (See (4.2.2), (4.2.3) and (4.6.11).) Let $\mathfrak{S}$ denote an algebraic stack of dimension d and let $x : X \rightarrow \mathfrak{S}$ denote a fixed atlas for $\mathfrak{S}$. Let $K^0(\mathfrak{S})$ denote the Grothendieck group of vector bundles on $\mathfrak{S}$ -see (1.5.0) for the definition.

(i) If $\mathfrak{S}$ is *smooth*, one obtains Chern-classes

$c_i : K^0(\mathfrak{S}) \rightarrow CH_{d-i}(\mathfrak{S}, x, 0)$, $i \geq 0$

which pull-back under any representable flat map $\mathfrak{S}' \rightarrow \mathfrak{S}$.

(ii) *Projective space bundle theorem.* Let $\mathcal{E}$ denote a vector bundle of rank r on the algebraic stack $\mathfrak{S}$ and let $\pi : \mathbb{P}(\mathcal{E}) = Proj(\mathcal{E}) \rightarrow \mathfrak{S}$ denote the associated projective space bundle. Let $x' = x \underset{\mathfrak{S}}{\times} Proj(\mathcal{E})$ denote the induced atlas for $Proj(\mathcal{E})$. Let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ denote the tautological bundle on $\mathbb{P}(\mathcal{E})$. This defines a class $\psi_{\mathcal{E}} \, \epsilon \, CH_{d+r-2}(\mathbb{P}(\mathcal{E}), x', 0)$. Then the map $\oplus_{i=0}^{r-1} CH_*(\mathfrak{S}, x, .) \rightarrow CH_*(\mathbb{P}(\mathcal{E}), x', .)$ sending $(a_0, ..., a_{r-1})$ to $\Sigma_i \pi^*(a_i).\psi_{\mathcal{E}}^i$ is an isomorphism.

(iii) *Covariance of the integral higher Chow groups.* Let $p : \mathfrak{S}' \rightarrow \mathfrak{S}$ denote a proper map of algebraic stacks that factors as the composition of a closed immersion and a projection $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathfrak{S}$ for some vector bundle $\mathcal{E}$ on the stack $\mathfrak{S}$. Let $x' = x \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ denote the induced atlas for $\mathfrak{S}'$. Then there exits a map $p_* : CH_*(\mathfrak{S}', x', .) \rightarrow CH_*(\mathfrak{S}, x, .)$.

(iv) For each integer i, let $\Gamma(i)$ denote a complex of abelian sheaves on the big smooth site of all algebraic stacks locally of finite type over k so that there exist universal Chern classes $C_i^p \, \epsilon \, \mathbb{H}_{\mathcal{C}}^{di}(B.GL_p, \Gamma(i))$. Let $\mathbf{K}(\mathfrak{S})$ denote the Waldhausen K-theory space of the category of perfect complexes on $\mathfrak{S}$ - see the definition in (1.5.1). Then one obtains *higher Chern classes*

$C_i(n) : \pi_n(\mathbf{K}(\mathfrak{S})) \rightarrow \mathbb{H}_{smt}^{di-n}(\mathfrak{S}, \Gamma(i))$

where d is an integer depending on the complex $\Gamma(i)$ and the right hand side denotes hypercohomology on the smooth site of the stack $\mathfrak{S}$. These pull-back under any representable map. In case the stack $\mathfrak{S}$ is smooth, there exist higher Chern classes $C_i(n) : \pi_n K(\mathfrak{S}) \rightarrow CH^{i-n}(\mathfrak{S}, x, i)$ where the right hand side is defined below. $\square$

Remark 0.1 The Chern classes in (iv) are also obtained in the thesis of Toen (see [Toe-1], [Toe-2]) for the K-theory of the exact category of vector bundles on $\mathfrak{S}$ by similar methods.

Let $\mathfrak{S}$ denote an algebraic stack *of finite dimension* over k. Recall that in [J-1](4.2.5), for each integer m and non-negative integer n, we defined an additive presheaf on the site $\mathfrak{S}_{res.\underline{smt}}$ as follows. (See (1.1) for the definition of this site.) For each U connected, $\mathcal{Z}_m^{\mathfrak{S}_{res.smt}}(U, n) =$ the free abelian group on dimension m cycles on $U \times \Delta_k[n]$ intersecting the faces properly. We define in section 3, an *additive* presheaf $\mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(\quad, n)$ on $\mathfrak{S}_{res.smt}$ by: $\mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(U, n) =$ the free abelian group on codimension i cycles on $U \times \Delta[n]$ intersecting all the faces of $U \times \Delta[n]$ properly, for each $u : U \rightarrow \mathfrak{S}$ in $\mathfrak{S}_{res.smt}$ and also *connected.* We let $\mathcal{Z}^*_{\mathfrak{S}_{res.smt}}(\quad, .) = \underset{i}{\oplus} \mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(\quad, .)$.

Let $\mathcal{Z}^i(\quad,.)$ denote the pre-sheafification of the higher cycle complex of codimension i on the site $(qp.schemes)_{Res.Zar}$. (See (1.1) again for the definition of this site.) We define an extension functor $E_{\mathfrak{S}}$ associated to each algebraic stack $\mathfrak{S}$ in section 2 so that $E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.))$ will be a presheaf on the site $\mathfrak{S}_{res.\underline{smt}}$. We also define an extension functor $E1$ extending presheaves on the site $(qp.schemes/k)_{Res.Zar}$ to the restricted Zariski site $(schemes/k)_{Res.Zar}$ of all schemes locally of finite type over k.

We define $\mathbf{CH}^i(\mathfrak{S},x,.) = \mathbb{H}_{et}(B_x\mathfrak{S}, E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.)))$ and $\mathbf{CH}^i(\mathfrak{S},x,.;\mathbb{Q}) = \mathbb{H}_{et}(B_x\mathfrak{S}, E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.))\otimes\mathbb{Q})$. If X is a scheme locally of finite type, we define $\mathbf{CH}^i(X,.) = \mathbb{H}_{Zar}(X, E1(\mathcal{Z}^i(\quad,.)))$. We let $CH^i(\mathfrak{S},x,n) = \pi_n\mathbf{CH}^i(\mathfrak{S},x,.)$ and $CH^i(X,n) = \pi_n\mathbf{CH}^i(X,.)$. Similarly $CH^i(\mathfrak{S},x,n;\mathbb{Q}) = \pi_n\mathbf{CH}^i(\mathfrak{S},x,.;\mathbb{Q})$. Let $\mathbb{Z}(i)[2i]$ denote the shifted motivic complex of weight i defined on the category of smooth separated schemes of finite type over k. We show in section 2, that $E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])$ defines an extension of this complex to the smooth site $\mathfrak{S}_{\underline{smt}}$ and that $E1(\mathbb{Z}(i)[2i])$ defines an extension to $(schemes/k)_{Res.Zar}$.

Theorem 2. (See (5.1), (5.3.10) and (3.5).)

(i) The groups $CH^i(\mathfrak{S},x,n)$, $i,n\geq 0$, are intrinsic to the stack and independent on the choice of the atlas for all smooth stacks $\mathfrak{S}$. In fact $CH^i(\mathfrak{S},x,n)\cong\pi_n(\mathbb{H}_{\underline{smt}}(\mathfrak{S},E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])))$ if $\mathfrak{S}$ is *smooth*. Moreover, if $\mathfrak{S}$ is any algebraic stack of finite type over k (not necessarily smooth), the groups $CH^i(\mathfrak{S},x,n;\mathbb{Q})$ are independent of the choice of the atlas x. If the stack is of finite dimension, $CH^i(\mathfrak{S},x,n)\cong\pi_n(\mathbb{H}_{et}(B_x\mathfrak{S},\mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(\quad,.)))$ as well.

(ii) If $f:\mathfrak{S}'\rightarrow\mathfrak{S}$ is a map of smooth algebraic stacks, one obtains an induced map $f^*: CH^*(\mathfrak{S},.)\rightarrow CH^*(\mathfrak{S}',.)$.

(iii) If $\mathfrak{S}$ is also *smooth*, one obtains an *intersection-pairing*:

$\cup: CH^i(\mathfrak{S},n)\otimes CH^j(\mathfrak{S},m)\rightarrow CH^{i+j}(\mathfrak{S},n+m)$.

In addition, if $f:\mathfrak{S}'\rightarrow\mathfrak{S}$ is a map of smooth algebraic stacks, the induced map $f^*: CH^*(\mathfrak{S},.))\rightarrow CH^*(\mathfrak{S}',.)$ is compatible with the above intersection pairing.

(iv) If $\mathfrak{S}$ is any smooth separated Deligne-Mumford stack of finite type over k, there exists an isomorphism $CH^*(\mathfrak{S},0;\mathbb{Q})\cong CH^*_{naive}(\mathfrak{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. The above intersection pairing on $CH^*(\mathfrak{S},0;\mathbb{Q})$ agrees with the known intersection pairing on the naive Chow group $CH^*_{naive}(\mathfrak{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$.

(v) Let G denote a linear algebraic group acting on a smooth quasi-projective scheme X so that the action is G-linearized (i.e. X admits a G-equivariant locally closed immersion into a projective space onto which the G-action extends to a linear action). Then one obtains an isomorphism $CH^*([X/G],.;\mathbb{Q})\cong CH^*_G(X,.;\mathbb{Q})$ preserving the multiplicative structure, where the right-hand-side is the G-equivariant intersection theory of X as in [EG]. $\square$

Remark 0.2. In view of Theorem 2 (i), we may omit the atlas x from the definition of the higher Chow groups for stacks satisfying the hypotheses there. The proof of (iv) involves showing the existence of λ-operations on the (higher) étale K-theory of any smooth Deligne-Mumford stack with rational coefficients. The intersection pairing in (iii) also extends to higher Chow groups with coefficients. Moreover, observe that we have extended the definition of the higher Chow groups to all stacks that are locally of finite type, but not necessarily of finite type over k: this is important since some of the more familiar stacks (see for example, Corollary 4 (v)) are only locally of finite type.

As a by-product of our techniques we also obtain a straightforward extension of intersection theory to all schemes of locally of finite type over a field. These results may be summarized in the following theorem. Let (schemes) denote the category of all schemes locally of finite type over a given field k.

Theorem 3. (See (4.2.2), (4.2.3) and (5.2.0).)

(i) Let X denote a scheme locally of finite type over a field k and of finite dimension d and let $\mathcal{E}$ denote a vector bundle of rank r on X. Let $\pi:\mathbb{P}(\mathcal{E}) = Proj(\mathcal{E})\rightarrow\mathfrak{S}$ denote the associated projective space bundle and let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ denote the tautological bundle on $\mathbb{P}(\mathcal{E})$. Let $\psi_{\mathcal{E}}\;\epsilon\;CH_{d+r-2}(\mathbb{P}(\mathcal{E}),0)$ denote the canonical class defined by this bundle. Then the map $\oplus_{i=0}^{r}CH_*(X,.)\rightarrow CH_*(\mathbb{P}(\mathcal{E}),.)$ sending $(a_0,...,a_{r-1})$ to $\Sigma_i\pi^*(a_i).\psi^i_{\mathcal{E}}$ is an isomorphism.

(ii) If $f : X' \to X$ is a map of smooth schemes locally of finite type over k, there exists an induced map $f^* : CH^*(X, .) \to CH^*(X', .)$.

(iii) If X is a smooth scheme as in (i), there exists an intersection pairing:

$CH^i(X, n) \otimes CH^j(X, m) \to CH^{i+j}(X, n + m)$.

Moreover, if $f : X' \to X$ is a map of smooth schemes as in (i), the induced map $f^* : CH^*(X, .) \to CH^*(X', .)$ is compatible with the above intersection pairing. $\square$

The following corollary is to serve as a sample of the applications of the above theorems.

Corollary 4. (i) Let X denote a *smooth* scheme locally of finite type over a field k and of finite dimension d provided with the action of an affine smooth group scheme G defined over k. Then there exist Chern classes

$c_i : K^0([X/G]) \to CH_{d-i}([X/G], X, 0)$ and $C_i(n) : \pi_n(K([X/G])) \to CH^{i-n}([X/G], n)$.

(ii) For each integer i, let $\Gamma(i)$ denote a complex of sheaves on the big smooth site of all algebraic stacks locally of finite type over k satisfying the hypotheses in Theorem 1 (iv). Assuming the hypotheses as in (i) except that X could be singular, there exist higher Chern classes

$C_i(n) : \pi_n(\mathbf{K}([X/G])) \to \mathbb{H}^{di-n}_{smt}([X/G], \Gamma(i))$.

(iii) Assuming the hypotheses of (i), there exists an intersection pairing

$\cup : CH^i([X/G], n) \otimes CH^j([X/G], m) \to CH^{i+j}([X/G], n + m)$.

(iv) Next assume X is separated and of finite type over k, the action is locally proper and the stabilizers are all finite and reduced and $\mathfrak{M}_{[X/G]}$ is a coarse moduli space (as in [J-1], (1.3.4)(vii)). Then one obtains an isomorphism $CH^*([X/G], .; \mathbb{Q}) \cong CH^*(\mathfrak{M}_{[X/G]}, .; \mathbb{Q})$ and hence an induced intersection pairing on the latter provided X is smooth. (The higher Chow groups of the algebraic space $\mathfrak{M}_{[X/G]}$ on the right hand side are defined as in [J-1] (4.6.6).)

(v) Let k be algebraically closed, X a smooth projective curve of genus g over k and $\mathcal{M}_G$ the stack of principal G-bundles over X with G as in (i). (See [LS] for example.) Then the stack $\mathcal{M}_G$ is smooth and there exists an intersection pairing

$\cup : CH^*(\mathcal{M}_G, .) \otimes CH^*(\mathcal{M}_G, .) \to CH^*(\mathcal{M}_G, .)$.

(vi) Let X denote a smooth projective variety which is *convex* in the sense of [F-P] p.6. Let $\bar{M}_{g,n}(X, \beta)$ denote the stack of stable families of maps of n-pointed genus g-curves to X and let $\overline{\mathfrak{M}}_{g,n}(X, \beta)$ denote the corresponding coarse-moduli space. Here β denotes a class in $CH^1(X)$. If this stack is smooth, (for example, $g = 0$ and $X = \mathbb{P}^r$ for some r), one obtains an intersection pairing $\cup : CH^*(\bar{M}_{g,n}(X, \beta), .) \otimes CH^*(\bar{M}_{g,n}(X, \beta), .) \to CH^*(\bar{M}_{g,n}(X, \beta), .)$. Moreover, one also obtains an isomorphism $CH^*(\bar{M}_{g,n}(X, \beta), .; \mathbb{Q}) \cong CH^*(\overline{\mathfrak{M}}_{g,n}(X, \beta), .; \mathbb{Q})$ and therefore an induced pairing $\cup : CH^*(\overline{\mathfrak{M}}_{g,n}(X, \beta), .; \mathbb{Q}) \otimes CH^*(\overline{\mathfrak{M}}_{g,n}(X, \beta), .; \mathbb{Q}) \to CH^*(\overline{\mathfrak{M}}_{g,n}(X, \beta), .; \mathbb{Q})$ when the stack $\bar{M}_{g,n}(X, \beta)$ is smooth. $\square$

The organization of the paper is as follows. The first section is a quick review of the first part of our work, recalling the basic framework. We also define the algebraic K-theory of algebraic stacks: for algebraic stacks, it seems best to adopt the definition of K-theory (G-theory) as the Waldhausen style K-theory of the category of perfect complexes (pseudo-coherent complexes with globally bounded cohomology, respectively) following [T-2]. Moreover, these are more convenient for us in section 4. The second section, which is purely technical, introduces a technique whereby we are able to extend results from the Zariski site of quasi-projective schemes over a field k to all algebraic spaces locally of finite type over a field and then to the restricted smooth site or the smooth site of any algebraic stack. For example, this way we extend the motivic complexes functorially to the smooth site of any algebraic stack. In the third section, we restrict to smooth algebraic stacks and obtain fundamental comparison theorems relating our higher Chow groups for stacks with hypercohomology computed with respect to appropriate extensions of the motivic complexes. As an immediate consequence, we show that our higher Chow groups are intrinsic to the stack for all smooth stacks: we also establish contravariant functoriality for maps between smooth stacks. We also obtain a comparison between our higher Chow groups and the equivariant higher Chow groups for quotient stacks associated to the actions of linear algebraic groups on smooth quasi-projective schemes. The higher Chern classes (along with a Chern character) are established in the fourth section by extending well known techniques. This includes higher Chern classes with values in our higher Chow groups for all smooth stacks.

The fifth section establishes an intersection pairing at the level of our higher Chow groups for all *smooth* algebraic stacks. We show, using standard techniques on the Adams operations, that this intersection pairing agrees modulo torsion with the known intersection pairing for all smooth separated Deligne-Mumford stacks. (This, in fact, involves showing the existence of λ and hence Adams operations on the rational étale K-theory of all smooth Deligne-Mumford stacks.) The intersection pairing for smooth stacks is established by invoking the comparison theorems of section 3. (This replaces the argument using DGAs associated to the cycle complex adopted in an earlier version. We thank Spencer Bloch for suggesting the idea of comparing with the motivic complexes as is done here.)

It ought to be pointed out that establishing an intersection pairing for *all smooth Artin stacks, in general*, even at the level of the Chow groups (i.e. in degree 0), has been an open problem: the main difficulty is that the diagonal map is no longer a local imbedding so that none of the standard techniques, including the ones using Gysin maps for lci morphisms apply in general. However, see [Kr] where he obtains an intersection pairing for Chow groups of all smooth stacks that have affine stabilizer groups.

The comparison theorem of section 3 also provides an intersection pairing for the higher Chow groups of all smooth schemes locally of finite type over k. Combined with the localization theorem for all schemes proved in [J-1], we obtain a complete theory of higher Chow groups for all schemes locally of finite type over a field, (at least modulo torsion).

1. Review of higher intersection theory on algebraic stacks: algebraic K-theory of algebraic stacks

(1.1) Throughout the paper we will adopt the conventions and terminology from the first part, i.e. [J-1]. Accordingly let k denote a fixed field of arbitrary characteristic. We will only consider objects (schemes, algebraic spaces and algebraic stacks) that are locally of finite type over k (and hence quasi-separated). The algebraic stacks we consider will be the ones in the sense of Artin. The category of all algebraic stacks (algebraic spaces) locally of finite type over k will be denoted $(alg.stacks/k)$ ($(alg.spaces/k)$, respectively). Now $(smt.alg.stacks/k)$ ($(smt.alg.spaces/k)$) will denote the corresponding full sub-categories of *smooth* objects. The big Zariski site of all quasi-projective schemes (schemes) over k will be denoted $(qp.schemes/k)_{Zar}$ ($(schemes/k)_{Zar}$, respectively). The big étale site (smooth site) of all algebraic spaces will be denoted $(alg.spaces/k)_{et}$ ($(alg.spaces/k)_{smt}$, respectively). Now $(smt.qp.schemes/k)_{Zar}$, $(smt.schemes/k)_{Zar}$, $(smt.alg.spaces/k)_{et}$ and $(smt.alg.spaces/k)_{smt}$ will denote the corresponding sites associated to smooth objects.

If $\mathcal{C}$ is any of these big sites we obtain the corresponding *restricted* sites which have the same objects and coverings but where the morphisms are restricted to be only flat maps. All *flat maps* will be required to be of some *fixed relative dimension*. i.e. if $f : \mathfrak{S}' \to \mathfrak{S}$ is a flat map of algebraic stacks we will also require that for each irreducible component T of $\mathfrak{S}$, every irreducible component of $f^{-1}(T) = T \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ is of dimension $= dim\mathfrak{S}' - dim\mathfrak{S}$. $(qp.schemes/k)_{Res.Zar}$, $(schemes/k)_{Res.Zar}$, $(alg.spaces/k)_{Res.et}$ and $(alg.spaces/k)_{Res.smt}$ will denote the corresponding restricted sites. Similarly $(alg.spaces/k)_{Res}$ ($(alg.stacks/k)_{Res}$) will denote the full sub-category of $(alg.spaces/k)$ ($(alg.stacks/k)$, respectively) where the morphisms are all required to be *flat* maps. If $\mathfrak{S}$ is a fixed algebraic stack, we have defined several sites associated to $\mathfrak{S}$: the smooth sites, $\mathfrak{S}_{smt}$, $\mathfrak{S}_{\underline{smt}}$, $\mathfrak{S}_{res.smt}$ and $\mathfrak{S}_{res.\underline{smt}}$, the étale site $\mathfrak{S}_{et}$ and for Deligne-Mumford stacks the étale site $\mathfrak{S}_{\underline{et}}$. (See [J-1] section 2 for details.)

(1.2) We will consider presheaves on any of these sites taking values in a complete pointed simplicial category as defined in [J-1] section 6. We defined in [J-1] (4.2.3) an additive presheaf $\mathcal{Z}_*(\quad,.) : (alg.stacks/k)_{Res.smt} \to$(chain complexes of abelian groups). If $\mathfrak{S}$ is an algebraic stack, the restriction of this presheaf to $\mathfrak{S}_{res.\underline{smt}}$ is denoted $\mathcal{Z}_*^{\mathfrak{S}_{res.smt}}(\quad,.)$. If $B_x\mathfrak{S}$ is the classifying simplicial space associated to an atlas $x : X \to \mathfrak{S}$, observe that the face maps of the simplicial space $B_x\mathfrak{S}$ are all smooth. This enables us to define the étale hypercohomology of $B_x\mathfrak{S}$ with respect to $\mathcal{Z}_*^{\mathfrak{S}_{res.smt}}(\quad,.)$. We define $\mathbf{CH}(\mathfrak{S},x,.) = \mathbb{H}_{et}(B_x\mathfrak{S}, \mathcal{Z}_*^{\mathfrak{S}_{res.\underline{smt}}}(\quad,.))$ and $CH_*(\mathfrak{S},x,n) = \pi_n(\mathbf{CH}(B_x\mathfrak{S},x,.))$. In case G is a smooth group scheme acting on a scheme X, the higher Chow groups of the quotient stack $[X/G]$ with respect to the atlas X will be denoted $CH_*([X/G],X,.)$. Often the atlas X will be omitted and the above group will be denoted simply by $CH_*([X/G],.)$.

(1.2)' If X is any scheme, we defined $\mathcal{Z}_*^{X_{Zar}}(\quad,.)$ to be the restriction of $\mathcal{Z}_*(\quad,.)$ to the Zariski site of X. We define $\mathbf{CH}(X,.) = \mathbb{H}_{Zar}(X, \mathcal{Z}_*^{X_{Zar}}(\quad,.)$ and $CH_*(X,n) = \pi_n(\mathbf{CH}(X,.))$.

(1.3) In order to be able to handle unbounded complexes and also the spaces of K-theory, we fix a *complete pointed simplicial category* $\mathbf{S}$ and consider only presheaves that take values in $\mathbf{S}$. (See [J-1] section 6 for details.) For the purposes of this paper, $\mathbf{S}$ will denote either the category of all complexes of sheaves of R-modules on a suitable site (where R is a commutative Noetherian ring with 1) or the category of presheaves of fibrant pointed simplicial sets (=spaces) on a suitable site. If P is such a presheaf $\pi_n(P)\,\tilde{}$ will denote the sheaf associated to the presheaf $U \rightarrow \mathcal{H}^{-n}(\Gamma(U,P))$ in the first case and the sheaf associated to the abelian presheaf of homotopy groups $U \rightarrow \pi_n(\Gamma(U,P))$ in the second case. A map $\alpha : P \rightarrow Q$ is a *quasi-isomorphism* (*a strong quasi-isomorphism*) if it induces an isomorphism on all $\pi_n(\quad)\,\tilde{}$ $(\pi_n\Gamma(U,\quad)$ for all U in the site and all n, respectively). Clearly every strong quasi-isomorphism is a quasi-isomorphism. If $\mathcal{C}$ is a site and $Presh(\mathcal{C},\mathbf{S})$ denotes the category of all presheaves on $\mathcal{C}$ with values in $\mathbf{S}$, the derived category associated to $Presh(\mathcal{C},\mathbf{S})$ is obtained by inverting maps that are quasi-isomorphisms. This will be denoted $D(Presh(\mathcal{C},\mathbf{S}))$.

(1.4) The only presheaves P we consider will be ones that are *additive* in the following sense: the natural map $\Gamma(U \sqcup V, P) \rightarrow \Gamma(U,P) \times \Gamma(V,P)$ is a quasi-isomorphism for all U and V in the category under consideration.

(1.5.0)**Definition**. Let $\mathfrak{S}$ denote an algebraic stack. A *quasi-coherent* (*coherent*, *coherent and locally free*) sheaf F on a stack $\mathfrak{S}$ is a sheaf of $\mathcal{O}_{\mathfrak{S}}$-modules on $\mathfrak{S}_{smt}$ so that for any map $x : X \rightarrow \mathfrak{S}$, with X an algebraic space, $x^*(F)$ is a quasi-coherent (coherent, coherent and locally free) sheaf on the algebraic space X.

Let $Mod_{coh}(\mathfrak{S})$ $(Mod_{l.free}(\mathfrak{S}))$ denote the category of all coherent (coherent and locally free, respectively) sheaves on the stack $\mathfrak{S}$. $K^0(\mathfrak{S})$ will denote the Grothendieck group of the latter category. (Both are also symmetric monoidal categories with the operation of direct sum of two sheaves. Therefore it is possible to consider the higher algebraic K-theory of these categories: nevertheless, it seems best for us to consider the more general definition of the higher K-theory and G-theory of algebraic stacks adopted below.)

(1.5.1) **Definition**. Let $\mathfrak{S}$ denote an algebraic stack and let $\mathcal{O}_{\mathfrak{S}}$ denote its structure sheaf. A complex of quasi-coherent $\mathcal{O}_{\mathfrak{S}}$-modules on $\mathfrak{S}_{\underline{smt}}$ is *strictly pseudo-coherent* (*strictly perfect*) if it is a strictly bounded above complex (bounded complex, respectively) of vector bundles. A complex of quasi-coherent $\mathcal{O}_{\mathfrak{S}}$-modules $\mathcal{E}^\bullet$ is *pseudo-coherent (perfect)* if it is locally quasi-isomorphic on the site $\mathfrak{S}_{\underline{smt}}$ (i.e. there exits a $u : U \rightarrow \mathfrak{S}$ in the site $\mathfrak{S}_{\underline{smt}}$ so that the restriction of $\mathcal{E}^\bullet$ to U is quasi-isomorphic) to a strictly pseudo-coherent (perfect, respectively) complex.

For each $u : U \rightarrow \mathfrak{S}$ in the site $\mathfrak{S}_{res.\underline{smt}}$ $(\mathfrak{S}_{\underline{smt}})$, let $Pseudocoh_{bdd}(U)$ $(Perf(U))$ denote the category of pseudo-coherent complexes with globally bounded cohomology (perfect complexes, respectively) on the algebraic space U. One may provide each of these categories with the structure of a complicial Waldhausen category (in the sense of [T-2]) by taking the cofibrations (weak-equivalences) to be maps of complexes that are degree-wise split injective (quasi-isomorphisms, respectively). This defines a lax-functor from the site $\mathfrak{S}_{res.\underline{smt}}$ $(\mathfrak{S}_{\underline{smt}}$, respectively) to the (large) category of complicial Waldhausen categories. Applying the functor sending a (small) complicial Waldhausen category to the corresponding space of algebraic K-theory, one obtains lax-functors $\mathbf{G} : \mathfrak{S}_{res.\underline{smt}} \rightarrow (spaces)$ $(\mathbf{K} : \mathfrak{S}_{\underline{smt}} \rightarrow (spaces)$, respectively) that sends $u : U \rightarrow \mathfrak{S}$ to $K(Pseudocoh_{bdd}(U))$, $K(Perf(U))$, respectively. We will *rigidify* these lax functors and obtain quasi-isomorphic (i.e. weakly equivalent) presheaves on the site $\mathfrak{S}_{\underline{smt}}$ $(\mathfrak{S}_{res.\underline{smt}}$, respectively) with values in the category of spaces. The corresponding presheaves will be denoted $\mathbf{G}$ and $\mathbf{K}$, respectively.

(1.5.2) Moreover, if T is a closed algebraic sub-stack of $\mathfrak{S}$, one defines $\mathbf{K}_T(\mathfrak{S})$ $(\mathbf{G}_T(\mathfrak{S}))$ as the homotopy fiber of the obvious restriction $\mathbf{K}(\mathfrak{S}) \rightarrow \mathbf{K}(\mathfrak{S} - T)$ $(\mathbf{G}(\mathfrak{S}) \rightarrow \mathbf{G}(\mathfrak{S} - T)$, respectively).

(1.6.1) *Remark*. On any Noetherian separated scheme with an ample family of line bundles, it is shown in [T-2] that the K-theory of perfect complexes is weakly equivalent to the K-theory of the symmetric monoidal category of locally free coherent sheaves. Moreover, it is shown there that, on any Noetherian separated scheme, the K-theory of pseudo-coherent complexes with bounded cohomology is weakly equivalent to the K-theory of the symmetric monoidal category of coherent sheaves. Therefore, as presheaves on the smooth site of an algebraic stack, the above presheaves may be replaced (up to stalkwise weak-equivalence) by the presheaves of the K-theory of vector bundles and the K-theory of coherent sheaves.

The following result shows that the correct version of algebraic K-theory for algebraic stacks is the one adopted above in terms of perfect complexes.

(1.6.2) **Theorem**(Poincaré duality). Let $\mathfrak{S}$ denote a *smooth* algebraic stack of finite type over k. Now the obvious map $\mathbf{K}(\mathfrak{S}) \to \mathbf{G}(\mathfrak{S})$ is a weak-equivalence.

Proof. Exactly the same proof as in (3.21) Theorem of [T-2] applies. However, for the sake of completeness we will sketch an outline of the proof. Recall the algebraic stack $\mathfrak{S}$ is assumed to be of finite type over a field and therefore automatically quasi-compact and Noetherian. By fixing a geometric point $\bar{x}$ of $\mathfrak{S}$ and by restricting to a small enough smooth neighborhood of $\bar{x}$, we may assume that $E^\bullet$ is a strictly pseudo-coherent complex with $\mathcal{H}^n(E^\bullet) = 0$ for $n \leq k$. Now $E^{n-2} \xrightarrow{d^{n-2}} E^{n-1} \to Z^n(E^\bullet) \to 0$ is exact, (where $Z^n(E^\bullet)$ denotes the cycles in degree n) showing $Z^n(E^\bullet)_{\bar{x}}$ is finitely presented for any $n \leq k$. Let the tor-dimension of $Z^k(E^\bullet)_{\bar{x}}$ over $\mathcal{O}_{\mathfrak{S},\bar{x}}$ be p. Let $\tau_{\geq k-p}E^\bullet$ be the complex defined by $(\tau_{\geq k-p}E^\bullet)^i = E^i$, $i > k-p$, $(\tau_{\geq k-p}E^\bullet)^i = Z^{k-p}(E^\bullet)$ if $i = k-p$ and $(\tau_{\geq k-p}E^\bullet)^i = 0$ for $i < k-p$. The hypothesis that $\mathcal{H}^n(E^\bullet) = 0$ for all $n \leq k$ shows that the obvious map $\tau_{\geq k-p}E^\bullet \to E^\bullet$ is a quasi-isomorphism and the part of $\tau_{\geq k-p}E^\bullet$ in degrees $\leq k-1$ is a resolution of $Z^k(E^\bullet)$. Therefore it suffices to show that $Z^{k-p}(E^\bullet)_{\bar{x}}$ is flat and finitely presented and hence a free $\mathcal{O}_{\mathfrak{S},\bar{x}}$-module. This follows by the assumption that the tor-dimension of $Z^k(E^\bullet)$ is p. $\square$

2. The extension technique

In this section we will discuss a technique for extending presheaves and maps between presheaves defined on the site $(qp.schemes/k)_{Res.Zar}$ to the étale site of all algebraic spaces locally of finite type over k and to the site $\mathfrak{S}_{res.\underline{smt}}$ associated to any algebraic stack $\mathfrak{S}$ locally of finite type over k. This will be used in the next sections to construct Chern classes from higher algebraic K-theory with values in various cohomology theories and to obtain a comparison with hypercohomology defined with respect to the motivic complexes. (In turn this will construct an intersection pairing for all smooth algebraic stacks at the level of our higher Chow groups and also establish contravariant functoriality for maps between smooth stacks at the level of our higher Chow groups.) Apart from its application in later sections of the paper, this technique may be of independent interest as it enables one to extend readily results from schemes to algebraic stacks.

(2.1.1) Let $\mathcal{C}$ denote a site closed under finite fibered products and arbitrary (small) sums, let $\mathcal{C}'$ denote a full sub-category of $\mathcal{C}$ that is closed under finite products in $\mathcal{C}$ and let $X \in \mathcal{C}$. We assume $\mathcal{C}'$ is a site so that if U belongs to $\mathcal{C}'$ and $\{u_i : U_i \to U | i\}$ is a covering in $\mathcal{C}'$, it is also a covering of U in $\mathcal{C}$. Let $\mathcal{U} = \{U_i \to X | i \in I\}$ and $\mathcal{V} = \{V_j \to X | j \in J\}$ be two coverings of the object X with each $U_i \to X$ and $V_j \to X$ in $\mathcal{C}$. A map of covers $\Phi : \mathcal{U} \to \mathcal{V}$ consists of a function $\phi : J \to I$ and for each $j \in J$ a morphism $f_j : V_j \to U_{\phi(j)}$ compatible with the projection to X. We will further assume that for every object U' in $\mathcal{C}'$, every $\mathcal{C}$-covering of U' has a refinement by a $\mathcal{C}'$-cover. ($\mathcal{V}$ is a refinement of $\mathcal{U}$ if there exists a map of covers $\mathcal{U} \to \mathcal{V}$.) Let P denote a presheaf on $\mathcal{C}'$ with values in a complete pointed simplicial category $\mathbf{S}$. We let $cosk_0(\mathcal{U}) = cosk_0^X(\sqcup_i U_i)$ denote the simplicial object defined in the usual manner.

(2.1.2) In the examples (2.3.1) and (2.3.2) below, all the structure maps of this simplicial object belongs to $\mathcal{C}$ whereas in the example (2.3.3) only the face maps belong to $\mathcal{C}$. In the former case $\Gamma(cosk_0(\mathcal{U}), P)$ is a cosimplicial object in $\mathbf{S}$. In the latter case we will assume that the category $\mathbf{S}$ is the category of co-chain complexes over a Noetherian ring. Then $\Gamma(cosk_0(\mathcal{U}), P)$ is a co-cochain complex trivial in negative degrees with the differential $\delta = \Sigma_i (-1)^i d^{i*}$. Now we apply a denormalizing functor DN as in ([J-1](3.6.3)) to obtain a cosimplicial object $DN(\Gamma(cosk_0(\mathcal{U}), P))$. We define $\check{\mathbb{H}}(\mathcal{U}, P) = \underset{\Delta}{\text{holim}} DN(\Gamma(cosk_0(\mathcal{U}), P))$ in this case. $\Gamma(cosk_0(\mathcal{V}), P)$ and $\check{\mathbb{H}}(\mathcal{V}, P)$ are defined similarly. One may readily verify that if $\Phi : \mathcal{U} \to \mathcal{V}$ is a map of covers, then there is an obvious induced map $\Gamma(cosk_0(\mathcal{U}, P)) \to \Gamma(cosk_0(\mathcal{V}), P)$ of cosimplicial objects in $\mathbf{S}$. In the setting of (2.3.1) and (2.3.2), if $\mathcal{A}$ is a filtered direct system of covers $\mathcal{U}_\alpha \in \mathcal{C}'$ of X, $\alpha \in \mathcal{A}$, we let $\check{\mathbb{H}}(\mathcal{A}; P) = \underset{\Delta}{\text{holim}} \underset{\mathcal{U}_\alpha \in \mathcal{A}}{\text{colim}} \Gamma(cosk_0(\mathcal{U}_\alpha), P)$. In the setting of (2.3.3), we let $\check{\mathbb{H}}(\mathcal{A}, P) = \underset{\Delta}{\text{holim}} \underset{\mathcal{U}_\alpha \in \mathcal{A}}{\text{colim}} DN\Gamma(cosk_0(\mathcal{U}_\alpha), P)$.

Assume the situation in (2.1.1). Now observe that the category of all coverings of a given object $X \in \mathcal{C}$ is *not filtered in general.* (This is filtered if there exists at most one map between two coverings. For example if X is a scheme, $\mathcal{C}'$ =the sub-category of quasi-projective schemes and the coverings are all in the Zariski topology of X, then the category of coverings of X by quasi-projective schemes is a filtered direct system. However coverings in the étale topology do not, in general, form a filtered category.) To remedy this we adopt the technique that restricts maps between coverings so that they form a filtered direct system.

(2.2.1) **Definition**. *The canonical filtered direct system of covers.* Assume the situation in (2.1.1). Let $\{\mathcal{U}_\lambda | \lambda \in \Lambda\}$ denote $Covers_{\mathcal{C}'}(X)$. We will assume this category is skeletally small. (Observe that this hypothesis is satisfied in all the applications we consider - see (2.3) below.) Let $\mathcal{F}$ denote the category of all finite subsets of Λ. Given such an F in $\mathcal{F}$, we define $\mathcal{U}_F = \mathcal{U}_{f_1} \times \mathcal{U}_{f_2} \times \ldots \times \mathcal{U}_{f_n}$, if $F = \{f_1, f_2, ..., f_n\}$ (i.e. with respect to some ordering of the elements of F and the product $\times$ denotes the fibered product over $\mathfrak{S}$.) If $F' \subseteq F$, the inclusion of F' into F induces an obvious projection $\mathcal{U}_F \rightarrow \mathcal{U}_{F'}$. Let $DCovers_{\mathcal{C}'}(X) = \{\mathcal{U}_F | F = \text{a finite subset of } \Lambda\}$ with a morphism $\mathcal{U}_F \rightarrow \mathcal{U}_{F'}$ defined as above. Clearly $DCovers_{\mathcal{C}'}(X)$ is a filtered direct system indexed by $\mathcal{F}$, since there is at most one map between any two objects.

(2.2.2) Observe that if X and Y are objects in $\mathcal{C}$ and $f : X \rightarrow Y$ is any map in $\mathcal{C}$, one obtains an induced map of direct systems $DCovers_{\mathcal{C}'}(Y) \rightarrow DCovers_{\mathcal{C}'}(X)$ that is natural in f. Observe also that if P is a presheaf on $\mathcal{C}'$ taking values in a complete pointed simplicial category, one obtains an induced map $\check{\mathbb{H}}(DCovers_{\mathcal{C}'}(Y), P) \rightarrow \check{\mathbb{H}}(DCovers_{\mathcal{C}'}(X), P)$ that is also natural in P.

(2.3) **Examples**. The main examples we will be interested in are the following:

(2.3.1) $\mathcal{C} = (schemes/k)_{Res.Zar}$, $\mathcal{C}' = (qp.schemes/k)_{Res.Zar}$. Now $Res1 : \mathcal{C} \rightarrow \mathcal{C}'$ will denote the obvious map of sites. If X is a scheme, $Covers_{qpschms}(X_{Zar})$ will denote the directed set of all coverings of X in the Zariski topology by quasi-projective schemes over k.

(2.3.2) $\mathcal{C} = (alg.spaces/k)_{Res.et}$, $\mathcal{C}' = (separated \quad schemes/k)_{Res.et}$. The coverings in the first (second) site are all *étale coverings* $y : Y \rightarrow X$, with both Y and X algebraic spaces (with both Y and X schemes.) Now $Res2 : \mathcal{C} \rightarrow \mathcal{C}'$ will denote the obvious map of sites. Given an algebraic space X, we will let $DCovers_{s.schms}(X_{et})$ denote the *canonical* directed set of étale coverings of X by *separated* schemes. As observed above this category is filtered: it is also skeletally small in view of our assumption that the algebraic spaces are locally of finite type over k. (The hypothesis of separation ensures that if $U \rightarrow X$ is an étale surjective map from a separated scheme, $cosk_0^X(U)$ is a simplicial scheme.)

(2.3.3) $\mathcal{C} = \mathcal{C}' = (alg.spaces/k)_{Res.smt}$. Next we consider the category of all atlases for a given algebraic stack $\mathfrak{S}$. Even-though the stack $\mathfrak{S}$ does not belong to the category $\mathcal{C}$, the definitions in (2.2.1) and (2.2.2) provide a *canonical filtered direct system of atlases* for $\mathfrak{S}$. This will be denoted $Datlases(\mathfrak{S})$. (i.e. In (2.2.1) let $Covers_{\mathcal{C}'}(\mathfrak{S})$ denote the category of all atlases for $\mathfrak{S}$. Since $\mathfrak{S}$ is assumed to be locally of finite type over k, this category is also skeletally small.)

(2.4.1) **Definition: the extension functors**. (i) Let P denote a presheaf on the site $(qp.schemes/k)_{Res.Zar}$ with values in a category $\mathbf{S}$ as before. We define its extension $E1(P)$ to $(schemes/k)_{Res.Zar}$ be the presheaf defined by $\Gamma(U, E1(P)) = \check{\mathbb{H}}(Covers_{qp.schms}(U_{Zar}), P)$.

(ii) Let Q denote a presheaf on the site $(schemes/k)_{Res.et}$. We define its extension $E2(Q)$ to $(alg.spaces/k)_{Res.et}$ be the presheaf defined by $\Gamma(U, E2(Q)) = \check{\mathbb{H}}(DCovers_{s.schms}(U_{et}), Q)$.

(2.4.2) **Remarks**. (i) Observe that if P is a presheaf on $(smt.qp.schemes/k)_{Res.Zar}$ (on $(smt.schemes/k)_{Res.et}$) the same extension functors provides an extension of P to $(smt.schemes/k)_{Res.Zar}$ ($(smt.alg.spaces/k)_{Res.et}$, respectively). This follows readily, because for a smooth object the filtered direct system of covers that is involved in the extension also consists of smooth objects.

(ii)Observe that in the above examples, $\mathcal{C}'$ is a full sub-category of $\mathcal{C}$. However these categories are *not* small categories; this makes the technique of *canonical Kan extensions* not applicable and justifies the definition of the extension functors as above.

(2.5) **Theorem**. Assume the above situation. (i) Then both the extension functors preserve strong quasi-isomorphisms.

(ii) There exist natural maps of presheaves $P \rightarrow Res1_*(E1(P))$ and $Q \rightarrow Res2_*(E2(Q))$. The second is a quasi-isomorphism always while the first is one for presheaves P that have cohomological descent on the Zariski site of affine schemes over k.

(iii) If P (Q) is already a presheaf on $(schemes/k)_{Res.Zar}$ ($(alg.spaces/k)_{Res.et}$) then there is a natural map $P \rightarrow E1(Res1_*(P))$ ($Q \rightarrow E2(Res2_*(Q))$). The first is a quasi-isomorphism for all presheaves that have cohomological descent on the Zariski site of affine schemes. The second is always a quasi-isomorphism under the same hypotheses as in (ii).

(iv) If P (Q) is contravariant for flat maps (for arbitrary maps), so is the extended presheaf $E1(P)$ ($E2(Q)$, respectively). In fact, if $f : X \rightarrow Y$ is a flat map of schemes, there exists an induced map $E1(P)_{|Y} \rightarrow$

$f_*(E1(P)_{|X})$ that is natural in the map f, where $E1(P)_{|Y}$ ($E1(P)_{|X}$) denotes the restriction of $E1(P)$ to the Zariski site of the scheme Y (X, respectively).) A corresponding assertion holds for the functor $E2$ and for arbitrary maps if P is contravariant for arbitrary maps.

(v) All the above assertions holds for presheaves defined on the corresponding sites of smooth objects.

Proof. (i) This follows from the observation that filtered colimits (see [J-1] (6.1.6)) and homotopy inverse limits preserve quasi-isomorphisms.

(ii) Observe that $Res1_*$ and $Res2_*$ are simply the appropriate restriction functors. Given any object U in the category $\mathcal{C}'$ and any covering $\mathcal{U} = \{U_i|i\}$ of it in $\mathcal{C}'$, there exists a natural map of cosimplicial objects in $\mathbf{S}$: $\Gamma(U,P) \to \Gamma(cosk_0(\mathcal{U}),P)$. Since this is natural in $\mathcal{U}$ one may take the direct limit of the constant system $\{\Gamma(U,P)|\mathcal{U}\}$ and of the direct system $\{\Gamma(cosk_0(\mathcal{U}),P)|\mathcal{U}\}$ as $\mathcal{U}$ varies in the category of all Zariski covers of U by quasi-projective schemes over k to obtain the map $P \to Res1_*(E1(P))$. One may similarly take the direct limit over all coverings of U by schemes in the directed system $DCovers_{s.schms}(U_{et})$ to obtain the map $Q \to Res2_*(E2(Q))$. The existence of the natural maps in (iii) also follows similarly.

To see that these are stalk-wise quasi-isomorphisms, one considers each case separately. In the first case it suffices to show that if U is a quasi-affine scheme over k, the natural map $\Gamma(U,P) \to \check{\mathbb{H}}(Covers_{qp.schms}(U_{Zar});P)$ is a quasi-isomorphism. This follows from [J-1](3.7.4)(ii) since P is assumed to have cohomological descent on the Zariski site of U and the Zariski site of U (= a quasi-affine scheme over k) has finite cohomological dimension with respect to abelian sheaves. In the second case observe that if U is a quasi-affine scheme over k, one obtains from [J-1](3.7.5) a natural quasi-isomorphism $\check{\mathbb{H}}(DCovers_{s.schms}(U_{et}),P) \simeq \mathbb{H}_{et}(U,P)$. (Recall that P is always additive by assumption.) Now it suffices to show that the natural map $P_{\bar{x}} \to \underset{\to}{\lim}\mathbb{H}_{et}(U,P)$ is a quasi-isomorphism where the last colimit is over all étale neighborhoods of any given geometric point $\bar{x}$ of X. Since the natural map $\mathbb{H}_{et}(U,P) \to \underset{\infty \leftarrow n}{\lim}\, \mathbb{H}_{et}(U,\tau_{\leq n}P)$ is a quasi-isomorphism by [J-1] (3.4.1) with $\phi = \Gamma$, one may now reduce to proving this for abelian sheaves where it is clear.

(iv) Let $f : X \to Y$ denote a flat map (arbitrary map). Given any cover $\mathcal{U} = \{U_i|i\}$ of Y, the inverse image $f^{-1}(\mathcal{U}) = \{U_i \underset{Y}{\times} X|i\}$ is a cover of X. Even if $\mathcal{U}$ is an open cover by Zariski open quasi-affine schemes (an étale cover by schemes) the inverse image need not satisfy the same condition, but may be refined to covers that are by quasi-affine open sub-schemes (to étale covers by schemes, respectively).

Let V be an object in the site associated to Y and let $U = V \underset{Y}{\times} X$. It is clear that we obtain an induced a map
$f^* : \check{\mathbb{H}}(Covers_{qp.schms}(V_{Zar}),P) \to \check{\mathbb{H}}(Covers_{qpschms}(U_{Zar},P)$
($f^* : \check{\mathbb{H}}(DCovers_{s.schms}(V_{et}),Q) \to \check{\mathbb{H}}(DCovers_{s.schms}(U_{et}),Q)$, respectively.)

Moreover, this map is natural in f. This essentially proves all the statements in (iv). Now the last assertion is clear since the filtered direct limits in the definition of the extension functors involve only smooth objects. □

Remark. Observe that the extension functor $E1$, in fact, may be viewed as a functor

$E1 : Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S}) \to Presh((schemes/k)_{Res.et}, \mathbf{S})$.

This follows from the observation that if P is a presheaf on the site $(qp.schemes/k)_{Res.Zar}$, $E1(P)$, by definition is a presheaf on $(schemes/k)_{Res.Zar}$ and hence is contravariant for all flat maps between schemes. Therefore, $E1(P)$, in fact, is a presheaf on the étale site of a given scheme (which has only étale maps as morphisms): hence $E1$ does extend presheaves to the site $(schemes/k)_{Res.et}$.

(2.6.1) **Proposition**. Let $E' = E2 \circ E1 : Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S}) \to Presh((alg.spaces/k)_{Res.et}, \mathbf{S})$.
(i) If X is an algebraic space locally of finite type over k, and $P \xrightarrow{\simeq} P'$ is a quasi-isomorphism of presheaves on $(qp.schemes/k)_{Res.Zar}$ both of which have cohomological descent on the Zariski site of any affine scheme over k, one obtains an induced quasi-isomorphism $\mathbb{H}_{et}(X,E'(P)) \xrightarrow{\simeq} \mathbb{H}_{et}(X,E'(P'))$.

(ii) Let $R' = Res1 \circ Res2 : (alg.spaces/k)_{Res.et} \to (qp.schemes/k)_{Res.Zar}$ denote the composition of the two restriction functors $Res2$ and $Res1$. Then the natural map $P \to E'(R'(P))$ induces a quasi-isomorphism $\mathbb{H}_{et}(X,P) \xrightarrow{\simeq} \mathbb{H}_{et}(X,E'(R'(P)))$ provided P has cohomological descent on the Zariski site of any affine scheme.

(iii) Any pairing $P \otimes P' \to P''$ of presheaves in $Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S})$ extends functorially to a pairing $E'(P) \otimes E'(P') \to E'(P'')$.

(iv) The corresponding assertions hold for presheaves defined only on smooth objects.

Proof. (i) It follows from (2.5)(ii) that the induced map $\Gamma(U, E'(P)) \to \Gamma(U, E'(P'))$ is a quasi-isomorphism for any quasi-affine scheme U over k. Therefore the map $E'(P) \to E'(P')$ is a quasi-isomorphism of presheaves and hence the induced map $\mathbb{H}_{et}(X, E'(P)) \xrightarrow{\simeq} \mathbb{H}_{et}(X, E'(P'))$ is also a quasi-isomorphism.

(ii) follows immediately from (2.5)(iii) and (iii) follows readily from the definition of the extension functor E'. (iv) follows as in the proof of (2.5)(v). $\square$

(2.6.2) **Corollary**. Let $\mathfrak{S}$ denote a fixed algebraic stack locally of finite type over k. Then there exist functors

$E_{\mathfrak{S}} : Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S}) \to Presh(\mathfrak{S}_{res.\underline{smt}}, \mathbf{S})$

$E_{\mathfrak{S}} : Presh((schemes/k)_{Res.et}, \mathbf{S}) \to Presh(\mathfrak{S}_{res.\underline{smt}}, \mathbf{S})$

having the following properties.

(i) If $u : U \to \mathfrak{S}$ is an object in $\mathfrak{S}_{res.\underline{smt}}$ with U an affine scheme and $P \,\epsilon\, Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S})$ has cohomological descent on the Zariski site of any affine scheme over k, the natural map $P_{|U_{et}} \to E_{\mathfrak{S}}(P)_{|U_{et}}$ is a quasi-isomorphism. A similar conclusion holds for the second extension functor with respect to any U in $\mathfrak{S}_{res.\underline{smt}}$ which is a scheme.

(ii) Let $P' \xrightarrow{\phi} P$ denote a map of presheaves on $(schemes/k)_{Res.Zar}$ which is a stalkwise quasi-isomorphism. If $E_{\mathfrak{S}}$ is the second functor, the induced map $E_{\mathfrak{S}}(\phi)$ induces a stalk-wise quasi-isomorphism on restriction to the étale site of any algebraic space U in $\mathfrak{S}_{res.\underline{smt}}$. Similar conclusions hold for the first functor if P' and P are presheaves on $(qp.schemes/k)_{Res.et}$ that have cohomological descent on the Zariski site of any affine scheme.

(iii) If P is a presheaf on $(alg.spaces/k)_{Res}$ that has cohomological descent on the Zariski site of any affine scheme and $P_{\mathfrak{S}}$ denotes the restriction of P to $\mathfrak{S}_{res.\underline{smt}}$, there exists a natural map $P_{\mathfrak{S}} \to E_{\mathfrak{S}}(R'(P))$ which is a quasi-isomorphism on restriction to the étale site of any algebraic space in the site $\mathfrak{S}_{res.\underline{smt}}$.

(iv) Let $f : \mathfrak{S}' \to \mathfrak{S}$ denote a flat map of algebraic stacks. Let P denote an object in $Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S})$ or in $Presh((schemes/k)_{Res.et}, \mathbf{S})$. Then there exists an induced map $E_{\mathfrak{S}}(P) \to Rf_* E_{\mathfrak{S}'}(P)$. In case P denotes an object in $Presh((qp.schemes/k)_{Zar}, \mathbf{S})$ or in $Presh((schemes/k)_{et}, \mathbf{S})$ and $f : \mathfrak{S}' \to \mathfrak{S}$ is an arbitrary *representable* map between algebraic stacks, there exists an induced map $E_{\mathfrak{S}}(P) \to Rf_* E_{\mathfrak{S}'}(P)$.

(v) The same extension functors define extensions with similar properties at the level of the corresponding unrestricted sites.

(vi) If $\mathfrak{S}$ is a smooth algebraic stack, there exist extension functors

$E_{\mathfrak{S}} : Presh((smt.qp.schemes/k)_{Res.Zar}, \mathbf{S}) \to Presh(\mathfrak{S}_{res.\underline{smt}}, \mathbf{S})$ and

$E_{\mathfrak{S}} : Presh((smt.schemes/k)_{Res.et}, \mathbf{S}) \to Presh(\mathfrak{S}_{res.\underline{smt}}, \mathbf{S})$ and

with similar properties. If, in either case, P denotes a presheaf on the corresponding unrestricted sites, the extension functors send P to a presheaf on $\mathfrak{S}_{\underline{smt}}$.

(vii) If $P \otimes P' \to P''$ is a pairing of presheaves either in $Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S})$, $Presh((qp.schemes/k)_{Res.et}, \mathbf{S})$ or the corresponding sites of smooth objects, there exists an induced pairing: $E_{\mathfrak{S}}(P) \otimes E_{\mathfrak{S}}(P') \to E_{\mathfrak{S}}(P'')$.

Proof. Recall $\mathfrak{S}_{res.\underline{smt}}$ has as objects all maps $x : X \to \mathfrak{S}$ that are smooth with X an algebraic space locally of finite type over k. Moreover, the morphisms in this site are restricted to be flat maps. Therefore this is a sub-category of the big site $(alg.spaces/k)_{Res.et}$ and any presheaf on the latter category defines by restriction a presheaf on $\mathfrak{S}_{res.\underline{smt}}$. Starting with a presheaf in $Presh((qp.schemes/k)_{Res.Zar}, \mathbf{S})$, the extension functor E' in (2.6.1) now provides the required extension of P to a presheaf on $(alg.spaces/k)_{Res.et}$. To see this, let P denote a presheaf on $(qp.schemes/k)_{Res.Zar}$. Now $E1(P)$ is a presheaf on $(schemes/k)_{Res.Zar}$. Since P is contravariant for flat maps so is $E1(P)$ which shows $E1(P)$ is in fact a presheaf on $(schemes/k)_{Res.et}$. Now $E'(P) = E2(E1(P))$ defines a presheaf on $(alg.spaces/k)_{Res.et}$. This defines the presheaf $E_{\mathfrak{S}}(P)$ by restriction to $\mathfrak{S}_{res.\underline{smt}}$. The remaining properties (i) through (iv) are all clear for the first extension functor $E_{\mathfrak{S}}$ from (2.5). The second extension functor is similarly defined by just $E2$ and the corresponding properties are again clear by (2.5). The fifth assertion is clear from the definition of the extension functors adopted above while the last two are clear from the above propositions. $\square$

3. (Lichtenbaum) Motivic cohomology vs. the higher Chow groups for smooth algebraic stacks

Much of the difficulty with the higher cycle complexes, for example, the difficulty with general contravariant functoriality or the difficulty with moving lemmas, may be circumvented, at least for all smooth schemes, by comparison with the motivic complexes. This follows from rather recent comparison theorems relating the higher Chow groups with motivic cohomology. (See [Fr-S] and [Voev].) This comparison, may be extended from smooth quasi-projective schemes to all smooth algebraic stacks using the extension techniques developed in the last section. It has to be pointed out that such a comparison was not available even for all smooth varieties until recently and therefore did not make it into the first version of this paper. We thank Spencer Bloch for first bringing such a comparison to our attention. We will restrict to smooth algebraic stacks throughout this section.

(3.0.1) For each integer $i \geq 0$, let $\mathbb{Z}(i)[2i]$ denote the shifted motivic complex of weight i defined on $(smt.qp.schemes/k)_{Zar}$. Similarly let $\mathcal{Z}^i(\quad,.)$ denote the pre-sheafification of the codimension i higher cycle complex on $(qp.schemes/k)_{Res.Zar}$. If $\mathfrak{S}$ denotes a smooth algebraic stack, we let $E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])$ and $E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.))$ denote the extension functor

$$E_{\mathfrak{S}} : Presh((smt.qp.schemes)_{Zar}, \mathbf{S}) \rightarrow Presh(\mathfrak{S}_{\underline{smt}}, \mathbf{S})$$

(see (2.6.2), where $\mathbf{S}$ = the category of all complexes of abelian groups as [J-1] (6.2.4)) applied to the above complexes. The former will be called the (shifted) *motivic complex of weight i on the stack* $\mathfrak{S}$: by (2.6.2)(vi) this defines a complex of sheaves on the smooth site $\mathfrak{S}_{\underline{smt}}$. We will let

(3.0.2) $H^{2i-n}_{\mathcal{M}}(\mathfrak{S}, i) = \pi_n(\mathbb{H}_{smt}(\mathfrak{S}, E_{\mathfrak{S}}(\mathcal{Z}(i)[2i])))$

and call it *the (Lichtenbaum) motivic cohomology of the stack* $\mathfrak{S}$. We define

(3.0.3) $\mathbf{CH}^i(\mathfrak{S}, x, .) = \mathbb{H}_{et}(B_x\mathfrak{S}, E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.)))$, $CH^i(\mathfrak{S}, x, n) = \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.))))$

for an atlas $x : X \rightarrow \mathfrak{S}$. Similarly we let

(3.0.3)' $\mathbf{CH}^i(X, .) = \mathbb{H}_{Zar}(X, E1(\mathcal{Z}^i(.\quad,.)))$, $i \geq 0$

where $E1 : (qp.schemes/k)_{Res.Zar} \rightarrow (schemes/k)_{Res.Zar}$ is the extension functor defined in (2.4.1).

Remark. An alternative to the above approach would be to extend the definition of the motivic complexes from schemes to algebraic spaces in the obvious manner (i.e. define a sheaf $\mathbb{Z}_{eq}(\mathbb{A}^n)$ on $(smt.alg.spaces/k)$ as in (A.3.1)) and use this to define the motivic cohomology of stacks. However, the use of the extension functor in the above definition seems to simplify proving such comparison results as in (3.1) Theorem, below.

(3.0.4) Let $\mathfrak{S}$ denote an algebraic stack of finite dimension. For each fixed non-negative integer i, presently we define an *additive* presheaf $\mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(\quad, n)$ on $\mathfrak{S}_{res.smt}$ by $\mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(U, n)$ = the free abelian group on codimension i cycles on $U \times \Delta[n]$ intersecting all the faces of $U \times \Delta[n]$ properly, for each $u : U \rightarrow \mathfrak{S}$ in $\mathfrak{S}_{res.smt}$ and also *connected*. We let $\mathcal{Z}^*_{\mathfrak{S}_{res.smt}}(\quad,.) = \underset{i\geq 0}{\oplus} \mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(\quad,.)$. (Similarly, if X is a scheme of finite dimension over k, we define an additive presheaf $\mathcal{Z}^i_{X_{Zar}}(.\quad, n)$ for each $i, n \geq 0$ on the Zariski site X_{Zar}. We let $\mathcal{Z}^*_{X_{Zar}}(.\quad,.) = \underset{c\geq 0}{\oplus} \mathcal{Z}^i_{X_{Zar}}(.\quad,.)$.)

(3.1) **Theorem**. (i) Let $\mathfrak{S}$ denote a smooth algebraic stack and let $x : X \rightarrow \mathfrak{S}$ denote a given atlas. Then there exists a quasi-isomorphism natural in $\mathfrak{S}$:

$$\mathbb{H}_{smt}(\mathfrak{S}, E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])) \simeq \mathbf{CH}^i(\mathfrak{S}, x, .).$$

Therefore one obtains the isomorphism $H^{2i-n}_{\mathcal{M}}(\mathfrak{S}, i) \cong CH^i(\mathfrak{S}, x, n)$ for all $i \geq 0$ and all n.

(ii) Let $\mathfrak{S}$ denote a smooth algebraic stack of finite dimension and let $x : X \rightarrow \mathfrak{S}$ denote a given atlas. Then there exists a quasi-isomorphism natural in $\mathfrak{S}$:

$$\mathbf{CH}^i(\mathfrak{S}, x, .) \simeq \mathbb{H}_{et}(B_x\mathfrak{S}, \mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(\quad,.))$$

Proof. Throughout the proof we will let $\mathcal{Z}^*_{(alg.spaces/k)_{Res.et}}(\quad.,.)$ denote the *additive* presheaf $U \rightarrow \mathcal{Z}^*(U, .)$ defined on the site $(alg.spaces/k)_{Res.et}$ by $\Gamma(U, \mathcal{Z}^*(\quad,.)) = \underset{i}{\oplus} z^i(U, .)$. (Here $z^i(U, .)$ is the higher cycle complex of codimension i cycles and U is connected.) Now the extension is provided by (2.6.1) and (2.6.2). One begins with the presheaves $\mathcal{Z}^i(\quad,.)$, $\mathbb{Z}(i)$ on $(smt.qp.schemes/k)_{Res.Zar}$; (A.3.3) and (A.5) in the appendix show that one obtains a natural quasi-isomorphism $\mathcal{Z}^i(\quad,.) \simeq \mathbb{Z}(i)[2i]$ on $(smt.qp.schemes/k)_{Res.Zar}$. Next one applies the extension functor $E_{\mathfrak{S}}$ from (2.6.2) to obtain a quasi-isomorphism $E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.)) \simeq E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])$ of presheaves on restriction to the étale site of any object $U \epsilon \mathfrak{S}_{res.\underline{smt}}$ and for any smooth algebraic stack $\mathfrak{S}$. Since we obtain a stalk-wise quasi-isomorphism $E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.)) \simeq E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])$ (on restriction to the étale site of any object $U \epsilon \mathfrak{S}_{res.\underline{smt}}$) we obtain the quasi-isomorphism:

$$\mathbb{H}_{et}(B_x\mathfrak{S}, E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])) \simeq \mathbb{H}_{et}(B_x\mathfrak{S}, E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.)))$$

However, observe from (2.6.2)(vi) that the complex of sheaves $E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])$ is in fact a complex of sheaves on the *smooth* site of the stack $\mathfrak{S}$. Therefore the comparison theorem in [J-1] (3.6.1) (see also [J-1](3.6.4)), shows that the left-hand-side identifies with $\mathbb{H}_{smt}(B_x\mathfrak{S}, E_{\mathfrak{S}}(\mathbb{Z}(i)[2i])) \simeq \mathbb{H}_{smt}(\mathfrak{S}, E_{\mathfrak{S}}(\mathbb{Z}(i)[2i]))$ proving the first assertion.

Now it suffices to prove that the presheaf $E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.))$ is quasi-isomorphic to the presheaf $\mathcal{Z}^i(\quad,.)_{\mathfrak{S}_{res.\underline{smt}}}$, on restriction to the étale site of any object U in $\mathfrak{S}_{res.\underline{smt}}$. However the presheaf $\mathcal{Z}^i_{\mathfrak{S}_{res.\underline{smt}}}(\quad,.)$ is the restriction of the presheaf $\mathcal{Z}^i_{alg.spaces/k_{Res.\underline{et}}}(\quad,.)$ and the latter restricts to $\mathcal{Z}^i(\quad,.)$ on $(smt.qp.schemes/k)_{Res.Zar}$. Therefore (2.6.2)(iii) shows that the presheaf $E_{\mathfrak{S}}(\mathcal{Z}^i(\quad,.))$ is quasi-isomorphic to $\mathcal{Z}^i_{\mathfrak{S}_{res.\underline{smt}}}(\quad,.)$ on restriction to the étale site of any $u: U \to \mathfrak{S}$ in $\mathfrak{S}_{res.\underline{smt}}$. This proves the second assertion. □

Remark. One also obtains a quasi-isomorphism in the case of (3.0.3)': $\mathbb{H}_{Zar}(X, \mathbb{Z}(i)[2i]) \simeq \mathbb{H}_{Zar}(X, \mathcal{Z}^i_{X_{Zar}}(\quad,.))$ for all smooth schemes X of finite type over the field k. However this is obvious from [Fr-S] section 11 or [Voev]. Moreover, one also obtains the quasi-isomorphism (for any smooth scheme locally of finite type over k)

$$\mathbb{H}_{Zar}(X, E1(\mathbb{Z}(i)[2i])) \simeq \mathbb{H}_{Zar}(X, E1(\mathcal{Z}^i(\quad,.))).$$

Once we define the motivic cohomology of the scheme X to be given by the left-hand-side, this provides the isomorphism $H^{2i-n}_{\mathcal{M}}(X,i) \cong CH^i(X,n)$ for any smooth scheme locally of finite type over k.

(3.2) **Corollary**. (*Independence on the choice of the atlas*)

The higher Chow-groups $CH^i(\mathfrak{S}, x, n)$ are in fact independent of the choice of the atlas and therefore intrinsic to the stack for all *smooth* algebraic stacks. The higher Chow groups $CH^i(\mathfrak{S}, x, n; \mathbb{Q})$ are independent of the choice of the atlas x for all *stacks of finite type* over the field k.

Proof. The first assertion is clear in view of the above theorem for all smooth stacks. Observe that if d is the dimension of the stack $\mathfrak{S}$, the complex $\mathcal{Z}^i_{\mathfrak{S}_{res.smt}}(\quad,.)$ is isomorphic to the complex $Z^{\mathfrak{S}_{res.smt}}_{d-i}(\quad,.)$. Therefore it suffices to show the higher Chow groups $CH_*(\mathfrak{S}, x, n; \mathbb{Q})$ are independent of the choice of the atlas x for any algebraic stack of finite type over k. We will prove the corollary in this case by Noetherian induction on the stack.

Let $\mathfrak{S}$ denote an algebraic stack of finite type over the given field and let $x: X \to \mathfrak{S}$, $x': X' \to \mathfrak{S}$ denote two atlases. Without loss of generality we may assume there exists a smooth surjective map $\alpha: X' \to X$ over $\mathfrak{S}$. In view of [J-1] (3.6.1)(iii), we may assume the stack is reduced and therefore one may find a *proper closed* sub-stack $\mathfrak{S}_1$ of $\mathfrak{S}$ with $\mathfrak{S}_0 = \mathfrak{S} - \mathfrak{S}_1$ smooth and non-empty. Now one obtains the commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc}
\mathbf{CH}_i(\mathfrak{S}_1, x'_1, .; \mathbb{Q}) & \longrightarrow & \mathbf{CH}_i(\mathfrak{S}, x', .; \mathbb{Q}) & \longrightarrow & \mathbf{CH}_i(\mathfrak{S}_0, x'_0, .; \mathbb{Q}) & \longrightarrow & \mathbf{CH}_i(\mathfrak{S}_1, x'_1, .; \mathbb{Q})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{CH}_i(\mathfrak{S}_1, x_1, .; \mathbb{Q}) & \longrightarrow & \mathbf{CH}_i(\mathfrak{S}, x, .; \mathbb{Q}) & \longrightarrow & \mathbf{CH}_i(\mathfrak{S}_0, x_0, .; \mathbb{Q}) & \longrightarrow & \mathbf{CH}_i(\mathfrak{S}_1, x_1, .; \mathbb{Q})[1]
\end{array}$$

Here $x'_i = x' \underset{\mathfrak{S}}{\times} \mathfrak{S}_i$ and $x_i = x \underset{\mathfrak{S}}{\times} \mathfrak{S}_i$, $i = 1, 2$. The inductive assumption and the first statement of the corollary show the first and last maps are quasi-isomorphisms; the last but one map is a quasi-isomorphism, since the stack $\mathfrak{S}_0$ is smooth. Therefore the second map is also a quasi-isomorphism. □

(3.3) **Remark**. Therefore, for all algebraic stacks satisfying the hypotheses above, we may omit the atlas in the definition of the higher Chow groups.

(3.4) **Corollary** (*Contravariant functoriality for maps between smooth stacks*)

Let $f: \mathfrak{S}' \to \mathfrak{S}$ denote a (possibly non-representable) map between *smooth* algebraic stacks. Then f induces a map

$f^* : CH^i(\mathfrak{S}, .) \to CH^i(\mathfrak{S}', .)$

Proof. Recall the motivic complexes $\mathbb{Z}(i)$ are in fact contravariantly functorial on the category $(smt.qp.schemes/k)$. Therefore (2.6.2)(v) and (vi) show their extensions to stacks are also contravariantly functorial for all representable maps between smooth stacks. This proves the corollary for representable maps.

Now we consider non-representable maps. Let $f : \mathfrak{S}' \to \mathfrak{S}$ denote a possibly non-representable map of algebraic stacks and let $x : X \to \mathfrak{S}$ denote an atlas for $\mathfrak{S}$. Let $X' = X \underset{\mathfrak{S}}{\times} \mathfrak{S}'$; since f is not necessarily representable, X' need not be an algebraic space. Let $X'' \to X'$ denote an atlas for the stack X' and let the composite map $X'' \to X' \to \mathfrak{S}'$ be denoted x''. If $B_x\mathfrak{S}$ ($B_{x''}\mathfrak{S}'$) denotes the classifying simplicial space for $\mathfrak{S}$ associated to x ($\mathfrak{S}'$ associated to x'', respectively), f induces a map $Bf : B_{x''}\mathfrak{S}' \to B_x\mathfrak{S}$ of simplicial spaces. (2.5)(iv) shows that one obtains a map $E(\mathbb{Z}(i)[2i])_{|B_x\mathfrak{S}} \to Bf_*(E(\mathbb{Z}(i)[2i])_{|B_{x''}\mathfrak{S}'})$ of presheaves on $B_x\mathfrak{S}_{et}$. This induces a map $E(\mathbb{Z}(i)[2i])_{|B_x\mathfrak{S}} \to R(Bf)_*(E(\mathbb{Z}(i)[2i])_{|B_{x''}\mathfrak{S}'})$ of presheaves on $B_x\mathfrak{S}_{et}$. Taking hypercohomology on the site $B_x\mathfrak{S}_{et}$, provides us with the required map $f^* : CH^i(\mathfrak{S}, .) \to CH^i(\mathfrak{S}', .)$ $\square$

We proceed to compare our higher Chow groups for quotient stacks associated to the action of linear algebraic groups on smooth quasi-projective schemes with corresponding Totaro-Edidin-Graham equivariant higher Chow groups.

(3.5) **Theorem**. Let G denote a linear algebraic group acting on a smooth quasi-projective scheme X so that the action is G-linearized, i.e. X admits a G-equivariant locally closed immersion into a projective space onto which the G-action extends to a linear action. Then there exists an isomorphism

$CH^*([X/G], .; \mathbb{Q}) \cong CH^*_G(X, .; \mathbb{Q})$

compatible with restriction to open G-stable sub-schemes of X and preserving the ring structures. (Here $CH^*_G(X, .; \mathbb{Q})$ denotes the G-equivariant higher Chow groups as defined in [EG] re-indexed by codimension.)

Proof. We will first show that one may reduce to the case G is a torus (in fact a split torus.) (This will occupy the first half of the proof.) Let H denote a closed algebraic subgroup of G. By faithfully flat descent one may establish an isomorphism between the stacks:

(3.5.1) $[X/H] \overset{\simeq}{\leftarrow} [(X \times G)/(H \times G)] \overset{\simeq}{\rightarrow} [(X \underset{H}{\times} G)/G]$.

The first (second) is induced by the projection to the first factor (the map sending $(x, g) \in X \times G$ to the class of (x, g) in $X \underset{H}{\times} G$, respectively). Clearly these are compatible with restrictions to open G-stable subschemes of X.

Next assume that H is a normal subgroup with $\bar{G} = G/H$ a finite group. Now the obvious map $X \underset{H}{\times} G \to X$ (sending $(x, g) \simeq (x \circ h^{-1}, h \circ g)$ to $x \circ g$) induces a map $[(X \underset{H}{\times} G)/G] \to [X/G]$ which will be a principal $\bar{G} = G/H$ bundle. Therefore, by [J-1] (3.7.11), and (3.5.1), (3.2) above it follows that

(3.5.2) $CH^*([X/H], .; \mathbb{Q}) \cong CH^*([X/G], .; \mathbb{Q})^{\bar{G}}$

where the isomorphism is compatible with restriction to open G-stable subschemes and with the given ring structures. On taking $H = G^o =$ the connected component of the identity, this reduces to the case when G is connected. If $R_u(G)$ is the unipotent radical of G, $\bar{G} = G/R_u(G)$ is reductive. The homotopy property of the equivariant higher Chow groups and our higher Chow groups enables one to now reduce to the case of a reductive group action. Moreover, since the chosen maximal torus splits over some finite Galois extension of the ground field k and we are working with $\mathbb{Q}$-coefficients, one may assume without loss of generality that T is in fact split. (See, for example, [J-1] (3.7.11).)

Let T denote a fixed maximal torus in G, $N(T) = N_G(T) =$ its normalizer in G and $W = N(T)/T =$ its Weyl group. By (3.5.1), we obtain the isomorphisms of the quotient stacks (where $N(T)$ acts diagonally on $X \times G$ in the last term):

$[X/N(T)] \simeq [(X \times G)/(N(T) \times G)] \simeq [(X \underset{N(T)}{\times} G)/G]$.

Therefore, (see (3.2)) the higher Chow groups, $CH^*(\ \ , .; \mathbb{Q})$ of these stacks are also isomorphic. Moreover, $CH^*([(X \underset{N(T)}{\times} G)/G], .; \mathbb{Q}) \cong \pi_*(\mathbb{H}_{et}(EG \underset{G}{\times} (X \underset{N(T)}{\times} G); \mathbb{Q}))$. One has a natural map $\pi : EG \underset{G}{\times} (X \underset{N(T)}{\times} G) \to EG \underset{G}{\times} X$

induced by the map $\pi_0 : X \underset{N(T)}{\times} G \to X$, sending $(x, g) \simeq (x \circ n^{-1}, n \circ g)$ to $x \circ g$. Clearly each π_n is flat (in fact, smooth) so that one obtains the second of the following two natural maps

(3.5.3) $\mathcal{Z}^*_{(EG \underset{G}{\times} X)_n}(\quad, .; \mathbb{Q}) \to R\pi_{n*}\pi_n^*(\mathcal{Z}^*_{(EG \underset{G}{\times} X)_n}(\quad, .; \mathbb{Q}) \to R\pi_{n*}(\mathcal{Z}^*_{(EG \underset{G}{\times} (X \underset{N(T)}{\times} G))_n}(\quad, .; \mathbb{Q}))$.

(Here $\mathcal{Z}^*(Y, .; \mathbb{Q})$ denotes $\mathcal{Z}^*(Y, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ for any scheme Y.) Clearly these maps being natural are compatible with restriction to open G-stable subschemes of X and also with the obvious multiplicative pairings. We proceed to show the composition above is a quasi-isomorphism for each $n \geq 0$: on taking hypercohomology (in the sense of [J-1] (3.6.4)), this will prove the isomorphism

(3.5.4) $CH^*([X/N(T)], .; \mathbb{Q}) \cong CH^*([(X \underset{N(T)}{\times} G)/G], .; \mathbb{Q}) \cong CH^*([X/G], .; \mathbb{Q})$

compatible with restriction to open G-stable sub-schemes of X and also with the given ring structures.

We will begin with the (Kunneth) quasi-isomorphism valid for any quasi-projective linear variety V (in the sense of [J-3]) and any quasi-projective scheme X:

(3.5.5) $\mathcal{Z}^*(V \times X, .) \simeq \mathcal{Z}^*(V, .) \overset{L}{\underset{\mathcal{Z}^*(Spec \quad k, .)}{\otimes}} \mathcal{Z}^*(X, .)$.

This follows from [J-3] Theorem 4.5. The Bruhat decomposition proves any connected reductive group is linear. Therefore, (3.5.5) applies to any connected reductive group G in the place of V. In general, any linear algebraic group G is the disjoint union over its connected components, which reduces to the case G is connected. In this case, the quotient $G/R_u(G)$ is connected reductive where $R_u(G)$ is the unipotent radical. Now the homotopy property of the cycle complex, therefore, proves (3.5.5) holds for any linear algebraic group G in the place of V.

Replacing X by $G^n \times X$ (and letting $N(T)$ act on $G^n \times X$ by its right-action on X), one may readily reduce to proving the composition of the maps in (3.5.3) is a quasi-isomorphism for $n = 0$. i.e. It suffices to show that the map $\mathcal{Z}^*_X(\quad, .; \mathbb{Q}) \to R\pi_{0*}(\mathcal{Z}^*_{X \underset{N(T)}{\times} G}(\quad, .; \mathbb{Q})$ is a quasi-isomorphism. The map π_0 and the obvious projection $pr_2 : X \underset{N(T)}{\times} G \to N(T)\backslash G$ induce a map $\phi : X \underset{N(T)}{\times} G \to X \times N(T)\backslash G$ which one may verify readily is bijective on points and hence purely inseparable. Therefore, it suffices to prove that $\mathcal{Z}^*_X(\quad, .; \mathbb{Q}) \to Rpr_{1*}(\mathcal{Z}^*_{X \times (N(T)\backslash G)}(\quad, .; \mathbb{Q})$ is a quasi-isomorphism where $pr_1 : X \times (N(T)\backslash G) \to X$ denotes the obvious projection.

Next observe that the formula

$(3.5.5)_{\mathbb{Q}}$ $\mathcal{Z}^*((N(T)\backslash G) \times U, .; \mathbb{Q}) \simeq \mathcal{Z}^*(N(T)\backslash G, .; \mathbb{Q}) \overset{L}{\underset{\mathcal{Z}^*(Spec \quad k, .; \mathbb{Q})}{\otimes}} \mathcal{Z}^*(U, .; \mathbb{Q})$

holds. To see this observe that $T\backslash G$ is a linear variety so that formula (3.5.5) holds for $T\backslash G$ in the place of V. Next observe $T\backslash N(T) \to U \times (T\backslash G) \to U \times (N(T)\backslash G)$ is a principal bundle with fiber $W = T\backslash N(T)$ for any scheme U. Therefore, (see [J-1] (3.7.11)), one obtains the formula in $(3.5.5)_{\mathbb{Q}}$ by taking the W-invariants from the corresponding formula with $T\backslash G$ in the place of $N(T)\backslash G$ which holds for all U. On the other hand, $B\backslash G$ is a projective smooth linear variety, (in fact stratified by affine spaces) so that $CH^*(T\backslash G, .; \mathbb{Q}) \cong CH^*(B\backslash G, .; \mathbb{Q}) \cong CH^*(B\backslash G, 0; \mathbb{Q})\underset{\mathbb{Q}}{\otimes}CH^*(Spec \quad k, .; \mathbb{Q})$. (This follows from the same argument as in [J-3] ; see also [J-4] Theorem (3.1).) Therefore, $CH^*(N(T)\backslash G, .; \mathbb{Q}) \cong CH^*(B\backslash G, 0; \mathbb{Q})^W \underset{\mathbb{Q}}{\otimes} CH^*(Spec \quad k, .; \mathbb{Q}) \cong CH^*(Spec \quad k, .; \mathbb{Q})$, since $CH^*(B\backslash G, 0; \mathbb{Q})^W \cong Q$. It follows readily from these observations that $R\pi_{0*}(\mathcal{Z}^*_{X \underset{N(T)}{\times} G}(\quad, .; \mathbb{Q})) \simeq \mathcal{Z}^*_X(\quad, .; \mathbb{Q})$ and therefore that $\mathcal{Z}^*_{(EG \underset{G}{\times} X)_n}(\quad, .; \mathbb{Q}) \simeq R\pi_{n*}(\mathcal{Z}^*_{(EG \underset{G}{\times} (X \underset{N(T)}{\times} G))_n}(\quad, .; \mathbb{Q}))$ for each $n \geq 0$. Therefore, we obtain the isomorphism in (3.5.4).

By (3.5.2), one obtains the isomorphism:

(3.5.6) $CH^*([X/N(T)], .; \mathbb{Q}) \cong CH^*([X/T], .; \mathbb{Q})^W$.

It is shown in [EG] Proposition 6, that

(3.5.7)$CH^*_G(X, .; \mathbb{Q}) \cong CH^*_T(X, .; \mathbb{Q})^W$.

(Strictly speaking this is shown only for the special case G is connected and reductive with a split maximal torus T and also not for the higher Chow groups; however, as in the arguments above, one may readily reduce the case of a general algebraic group to the case of a connected reductive group. The extension to the higher Chow groups is clear.) From the definition of the equivariant higher Chow groups in terms of schemes approximating $EG\underset{G}{\times}X$ and $ET\underset{T}{\times}X$ (through a certain range), it is clear this isomorphism is compatible with restriction to open G-stable sub-schemes of X and also compatible with the ring structures.

In view of (3.5.4), (3.5.6) and (3.5.7), we reduce to showing that there exists an isomorphism

(3.5.8) $CH^*([X/T], .; \mathbb{Q}) \cong CH^*_T(X, .; \mathbb{Q})$

compatible with restriction to open T-stable subschemes of X and the given ring structures on either side. For this we will identify the term $CH^q([X/T], 2q-p; \mathbb{Q})$ on the left-hand-side with the motivic cohomology, $H^p_{et}([X/T], .; \mathbb{Q}(q)) \cong H^0_{et}(ET\underset{T}{\times}X, .; \mathbb{Q}(q)[p])$ where $\mathbb{Q}(q)$ denotes the rational motivic complex of weight q. One identifies the latter with $Hom_{H(Spec \quad k)}(ET\underset{T}{\times}X, K(\mathbb{Q}(q)[p]))$ where $H(Spec \quad k)$ denotes the A^1-homotopy category of smooth schemes over $Spec \quad k$ and $K(\mathbb{Q}(q)[p])$ is the simplicial sheaf corresponding to the motivic complex $\mathbb{Q}(q)[p]$. (See [M-V].) (To see this first observe that the motivic complex $\mathbb{Q}(q)$ is homotopy invariant, showing $K(\mathbb{Q}(q)[p])$ is A^1-local. Let $SimplSh(Spec \quad k)_{Nis,\mathbb{A}^1}$ denote the category of all simplicial sheaves on the big Nisnevich site of smooth schemes over k provided with the simplicial model structure where cofibrations (fibrations, weak-equivalences) are monomorphisms (A^1-fibrations, A^1-weak-equivalences, respectively) as in [M-V] (3.1). Since each term, $(ET\underset{T}{\times}X)_n = T^n \times X$ is a smooth quasi-projective scheme, one obtains a natural quasi-isomorphism $\{\mathbb{H}_{et}((ET\underset{T}{\times}X)_n, \mathbb{Q}(q)[p])|n\} \simeq \{\mathbb{H}_{Nis}((ET\underset{T}{\times}X)_n, \mathbb{Q}(q)[p])|n\} \simeq \{Map((ET\underset{T}{\times}X)_n, K(\mathbb{Q}(q)[p]))|n\}$ where Map denotes the bi-functor

$$SimplSh(Spec \quad k)^{op}_{Nis,\mathbb{A}^1} \times SimplSh(Spec \quad k)_{Nis,\mathbb{A}^1} \to (simpl.sets),$$

sending (M, N) to $Map(M,N)_n = Hom(M \times \Delta[n], N)$. Taking the homotopy inverse limit over Δ and taking the homotopy groups, one obtains the required isomorphism.)

We proceed to consider the right-hand-side of (3.5.8). Let $T = \mathbb{G}_m^n$ and let $U_\infty = \mathbb{A}^\infty - 0$; there is a natural principal $\mathbb{G}_m$-bundle $U_\infty \to \mathbb{P}^\infty$. The term on the right-hand-side of (3.5.8) (in bi-degree q and $2q-p$) now identifies with $CH^q((U_\infty)^n\underset{T}{\times}X, 2q-p; \mathbb{Q})$. (For each fixed q and p, $CH^q((U_\infty)^n\underset{T}{\times}X, 2q-p; \mathbb{Q}) = CH^q((U_N)^n\underset{T}{\times}X, 2q-p; \mathbb{Q})$ if N is sufficiently large and $U_N \to \mathbb{P}^N$ is the corresponding $\mathbb{G}_m$-principal bundle.) Therefore, one may identify this with $Hom_{H(Spec \quad k)}((\mathbb{U}_\infty)^n\underset{T}{\times}X, K(\mathbb{Q}(q)[p]))$ as well.

We proceed to show that the two simplicial sheaves $ET\underset{T}{\times}X$ and $(U_\infty)^n\underset{T}{\times}X$ are isomorphic in the homotopy category $H(Spec \quad k)$ and that this isomorphism is natural in X. First it follows from [M-V] section 4, Proposition (3.7), that the two simplicial sheaves $(\mathbb{P}^\infty)^n$ and BT are naturally isomorphic in the homotopy category $H(Spec \quad k)$. The required isomorphism may be obtained readily from this: however, here are some details. For this we consider the simplicial sheaf $(ET \times X)\underset{T}{\times}U_\infty$ where T acts diagonally on $ET \times X$. One sees readily that the natural map $p_1 : (ET \times X)\underset{T}{\times}U_\infty \to X\underset{T}{\times}U_\infty$ is a *local fibration* with fibers the contractible simplicial sheaf ET (in the sense of [M-V] p. 51). (Recall that a local fibration is a map of simplicial sheaves that is a fibration in the usual sense at each stalk.) Therefore, p_1 is a weak-equivalence, hence an A^1-equivalence (see [M-V] p. 71) and therefore an isomorphism in the homotopy category $H(Spec \quad k)$.

Let $p_2 : (ET\times X)\underset{T}{\times}U_\infty \to ET\underset{T}{\times}X$ denote the obvious projection: it suffices to show this is also an isomorphism in the homotopy category $H(Spec \quad k)$. First one observes that p_2 is also a local fibration with fibers U_∞ and that if Ex_{A^1} is the A^1-resolution functor (as in [M-V] p.110), then it sends local fibrations to A^1-fibrations. (See [M-V] p. 79. A map of simplicial sheaves is called an A^1-fibration if it has the right-lifting property with respect to maps which are monomorphisms and A^1-weak-equivalences. Since each map which is a stalk-wise weak-equivalence is automatically an A^1-weak-equivalence, every A^1-fibration is automatically a local fibration.) Therefore, applying this functor to p_2 produces an A^1-fibration $p_2' : E' \to B'$ with fiber $F' = Ex_{A^1}(U_\infty)$, since U_∞ is the fiber of p_2. It follows that F' is A^1-weakly-equivalent to U_∞. Now recall from [M-V] p.134 that U_∞ is contractible in the homotopy category $H(Spec \quad k)$. Therefore, F' is also contractible in the A^1-homotopy category, showing that p_2' and hence p_2 is an isomorphism in the homotopy category $H(Spec \quad k)$.

It follows that we obtain an isomorphism

$CH^*([X/T], .; \mathbb{Q}) \cong CH^*_T(X, .; \mathbb{Q})$

preserving the ring structures and compatible with restriction to open T-stable subschemes. (Observe that the ring structures on these groups are induced from the same product structure on the motivic complexes in the homotopy category $H(Spec \quad k)$.) This proves the theorem. □

Remark. As a further application of the comparison theorem, (3.1), we will also establish higher Chern classes for smooth stacks with values in our higher Chow groups and an intersection pairing for all smooth stacks. These are discussed in sections 4 and 5, respectively.

4. Homotopy property, Chern classes and the projective space bundle theorem

(4.1) **Theorem**. Let $\mathfrak{S}$ denote an algebraic stack and let $\pi : \mathcal{E} \to \mathfrak{S}$ denote a vector bundle of rank r. Let $x : X \to \mathfrak{S}$ denote an atlas for $\mathfrak{S}$ and let $x' : X \underset{\mathfrak{S}}{\times} \mathcal{E} \to \mathcal{E}$ denote the induced atlas. Then one obtains the isomorphisms for each $c \geq 0$:

$CH_*(\mathfrak{S}, x, .) \underset{\simeq}{\xrightarrow{\pi^*}} CH_{*+r}(\mathcal{E}, x', .)$

Proof. Let $n \geq 0$ be a fixed integer and let $\mathcal{Z}_*^{B_x\mathfrak{S}_{n,et}}(\quad , .)$ $(\mathcal{Z}_*^{B_{x'}\mathcal{E}_{n,et}}(\quad , .))$ denote the presheaf on $B_x\mathfrak{S}_{n,et}$ ($B_{x'}\mathcal{E}_{n,et}$, respectively) which are the restrictions of the corresponding presheaves on the restricted smooth sites of $\mathfrak{S}$ and $\mathcal{E}$ respectively. Observe that one obtains a natural map $\mathcal{Z}_*^{B_x\mathfrak{S}_{n,et}}(\quad , .) \to R\pi_*\pi^*\mathcal{Z}_*^{B_x\mathfrak{S}_{n,et}}(\quad , .)$. Since π is flat of relative dimension r, it induces a map of presheaves $\pi^*\mathcal{Z}_*^{B_x\mathfrak{S}_{n,et}}(\quad , .) \to \mathcal{Z}_{*+r}^{B_{x'}\mathcal{E}_{n,et}}(\quad , .)$. Combining the above maps, we obtain the map

$\mathcal{Z}_*^{B_x\mathfrak{S}_{n,et}}(\quad , .) \to R\pi_*\mathcal{Z}_*^{B_{x'}\mathcal{E}_{n,et}}(\quad , .)$

of presheaves on $B_x\mathfrak{S}_{n,et}$. In order to prove the theorem, it suffices to show that this map is a quasi-isomorphism. By (the trivial smooth-base change), it suffices to prove this when $\mathfrak{S}$ is replaced by a scheme $u : U \to (B_x\mathfrak{S})_n$ in the site $B_x\mathfrak{S}_{n,et}$. This is clear by the homotopy property established for the higher Chow groups in [Bl] Theorem (2.1). □

Next we proceed to construct Chern classes. For the cycle complex we will first construct only *ordinary* Chern classes defined on the Grothendieck group of vector bundles of stacks. Next we extend these to higher Chern classes defined on the higher K-theory of stacks and taking values in cohomology theories of stacks that are defined by complexes of sheaves that are sheaves on the big site $(alg.stacks/k)_{smt}$. The difficulty of doing a similar construction for the higher cycle complex comes from the fact it defines only a (pre-)sheaf on the corresponding *restricted* sites. However, as we show, hypercohomology with respect to the higher cycle complex may be replaced by hypercohomology with respect to the motivic complexes provided one restricts to smooth objects. This observation enables us to obtain a theory of higher Chern classes *for smooth stacks* with values in our higher intersection theory.

(4.2.1) Let $\mathcal{E}$ denote a vector bundle of rank r on an algebraic stack $\mathfrak{S}$ of dimension d and let $Proj(\mathcal{E})$ denote the associated *projective bundle stack* over $\mathfrak{S}$. Associated to $\mathcal{O}_{Proj(\mathcal{E})}(1)$ now one may define a canonical cycle $\psi_{\mathcal{E}} \in \mathcal{Z}_{d+r-2}(Proj(\mathcal{E}, 0))$ in the usual manner. Observe that if $x : X \to \mathfrak{S}$ is a map from an algebraic space and $x^*(\mathcal{E})$ is the associated vector bundle on X, $x^*(\psi_{\mathcal{E}}) = \psi_{x^*\mathcal{E}}$ is the corresponding divisor in $\mathcal{Z}_*(Proj(x^*\mathcal{E}), 0)$. Now it is easy to see that intersecting with the divisor $\psi_{\mathcal{E}}$ defines a map $\circ\psi_{\mathcal{E}} : \mathcal{Z}_n(Proj(\mathcal{E}, .) \to \mathcal{Z}_{n-1}(Proj(\mathcal{E}, .)$ for each n. (To see this proceed as follows: let $x : X \to \mathfrak{S}$ denote an atlas and let $p_i : X \underset{\mathfrak{S}}{\times} X \to X$ denote the projection to the i-th factor. If Z is an integral sub-stack of $\mathfrak{S}$ of dimension n, $p_i^*(x^*(Z) \circ x^*(\psi_{\mathcal{E}})) = p_i^*x^*(Z) \circ p_i^*(x^*(\psi_{\mathcal{E}}))$. Therefore, $p_1^*(x^*(Z) \circ x^*(\psi_{\mathcal{E}})) = p_2^*(x^*(Z) \circ x^*(\psi_{\mathcal{E}}))$. It follows that $x^*(Z) \circ x^*(\psi_{\mathcal{E}})$ descends to a class in $\mathcal{Z}_{n-1}(Proj(\mathcal{E}), .)$ thereby defining the operation $\circ\psi_{\mathcal{E}}$.)

(4.2.2) **Theorem**. Let $\mathfrak{S}$ denote an algebraic stack of finite dimension and let $\pi_0 : \mathcal{E} \to \mathfrak{S}$ denote a vector bundle of rank r. Let $\pi : \mathbb{P}(\mathcal{E}) = Proj(\mathcal{E}) \to \mathfrak{S}$ denote the associated projective space bundle. Let $z : Z \to \mathfrak{S}$ denote a fixed atlas for $\mathfrak{S}$ and let $z' : z \underset{\mathfrak{S}}{\times} Proj(\mathcal{E})$ denote the induced atlas for $Proj(\mathcal{E})$. Let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ denote the tautological bundle on $\mathbb{P}(\mathcal{E})$. Then the map $\oplus_{i=0}^r CH_*(\mathfrak{S}, z, .) \to CH_*(\mathbb{P}(\mathcal{E}), z', .)$ sending $(a_0, ..., a_{r-1})$ to $\Sigma_i \pi^*(a_i) \circ \psi_{\mathcal{E}}^i$ is an isomorphism. The corresponding statements hold when X is any scheme of finite dimension over k and $\mathcal{E}$ is a vector bundle of rank r.

Proof. Since the proof of the second assertion is entirely similar we will skip it. Let $n \geq 0$ denote a fixed integer. Observe that the vector bundle $\mathcal{E}$ pulls back to a vector bundle over each object $x : X \rightarrow (B_z\mathfrak{S})_n$ in the site $B_z\mathfrak{S}_{n,et}$. Let $\mathcal{Z}_*^{B_z\mathfrak{S}_{n,et}}(\quad,.)$ $(\mathcal{Z}_*^{B_{z'}\mathbb{P}(\mathcal{E})_{n,et}}(\quad,.))$ denote the restriction of $\mathcal{Z}_*(\quad,.)$ on $B_z\mathfrak{S}_{n,et}$ $(B_{z'}\mathbb{P}(\mathcal{E})_{n,et})$ as before. Now it suffices to show that the map:

$$\underset{0 \leq i \leq r-1}{\oplus} \mathcal{Z}_*^{B_z\mathfrak{S}_{n,et}}(\quad,.) \rightarrow R\pi_*\mathcal{Z}_*^{B_{z'}\mathbb{P}(\mathcal{E})_{n,et}}(\quad,.)$$

given by $(a_0, ..., a_{r-1}) \mapsto \Sigma_i \pi^*(a_i) \circ \psi_{\mathcal{E}}^i$ is a quasi-isomorphism. In order to establish this, it suffices to work locally on the site $B_z\mathfrak{S}_{n,et}$: i.e. one may replace $\mathfrak{S}$ by a scheme X that belongs to the site $B_z\mathfrak{S}_{n,et}$. In other words, we reduce to proving the theorem when the stack $\mathfrak{S}$ is replaced by a scheme. Now the result follows as in [Bl-1] Theorem (3.1). $\square$

(4.2.3) **Corollary**. Assume the situation of (4.2.1). Then one obtains a push-forward map

$\pi_* : CH_*(\mathbb{P}(\mathcal{E}), z', .) \rightarrow CH_*(\mathfrak{S}, z, .)$ if $\mathcal{E}$ is a vector bundle over the stack $\mathfrak{S}$

and a push-forward map

$\pi_* : CH_*(\mathbb{P}(\mathcal{E}), .) \rightarrow CH_*(X, .)$ if $\mathcal{E}$ is a vector bundle over the scheme X.

Proof. We define π_* as the projection onto the 0-th summand. $\square$

(4.3.1) With the above theorem at our disposal we immediately obtain the *covariance* of the *integral* higher Chow groups with respect to maps $f : \mathfrak{S} \rightarrow \mathfrak{S}'$ of algebraic stacks that are *strongly projective*, i.e. maps that can be factored as the composition of a closed immersion followed by projection from a projective space $Proj(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle over the stack $\mathfrak{S}'$.

(4.3.2) **Proposition**(*Base-change*). Consider the cartesian square

$$\begin{tikzcd}
\bar{\mathfrak{S}}' \arrow[r, "p'"] \arrow[d, "\bar{f}"'] & \mathfrak{S}' \arrow[d, "f"] \\
\bar{\mathfrak{S}} \arrow[r, "p"] & \mathfrak{S}
\end{tikzcd}$$

with $\mathfrak{S}$ an algebraic stack of finite dimension, f flat of relative dimension m and p strongly projective. Let $z : Z \rightarrow \mathfrak{S}$ denote a given atlas for $\mathfrak{S}$ and let $z' = z\underset{\mathfrak{S}}{\times}\mathfrak{S}'$, $\bar{z} = z\underset{\mathfrak{S}}{\times}\bar{\mathfrak{S}}$ and $\bar{z}' = \bar{z}\underset{\bar{\mathfrak{S}}}{\times}\bar{\mathfrak{S}}'$ denote the induces atlases. Now we obtain the commutative square:

$$\begin{tikzcd}
CH_{k+m}(\bar{\mathfrak{S}}', \bar{z}'; n) \arrow[r, "p'_*"] & CH_{k+m}(\mathfrak{S}', z'; n) \\
CH_k(\bar{\mathfrak{S}}, \bar{z}, n) \arrow[r, "p_*"] \arrow[u, "\bar{f}^*"] & CH_k(\mathfrak{S}, z, n) \arrow[u, "f^*"']
\end{tikzcd}$$

More generally, the same conclusion holds modulo torsion, if p is any proper representable morphism. A corresponding assertion holds in case the above square is one of schemes over k.

Proof. As before we will skip the case of schemes. We will consider the case when p is a closed immersion first. Let $n \geq 0$ denote a fixed integer. In this case it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
Rf_*f^{-1}p_*\mathcal{Z}_*^{B_{\bar{z}}\bar{\mathfrak{S}}_{n,et}}(\quad,n) \arrow[r, "Rf_*f^{-1}(p_*)"] \arrow[d, "\simeq"'] & Rf_*f^{-1}\mathcal{Z}_*^{B_z\mathfrak{S}_{n,et}}(\quad,n) \arrow[dd, "f^*"] \\
Rf_*p'_*\bar{f}^{-1}\mathcal{Z}_*^{B_{\bar{z}}\bar{\mathfrak{S}}_{n,et}}(\quad,n) \arrow[d, "\bar{f}^*"'] & \\
Rf_*p'_*\mathcal{Z}_{*+m}^{B_{\bar{z}'}\bar{\mathfrak{S}}'_{n,et}}(\quad,n) \arrow[r, "p'_*"'] & Rf_*\mathcal{Z}_{*+m}^{B_{z'}\mathfrak{S}'_{n,et}}(\quad,n)
\end{tikzcd}$$

of presheaves. One may readily reduce the commutativity of the above diagram to that of the square:

$$
(4.3.2.*)\qquad
\begin{array}{ccc}
\mathcal{Z}_*^{B_{\bar{z}}\bar{\mathfrak{S}}_n}(p^{-1}(V),n) & \xrightarrow{p_*} & \mathcal{Z}_*^{B_z\mathfrak{S}_n}(V,n)\\
\bar{f}^*\big\downarrow & & \big\downarrow f^*\\
\mathcal{Z}_{*+m}^{B_{\bar{z}'}\bar{\mathfrak{S}}'_n}(p'^{-1}f^{-1}(V),n) & \xrightarrow{p'_*} & \mathcal{Z}_{*+m}^{B_{z'}\mathfrak{S}'_n}(f^{-1}(V),n)
\end{array}
$$

where V belongs to the site $B_z\mathfrak{S}_{n,et}$, $f^{-1}(V) = \mathfrak{S}'\underset{\mathfrak{S}}{\times}V$, $p^{-1}(V) = \bar{\mathfrak{S}}\underset{\mathfrak{S}}{\times}V$ and $p'^{-1}(f^{-1}(V)) = \bar{\mathfrak{S}}'\underset{\mathfrak{S}'}{\times}f^{-1}(V)$. This is clear from the usual properties of the cycle complex. (See [J-1] section 4 for example.) This completes the proof when p is a closed immersion. The proof when p is a projection from $\mathbb{P}(\mathcal{E})$ follows immediately from (4.2.2).

Next we consider the case when p is an arbitrary proper and representable morphism. (3.6.7) Proposition of [J-1] shows that the higher Chow groups with rational coefficients may be computed using hypercoverings. (4.6.3) Theorem in [J-1] in fact shows that p induces a push-forward map p_*. One may now reduce the commutativity of the square in the Proposition to the commutativity of the square (4.3.2.*) above. (Observe that if $V = U_{n,n}$, for a hypercovering $U_{\bullet,\bullet} \to B_x\mathfrak{S}$, the corresponding squares in (4.3.2.*) as n varies, are compatible with respect to the face maps of the simplicial space $\Delta U_{\bullet,\bullet}$.) $\square$

(4.3.3)**Definition**. (Chern classes) Let $\mathcal{E}$ denote a vector bundle of rank r on a *smooth* algebraic stack $\mathfrak{S}$ of dimension d. We define the i-th *Chern class* of $\mathcal{E}$ to be the element of $CH_{d-i}(\mathfrak{S},0)$ defined as follows. Observe from (5.1) (see below) that $CH^*(\mathfrak{S},0) \cong CH_*(\mathfrak{S},0)$ has a ring structure. Now (4.2.2) shows that $CH^*(Proj(\mathcal{E}),0)$ is a free module of rank r over $CH^*(\mathfrak{S},0)$ with $1,\ldots,\ \psi_{\mathcal{E}}^{r-1}$ as a basis. Therefore, we may define $c_i(\mathcal{E})\ \epsilon\ CH_{d-i}(\mathfrak{S},0)$ so that $\Sigma_{i=0}^{r}(-1)^i\pi^*(c_i(\mathcal{E}))\circ\psi_{\mathcal{E}}^{r-i} = 0$. (One may readily verify these pull back under representable flat maps.)

Next we proceed to define *higher Chern classes* in cohomology theories that are defined by complexes of presheaves (in the sense of [Gi-1] section 1) on the big smooth site of all algebraic stacks over k. Observe that the higher cycle complex being contravariantly functorial only for flat maps, is *not* a presheaf on this site.

(4.4.1) Let $(stacks/k)_{smt}$ denote the *big smooth site* of all stacks locally of finite type over k and let $\Gamma(i)$ denote a complex of sheaves on this site for each integer i. Alternatively let $(smt.stacks/k)_{smt}$ denote the *big smooth site* of all *smooth* stacks locally of finite type over k and let $\Gamma(i)$ denote a complex of sheaves on this site for each integer i. Now $\Gamma(i)$ defines by restriction a complex of presheaves on the quotient stack [Spec k$/GL_n$]. (Since this stack is also smooth, one may consider either of the above two situations.) The classifying simplicial scheme for this stack with respect to the atlas Spec k is $B.GL_n$. The pull-back of $\Gamma(i)$ to $B.GL_n$ will also be denoted by $\Gamma(i)$.

(4.4.2) Let $\mathcal{C}$ denote either the Zariski site or the étale site of $B.GL_n$. We will next make the assumption that there exist *universal Chern classes* $C_i^p\ \epsilon\ \mathbb{H}_{\mathcal{C}}^{di}(B.GL_p;\Gamma(i))$, $1 \le i \le p$. Here d is an integer depending on the complex $\Gamma(i)$.

One may consider the following *examples* of the above set-up. In each case $\mathfrak{S}$ will denote an algebraic stack locally of finite type over k and $\Gamma(i)_{|\mathfrak{S}}$ will denote the restriction of $\Gamma(i)$ to the smooth site of $\mathfrak{S}$.

(4.5.1) *Algebraic de Rham cohomology.* $\Gamma(i)_{|\mathfrak{S}} = \Omega^{\bullet}_{\mathfrak{S}/k}$ for all $i \ge 0$ and $= 0$ for $i < 0$. Here $\Omega^{\bullet}_{\mathfrak{S}/k}$ denotes the de Rham complex on $\mathfrak{S}$. (Here $d = 2$.)

(4.5.2) *Gersten complex.* $\Gamma(i)_{|\mathfrak{S}} = \pi_i(\mathbf{K}(\quad))_{|\mathfrak{S}}$, $i \ge 0$ and $= 0$ for $i < 0$. (Now $d = 1$.)

(4.5.3) *Motivic complexes for (smooth objects).* Here $\Gamma(i) = \mathbb{Z}(i)[2i] =$ the (shifted) motivic complex of weight i considered in the appendix.

In all of the above cases, the universal Chern classes live in the *Zariski* hypercohomology of $B.GL_n$.

(4.5.4) Now we will start with a complex $\Gamma(i)$ of sheaves so that there exist universal Chern classes $c_i\ \epsilon\ \mathbb{H}_{et}^{2i}(B.GL_n,\Gamma(i))$. (Therefore, $d = 2$ here.) As examples of the complexes $\Gamma(i)$ we may consider the following. Fix an integer l and let $(alg.stacks/\mathbb{Z}[1/l])$ denote the category of algebraic stacks locally of finite type over k with l invertible in the structure sheaf. Let $\Gamma(i) = \mu_l(i) =$ the sheaf of l-th roots of unity twisted i-times. If A is any abelian group, let $\underline{A}[0]$ denote the obvious complex of sheaves concentrated in degree 0 where it is the constant sheaf associated to A. Now one may also consider the complex of sheaves $\Gamma(i) = A[0]$ for all i.

For example let G denote a smooth group scheme acting on a scheme X locally of finite type over a field k. Observe that the smooth cohomology of the stack $[X/G]$ with respect to any of the above sheaves may identified with the equivariant cohomology of X: therefore our constructions provide *higher equivariant Chern classes with values in equivariant étale cohomology* for smooth group scheme actions.

(4.6.1) For each scheme (smooth scheme) X, let h_X denote the contravariant functor from the category $(schemes/k)$ $((smt.schemes/k)$, respectively) to sets represented by X. Let P denote a presheaf of abelian groups on the above category. Then the Yoneda lemma provides the isomorphism $\Gamma(X, P) \cong Hom(h_X, U(P))$ where Hom denotes the (external) Hom in the category of presheaves of sets on $(schemes/k)$ $((smt.schemes/k)$, respectively) and U is the forgetful functor sending an abelian presheaf to the associated presheaf of sets. Let $\mathbb{Z}h_X$ denote the contravariant functor from $(schemes/k)$ $((smt.schemes/k))$ to abelian groups defined by $\mathbb{Z}h_X(Y) = \mathbb{Z}(Hom_{(schemes/k)}(Y, X))$ = the free abelian group on $Hom_{(schemes/k)}(Y, X)$
$(= \mathbb{Z}(Hom_{(smt.schemes/k)}(Y, X))$ = the free abelian group on $Hom_{(smt.schemes/k)}(Y, X)$, respectively). Now one also obtains the isomorphism

(4.6.2) $\Gamma(X, P) = Hom(\mathbb{Z}h_X, P)$

where Hom on the right denotes the (external) Hom in the category of abelian presheaves on $(schemes/k)$ $((smt.schemes/k)$, respectively).

Therefore, for each fixed $l \geq 0$, $p \geq 0$,

(4.6.3) $\mathbb{H}_{\mathcal{C}}(B_l GL_p, P) = \Gamma(B_l GL_p, \mathcal{G}P) = Hom(\mathbb{Z}h_{B_l GL_p}, \mathcal{G}P)$

where $\mathcal{C}$ denotes either the big sites $(schemes/k)_{Zar}$, $(schemes/k)_{et}$, $(smt.schemes/k)_{Zar}$ or $(smt.schemes/k)_{et}$ and P denotes a presheaf of complexes of abelian groups on the above site, $\mathcal{G}P$ is the Godement resolution as in [J-1](3.3.1) and Hom denotes the external Hom in the category of all (unbounded) complexes of abelian presheaves on the above site. (Observe that hypercohomology computed on the big site $(schemes/k)_{Zar}$ or $(schemes/k)_{et}$ is isomorphic to the hypercohomology computed on the corresponding small site.) Recall that for each fixed l, $\mathbb{Z}h_{B_l GL_p}$ is a contravariant functor from $(schemes/k)$ (or from $(smt.schemes/k)$) to abelian groups which we may imbed in the category of all simplicial abelian groups. Therefore, one may view $\{\mathbb{Z}h_{B_l GL_p}|l\}$ as a contravariant functor from $(schemes/k)$ $((smt.schemes/k))$ to the category of double simplicial abelian groups. Now the homotopy colimit of the double simplicial abelian presheaf $\{\mathbb{Z}h_{B_l GL_p}|l\}$ may be identified with its diagonal; since the above double simplicial object is constant in one direction, this is simply the same simplicial abelian presheaf $\{\mathbb{Z}h_{B_l GL_p}|l\}$. This simplicial abelian presheaf may be identified with the corresponding chain complex: we will denote this presheaf on $(schemes/k)$ (or on $((smt.schemes/k)$, respectively) by

(4.6.4) $\mathbb{Z}h_{B.GL_p}$.

Now observe that $Hom(\mathbb{Z}h_{B_l GL_p}, \mathcal{G}P) = \Gamma(B_l GL_p, \mathcal{H}om(\mathbb{Z}h_{B_l GL_p}, \mathcal{G}P))$ where $\mathcal{H}om$ is the internal Hom in the complete pointed simplicial category of all complexes of abelian presheaves on the above site, i.e. $(schemes/k)_{Zar}$, $(schemes/k)_{et}$, $(smt.schemes/k)_{Zar}$ or $(smt.schemes/k)_{et}$. Therefore [J-1](6.1.5) shows that on taking the homotopy inverse limit of the cosimplicial object $\{\mathbb{H}_{\mathcal{C}}(B_l GL_p, P)|l\}$ one obtains

$\Gamma(B.GL_p, \mathcal{H}om(\mathbb{Z}h_{B.GL_p}, \mathcal{G}P))$.

Next let $\Gamma(i)$ denote a complex as in (4.4.1) and let $\Gamma(i)_{schemes/k}$ denote its restriction to the category of all schemes (locally of finite type) over k with *universal Chern class* $C_i^p \in \mathbb{H}_{\mathcal{C}}^{di}(B.GL_p, \mathcal{G}\Gamma_{schemes/k}(i)) = \mathbb{H}_{\mathcal{C}}^0(B.GL_p, \Gamma(i)[di]) = \pi_0(\Gamma(B.GL_p, \mathcal{H}om(\mathbb{Z}h_{B.GL_p}, \Gamma(i)_{schemes/k}[di]))$. The above discussion shows that the universal Chern class C_i^p corresponds to a map

(4.6.5) $\hat{C}_i^p : \mathbb{Z}h_{B.GL_p} \to \Gamma(i)_{schemes/k}[di]$

in an appropriate derived category of presheaves on $(schemes/k)$. Alternatively if $\Gamma(i)$ denotes a complex as in (4.4.1) and let $\Gamma(i)_{smt.schemes/k}$ denote its restriction to the category of all smooth schemes locally of finite type over k, the universal Chern class C_i^p will correspond to a map

(4.6.5)' $\hat{C}_i^p : \mathbb{Z}h_{B.GL_p} \to \Gamma(i)_{smt.schemes/k}[di]$

in an appropriate derived category of presheaves on $(smt.schemes/k)$.

The stability of the universal Chern classes as p increases shows that the following diagram

$$\begin{tikzcd} (4.6.6)\mathbb{Z}h_{B.GL_p} \arrow[rr, "\hat{C}_i^p"'] \arrow[dr] & & \Gamma(i)_{schemes/k}[di] \\ & \mathbb{Z}h_{B.GL_{p+p'}} \arrow[ur, "\hat{C}_i^{p+p'}"'] & \end{tikzcd}$$

of presheaves commutes in the appropriate derived category.

Let $\mathbb{Z}h_{B.GL_p}$ and $\Gamma(i)$ denote the corresponding presheaves defined on $\mathfrak{S}_{\underline{smt}}$ similarly. (Observe that the above presheaves are contravariant for all maps, and therefore define presheaves on $\mathfrak{S}_{\underline{smt}}$ in fact.) By pull-back to $(B_x\mathfrak{S})_{\underline{smt}}$ (and restriction) we obtain presheaves on $(B_x\mathfrak{S})_{et}$ which will be denoted $\mathbb{Z}h_{B.GL_{p}{}_{B_x\mathfrak{S}_{et}}}$ and $\Gamma(i)_{B_x\mathfrak{S}_{et}}$ (or just $\Gamma(i)$ itself). Now we apply one of the extension functors in (2.6.2) to obtain functorial extensions of the maps in (4.6.5) to the functorial extensions of the above presheaves to the site $\mathfrak{S}_{\underline{smt}}$ for any algebraic stack $\mathfrak{S}$. i.e. We obtain:

$$\begin{tikzcd} E_{\mathfrak{S}}(\mathbb{Z}h_{B.GL_p}) \arrow[r, "E(\hat{C}_i^p)"] & E_{\mathfrak{S}}(\Gamma(i)_{schemes/k}[di]) \\ \mathbb{Z}h_{B.GL_{p}{}_{B_x\mathfrak{S}_{et}}} \arrow[u] & \Gamma(i)[di] \arrow[u] \end{tikzcd}$$

Observe from (2.6.2)(iii) that the natural map $\Gamma(i)_{schemes/k} \to E_{\mathfrak{S}}(\Gamma(i)_{schemes/k})$ is a quasi-isomorphism on restriction to the étale site of any algebraic space in $\mathfrak{S}_{\underline{smt}}$. (Clearly both of the above assertions hold for complexes $\Gamma(i)$ that are only defined for smooth objects provided the stack $\mathfrak{S}$ we consider is also smooth.) Therefore the right vertical map in the above diagram is a quasi-isomorphism: we will therefore identify $E_{\mathfrak{S}}(\Gamma(i)_{schemes/k}[di])$ with $\Gamma(i)[di]$. Therefore we obtain the map

(4.6.7) $C_i^p : \mathbb{Z}h_{B.GL_{p}{}_{B_x\mathfrak{S}_{et}}} \to \Gamma(i)[di]$

(in an appropriate derived category) of presheaves on $(B_x\mathfrak{S})_{et}$. Moreover, if $GL_p \to GL_{p+p'}$ is the obvious closed immersion, sending GL_p to the first p rows and columns, the diagram

$$\begin{tikzcd} \mathbb{Z}h_{B.GL_{p}{}_{B_x\mathfrak{S}_{et}}} \arrow[rr, "C_i^p"'] \arrow[dr] & & \Gamma(i)[di] \\ & \mathbb{Z}h_{B.GL_{p+p'}{}_{B_x\mathfrak{S}_{et}}} \arrow[ur, "C_i^{p+p'}"'] & \end{tikzcd}$$

commutes in the appropriate derived category of sheaves on $B_x\mathfrak{S}_{et}$, since the diagram (4.6.6) commutes. Observe also that if $f : \mathfrak{S}' \to \mathfrak{S}$ is a flat representable map of algebraic stacks, one obtains a commutative square:

$$(4.6.8) \qquad \begin{tikzcd} Rf_*(\mathbb{Z}h_{B.GL_{p}{}_{B_{x'}\mathfrak{S}'_{et}}}) \arrow[r, "Rf_*(C_i^p)"] & Rf_*(\Gamma(i)[di]_{B_{x'}\mathfrak{S}'_{et}}) \\ \mathbb{Z}h_{B.GL_{p}{}_{B_x\mathfrak{S}_{et}}} \arrow[u, "f^*"] \arrow[r, "C_i^p"] & \Gamma(i)[di]_{B_x\mathfrak{S}_{et}} \arrow[u, "f^*"'] \end{tikzcd}$$

(Here $x : X \to \mathfrak{S}$ is an atlas and $x' : X' \to \mathfrak{S}'$ is the induced atlas.) This follows from (2.6.2)(iv).

Next observe that $\mathbb{H}_{smt}(\mathfrak{S}, \Gamma(i)) = \mathbb{H}_{et}(B_x\mathfrak{S}, \Gamma(i))$. Therefore, taking the hypercohomology on $(B_x\mathfrak{S})_{et}$, taking the resulting homotopy groups and letting $m \to \infty$, we obtain maps

(4.6.9) $C_i(n)' : \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, \mathbb{Z}h_{B.GL_{(B_x\mathfrak{S})_{et}}})) = \lim_{m\to\infty} \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, \mathbb{Z}h_{B.GL_{m}{}_{(B_x\mathfrak{S})_{et}}}))$

$\to \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, \Gamma(i)[di])) = \pi_{n-di}(\mathbb{H}_{smt}(\mathfrak{S}, \Gamma(i))) = \mathbb{H}_{smt}^{di-n}(\mathfrak{S}, \Gamma(i))$

If one considers $\mathbb{Z}h_{B.GL}$ as a presheaf of simplicial abelian groups instead, the obvious map $h_{B.GL} \to \mathbb{Z}h_{B.GL}$ of simplicial presheaves factors through the Bousfield-Kan integral completion $\mathbb{Z}_\infty(h_{B.GL})$. Next we observe the weak-equivalence of simplicial presheaves

(4.6.10) $\mathbf{K} \simeq \mathbb{Z} \times \mathbb{Z}_\infty B.GL \simeq \lim_{m\to\infty} \mathbb{Z} \times \mathbb{Z}_\infty B.GL_m$

on the site $(B_x\mathfrak{S})_{et}$ (and also on the Zariski site of any scheme X). Here $\mathbf{K}$ is the presheaf of spaces defined by $U \to K(U)$ which is the K-theory space of the symmetric monoidal category of locally free coherent sheaves on U. It is shown in [Gi-1] Proposition (2.15) that if X is a Noetherian scheme of finite Krull dimension, one obtains a weak-equivalence $\mathbf{K}(X) \simeq \mathbb{Z} \times \mathbb{Z}_\infty B.GL(\mathcal{O}_X)$ which is *natural* in X. Since both sides commute with filtered colimits in X, it follows readily that one obtains the quasi-isomorphism of the presheaves in (4.6.10) on the site $(B_x\mathfrak{S})_{et}$. Moreover, the two presheaves $\mathbb{Z}_\infty h_{B.GL}$ and $\mathbb{Z}_\infty B.GL$ are identical.

Now one obtains a natural map $\mathbf{K}(\mathfrak{S}) \to \mathbb{H}_{et}(B_x\mathfrak{S}, \mathbf{K}) \simeq \mathbb{H}_{et}(B_x\mathfrak{S}, \mathbb{Z} \times \mathbb{Z}_\infty B.GL)$. Since the $C_i(n)$ are constant on the factor $\mathbb{Z}$, it follows that we obtain Chern classes:

(4.6.11) $C_i(n) : \pi_n(\mathbf{K}(\mathfrak{S})) \to \mathbb{H}^{di-n}_{smt}(\mathfrak{S}, \Gamma(i))$, if $\mathfrak{S}$ is any algebraic stack over k.

This is defined as the composition

$$\pi_n(\mathbf{K}(\mathfrak{S})) \to \pi_n(\underset{\Delta}{\mathrm{holim}}\mathbf{K}(B_x\mathfrak{S})) \to \pi_n(\mathbb{H}_{et}(\mathfrak{S}, \mathbf{K})) \cong \pi_n(\mathbb{H}_{et}(B_{\mathfrak{S}}, \mathbb{Z} \times \mathbb{Z}_\infty B.GL))$$

$$\to \pi_n(\mathbb{H}_{et}(B_{\mathfrak{S}}, \mathbb{Z} \times \mathbb{Z}h_{B.GL_{(B_x\mathfrak{S})_{et}}})) \xrightarrow{C_i(n)'} \mathbb{H}^{di-n}_{smt}(\mathfrak{S}, \Gamma(i))$$

These are the required higher Chern-classes. Since both the presheaves $h_{B.GL_n}$ and $\Gamma(i)$ are in fact presheaves on the big site $(alg.stacks/k)_{smt}$ it follows readily that the above Chern classes pull back under any representable map of stacks. If $n = 0$, we will let the classes $C_i(0)$ be denoted simply by c_i. It follows that if $\mathcal{E}$ is a vector bundle on the algebraic stack $\mathfrak{S}$, one has Chern-classes $c_i(\mathcal{E}) \in \mathbb{H}^{di}_{smt}(\mathfrak{S}; \Gamma(1))$.

(4.7) Next we outline how to define a Chern-character map $\pi_n(\mathbf{K}(\mathfrak{S})) \to \underset{i}{\oplus}\mathbb{H}^{di-n}(\mathfrak{S}, \Gamma(i))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ as in [Bl] section 8 (or [Sou] section 7). For this we start with classes $ch^p_i \in \mathbb{H}^{di}_C(\mathfrak{S}, \Gamma(i))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ instead of the classes C^p_i in (4.4.2): these are defined as in [SGA] 6, Exposé 0, appendix. Now one applies the above arguments to obtain maps

$$ch_i : \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, 0 \times \mathbb{Z}_\infty B.GL)) \to \mathbb{H}^{di-n}_{smt}(\mathfrak{S}, \Gamma(i))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$$

One defines $\overline{ch} : \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, 0 \times \mathbb{Z}_\infty B.GL)) \to \underset{i}{\oplus}\mathbb{H}^{di}_{smt}(\mathfrak{S}, \Gamma(i))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ as $\underset{i}{\oplus} ch_i$. Finally observe that $\pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, \mathbb{Z} \times \mathbb{Z}_\infty B.GL)) \cong \underset{r\, \epsilon\, \mathbb{Z}}{\oplus} \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, r \times \mathbb{Z}_\infty B.GL))$. Therefore one defines the Chern-character

$$\widehat{ch} : \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, \mathbb{Z} \times \mathbb{Z}_\infty B.GL)) \to \underset{i}{\oplus}\mathbb{H}^{di-n}_{smt}(\mathfrak{S}, \Gamma(i))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$$

by defining it to be $r + \overline{ch}$ on the rank r component. Finally composing with the obvious map $\pi_n(\mathbf{K}(\mathfrak{S})) \to \pi_n(\mathbb{H}_{et}(B_x\mathfrak{S}, \mathbb{Z} \times \mathbb{Z}_\infty B.GL))$ we obtain the Chern-character

(4.7.*) $ch : \pi_n(\mathbf{K}(\mathfrak{S})) \to \underset{i}{\oplus}\mathbb{H}^{di-n}_{smt}(\mathfrak{S}, \Gamma(i))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$

(4.7.1) Since the map $\widehat{ch}$ is natural in $\mathfrak{S}$, one may readily verify that it (not ch) becomes an isomorphism on tensoring the left-hand-side with $\mathbb{Q}$. One may define the chern character with values in $\underset{i}{\oplus}\mathbb{H}^{di-n}_{smt}(\mathfrak{S}, \Gamma(i)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$ in a similar manner.

Remark. Clearly the above assertions hold for smooth stacks if one considers complexes $\Gamma(i)$ defined only for smooth objects. In particular, taking $\Gamma(i) =$ the motivic complex $\mathbb{Z}(i)[2i]$, and making use of the comparison theorem (3.1), we obtain higher Chern classes for all smooth algebraic stacks with values in our higher intersection theory.

5. Intersection theory for smooth algebraic (Artin) stacks via motivic cohomology

Once again, we will restrict to smooth algebraic stacks throughout this section. Let $\mathfrak{S}$ denote a smooth algebraic stack with $x : X \to \mathfrak{S}$ a given atlas. For each integer $c \geq 0$, we let $\mathbf{CH}^c(\mathfrak{S}, x, .)$ denote the object defined in (3.0.3). (Recall from (3.3) that we may in fact omit the atlas x from the definition: therefore, the above object will be henceforth denoted $\mathbf{CH}^c(\mathfrak{S}, .)$.) We begin with the following theorem which establishes an intersection pairing on these higher Chow groups. (Observe that we are in fact working *integrally.*)

(5.1) **Theorem**. Now there exists an intersection pairing:

$\mathbf{CH}^i(\mathfrak{S},.) \otimes \mathbf{CH}^j(\mathfrak{S},.) \to \mathbf{CH}^{i+j}(\mathfrak{S},.)$

and hence an induced pairing $CH^i(\mathfrak{S},n) \otimes CH^j(\mathfrak{S},m) \to CH^{i+j}(\mathfrak{S},n+m)$ of the higher Chow groups. If $f : \mathfrak{S}' \to \mathfrak{S}$ is a map with $\mathfrak{S}'$ also a smooth algebraic stack, the map f^* is compatible with the above intersection products.

Proof. We recall from (A.4) that there exists a natural pairing $\mathbb{Z}(i)[2i]\otimes\mathbb{Z}(j)[2j] \to \mathbb{Z}(i+j)[2(i+j)]$ of the motivic complexes on the category of all smooth quasi-projective schemes over k. (Here we use the convention that $\mathbb{Z}(i) = \mathbb{Z}(i)^{F-S}$.) By (2.6.2)(vii) this extends to a pairing: $E_{\mathfrak{S}}(\mathbb{Z}(i)[2i]) \otimes E_{\mathfrak{S}}(\mathbb{Z}(j)[2j]) \to E_{\mathfrak{S}}(\mathbb{Z}(i+j)[2(i+j)])$ of the associated complexes on the smooth site of the stack $\mathfrak{S}$. Next we take hypercohomology on the smooth site of $\mathfrak{S}$ and invoke (3.1) Theorem, to obtain the pairing $\mathbf{CH}^i(\mathfrak{S},.) \otimes \mathbf{CH}^j(\mathfrak{S},.) \to \mathbf{CH}^{i+j}(\mathfrak{S},.)$. Taking the corresponding homotopy groups, we obtain the pairing of the higher Chow groups. The last assertion follows from the observation that the pairing in (2.6.2)(vii) is compatible with the induced map as in (2.6.2)(iv) and (3.4). $\square$

(5.2.0) *Remark.* The remark following (3.1) shows one obtains an intersection pairing on the higher Chow groups of all smooth schemes (locally of finite type) over k in a similar manner. This, combined with our localization theorem for all schemes locally of finite type over k (see [J-1] Theorem 4) provides a complete extension of higher Chow groups (at least modulo torsion) to all schemes locally of finite type over k and of finite dimension: previously this was available only for quasi-projective schemes over k.

In the rest of the paper we will show that if $\mathfrak{S}$ is a smooth separated Deligne-Mumford stack of finite type over k, there exists an isomorphism between $CH^*(\mathfrak{S},0;\mathbb{Q})$ and $CH^*_{naive}(\mathfrak{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ sending the above ring-structure to a previously known ring-structure on the latter. For this we need to show the existence of Adams operations on the étale K-theory of Deligne-Mumford stacks tensored with $\mathbb{Q}$.

(5.2.1) Given a presheaf P of spaces, we will let $P_{\mathbb{Q}}$ denote the localization of P at $\mathbb{Q}$ in the sense of [B-K] chapter V. (When P is the zero-th term of a presheaf of Ω-spectra, P is a presheaf of nilpotent spaces and therefore one obtains the isomorphism $\pi_*(P_{\mathbb{Q}}) \cong \pi_*(P)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ of presheaves. Observe that the presheaves of K-theory and G-theory spaces therefore have this property.)

(5.3.1) **Proposition**. (i) Let $\mathfrak{S}$ denote a Deligne-Mumford stack of finite type over k, let $B\mathfrak{S}$ denote the classifying simplicial scheme associated to a given atlas $x : X \to \mathfrak{S}$. Assume that X is a separated scheme so that each $(B\mathfrak{S})_n$ is also separated. Let $\mathbf{G}_{\mathbb{Q}}(B\mathfrak{S})$ denote the space $\underset{\Delta}{\operatorname{holim}}\{\mathbf{G}_{\mathbb{Q}}(B\mathfrak{S}_n)|n\}$ where $\mathbf{G}(B\mathfrak{S}_n)$ denotes the K-theory space of the complicial Waldhausen category of pseudo-coherent complexes with globally bounded cohomology on $(B\mathfrak{S})_n$. Then the obvious augmentation $\pi_*(\mathbf{G}_{\mathbb{Q}}(B\mathfrak{S})) \to \pi_*(\mathbb{H}_{et}(B\mathfrak{S},\mathbf{G}_{\mathbb{Q}})) \cong \pi_*(\mathbb{H}_{et}(\mathfrak{S},\mathbf{G}_{\mathbb{Q}}))$ is an isomorphism. Similarly if $T \subseteq \mathfrak{S}$ is a closed algebraic sub-stack of $\mathfrak{S}$ and BT is the associated simplicial algebraic space, the obvious augmentation $\pi_*(\mathbf{G}_{\mathbb{Q},BT}(B\mathfrak{S})) \to \pi_*(\mathbb{H}_{BT,et}(B\mathfrak{S},\mathbf{G}_{\mathbb{Q}}))$ is an isomorphism. (We have let $\mathbf{G}_{\mathbb{Q},BT}(B\mathfrak{S})$ = the canonical homotopy fiber of the obvious map $\mathbf{G}_{\mathbb{Q}}(B\mathfrak{S}) \to \mathbf{G}_{\mathbb{Q}}(B\mathfrak{S} - BT)$.)

(ii) Moreover, the obvious map $\lim_{N\to\infty} \pi_*(\mathbb{H}_{BT,et}(B\mathfrak{S},(\mathbb{Z}\times\mathbf{B}GL_N^+)_{\mathbb{Q}})) \to \pi_*(\mathbb{H}_{BT,et}(B\mathfrak{S},(\mathbb{Z}\times\mathbf{B}GL^+)_{\mathbb{Q}}))$ is an isomorphism. (Here $\mathbf{B}GL_N$ ($\mathbf{B}GL$) are the obvious presheaves of simplicial sets and the + denotes the + construction applied to these. One may identify the presheaf $\mathbf{K}$ with $\mathbb{Z}\times\mathbf{B}GL^+$ upto natural quasi-isomorphism.)

Proof. (i) Since the homotopy inverse limits preserve quasi-isomorphism, it suffices to show that for each fixed integer $n \geq 0$, the obvious augmentation $\pi_*(\mathbf{G}_{\mathbb{Q},(BT)_n}((B\mathfrak{S})_n)) \to \pi_*(\mathbb{H}_{(BT)_n,et}(B\mathfrak{S}_n,\mathbf{G}_{\mathbb{Q}}))$ is an isomorphism. This follows from [T-1] Theorem (2.15) since each $B\mathfrak{S}_n$ is a separated scheme of finite type over k.

Next we consider (ii). It follows from [Sou] p. 510 that, on taking the direct limit as $N \to \infty$, the map of presheaves $\mathbf{Z}\times\mathbf{B}GL_N^+ \to \mathbf{Z}\times\mathbf{B}GL^+$ induces a stalk-wise weak-equivalence.

Let P denote a presheaf of spaces on $\mathfrak{S}_{et}$. Now we observe that in the spectral sequence

$E_2^{s,t} = H^s_{et}(B\mathfrak{S},\pi_{-t}(P_{\mathbb{Q}})) \Rightarrow \pi_{-s-t}\mathbb{H}_{et}(B\mathfrak{S},P_{\mathbb{Q}})$

$E_2^{s,t} = 0$ for all but finitely many s, independent of t. In particular, the above spectral sequence converges strongly. (One may easily prove this when the stack is the quotient stack associated to the action of a finite group. Now devissage (see [J-1] (5.1.2.*) or [L-MB1] Theorem (10.2) and Corollaire (10.2.1)) enables us to reduce to the case of a general Deligne-Mumford stack to this.)

Now we obtain a natural map of the spectral sequences:

$$E_2^{s,t} = \lim_{N\to\infty} H^s_{et}(B\mathfrak{S}, \pi_{-t}((\mathbb{Z}\times \mathbf{B}GL_N^+)_{\mathbb{Q}})) \Rightarrow \lim_{N\to\infty} \pi_{-s-t}(\mathbb{H}_{et}(B\mathfrak{S}, (\mathbb{Z}\times \mathbf{B}GL_N^+)_{\mathbb{Q}})) \text{ and}$$

$$E_2^{s,t} = H^s_{et}(B\mathfrak{S}, \pi_{-t}((\mathbb{Z}\times \mathbf{B}GL^+)_{\mathbb{Q}})) \Rightarrow \pi_{-s-t}(\mathbb{H}_{et}(B\mathfrak{S}, (\mathbb{Z}\times \mathbf{B}GL^+)_{\mathbb{Q}}))$$

Since both the spectral sequences converge strongly, it suffices to show that one obtains an isomorphism at the E_2-terms. This in turn follows from the observation that the étale cohomology of a Deligne-Mumford stack with respect to an abelian sheaf (tensored with $\mathbb{Q}$) commutes with filtered colimits of such sheaves. Once again one may reduce this assertion to the case of quotient stacks associated to finite group actions, where it is clear. □

Remark. It may be worth pointing out that the result (ii) of the last proposition is essential in establishing a λ-ring structure on $\pi_*(\mathbf{K}_{\mathbb{Q}}(B\mathfrak{S}))$ at least if one wishes to apply the arguments in [Sou]. This would be clearly false if did not use rational K-theory of the simplicial space $B\mathfrak{S}$.

(5.3.2) **Corollary**. Let $\mathfrak{S}$ denote a *smooth* Deligne-Mumford stack of finite type over k and let $T \subseteq \mathfrak{S}$ denote an integral algebraic sub-stack. (i) Then $\pi_*(\mathbb{H}_T(\mathfrak{S}, \mathbf{K}_{\mathbb{Q}})) \cong \pi_*(\mathbb{H}_{BT,et}(B\mathfrak{S}, \mathbf{K}_{\mathbb{Q}})) \cong \pi_*(\mathbf{K}_{\mathbb{Q},BT}(B\mathfrak{S}))$ has the structure of a graded λ-ring, there exists an associated γ-filtration on $\pi_*(\mathbf{K}_{\mathbb{Q},BT}(B\mathfrak{S}))$ and one also has Adams operations $\{\psi^k | k\}$ defined on it. (ii) There exists a spectral sequence

$$E_1^{s,t} = \underset{p\ \epsilon\ \mathfrak{S}^{(s)}}{\oplus} \pi_{-s-t}(\mathbf{K}_{\mathbb{Q},Bx}(B\mathfrak{S})) \Rightarrow \pi_{-s-t}(\mathbf{K}_{\mathbb{Q}}(B\mathfrak{S})) \cong \pi_{-s-t}(\mathbb{H}_{et}(\mathfrak{S}, \mathbf{K}_{\mathbb{Q}}))$$

where the direct sum varies over all closed integral sub-stacks of codimension p. (iii) The Adams operations act on the $E_r^{s,t}$-terms of the above spectral sequence and are compatible with the differentials.

Proof. Since the stack $\mathfrak{S}$ is smooth, observe that one may identify the presheaves $\mathbf{G}$ and $\mathbf{K}$ up to quasi-isomorphism. (See (1.6.2).) Moreover, we may choose an atlas $x : X \to \mathfrak{S}$ so that X is a separated scheme (for example an affine or quasi-projective scheme). Therefore (5.3.1) provides the isomorphisms in (i). Now the proof of the remaining assertion in (i) follows immediately from the isomorphisms in (i) since one can now define λ-operations as in [Sou] section 4. We will briefly recall this for the sake of completeness. Let $\rho : GL_N \to GL_M$ denote a representation of the group scheme GL_N. If $\mathbf{B}GL_N$ and $\mathbf{B}GL_M$ denote the associated simplicial sheaves on $\mathfrak{S}_{et}$ or on the étale site of the simplicial scheme $B\mathfrak{S}$, ρ induces a map $\mathbf{B}GL_N \to \mathbf{B}GL_M$. Recall these are presheaves of simplicial groups on $\mathfrak{S}_{et}$ or equivalently on the étale site of the simplicial scheme $B\mathfrak{S}$. (i.e. We may take the complete pointed simplicial category as in [J-1] section 6 to be the category of fibrant pointed simplicial sets.) Composing with the obvious map to $\mathbf{B}GL \to \mathbf{B}GL^+$, ρ induces a map $\rho : \mathbf{Z}\times \mathbf{B}GL_N \to \mathbf{Z}\times \mathbf{B}GL^+$. i.e. one obtains a map of abelian groups $R_{\mathbb{Z}}(GL_N) \to \pi_0(RMap(\mathbf{Z}\times \mathbf{B}GL_N, \mathbf{Z}\times \mathbf{B}GL^+))$ where $RMap(\mathbf{Z}\times \mathbf{B}GL_N, \mathbf{Z}\times \mathbf{B}GL^+))$ is defined to be $\underset{\Delta}{\text{holim}} Map(\mathbf{Z}\times \mathbf{B}GL_N^+, \mathcal{G}\cdot \mathbf{Z}\times \mathbf{B}GL^+)$ and $R_{\mathbb{Z}}(GL_N)$ denotes the integral representation ring of the group scheme GL_N. (The functor Map : (*simplicial presheaves on* $\mathfrak{S}_{et}$) $\times$ (*simplicial presheaves on* $\mathfrak{S}_{et}$) $\to$ (*simplicial sets*) is defined by $Map(F, K)_n = Hom_{simplicial\ \ presheaves}(F \times \Delta[n], K)$.)

Therefore one obtains an *additive* homomorphism $r : \lim_{\infty\leftarrow N} R_{\mathbb{Z}}(GL_N) \to \lim_{\infty\leftarrow N} \pi_0(RMap(\mathbf{Z}\times \mathbf{B}GL_N, \mathbf{Z}\times \mathbf{B}GL^+)) \cong \lim_{\infty\leftarrow N} \pi_0(RMap(\mathbf{Z}\times \mathbf{B}GL_N^+, \mathbf{Z}\times \mathbf{B}GL^+)$ where the last isomorphism follows from the universal property of the + construction. Now $r(\{\lambda^n(id_{GL_N} - N)|N\})$ defines a compatible collection of homotopy classes of maps $\lambda^n : \mathbf{Z}\times \mathbf{B}GL_N^+ \to \mathbf{Z}\times \mathbf{B}GL^+$ of presheaves.

By (5.3.1)(ii) we observe that the obvious map

$$\lim_{N\to\infty} \pi_*(\mathbb{H}_{BT,et}(B\mathfrak{S}, (\mathbf{Z}\times \mathbf{B}GL_N^+)_{\mathbb{Q}})) \to \pi_*(\mathbb{H}_{BT,et}(B\mathfrak{S}, (\mathbf{Z}\times \mathbf{B}GL^+)_{\mathbb{Q}}))$$

is an isomorphism. It follows that on taking $\pi_*\mathbb{H}_{BT,et}(B\mathfrak{S},\quad)$ the map λ^n induces the lambda operation $\lambda^n : \pi_*(\mathbf{K}_{\mathbb{Q},BT}(B\mathfrak{S})) \to \pi_*(\mathbf{K}_{\mathbb{Q},BT}(B\mathfrak{S}))$. To prove that one obtains the usual relations on the λ^ns, one reduces to showing they hold on $\lim_{\infty\leftarrow N} \pi_0(RMap(\mathbf{Z}\times \mathbf{B}GL_N^+, \mathbf{Z}\times \mathbf{B}GL^+))$: this follows readily from the fact they hold on the representation ring $\lim_{\infty\leftarrow N} R_{\mathbb{Z}}(GL_N)$. The existence of the Adams operations is now a formal consequence.

Each closed integral sub-stack Y of $\mathfrak{S}$ defines a closed sub-simplicial scheme of $B\mathfrak{S}$. Now we obtain the fibration sequence:

$$\underset{Y\ \epsilon\ M_{s+1}}{\text{colim}} \mathbf{K}_{\mathbb{Q},BY}(B\mathfrak{S}) \to \underset{Y'\ \epsilon\ M_s}{\text{colim}} \mathbf{K}_{\mathbb{Q},BY'}(B\mathfrak{S}) \to \underset{p\ \epsilon\ \mathfrak{S}^{(s)}}{\oplus} \mathbf{K}_{\mathbb{Q},Bp}(B\mathfrak{S})$$

where M_k denotes the full sub-category of closed integral sub-stacks of codimension $\leq k$ and BY (BY', Bp) denotes the obvious classifying simplicial scheme. Moreover, $\mathfrak{S}^{(p)}$ denotes all the points of codimension exactly p; these are the generic points of codimension p integral sub-stacks of $\mathfrak{S}$. Taking the homotopy groups we obtain an associated long-exact sequence which provides the spectral sequences in (ii). The identifications $\underset{Y \,\epsilon\, M_k}{\operatorname{colim}}\, \pi_*(\mathbf{K}_{\mathbb{Q},BY}(B\mathfrak{S})) \simeq \underset{Y \,\epsilon\, M_k}{\operatorname{colim}}\, \pi_*\mathbb{H}_{et,Y}(\mathfrak{S}, (\mathbf{Z}\times\mathbf{B}GL^+)_{\mathbb{Q}})$ and the observation that each λ^n is actually induced by a map $\mathbf{Z}\times\mathbf{B}GL_N^+ \rightarrow \mathbf{Z}\times\mathbf{B}GL^+$ shows that the differentials in the spectral sequence are compatible with the λ operations and hence with the Adams operations. This completes the proof of (ii) and (iii). $\square$

(5.3.3) *Gersten resolutions on smooth Deligne-Mumford stacks* (See [Gi-2](4.7).)

Let $\mathfrak{S}$ denote a smooth Deligne-Mumford stack. For each integer p, $\mathcal{R}^*(t)$ is the presheaf on the étale site of $\mathfrak{S}$ defined as follows. For each $U \rightarrow \mathfrak{S}$ in the étale site of $\mathfrak{S}$, and each point u of U, we let $i_u : Spec \quad k(u) \rightarrow U$ denote the obvious map, where $k(u)$ denotes the generic point of u. We let $\mathcal{R}^*(t)_{|U_{et}}$ be the complex of sheaves on U_{et} defined by

$$(5.3.3.1)\quad \underset{u \,\epsilon\, U^{(0)}}{\oplus} i_{u*}K_t(k(u)) \rightarrow \dots \rightarrow \underset{u \,\epsilon\, U^{(i)}}{\oplus} i_{u*}K_{t-i}(k(u)) \rightarrow \dots \rightarrow \underset{u \,\epsilon\, U^{(t)}}{\oplus} i_{u*}K_0(k(u)).$$

Here $U^{(i)}$ denotes the set of points of codimension i in U. Since each U is a smooth scheme, it follows from Quillen's proof of Gersten's conjecture that the obvious map $\pi_t(\mathbf{K}) \rightarrow \mathcal{R}^*(t)$ is a *resolution* on the étale site of the stack $\mathfrak{S}$. i.e. if $U \rightarrow \mathfrak{S}$ belongs to the étale site of the stack $\mathfrak{S}$, $\mathcal{R}^*(t)_{|U_{et}}$ is a resolution of the sheaf $\pi_t(\mathbf{K})_{|U_{et}}$. It follows that $\mathcal{R}^*(t)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is a resolution of the sheaf $\pi_t(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. If $x : X \rightarrow \mathfrak{S}$ is an atlas for the stack, it follows that $\mathcal{R}^*(t)_{B_x\mathfrak{S}_n}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is a resolution of the sheaf $\pi_t(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ on $B_x\mathfrak{S}_{n,et}$. Observe that $\mathcal{R}^*(t)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q} = \{\mathcal{R}^*(t)_{B_x\mathfrak{S}_n}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}|n\}$ is now a complex of sheaves on the étale site $B_x\mathfrak{S}_{et}$ of the simplicial scheme $B_x\mathfrak{S}$. Therefore $CH^*_{naive}(\mathfrak{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \cong \underset{s}{\oplus}\mathbb{H}^s_{et}(B_x\mathfrak{S}, \mathcal{R}^*(s)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \cong \underset{s}{\oplus}H^s_{et}(B_x\mathfrak{S}, \pi_s(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \cong \underset{s}{\oplus}H^s_{et}(\mathfrak{S}, \pi_s(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$.

Next we proceed to interpret the E_1-terms of the spectral sequence in (5.3.2) in terms of such a Gersten resolution. For this one begins with a punctual sub-stack p of codimension p in the stack $\mathfrak{S}$. One may now observe that, since p is punctual, it is regular and a closed sub-stack of an open sub-stack U of $\mathfrak{S}$ in (5.3.2)(ii)). Moreover, now

$$(5.3.3.2)\ \pi_{-s-t}(\mathbf{K}_{\mathbb{Q},Bp}(BU)) \cong \pi_{-s-t}(\mathbf{K}_{\mathbb{Q}}(Bp)) \cong \mathbb{H}^0_{et}(p, \pi_{-s-t}(\mathbf{K}_{\mathbb{Q}}))$$
$$\cong \mathbb{H}^0_{et}(p, \pi_{-s-t}(\mathbf{K}))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \cong \pi_{-s-t}(\mathbf{K}(k(p)))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} = K_{-s-t}(k(x))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}.$$

Here $k(p) = H^0(p, \mathcal{O}_p)$ is the function field of the (punctual) stack p and the last isomorphism follows from (5.3.9)(i) below. Therefore the E_1-terms of the spectral sequence in (5.3.2) form the complex of abelian groups:

$$(5.3.3.3)\ R^*(t)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} : \underset{p \,\epsilon\, \mathfrak{S}^{(0)}}{\oplus} K_t(k(p))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \rightarrow \cdots \rightarrow \underset{p \,\epsilon\, \mathfrak{S}^{(i)}}{\oplus} K_{t-i}(k(p))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \rightarrow \cdots \rightarrow \underset{p \,\epsilon\, \mathfrak{S}^{(p)}}{\oplus} \underline{\mathbb{Q}}$$

where $\mathfrak{S}^{(i)}$ denotes the set of points (=integral closed sub-stacks) of codimension i in $\mathfrak{S}$. One proceeds to compare the above complex with $\mathcal{R}^*(t)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$.

Observe that a closed integral sub-stack T of $\mathfrak{S}$ corresponds to a closed integral subscheme Y of the atlas X so that $d_0^{-1}(Y) = d_1^{-1}(Y)$ (where $d_i^{-1}(Y) = Y\underset{X}{\times}(B_x\mathfrak{S})_1$ with $d_i : B_x\mathfrak{S}_1 \rightarrow B_x\mathfrak{S}_0 = X$ denotes the obvious map). Moreover it is shown in [Gi-2] p.219 that $K_t(k(p))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} = ker(\delta^0 - \delta^1 : K_t(k(x_0))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \rightarrow K_t(k(x_1))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ where x_0 is a point of the atlas X lying above the point p and $x_1 = d_0^{-1}(p) = d_1^{-1}(p)$. It follows that

$$(5.3.3.4)\ R^*(t)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} = ker((\delta^0 - \delta^1) : \Gamma(B_x\mathfrak{S}_0, \mathcal{R}^*(t)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \rightarrow \Gamma(B_x\mathfrak{S}_1, \mathcal{R}^*(t)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}))$$

We proceed to show that the s-th cohomology of the complex on the right-hand-side above may be identified with $H^s_{et}(B_x\mathfrak{S}, \mathcal{R}^*(t)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$. We begin with the following lemma.

(5.3.3.5) **Lemma**. Assume, in addition, that the stack $\mathfrak{S}$ is separated. Let $\bar{x}_* : Sh(B_x\mathfrak{S}_{et}) \rightarrow Sh(\mathfrak{S}_{et})$ be defined by $\bar{x}_*(F) = ker(\delta^0 - \delta^1 : x_{0*}(F_0) \rightarrow x_{1*}(F_1))$ where $x_i : B_x\mathfrak{S}_i \rightarrow \mathfrak{S}$ is the obvious map induced by the atlas $x : X \rightarrow \mathfrak{S}$. Then the functor $\bar{x}_*$ is exact for sheaves of $\mathbb{Q}$-vector spaces.

Proof. Let $\mathfrak{M}$ denote a coarse moduli space for the stack $\mathfrak{S}$ and let $\pi : \mathfrak{S} \to \mathfrak{M}$ denote the obvious map. Clearly this is finite since the stack $\mathfrak{S}$ is separated and Deligne-Mumford. Next we prove that the composite functor $R\pi_* \circ R\bar{x}_* = R(\pi \circ \bar{x})_*$ is exact for sheaves of $\mathbb{Q}$-vector spaces. For each $i \geq 0$, let $x_i : B_x\mathfrak{S}_i \to \mathfrak{S}$ be the obvious structure map. Now $\{x_{i_*}(F_i)|i\}$ is a cosimplicial object of sheaves on $\mathfrak{S}_{et}$ and $\bar{x}_*(F) = \lim_{\Delta}\{x_{i*}(F_i)|i\}$ for any sheaf $F = \{F_i|i\}$ on $B_x\mathfrak{S}_{et}$. We let $\bar{x}_\bullet : B_x\mathfrak{S} \to \mathfrak{S}$ denote the map of simplicial objects given in degree i by x_i.

One may first reduce to the case $\mathfrak{S}$ is in fact a quotient stack $[X/G]$ for a finite group; in this case the geometric fibers of the map $\pi \circ \bar{x}_\bullet$ may be identified with the classifying simplicial schemes of the stabilizers. Therefore one may identify the stalks of $R^n(\pi \circ \bar{x})_*$ with the cohomology of the classifying spaces of finite groups with rational coefficients; hence these are trivial for all $n > 0$. (See [J-2] for more details.)

Since π is finite, one may observe readily that π_* is an exact functor. It follows that the composite functor $\pi_* \circ \bar{x}_*$ as well π_* are exact and that $R^n\pi_*(F) = 0 = R^n(\pi \circ \bar{x})_*(F)$ for $n > 0$ and for sheaves of $\mathbb{Q}$-vector spaces F. Now consider the spectral sequence $E_2^{s,t} = R^s\pi_*R^t\bar{x}_*(F) \Rightarrow R^{s+t}(\pi \circ \bar{x})_*(F)$. $E_2^{s,t} = 0$ for $s > 0$ and therefore $\pi_*R^t(\bar{x})_*(F) \cong R^t(\pi \circ \bar{x})_*(F) \cong 0$ for $t > 0$ and for any sheaf F of $\mathbb{Q}$-vector spaces. Since π is finite, it follows that $R^t(\bar{x})_*(F) = 0$ for $t > 0$ and any sheaf F of $\mathbb{Q}$-vector spaces. $\square$

If p denotes a point of the stack $\mathfrak{S}$, we will identify it with the corresponding punctual sub-stack; now $i_p : p \to \mathfrak{S}$ will denote the obvious map. One may observe that $i_{p*}\pi_p(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is a sheaf on $\mathfrak{S}_{et}$; this is flabby by (5.3.9) below.

(5.3.3.6) **Proposition** Let $F_i = \mathcal{R}^*(t)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}(i)$ denote the i-th term of the complex $\mathcal{R}^*(t)_{B_x\mathfrak{S}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. This is a flabby sheaf on $B_x\mathfrak{S}_{et}$ in the sense that $H^n_{et}(B_x\mathfrak{S}, F_i) = 0$ for $n > 0$.

Proof. Observe that $H^n_{et}(B_x\mathfrak{S}, F_i) = H^n_{et}(\mathfrak{S}, R\bar{x}_*(F_i)) \cong H^n_{et}(\mathfrak{S}, \bar{x}_*(F_i))$ by the above lemma. Observe next that $\bar{x}_*(F_i) = \underset{p\,\epsilon\,\mathfrak{S}^{(i)}}{\oplus} i_{p*}\pi_{t-i}(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. By the observations above, it follows that $\bar{x}_*(F_i)$ is flabby. $\square$

(5.3.3.7) **Corollary** The $E_2^{s,-t}$-term of the spectral sequence in (5.3.2) identifies with $H^s_{et}(\mathfrak{S}, \mathcal{R}^*(t)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \cong H^s_{et}(\mathfrak{S}, \pi_t(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$.

Proof. This is clear in view of the above discussion. $\square$.

Let $\mathfrak{S}$ be smooth and let $i : T \to \mathfrak{S}$ denote the closed immersion of a smooth algebraic sub-stack. Then the normal bundle, N, associated to i exists as a vector bundle of rank p if the codimension of T in $\mathfrak{S}$ is p. Observe from [L-MB1] (4.2.3) that the construction of the *Proj* of a coherent sheaf on stacks is a *local construction.* This shows that the techniques of blow-ups and deformation to the normal cone (see [BFM] section 2) extend to algebraic stacks. Recall the deformation to the normal cone is constructed from the diagram

$$(5.3.4)\quad \begin{array}{ccccc} T & \xrightarrow{j_1} & T \times \mathbb{A}^1 & \xleftarrow{j_0} & T \\ {\scriptstyle i}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \bar{i}}\downarrow \\ \mathfrak{S} & \xrightarrow{k_1} & W & \xleftarrow{k_0} & N \end{array}$$

where W is the blow-up of $\mathfrak{S} \times \mathbb{A}^1$ along $T \times 0$. The naturality of the above deformation shows that if $B\mathfrak{S}$ is a classifying simplicial scheme associated to the stack $\mathfrak{S}$ one obtains a similar diagram of simplicial schemes:

$$(5.3.5)\quad \begin{array}{ccccc} BT & \xrightarrow{Bj_1} & BT \times \mathbb{A}^1 & \xleftarrow{Bj_0} & BT \\ {\scriptstyle Bi}\downarrow & & {\scriptstyle B\psi}\downarrow & & {\scriptstyle B\bar{i}}\downarrow \\ B\mathfrak{S} & \xrightarrow{Bk_1} & BW & \xleftarrow{Bk_0} & BN \end{array}$$

One defines the algebraic K-theory of a simplicial scheme $X.$ by $\underset{\Delta}{\mathrm{holim}}\{K(X_n)|n\}$ which is the homotopy inverse limit of the cosimplicial spectrum $\{K(X_n)|n\}$, where each $K(X_n)$ is defined as in (1.5.1). Similarly if $Y.$ is a closed sub-simplicial scheme of $X.$, one defines $K_{Y.}(X.) = \underset{\Delta}{\mathrm{holim}}\{K_{Y_n}(X_n)|n\}$ with each $K_{Y_n}(X_n)$ defined as in (1.5.2). Since all the vertices in the above diagram are schemes, one obtains the excision isomorphisms (see for example [T-2] (3.19)):

$$(5.3.6)\quad \pi_*(K_{BT}(B\mathfrak{S})) \xleftarrow{k_1^*} \pi_*(K_{BT\times\mathbb{A}^1}(BW)) \xrightarrow{k_0^*} \pi_*(K_{BT}(BN)).$$

Observe also that the pull-back maps are compatible with the action of the Adams operations as in (5.3.2).

Let $\pi : N \to T$ denote the projection from the normal bundle to T and let $\Lambda(\pi^*(N^\vee))$ denote the Koszul-Thom class associated to $N^\vee =$ the co-normal bundle associated to i. This is a class in $\pi_0(K_T(N))$. Now $\Lambda(\pi^*(N^\vee))$ is a complex of vector bundles on N with support contained in T. Let BN denote the normal bundle to the closed immersion of BT in $B\mathfrak{S}$ and let $B\pi : BN \to BT$ denote the obvious map. The pull-back of $\pi^*(N^\vee)$ to BN is the bundle $B\pi^*BN^\vee$ and one may verify the Koszul-Thom class $\Lambda(\pi^*(N^\vee))$ pulls-back to the Koszul-Thom class $\Lambda(B\pi^*(BN^\vee))$ which represents a class in $\pi_0(K_{BT}(BN))$. Recall we have used the Waldhausen style K-theory of perfect complexes to define the K-groups. (See (1.5.2).) The map $E^\cdot \to \pi^*(E^\cdot) \cup \Lambda(\pi^*(N^\vee))$ $(E^\cdot \to B\pi_n^*(E^\cdot) \cup \Lambda(B\pi_n^*(BN_n^\vee)))$ defines a map

(5.3.7) $K(T)) \to K_T(N)$ $(K(BT_n) \to K_{BT_n}(BN_n)$ for each n, respectively)

Moreover, since $\pi^*(E^\cdot) \cup \Lambda(\pi^*(N^\vee)) \to \bar{i}_*(E^\cdot)$ is a natural map in $E^\cdot$ which is a quasi-isomorphism one may verify that the first map may be identified with $\bar{i}_*$. Similarly the second map above may be identified with $B_n\bar{i}_*$ for each n.

(5.3.8) **Lemma**. Let the codimension of T in $\mathfrak{S}$ be c. Let $\Lambda \,\epsilon\, \pi_0(K_T(N))$ denote the class of the Koszul-Thom class $\Lambda(\pi^*(N^\vee))$. Now $\psi^k(\Lambda) = k^c\Lambda$. Similarly, if $\bar{\Lambda}_n \,\epsilon\, \pi_0(K_{BT_n}(BN_n))$ denotes the class of $\Lambda(B\pi^*(BN_n^\vee))$, $\psi^k(\bar{\Lambda}_n) = k^c\bar{\Lambda}_n$.

Proof. We can adopt the proof in [G-S] (4.12.1) and Lemma (4.12) verbatim. For this we use the inductive formula

$$\psi^k - \psi^{k-1} \cup \lambda^1 + ... + (-1)^{k-1}\psi^1 \cup \lambda^{k-1} + (-1)^k k\lambda^k = 0,\ k \geq 1.$$

as in [G-S] (4.12.1). One may reduce immediately to the case where the Koszul-Thom complex $\Lambda(\pi^*(\mathcal{N}))$ is replaced by the complex: $\mathcal{O}_N \xrightarrow{a} \mathcal{O}_N$ where a corresponds to an element in $\mathcal{O}$ locally. Now it suffices to show that λ^k applied to the above complex = the same complex shifted $[k-1]$ times to the left, where λ^k denotes the k-th exterior power operation. The same proof as in [G-S] lemma (4.12) applies verbatim to establish this. $\square$

(5.3.9) **Lemma**. (i) Let p denote a punctual Deligne-Mumford stack. If $k(p) = H^0(p, \mathcal{O}_p)$, one obtains the isomorphism $H^i_{et}(p, \pi_p(\mathbf{K}))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \cong \pi_p(\mathbf{K}(k(p)))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$, if $i = 0$ and $\cong 0$ otherwise. There exist Adams operations on both of the above terms and the above isomorphisms are compatible with the action of the Adams operations.

(ii) Assume in addition that $i : p \to U$ is the closed immersion of a punctual sub-stack of codimension p into a regular stack U. Then the map Bi_* induces an isomorphism $\pi_*K(Bp) \xrightarrow{\cong} \pi_*(K_{Bp}(BU))$.

(iii) Let N denote the normal bundle associated to the closed immersion in (ii). If $\bar{i} : p \to N$ is the zero section, it induces an isomorphism as in (ii). The map $B\bar{i}_*$ may be identified with the map $\alpha \to \pi^*(\alpha) \cup \Lambda(\pi^*(BN^\vee))$ modulo torsion.

Proof. (i) The isomorphism in (i) is established in [Gi-2] (5.3) using the degeneration of the spectral sequence (from the E_1-terms onwards): $E_1^{i,j} = H^j_{et}((Bp)_i; \pi_p(\mathbf{K})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \Rightarrow H^{i+j}_{et}(Bp, \pi_p(\mathbf{K}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}))$. i.e. $E_1^{i,j} = 0$ if $j \neq 0$. (Observe that we keep p fixed throughout.) Recall the Adams operations may be realized as maps of the presheaves $\mathbf{Z} \times \mathbf{B}GL_N^+ \to \mathbf{Z} \times \mathbf{B}GL^+$; therefore they are compatible with the differentials of the above spectral sequence. This proves (i).

Observe that it suffices to show each of the maps $B\bar{i}_{n*} : K(Bp_n) \to K_{Bp_n}(BN_n)$ is a weak-equivalence. Next one observes readily that each $(Bp)_n$ is isomorphic to a finite disjoint union of the spectra of fields. Therefore, one obtains the weak-equivalences: $K((Bp)_n) \simeq G((Bp)_n)$ for all n and $K_{Bp_n}(BN_n) \simeq G(Bp_n)$ for all n. To see the map $B\bar{i}_{n*} : K(Bp_n) \simeq G(Bp_n) \to K_{Bp_n}(BN_n)$ is a weak-equivalence, one just compares fibration sequence provided by Quillen's localization theorem with fibration sequence defining $K_{Bp_n}(BN_n)$. This proves (ii).

That the map $\bar{i}$ induces an isomorphism as in (ii) is clear from (ii) applied to the closed immersion $\bar{i}$. To see this may be identified with the map as in (iii) we observe the existence of a natural quasi-isomorphism $\pi^*(E^\cdot) \cup \Lambda(B\pi^*BN^\vee) \to B\bar{i}_{n*}(E^\cdot)$ for any complex of vector bundles $E^\cdot$ on BT_n. It follows from standard results on Waldhausen K-theory that the map $B\bar{i}_*$ may now be identified with taking cup product with $\Lambda(B\pi^*BN^\vee)$. $\square$

(5.3.10) **Theorem** Let $\mathfrak{S}$ denote a smooth separated Deligne-Mumford stack of finite type over k. Now one obtains isomorphisms

$$CH^*(\mathfrak{S}, 0; \mathbb{Q}) \cong \underset{s}{\oplus} H^s_{et}(\mathfrak{S}, \pi_s(\mathbf{K}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \cong CH^*_{naive}(\mathfrak{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

preserving the ring structures on each.

Proof. We proceed to show (see [Sou] pp. 524-525) that the differentials $d_r : E_r^{s-r,-s+r-1} \to E_r^{s,-s}$, $r \geq 2$ are trivial modulo torsion. For this one begins with a punctual sub-stack p of codimension s in the stack $\mathfrak{S}$. One may now observe that, since p is punctual, it is regular and a closed sub-stack of an open sub-stack U of $\mathfrak{S}$ in (5.3.2)(ii)). Moreover, recall

$$\pi_{-s-t}(\mathbf{K}_{\mathbb{Q},Bp}(BU)) \cong \pi_{-s-t}(\mathbf{K}_{\mathbb{Q}}(Bp))$$
$$\cong \mathbb{H}^0_{et}(p, \pi_{-s-t}(\mathbf{K}_{\mathbb{Q}})) \cong \mathbb{H}^0_{et}(p, \pi_{-s-t}(\mathbf{K})) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \pi_{-s-t}(\mathbf{K}(k(p))) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}.$$

For $-s-t = 0, -1$, or -2, one knows that the Adams operation ψ^k acts on $\pi_{-s-t}(\mathbf{K}(k(p)))$ by k^{-s-t}. The first isomorphism follows from (5.3.9) (ii); moreover (5.3.9)(iii) shows we may assume this isomorphism is given by cup-product with a Koszul-Thom class associated to the normal bundle for the closed immersion of p in U. Therefore, by (5.3.8), one observes that ψ^k acts on $E_1^{u,v} = \pi_{-u-v}(\mathbf{K}_{Bp}(B\mathfrak{S})) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (which is the spectral sequence in (5.3.2)) by multiplication with k^{-v}. Since the differentials are compatible with the action of the Adams operations, it follows that the action of ψ^k on $E_r^{u,v}$ is by k^{-v} for $r \geq 1$.

Next let $\alpha \,\epsilon\, E_r^{u,v}$. Now $k^{-v} d_r(\alpha) = d_r(k^{-v}\alpha) = d_r(\psi^k(\alpha)) = \psi^k(d_r(\alpha)) = k^{-v+r-1} d_r(\alpha)$, since $d_r(\alpha) \,\epsilon\, E_r^{u+r,v-r+1}$. Letting $u = s - r$ and $v = -s + r - 1$, it follows that $d_r(\alpha) = 0$ modulo torsion, if $d_r : E_r^{s-r,-s+r-1} \to E_r^{s,-s}$, $r \geq 2$. Observe that the differentials $d_r : E_r^{s,-s} \to E_r^{s+r,-s-r+1}$ are trivial since $E_1^{s+r,-s-r+1} = 0$ for all $r \geq 2$ and all s. It follows that $E_2^{s,-s} \cong E_\infty^{s,-s}$ which is the s-th associated graded piece of $\pi_0(\mathbb{H}_{et}(\mathfrak{S}, \mathbf{K}_{\mathbb{Q}}))$.

Recall $CH^s_{naive}(\mathfrak{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = H^s_{et}(B\mathfrak{S}, \mathcal{R}^*(s) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \cong H^s_{Zar}(B\mathfrak{S}, \mathcal{R}^*(s) \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ where the last isomorphism follows by [Gi-2] proof of Theorem (6.1). One may identify the last term with $H^s_{et}(\mathfrak{S}, \pi_s(\mathbf{K}) \otimes \mathbb{Q}) \cong E_2^{s,-s}$ by (5.3.3.7). The ring structure on $CH^*_{naive}(\mathfrak{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is obtained from the obvious one on $\underset{s}{\oplus} H^s_{et}(\mathfrak{S}, \pi_s(\mathbf{K}_{\mathbb{Q}}))$.

Now it suffices to prove the first isomorphism in the theorem preserves the ring structure. (Clearly the other isomorphisms preserve the obvious ring structures.) For this one observes that the Chern-character induces a quasi-isomorphism (stalk-wise) of the presheaves $U \to \mathbf{K}(U)_{\mathbb{Q}}$ and $U \to \mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. (Though the Riemann-Roch theorem in [Bl-1] is stated as an isomorphism between the corresponding homotopy groups, as we observed in [J-1] (4.4.5), the Chern-character can be made to be a map of presheaves on the étale site of a given scheme.) Therefore one obtains the isomorphisms:

$$CH^*(\mathfrak{S}, 0; \mathbb{Q}) \cong \pi_0(\mathbb{H}_{et}(\mathfrak{S}, \mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})) \cong \pi_0(\mathbb{H}_{et}(\mathfrak{S}, \mathbf{K}_{\mathbb{Q}})) \cong \underset{s}{\oplus} H^s_{et}(\mathfrak{S}, \pi_s(\mathbf{K}_{\mathbb{Q}})) = CH^*_{naive}(\mathfrak{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

Finally it suffices to show that the middle isomorphism preserves the ring structures. For this, first recall that the Chern-character induces a quasi-isomorphism of the presheaves: $\mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \overset{\simeq}{\leftarrow} \mathbf{K}_{\mathbb{Q}}$. One may obtain a spectral sequence as in (5.3.2) for the complex $\mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ instead of $\mathbf{K}_{\mathbb{Q}}$ and the quasi-isomorphism extends to an isomorphism of the two spectral sequences from the E_2-terms onwards. Therefore, the differential $d_r^{s,t} = 0$ for the spectral sequence for the presheaf $\mathbf{K}_{\mathbb{Q}}$ implies the corresponding differential is also trivial for the spectral sequence for $\mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Therefore $\pi_0(\mathbb{H}_{et}(\mathfrak{S}, \mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})) \cong \underset{s}{\oplus} E_2^{s,-s}$ which are the E_2-terms of the corresponding spectral sequence. Moreover, this isomorphism is compatible with the ring structures. Now it suffices to show that the isomorphism $H^s_{et}(\mathfrak{S}, \pi_s(\mathbf{K}_{\mathbb{Q}})) \simeq H^s_{et}(\mathfrak{S}, \pi_s(\mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}))$ is compatible with the obvious pairings. This follows from the observation that the Chern-character map $ch : \pi_*(\mathbf{K}_{\mathbb{Q}}) \to \pi_*(\mathcal{Z}^{\cdot}(\quad, .)_{\mathfrak{S}_{res.smt}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ is clearly multiplicative as a map of presheaves of $\mathbb{Q}$ vector spaces on the site $\mathfrak{S}_{et}$. This completes the proof of the theorem. □

Proof of Theorems 1, 2 and 3. The statements in Theorem 1 all follow readily from (4.2.2), (4.2.3) and (4.6.11), while those in Theorem 2 follow from (5.1), (5.3.10) and (3.5). The statements in Theorem 3 follow from (4.2.2), (4.2.3) and (5.2.0). □

Proof of Corollary 4. The Chern classes in (i) are provided by (4.3.3) while the higher Chern classes in (ii) are provided by (4.6.11). Now we consider (iii). Observe that X is an atlas for the quotient stack $[X/G]$;

therefore if X is smooth so is the stack. Therefore the intersection pairing $CH_*([X/G], .) \otimes CH_*([X/G], .) \rightarrow CH_*([X/G], .)$ is provided by Theorem 2(iii). In case the action is locally proper with finite reduced stabilizers the stack $[X/G]$ is known to be Deligne-Mumford. Therefore Theorem 2 (iii) along with [J-1] (5.1.2) proves the assertion in (iv). Observe that the stack $\mathcal{M}_G$ is smooth. (See [LS] Proposition (3.4).) Therefore Theorem 2(iii) again provides the required intersection pairings in (v). Observe from [B-M] (see also [F-P] or [C-K] p. 170) that the stack $\overline{\mathfrak{M}}_{g,n}(X, \beta)$ is a separated (in fact, proper) Deligne-Mumford stack. Therefore, Theorem 2 (iii) and [J-1] (5.1.2) complete the proof of (vi). $\square$

Appendix: The motivic complexes

Throughout this appendix, we will freely adopt the terminology from [Fr-S], [F-S-V] and [Voev]. Occasionally we will differ from their terminology so as to be consistent with the ones adopted in the rest of this paper.

(A.0) Given a field k, $(smt.schemes/k)$ will denote the category of all smooth separated schemes of finite type over k. We will consider three *big topologies* on this category: namely, the Zariski, the Nisnevich and étale topologies. $(smt.schemes/k)_{Zar}$, $(smt.schemes/k)_{Nis}$ and $(smt.schemes/k)_{ET}$ will denote these sites. For any presheaf P of abelian groups on $(smt.schemes/k)$, we will let $P_{Zar}\ \tilde{}$, $P_{Nis}\ \tilde{}$ and $P_{et}\ \tilde{}$ denote the associated sheaves on the Zariski, the Nisnevich and étale sites. Now $\Delta[n]$ will denote the standard n-simplex defined as usual. As n varies, $\{\Delta[n]|n\}$ defines a cosimplicial scheme. Given an abelian presheaf P on $(smt.schemes/k)$, we extend P to a simplicial presheaf as in [Fr-S]: i.e. we let $C_*(P)$ denote this simplicial presheaf defined by $\Gamma(U, C_n(P)) = \Gamma(U \times \Delta[n], P)$. This simplicial abelian presheaf may be replaced by the corresponding chain complex (i.e. with a differential of degree -1): this complex will also be denoted $C_*(P)$. Now we may re-index $C^i(P) = C_{-i}(P)$ to obtain a co-chain complex of presheaves that are trivial in positive degrees.

(A.1) Given $X,\ Y \,\epsilon\, (smt.schemes/k)$, we define $Cor(Y, X)$ be the free abelian group generated by closed integral subschemes $Z \subseteq Y \times X$ that are quasi-finite and dominant over each component of Y.

(A.2) Given a fixed $X \,\epsilon\, (smt.schemes/k)$, one defines a presheaf with transfers, $\mathbb{Z}_{eq}(X)$, by $\Gamma(U, \mathbb{Z}_{eq}(X)) = Cor(U, X)$. It is observed in [F-S-V], that this defines a *sheaf with transfer* on $(smt.schemes)_{et}$, $(smt.schemes/k)_{Nis}$ and $(smt.schemes/k)_{Zar}$.

The motivic complexes $\mathbb{Z}(n)$.

(A.3.1) For us it is often convenient to use the ones introduced in [Fr-S], section 11 as they compare readily with the cycle complexes of Bloch. We let $\mathbb{Z}(n)^{F-S}$ for the corresponding motivic complex of weight n; recall this is defined as $C^*(\mathbb{Z}_{eq}(\mathbb{A}^n))[-2n]$. The main advantage of this complex is that there is an imbedding $C^*(\mathbb{Z}_{eq}(\mathbb{A}^n)) \to \mathcal{Z}^n(\quad \times \mathbb{A}^n, .)$ preserving the degree. We will also show that it preserves pairings.

(A.3.2) On the other hand the comparison theorem of [Voev] shows that the complex $\mathbb{Z}(n)^{F-S}$ is quasi-isomorphic to the motivic complex denoted $\mathbb{Z}(n)$ in [F-S-V]. For the sake of completeness we will also recall its definition. Let $\mathbb{G}_m$ denote the multiplicative group scheme $\mathbb{A}^1 - 0$. For each integer $n \geq 1$, let $\mathbb{G}_m^{\times n}$ be the n-fold product of $\mathbb{G}_m$ with itself. We will now consider $\mathbb{Z}_{eq}(\mathbb{G}_m^{\times n})$. We let D_n denote the sum of the images of the homomorphisms $\mathbb{Z}_{eq}(\mathbb{G}_m{}^{\times n-1})$ given by the imbeddings of the form $(x_1, ..., x_{n-1}) \mapsto (x_1, ..., 1, ..., x_{n-1})$. D_n is a sub-sheaf of $\mathbb{Z}_{eq}(\mathbb{G}_m^{\times n})$ which is in fact a direct summand. We let $\mathbb{Z}_{eq}(\mathbb{G}_m^{\wedge n}) = \mathbb{Z}_{eq}(\mathbb{G}_m^{\times n})/D_n$. $\mathbb{Z}(n)$ on $(smt.schemes/k)$ is the complex $C^*(\mathbb{Z}_{eq}(\mathbb{G}_m^{\wedge n}))[-n]$.

(A.3.3) Since the two complexes in (A.3.1) and (A.3.2) are quasi-isomorphic (see [Voev]) we may use either one. However, for explicit comparison with the cycle complex, it is preferable to use the one in (A.3.1). Unless the choice is important for us, we will let $\mathbb{Z}(n)$ denote either of these complexes.

(A.3.4) We define $\mathbb{Q}(n) = \mathbb{Z}(n) \otimes \mathbb{Q}$.

It is observed in [F-S-V], that, $\mathbb{Z}(n)$ and $\mathbb{Q}(n)$ define complexes of sheaves with transfers and homotopy invariant cohomology presheaves on $(smt.schemes/k)_{et}$, $(smt.schemes/k)_{Nis}$ and $(smt.schemes/k)_{Zar}$.

Tensor structure on the motivic complexes.

(A.4) **Theorem**. There exists a natural pairing $\mathbb{Z}(n)^{F-S} \otimes \mathbb{Z}(m)^{F-S} \to \mathbb{Z}(n+m)^{F-S}$ that is compatible with the intersection pairing on the cycle complex for smooth schemes.

Proof. Observe that both the pairings (i.e. the one on the motivic complexes and the one on the cycle complex) are induced by first taking an external product and then pulling back by the diagonal. It involves the use of the *easy moving lemma* in the case of the cycle complex while in the case of the motivic complexes no such moving is needed. To see this, recall that the complex $\mathbb{Z}(n)^{F-S}[2n] = C^*(\mathbb{Z}_{eq}(\mathbb{A}^n))$. For a given smooth scheme U,

$$\Gamma(U, C^{-p}(\mathbb{Z}_{eq}(\mathbb{A}^n))) = Cor(U \times \Delta[p], \mathbb{A}^n) \text{ and } \Gamma(U, C^{-q}(\mathbb{Z}_{eq}(\mathbb{A}^m))) = Cor(U \times \Delta[q], \mathbb{A}^m).$$

Therefore, the pairing $\mathbb{Z}(n)^{F-S}[-2n] \otimes \mathbb{Z}(m)^{F-S}[-2m] \to \mathbb{Z}(n+m)^{F-S}[-2n-2m]$ is defined as follows. First take the external product of two correspondences $Z_1 \,\epsilon\, Cor(U \times \Delta[p], \mathbb{A}^n)$ and $Z_2 \,\epsilon\, Cor(U \times \Delta[q], \mathbb{A}^m)$ to obtain $Z_1 \times Z_2 \,\epsilon\, Cor(U \times U \times \Delta[p] \times \Delta[q], \mathbb{A}^{m+n})$. Now the projection of the cycle $Z_1 \times Z_2$ onto $U \times U \times \Delta[p] \times \Delta[q]$ is finite and surjective onto each component. Therefore, if $T : \Delta[p+q] \to \Delta[p] \times \Delta[q]$ is given by a triangulation as in [Bl-1], the pull-back by T defines a cycle $T^*(Z_1 \times Z_2) \,\epsilon\, Cor(U \times U \times \Delta[p+q], \mathbb{A}^{m+n})$. Finally pulling back by the diagonal $\Delta \times id : U \times \mathbb{A}^{m+n} \to U \times U \times \mathbb{A}^{m+n}$ defines the required class $Z_1 \circ Z_2 \,\epsilon\, Cor(U \times \Delta[p+q], \mathbb{A}^{m+n})$. (See the remarks below.)

It is also clear that the imbedding of the above complex into $z^n(\quad \mathbb{A}^n, .)$ is now compatible with the intersection pairing on the cycle complex. To see this consider:

$$z^n(U, .) \otimes z^m(U, .) \simeq z^n(U \times \mathbb{A}^n, .) \otimes z^m(U \times \mathbb{A}^m, .)$$

$$\xrightarrow{\times} z^{n+m}(U \times U \times \mathbb{A}^{n+m}, .) \simeq z^{n+m}_{\Delta}(U \times U \times \mathbb{A}^{n+m}, .) \xrightarrow{\Delta^*} z^{n+m}(U \times \mathbb{A}^{n+m}, .)$$

The sub-complex $z^{n+m}_{\Delta}(U \times U \times \mathbb{A}^{n+m}, .)$ consists of those cycles in $z^{n+m}(U \times U \times \mathbb{A}^{n+m}, .)$ that intersect the diagonal properly. The quasi-isomorphism $z^{n+m}(U \times U \times \mathbb{A}^{n+m}, .) \simeq z^{n+m}_{\Delta}(U \times U \times \mathbb{A}^{n+m}, .)$ makes use of the *easy moving lemma.* $\square$

(A.5) It is shown in [Fr-S] section 11, that the natural imbedding of the complex $\mathbb{Z}(n)^{F-S}[2n]$ into the cycle complex $\mathcal{Z}^n(\quad \times \mathbb{A}^n, .)$ is a quasi-isomorphism on the Zariski site of any smooth scheme.

(A.6) *Remarks.*(i) It is shown in [F-S-V] that there also exist pairings $\mathbb{Z}(n) \otimes \mathbb{Z}(m) \to \mathbb{Z}(n+m)$. Therefore if one is only interested in providing a ring structure on our higher Chow groups for algebraic stacks, we may make use of this pairing. However, from the definition of the quasi-isomorphism of this complex with the cycle complex, it is not clear a priori that this pairing is compatible with the one on the cycle complexes for smooth schemes.

(ii) The extension functors of section 2 enable one to extend the comparison result in (A.5) to algebraic stacks.

(iii) In using the triangulation of $\Delta[p] \times \Delta[q]$, one has to show that it is possible to choose these triangulations compatibly so that pull-back by T defines a map of complexes:

$$T^* : C^*(\mathbb{Z}_{eq}(\mathbb{A}^n))(U) \otimes C^*(\mathbb{Z}_{eq}(\mathbb{A}^m))(U) \to C^*(\mathbb{Z}_{eq}(\mathbb{A}^{n+m}))(U \times U)$$

Since n and m are fixed here, one may invoke exactly the same arguments as in [Bl-1] section 5 showing that one may find a triangulation T, for all $\Delta[p] \times \Delta[q]$ so that it defines a map of complexes: $\mathbb{Z}(n)^{F-S} \otimes \mathbb{Z}(m)^{F-S} \to \mathbb{Z}(n+m)^{F-S}$. An alternative to this is the use of cubical complexes. (See [J-3].)

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