Künneth decompositions for quotient varieties

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ABSTRACT

In this paper we discuss Künneth decompositions for finite quotients of several classes of smooth projective varieties. The *main result is the existence of an explicit (and readily computable) Chow–Künneth decomposition in the sense of Murre with several pleasant properties* for finite quotients of abelian varieties. This applies in particular to symmetric products of abelian varieties and also to certain smooth quotients in positive characteristics which are known to be not abelian varieties, examples of which were considered by Enriques and Igusa. We also consider briefly *a strong Künneth decomposition* for finite quotients of projective smooth linear varieties.

1. INTRODUCTION

Chow–Künneth decompositions are conjectured (by optimists) to exist over $\mathbf{Q}$ for all smooth projective varieties X; at present, they are known to exist for curves [18], surfaces [21], projective spaces [18], abelian varieties [7] and other special types of varieties (see, for example, [5,6]). Igusa [11] gives a construction (possibly discovered earlier by Enriques) of a finite group action on an abelian variety such that the quotient variety is smooth, but not an abelian variety. It is natural to ask whether varieties of this sort – or more generally, quotients of abelian varieties by a finite group action – admit a Chow–Künneth decomposition. Indeed, even when the quotient variety is singular, one may still speak of motives and

correspondences (and hence Chow–Künneth decompositions) using the reasoning of [8, Examples 8.3.12 and 16.1.13].

In this paper we construct a Chow–Künneth decomposition for quotients of abelian varieties by a finite group action by descending the canonical Chow–Künneth decomposition for abelian varieties given by Beauville, Deninger and Murre [7]. We obtain explicit formulae for the Chow–Künneth projectors for the quotient that are readily computable in terms of the Chow–Künneth projectors for the abelian variety. *In view of the close relation our Chow–Künneth projectors bear to the Deninger–Murre projectors for the original abelian variety, our projectors have very pleasant properties: they are easily defined explicitly and also act in the expected manner on the rational Chow groups of the quotient variety thereby verifying the existence of a conjectured filtration for all quotients of abelian varieties (by actions of finite groups) up to dimension 4 or less.* We also sketch in outline that strong Künneth decompositions – which are known to exist for linear varieties (cf. [29]) – may be descended to finite quotients of such. This provides a means of computing the higher (rational) Chow groups of such quotient varieties. Needless to say, we work throughout with rational coefficients.

Here is an outline of the paper. In the rest of this section we set up the basic terminology and conventions for the rest of the article. In particular, we recall the definitions of Chow–Künneth decomposition and strong Künneth decomposition. In the next section we prove the following theorem:

Theorem 1.1. (See Theorem 2.3). *Let A be an abelian variety of dimension d over a field k and G a finite group acting on A. Let $f : A \to A/G$ be the quotient map. Suppose $[\Delta_A] = \sum_{i=0}^{2d} \pi_i$ is the Beauville–Deninger–Murre Chow–Künneth decomposition for A and let $\eta_i = \frac{1}{|G|}(f \times f)_* \pi_i$.*
Then

$$[\Delta_{A/G}] = \sum_{i=0}^{2d} \eta_i$$

is a Chow–Künneth decomposition for A/G. This Chow–Künneth decomposition satisfies Poincaré duality: that is, for any i, $\eta_{2d-i} = {}^t\eta_i$.

In addition, η_i acts as zero on $CH^j_{\mathbf{Q}}(A/G)$ for $i < j$ and also for $i > j + d$ in general. In case $d \leqslant 4$, we may also conclude that η_i acts trivially on $CH^j_{\mathbf{Q}}(A/G)$ for $i < j$ and also for $i > 2j$.

While the techniques are a modification of those of Deninger–Murre [7] and also [3], there are several non-trivial issues that need to be considered as will become clear from the proof. To begin with, the quotient of the abelian variety by the action of a finite group need not be an abelian variety nor even a nonsingular variety, so there is no analogue of the Poincaré bundle which plays a key role in the Deninger–Murre proof for abelian varieties.

We first prove the theorem under the hypothesis that for each $g \in G$, $g(0)$ is a torsion point of the abelian variety; this is equivalent to requiring that the action

of the group preserve the torsion points of the abelian variety. It is clear that this hypothesis is not always satisfied: for example, choose $a \in A(k)$ to be a point of infinite order and take $G = \mathbf{Z}/2\mathbf{Z} = \{1, g\}$ with the action $g \cdot x = -x + a$. However, we reduce the general case to this one by means of a technique in group cohomology; hence we are able to obtain an explicit formula for the Chow–Künneth projectors for *all* actions of finite groups on *all* abelian varieties.

One may observe that the existence of such a Chow–Künneth decomposition is guaranteed by the work of Kimura [15] and [9] on finite-dimensional motives. However, there are several clear advantages to our construction of the Chow–Künneth projectors. *For example, the statement that the projectors η_i act as zero on $CH_{\mathbf{Q}}^j(A/G)$ for $i < j$ and $i > j + d$ already verifies part of the conjectured formalism for the Bloch–Beilinson filtration on Chow groups. Our explicit and easily stated formula enables us to obtain such results readily by exploiting the close relation our Chow–Künneth projectors bear to the Deninger–Murre Chow–Künneth projectors on the original abelian variety.* In addition, the Poincaré duality property for the projectors is also immediate from our formula. For the convenience of the reader and for comparison purposes, we have also included a discussion of construction of Chow–Künneth projectors via finite dimensionality of motives. See Section 2.2. We show that the complexity of these calculations increases exponentially with respect to the dimension of the abelian variety. Our techniques are more explicit and result in a much cleaner formula, irrespective of the dimension of the abelian variety.

In the third section, we extend the strong Künneth decomposition to finite quotients of projective smooth linear varieties. It is shown here, essentially using elementary methods, that if X is a variety possessing a strong Künneth decomposition and $f : X \to Y$ is a finite surjective map, then Y possesses a strong Künneth decomposition which we may describe explicitly in terms of that for X. As an application, we describe a strong Künneth decomposition for symmetric products of projective spaces. We also discuss some formal consequences of such a strong Künneth decomposition: we show that a strong Künneth decomposition implies a Chow–Künneth decomposition and that the rational higher Chow groups are determined by the rational higher Chow groups of the base field and the rational (ordinary) Chow groups of the given variety.

We discuss, in an appendix, functoriality properties of cohomology theories for pseudo-smooth schemes as well as consequences of the existence of strong Künneth decompositions for the quotient varieties we consider in this paper. The latter discussion is done in a somewhat more general setting so that it applies not only to determine the higher rational Chow groups of finite quotients of projective smooth linear varieties, but also possibly to other situations not considered in the body of the paper.

Throughout the paper we will fix a field k of arbitrary characteristic and restrict to the category of quasi-projective schemes over k.

1.1. Group scheme actions and quotients

Following the treatment in [20], we review some definitions and results related to group scheme actions.

Definition 1.1. Let G be a group scheme over a field k with identity section $e : \operatorname{Spec} k \to G$ and multiplication $m : G \times_k G \to G$.

An *action* of G on a scheme X is a morphism $\mu : G \times X \to X$ such that:

(1) The composite

$$X \cong \operatorname{Spec} k \times_k X \overset{e \times 1_X}{\to} G \times_k X \overset{\mu}{\to} X$$

is the identity map.

(2) The diagram below commutes:

$$
\begin{array}{ccc}
G \times_k G \times_k X & \overset{m \times 1_X}{\longrightarrow} & G \times_k X \\
\big\downarrow{\scriptstyle 1_G \times \mu} & & \big\downarrow{\scriptstyle \mu} \\
G \times_k X & \overset{\mu}{\longrightarrow} & X
\end{array}
$$

In the future, we will identify elements $g \in G$ with the morphism $\mu_g : X \to X$ defined by $\mu_g(x) = \mu(g, x)$ and (by abuse of notation) refer to this morphism simply as g.

If G is a finite group ($=$ a constant étale finite group scheme) over k acting on a quasi-projective scheme X (also over k), there exists a quasi-projective variety Y together with a finite, surjective G-invariant morphism $f : X \to Y$ universal for G-invariant morphisms $X \to Z$. The scheme Y is called the *quotient* of X by G, and is typically denoted $Y = X/G$.

Definition 1.2. We say a scheme X is *pseudo-smooth* (over k) if it is the quotient of a smooth scheme (over k) by the action of a finite group.

1.1.1. Notation and terminology: review of correspondences

In this section we define the category of *rational correspondences* and *rational Chow motives* for pseudo-smooth projective varieties.

Let k be a field and $\mathcal{V}_k$ the category of schemes pseudo-smooth and projective over k. If X, Y are objects of $\mathcal{V}_k$ and X has pure dimension d, we define the group of *degree r correspondences* from X to Y by $Corr^r(X, Y) = CH^{d+r}(X \times_k Y) \otimes \mathbb{Q}$, the group of codimension $d + r$ (rational) cycles on $X \times_k Y$ modulo rational equivalence. In general, let $X_1, \ldots, X_n$ be the irreducible components of X; we then define $Corr^r(X, Y) = \bigoplus_{i=1}^{n} Corr^r(X_i, Y)$. When $\alpha \in Corr^r(X, Y)$ and $\beta \in Corr^s(Y, Z)$, we define their composition $\beta \bullet \alpha \in Corr^{r+s}(X, Z)$ by the formula

$$\beta \bullet \alpha = (p_{13})_* \left(p_{12}^* \alpha \cdot p_{23}^* \beta \right);$$

here p_{ij} represents projection of $X \times_k Y \times_k Z$ on the ith and jth factors. Moreover given a correspondence α as above and a class $[x] \in CH^*(X) \otimes \mathbb{Q}$, we obtain a new class $\alpha \bullet [x] = p_{2*}(\alpha.p_1^*([x]))$. Given the correspondence β as above, one verifies readily that $[x] \bullet (\alpha \bullet \beta) = ([x] \bullet \alpha) \bullet \beta$.

One then constructs a new category $\mathcal{M}_k(\mathbb{Q})$, the category of (rational) *Chow motives of pseudo-smooth projective varieties*. The objects of $\mathcal{M}_k(\mathbb{Q})$ are pairs (X, π), where X is an object of $\mathcal{V}_k$ of dimension d and $\pi \in Corr^0(X, X)$ is a *projector*; that is, an element satisfying $\pi \bullet \pi = \pi$. For any two Chow motives (X, π) and (Y, ρ), one then defines

$$Hom_{\mathcal{M}_k}\big((X, \pi), (Y, \rho)\big) = \bigoplus_j \rho \bullet Corr^0(X, Y) \bullet \pi.$$

If Δ_X is the diagonal of $X \times_k X$ and $[\Delta_X]$ its class in $CH^*(X \times_k X) \otimes \mathbb{Q}$, a straightforward computation shows that $[\Delta_X]$ is a projector, and furthermore that $\alpha = \alpha \bullet [\Delta_X]$, $\alpha = [\Delta_Y] \bullet \alpha$ for any pseudo-smooth projective scheme Y and $\alpha \in Corr^*(X, Y)$. (See (H.5) in Appendix for more details.) Thus, there is a functor $h: \mathcal{V}_k^{opp} \to \mathcal{M}_k(\mathbb{Q})$ defined on objects by $h(X) = (X, [\Delta_X])$ and on morphisms by $h(X \xrightarrow{f} Y) = \Gamma_f$, where $[\Gamma_f] \in Hom_{\mathcal{M}_k(\mathbb{Q})}(h(Y), h(X))$ is the class of the graph of f. Furthermore, letting $\coprod$ denote disjoint union (of schemes), one may define the sum $\oplus$ and product $\otimes$ of motives:

$$(X, p) \oplus (Y, q) = \big(X \coprod Y, p \coprod q\big),$$
$$(X, p) \otimes (Y, q) = (X \times_k Y, p \otimes q)$$

where $p \otimes q = s^*(p \times q)$ and $s: X \times_k Y \times_k X \times_k Y \to X \times_k X \times_k Y \times_k Y$ is the map exchanging the middle two factors.

We denote by $\mathbb{I}$ the "trivial" motive $h(\operatorname{Spec} k)$, a neutral element for $\otimes$, and by $\mathbb{L}$ the "Lefschetz motive" $(\mathbf{P}_k^1, \mathbf{P}_k^1 \times_k \{x\})$; here $x \in \mathbf{P}_k^1$ is any rational point. Finally, if $\alpha \in Corr^*(X, Y)$ is any correspondence, we define its "transpose" ${}^t\alpha = s^*(\alpha) \in Corr^*(Y, X)$, where $s: X \times_k Y \to Y \times_k X$ is the exchange of factors. For further discussion of motives, we refer the reader to [27]. Also see [8, Examples (8.3.12) and (16.1.12)] for discussion that shows one can in fact define a category of Chow motives for pseudo-smooth schemes as we have done. In fact it is possible to consider the above theory for all smooth Deligne–Mumford stacks over k; some of our results extend to this situation readily.

1.2. Abelian varieties

In this section we establish notation and cite a rigidity property for abelian varieties necessary in the sequel. A comprehensive treatment of abelian varieties may be found in [20] or [19].

Let k be a field and A an abelian variety over k. Following [19], we denote by $m: A \times_k A \to A$ the morphism representing composition on the group scheme A and use additive notation for this (commutative) operation. For any $a \in A(k)$, we denote by $\tau_a: A \to A$ (translation by a) the map defined by $\tau_a(x) = x + a$.

A morphism $f : A \to B$ between abelian varieties is called a *homomorphism* if for every $a, a' \in A$, $f(a + a') = f(a) + f(a')$. When $n \in \mathbf{Z}$ we define $\mathbf{n} : A \to A$ by $\mathbf{n}(a) = na$ and set $A[n] = \mathrm{Ker}(A \xrightarrow{\mathbf{n}} A)$, the (group scheme of) n-torsion points on A. For clarity of notation, we write σ instead of $-\mathbf{1}$.

The following important result is a consequence of a general rigidity principle; see [19, Corollary 2.2] for details:

Proposition 1.3. *Let $h : A \to B$ be a morphism of abelian varieties. Then there exists a homomorphism $h_0 : A \to B$ and an element $a \in A(k)$ such that $h = \tau_a \circ h_0$.*

Here and everywhere else, $\circ$ will denote composition of morphisms. We remark that h_0 and a are in fact unique. Indeed, one must have $a = h(0)$; uniqueness of h_0 then follows immediately.

Let $\hat{A}$ be the dual abelian variety; we will denote by $\mathcal{L}$ the Poincaré bundle and ℓ its class in $CH^1_\mathbf{Q}(A \times_k \hat{A})$.

We conclude this section by recalling the definition of strong Künneth and Chow–Künneth decompositions.

Definition 1.4. Suppose X is projective and pseudo-smooth, of dimension d over k. We say that X has a *Chow–Künneth decomposition* if there exist elements $\pi_0, \ldots, \pi_{2d} \in CH^d_\mathbf{Q}(X \times_k X)$ such that:

- $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$.
- For every i, $\pi_i \bullet \pi_i = \pi_i$ and for all $j \neq i$, $\pi_i \bullet \pi_j = 0$. (Thus, $\pi_0, \ldots, \pi_{2d}$ form a system of mutually orthogonal projectors.)
- Let H be a Weil cohomology theory H^* (cf. [16]) and, for any k-scheme Y, let $cl_Y : CH^*_\mathbf{Q}(Y) \to H^*(Y)$ denote the cycle map. We require that $cl_{X \times_k X}(\pi_i) = \Delta(i)$, where $\Delta(i)$ is the codimension i Künneth component of the class of $[\Delta_X]$ in $H^*(X \times_k X)$. (We will show later that any Weil cohomology theory admits an extension to the category of pseudo-smooth schemes.)

Next let X be any scheme of pure dimension d over a field k.

We say that X possesses a *strong Künneth decomposition* if there exist elements $a_{i,j}, b_{i,j} \in CH^i_\mathbf{Q}(X)$ such that

$$(1.1) \qquad [\Delta_X] = \sum_i \sum_j a_{i,j} \times b_{d-i,j}.$$

Observe that if X is projective, X having a Chow–Künneth decomposition is equivalent to asserting that $h(X) \cong \bigoplus_{i=0}^{2d} h^i(X)$ where $h^i(X)$ is the motive (X, π_i).

2. CHOW–KÜNNETH DECOMPOSITION FOR QUOTIENTS OF ABELIAN VARIETIES

Our goal in this section is to exhibit an explicit Chow–Künneth decomposition for the quotient of an abelian variety A by the action of a finite group G, assuming

only that $g(0)$ is a torsion point for each $g \in G$. As before, the quotient A/G may be singular. We rely on the following result, originally due to Shermenev [28], but later proved in a somewhat more functorial setting by Deninger and Murre [7, Theorem 3.1]; in this latter source the result is proved more generally for abelian schemes over a smooth quasi-projective base:

Theorem 2.1. *Let A be an abelian variety of dimension d over a field k. Then there exist projectors $\pi_i \in CH^d(A \times_k A)$, $i = 0, \ldots, 2d$, giving a Chow–Künneth decomposition for A:*

$$[\Delta_A] = \sum_{i=0}^{2d} \pi_i.$$

Since we need to make explicit use of the projectors π_i, we will presently review their construction. First, consider $A \times_k A$ as an abelian A-scheme via projection on the first factor; with respect to this structure, the dual abelian scheme is $A \times_k \hat{A}$. Consider then the Fourier transform (cf. [7, 2.12; 17, 1.3]):

$$F_{CH} : CH^*_{\mathbf{Q}}(A \times_k A) \to CH^*_{\mathbf{Q}}(A \times_k \hat{A})$$

defined by $F_{CH}(\alpha) = p_{13*}(p_{12}{}^*\alpha \cdot F)$, where

$$F = 1 \times \sum_{i=0}^{\infty} \frac{\ell^i}{i!} \in CH_{\mathbf{Q}}(A \times_k A \times_k \hat{A})$$

and the various p_{ij} represent projections from $A \times_k A \times_k \hat{A}$ on the ith and jth factor. Note that the sum defining F is actually finite.

Dualizing this construction, we may define

$$\hat{F}_{CH} : CH^*_{\mathbf{Q}}(A \times_k \hat{A}) \to CH^*_{\mathbf{Q}}(A \times_k A)$$

by $\hat{F}_{CH}(\gamma) = q_{13*}(q_{12}^*\gamma \cdot \hat{F})$, where

$$\hat{F} = 1 \times \sum_{i=0}^{\infty} \frac{{}^t\ell^i}{i!} \in CH^*_{\mathbf{Q}}(A \times_k \hat{A} \times_k A)$$

and q_{ij} represent the various projections from $A \times_k \hat{A} \times_k A$. By switching the last two factors and changing notation appropriately, we see that in fact

$$\hat{F}_{CH}(\gamma) = p_{12*}(p_{13}^*\gamma \cdot F).$$

An argument involving the theorem of the square (cf. [7, Corollary 2.22], also [3, Proposition 3]) then shows that $\hat{F}_{CH}(F_{CH}(\alpha)) = (-1)^d \sigma^*\alpha$ for all $\alpha \in CH^*(A \times_k A)$, and similarly for the other composition.

Observe that $[\Delta_A] \in CH^d(A \times_k A)$, and write $F_{CH}([\Delta_A]) = \sum_{i=0}^{2d} \beta_i$, where $\beta_i \in CH^i_{\mathbf{Q}}(A \times_k \hat{A})$. It is a fact [7, pp. 214–216] that $(1 \times \mathbf{n})^* \beta_i = n^i \beta_i$. Now define

$$(2.1) \qquad \pi_i = (-1)^d \sigma^* \hat{F}_{CH}(\beta_i).$$

The following result will allow us to make some helpful reductions:

Lemma 2.2.

(1) *If Theorem 1.1 holds when k is radicially closed, it holds for arbitrary k.*
(2) *If Theorem 1.1 holds after extending the base field k by some finite Galois extension L/k, then it holds in general.*

Proof. The first assertion follows from the fact that the rational Chow groups are invariant under radicial (i.e. purely inseparable) extensions of the base field. Now let L/k be a Galois extension and let $H = \text{Gal}(L/k)$. For any k-scheme X, we denote by $j_X : X_L \to X$ the corresponding base extension morphism. If $[\Delta_A] = \sum_{i=0}^{2d} \pi_i$ is the Beauville–Deninger–Murre Chow–Künneth decomposition for A, then it is easy to check that $[\Delta_{A_L}] = \sum_{i=0}^{2d} j^*_{A \times A} \pi_i$ is the corresponding decomposition for A_L. Moreover, by base extension we may interpret $G = G_L$ as a finite étale constant group scheme over L acting on A_L; we then have $(A/G)_L \cong A_L/G_L$. Since the theorem holds in L,

$$[\Delta_{(A/G)_L}] = \sum_{i=0}^{2d} (f_L \times f_L)_* j^*_{A \times A} \pi_i = j^*_{A/G \times A/G} (f \times f)_* \pi_i$$

is a Chow–Künneth decomposition for $(A/G)_L$. Furthermore, the last expression shows that the Chow–Künneth projectors are H-invariant. Thus, by [27, Lemma 1.17], we may descend the Chow–Künneth decomposition to k; that is, we may conclude that

$$[\Delta_{A/G}] = \sum_{i=0}^{2d} \frac{1}{|H|} j_{A/G \times A/G *} j_{A/G \times A/G}^* (f \times f)_* \pi_i = \sum_{i=0}^{2d} (f \times f)_* \pi_i$$

is a Chow–Künneth decomposition for A/G, as desired. The second statement now follows from the observation. Let $\alpha \in CH^j_{\mathbf{Q}}(A)$ for j as above. Now

$$(f \times f)_*(\pi_i) \bullet \alpha = f_*\left(\pi_i \bullet f^*(\alpha)\right).$$

Observe $f^*(\alpha) \in CH^j_{\mathbf{Q}}(A_L)^H$ and therefore we reduce to proving the last statement of the theorem to the situation where A is replaced by A_L. This completes the proof of theorem. $\square$

Next we proceed to prove the following weakened form of the main theorem:

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Theorem 2.3. *Let A be an abelian variety of dimension d over a field k and G a finite group acting on A such that $g(0) \in A(k)$ is a torsion point for each $g \in G$. Let $f : A \to A/G$ be the quotient map. Suppose $[\Delta_A] = \sum_{i=0}^{2d} \pi_i$ is the (Beauville–Deninger–Murre) Chow–Künneth decomposition for A considered above and let $\eta_i = \frac{1}{|G|}(f \times f)_* \pi_i$.*
Then

$$[\Delta_{A/G}] = \sum_{i=0}^{2d} \eta_i$$

is a Chow–Künneth decomposition for A/G.

Moreover, the G-action commutes with multiplication by n for all n in the infinite subset E chosen as in 2.6 and η_i acts as zero on $CH_{\mathbf{Q}}^j(A/G)$ in the following cases: (i) $i < j$ or $i > j + d$ in general, and (ii) if $j = 0, 1, d - 2, d - 1, d$ and $i > 2j$.

Remark. *The hypothesis that $g(0)$ be a torsion point of A is not always satisfied.* For example, if $a \in A(k)$ is any point of infinite order, then the automorphism $g : x \mapsto -x + a$ defines an action of $\mathbf{Z}/2\mathbf{Z}$ on A for which $g(0) = a$ is not a torsion point. However, if k is an algebraic extension of a finite field, then this hypothesis is always satisfied.

Observe also that this hypothesis is *equivalent to requiring that G preserves the torsion points of A.* This hypothesis seems quite helpful for being able to descend the Chow–Künneth projectors of the abelian variety A to A/G, which, a priori, has no other structure other than that of an algebraic variety. We will show later that one may reduce the general case to this situation.

Our method of proof is based on that of [7, Theorem 3.1]; however, there are further technicalities which complicate it somewhat. The content of the proof is, of course, to show that the elements $\frac{1}{|G|}(f \times f)_* \pi_i$, $0 \leqslant i \leqslant 2d$, are mutually orthogonal projectors. The map f^* establishes an isomorphism [8, Example 1.7.6]:

$$CH_{\mathbf{Q}}^*(A/G) \to CH_{\mathbf{Q}}^*(A)^G$$

with inverse $\frac{1}{|G|} f_*$. (See also (H.0) in the appendix for an explanation of this from the point of view of equivariant Chow groups.) Thus, we will work in the group $CH_{\mathbf{Q}}^*(A)^G$, constructing mutually orthogonal $G \times G$-invariant elements which may be descended to elements of $CH_{\mathbf{Q}}^*(A/G)$ by the following device:

Lemma 2.4. *Suppose X is a pseudo-smooth projective variety of dimension d and G a finite group of automorphisms of X. Let $f : X \to Y = X/G$ be the quotient map and suppose*

$$\sum_{g,h \in G} (g \times h)^* [\Delta_X] = \sum_{i=0}^{2d} \rho_i$$

where $\rho_i \bullet \rho_j = 0$ if $i \neq j$, $\rho_i \bullet \rho_i = |G|^2 \rho_i$ and the ρ_i are $G \times G$-invariant, i.e. for any $g, h \in G$, $(g \times h)^* \rho_i = \rho_i$.

Then

$$[\Delta_Y] = \sum_{i=0}^{2d} \frac{1}{|G|^3} (f \times f)_* \rho_i$$

is a Chow–Künneth decomposition for Y.

Proof. We have (making use of the identifications from the appendix identifying $CH_\mathbb{Q}^*(A/G)$ with the rational G-equivariant Chow groups of A)

$$(f \times f)_*(f \times f)^* = |G|^2, \qquad \sum_{g,h \in G} (g \times h)^* = (f \times f)^*(f \times f)_*$$

$$\text{and} \quad (f \times f)_*[\Delta_X] = |G|[\Delta_Y],$$

and therefore:

$$|G|^2 (f \times f)_*[\Delta_X] = (f \times f)_* \sum_i \rho_i.$$

Hence

$$[\Delta_Y] = \frac{1}{|G|^3} \sum_i (f \times f)_* \rho_i.$$

It remains to show that $\frac{1}{|G|^3}(f \times f)_* \rho_i$ are mutually orthogonal idempotents. As in Proposition 3.4, we add subscripts and superscripts to p (respectively, q) to denote the various projections between products of X (respectively, Y), and for convenience of notation set $r = (f \times f \times f) : X \times_k X \times_k X \to Y \times_k Y \times_k Y$. Now,

$$(2.2) \qquad (f \times f)_* \rho_i \bullet (f \times f)_* \rho_j = q_{13\,*}^{123} \left(q_{12}^{123*}(f \times f)_* \rho_i \cdot q_{23}^{123*}(f \times f)_* \rho_j \right).$$

Since the degree of r is $|G|^3$, $r_* r^*$ corresponds to multiplication by $|G|^3$, and therefore, the last expression equals:

$$(2.3) \qquad \frac{1}{|G|^3} q_{13\,*}^{123} \left(r_* r^* q_{12}^{123*}(f \times f)_* \rho_i \cdot q_{23}^{123*}(f \times f)_* \rho_j \right).$$

Because $q_{12}^{123} \circ r = (f \times f) \circ p_{12}^{123}$, the above simplifies to:

$$\frac{1}{|G|^3} q_{13\,*}^{123} \left(r_* p_{12}^{123*}(f \times f)^*(f \times f)_* \rho_i \cdot q_{23}^{123*}(f \times f)_* \rho_j \right).$$

Because the ρ_i are $G \times G$-invariant, we have $(f \times f)^*(f \times f)_*$ is multiplication by $|G|^2$, so the expression equals:

$$\frac{1}{|G|^3} q_{13\,*}^{123} \left(r_* p_{12}^{123*} |G|^2 \rho_i \cdot q_{23}^{123*}(f \times f)_* \rho_j \right).$$

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Finally, applying the projection formula, the formula $q_{ij}^{123} \circ r = (f \times f) \circ p_{ij}^{123}$ and $(G \times G)$-invariance of the ρ_i, one may identify the last expression with:

$$\frac{1}{|G|} q_{13}^{123}{}_* r_* \left(p_{12}^{123^*} \rho_i \cdot r^* q_{23}^{123^*} (f \times f)_* \rho_j \right)$$

$$= \frac{1}{|G|} (f \times f)_* p_{13}^{123}{}_* \left(p_{12}^{123^*} \rho_i \cdot p_{23}^{123^*} (f \times f)^* (f \times f)_* \rho_j \right)$$

$$= |G| (f \times f)_* p_{13}^{123}{}_* \left(p_{12}^{123^*} \rho_i \cdot p_{23}^{123^*} \rho_j \right)$$

$$= |G| (f \times f)_* (\rho_i \bullet \rho_j).$$

Because the ρ_k are mutually orthogonal projectors, this last expression equals 0 if $i \neq j$ or $|G|^3 (f \times f)_* \rho_i$ if $i = j$. $\square$

In [7], the crucial step in the proof of the Chow–Künneth decomposition for abelian varieties is the following computation, which may be proved using the seesaw theorem [19, Corollary 5.2]:

Proposition 2.5. *[7, 2.15]. For any integer n,*

$$(1 \times \mathbf{n})^* \ell = n\ell$$

where $\mathbf{n}$ denotes multiplication by the integer n.

The analogous strategy in our context would seem to be to study the action of $(1 \times \mathbf{n})^*$ on $(g \times h)^* \ell$; however, the induced action by G on $\hat{A}$ is by pulling back line bundles by elements of G. Therefore, the induced action by elements of G on $\hat{A}$ is by homomorphisms and hence on double-dualizing this provides an action on A which may differ from the original action. (If G acts on A "by isogenies"; that is, if all of the maps $g: A \to A$ are in fact homomorphisms of A, then duality gives a natural action of G on $\hat{A}$, but we are not assuming this.) Instead, we rely on the fact [19, p. 119] that the Poincaré bundle on $\hat{A} \times_k A$ is the transpose of the Poincaré bundle on $A \times_k \hat{A}$. Hence:

$$(2.4) \qquad (\mathbf{n} \times 1)^* \ell = {}^t\left((1 \times \mathbf{n})^* {}^t \ell \right) = {}^t\left(n {}^t \ell \right) = n\ell$$

and we prove the following:

Proposition 2.6. *There is an infinite subset $E \subset \mathbf{N}$ such that for all $n \in E$ and all $g \in G$:*

$$(2.5) \qquad \begin{aligned} (g \times 1) \circ (\mathbf{n} \times 1) &= (\mathbf{n} \times 1) \circ (g \times 1), \quad \text{and} \\ (\mathbf{n} \times 1)^* (g \times 1)^* \ell &= n(g \times 1)^* \ell. \end{aligned}$$

Moreover, one has:

$$(2.6) \qquad \begin{aligned} (\tau_{-2g(0)} \times 1) \circ (g \times 1) \circ (\mathbf{n} \times 1) &= (\mathbf{n} \times 1) \circ (\tau_{-2g(0)} \times 1) \circ (g \times 1), \quad \text{and} \\ (\mathbf{n} \times 1)^* (g \times 1)^* (\tau_{-2g(0)} \times 1)^* \ell &= n(g \times 1)^* (\tau_{-2g(0)} \times 1)^* \ell. \end{aligned}$$

Proof. For each $g \in G$, write $g = a_g \circ g_0$ as in Proposition 1.3. Let m_g be the order of $a_g = g(0)$; this is guaranteed to be finite by our hypothesis in Theorem 2.3. Next, let $m = \prod_{g \in G} m_g$, and

$$E = \{n \in \mathbf{Z}: n \equiv 1 \pmod{m}, \ n \neq \pm 1\}.$$

Note that if $n \in E$, m_g divides $n - 1$ (for any g), so $na_g = a_g$.
Now, if $n \in E$, we have

$$(2.7) \qquad (g \times 1) \circ (\mathbf{n} \times 1) = (\tau_{a_g} \times 1) \circ (g_0 \times 1) \circ (\mathbf{n} \times 1).$$

Since g_0 is a homomorphism, $\mathbf{n} \circ g_0 = g_0 \circ \mathbf{n}$; therefore the last expression equals $(\tau_{a_g} \times 1) \circ (\mathbf{n} \times 1).(g_0 \times 1)$. Since $a_g = na_g$, this equals $(\mathbf{n} \times 1) \circ (\tau_{a_g} \times 1) \circ (g_0 \times 1) = (\mathbf{n} \times 1) \circ (g \times 1)$. This proves the first equality in the proposition.
Therefore,

$$(\mathbf{n} \times 1)^*(g \times 1)^*\ell = (g_0 \times 1)^*(\tau_{a_g} \times 1)^*(\mathbf{n} \times 1)^*\ell.$$

By (2.4) the last term equals

$$n(g_0 \times 1)^*(\tau_{a_g} \times 1)^*\ell = n(g \times 1)^*\ell.$$

This completes the proof of the first statement. The second follows similarly. □

The next step in the proof of Theorem 2.3 is to construct the elements ρ_i appearing in Lemma 2.4; for each i, we simply set

$$\rho_i = \sum_{g,h \in G} (g \times h)^* \pi_i$$

where π_i are the Chow–Künneth components of $[\Delta_A]$ from Theorem 2.1. It is clear from the formula that the ρ_i are $G \times G$-invariant and that $\sum_{i=0}^{2d} \rho_i = \sum_{g,h \in G}(g,h)^*[\Delta_A]$; so it remains to prove that they are mutually orthogonal. In preparation for this, we study the action of $(1 \times \mathbf{n})^*$ on ρ_i:

Proposition 2.7. *For $n \in E$, $(1 \times \mathbf{n})^*(g \times h)^*\pi_i = n^i(g \times h)^*\pi_i$. Hence, $(1 \times \mathbf{n})^* \rho_i = n^i \rho_i$.*

Proof. Observe that $(1 \times \mathbf{n})^*(g \times h)^*\pi_i = (1 \times \mathbf{n})^*(g \times 1)^*(1 \times h)^*\pi_i = (g \times 1)^*(1 \times \mathbf{n})^*(1 \times h)^*\pi_i$, so it suffices to consider the case $g = 1$.
We recall the construction of π_i from (2.1):

$$(2.8) \qquad (1 \times \mathbf{n})^*(1 \times h)^*\pi_i = (-1)^d(1 \times \mathbf{n})^*(1 \times h)^*\sigma^*\hat{F}_{CH}(\beta_i).$$

One may verify, by writing $h = \tau_{h(0)} \circ h_0$ as in the proof of the last proposition, that $\sigma \circ (1 \times h) = (1 \times \tau_{-2h(0)}) \circ (1 \times h) \circ \sigma$. Therefore, using the definition of $\hat{F}_{CH}$ the last expression identifies with:

$$(2.9) \qquad (-1)^d \sigma^*(1 \times \mathbf{n})^*(1 \times h)^*(1 \times \tau_{-2h(0)})^* p_{12*}\left(p_{13}^*\beta_i \cdot \left(1 \times \sum_{i=0}^{\infty} \frac{(\ell^\mu)}{\mu!}\right)\right).$$

Since $\hat{F}_{CH}(\beta_i)$ has degree d, one may readily see that all terms in $\hat{F}_{CH}(\beta_i)$ (and hence in the expression above) except for $\mu = i$ are trivial (see, for example, [7, Lemma 2.8 and Theorem 2.19]). Next, in view of the Cartesian square

$$
\begin{array}{ccc}
A \times_k A \times_k \hat{A} & \xrightarrow{p_{12}} & A \times_k A \\
{\scriptstyle (1 \times \tau_{-2h(0)} \times 1).(1 \times h \times 1)}\Big\downarrow & & \Big\downarrow{\scriptstyle (1 \times \tau_{-2h(0)}).(1 \times h)} \\
A \times_k A \times_k \hat{A} & \xrightarrow{p_{12}} & A \times_k A
\end{array}
$$

the above expression becomes:

$$(-1)^d \sigma^*(1 \times \mathbf{n})^* p_{12*}(1 \times h \times 1)^*(1 \times \tau_{-2h(0)} \times 1)^*\left(p_{13}^*\beta_i \cdot \left(1 \times \frac{(\ell^i)}{i!}\right)\right).$$

Now using another Cartesian square

$$
\begin{array}{ccc}
A \times_k A \times_k \hat{A} & \xrightarrow{p_{12}} & A \times_k A \\
{\scriptstyle 1 \times \mathbf{n} \times 1}\Big\downarrow & & \Big\downarrow{\scriptstyle 1 \times \mathbf{n}} \\
A \times_k A \times_k \hat{A} & \xrightarrow{p_{12}} & A \times_k A
\end{array}
$$

this equals

$$(-1)^d \sigma^* p_{12*}(1 \times \mathbf{n} \times 1)^*(1 \times h \times 1)^*(1 \times \tau_{-2h(0)} \times 1)^*\left(p_{13}^*\beta_i \cdot \left(1 \times \frac{(\ell^i)}{i!}\right)\right).$$

Since p_{13} leaves the second factor unchanged, this expression identifies with:

$$(-1)^d \sigma^* p_{12*}\left(p_{13}^*\beta_i \cdot (1 \times \mathbf{n} \times 1)^*(1 \times h \times 1)^*(1 \times \tau_{-2h(0)} \times 1)^*\left(1 \times \frac{(\ell^i)}{i!}\right)\right)$$

$$= (-1)^d \sigma^* p_{12*}\left(p_{13}^*\beta_i \cdot \left(1 \times \frac{1}{i!}\left((\mathbf{n} \times 1)^*(h \times 1)^*(\tau_{-2h(0)} \times 1)^* \ell^i\right)\right)\right).$$

By the second statement in Proposition 2.6 the last term is given by

$$(2.10) \qquad n^i(-1)^d \sigma^* p_{12*}\left(p_{13}^*\beta_i \cdot \left(1 \times \frac{1}{i!}\left((h \times 1)^*(\tau_{-2h(0)} \times 1)^* \ell^i\right)\right)\right).$$

By applying the same steps above in essentially the opposite order one obtains the identification of the last expression with (observe again that all terms except the one with $\mu = i$ are trivial):

$$n^i(-1)^d \sigma^* p_{12*}\left(p_{13}^*\beta_i \cdot (1 \times h \times 1)^*(1 \times \tau_{-2h(0)} \times 1)^*\left(1 \times \left(\sum_{\mu=0}^{\infty} \frac{(\ell^\mu)}{\mu!}\right)\right)\right)$$

$$= n^i(-1)^d \sigma^* p_{12*}(1 \times h \times 1)^*(1 \times \tau_{-2h(0)} \times 1)^*\left(p_{13}^*\beta_i \cdot \left(1 \times \sum_{\mu=0}^{\infty} \frac{(\ell^\mu)}{\mu!}\right)\right)$$

$$= n^i(-1)^d \sigma^*(1 \times h)^*(1 \times \tau_{-2h(0)})^* p_{12*}\left(p_{13}^*\beta_i \cdot \left(1 \times \sum_{\mu=0}^{\infty} \frac{(\ell^\mu)}{\mu!}\right)\right)$$

$$= n^i(1 \times h)^*(-1)^d \sigma^* p_{12*}\left(p_{13}^*\beta_i \cdot \left(1 \times \sum_{\mu=0}^{\infty} \frac{(\ell^\mu)}{\mu!}\right)\right)$$

$$= n^i(1 \times h)^*\pi_i. \qquad \square$$

To prove orthogonality of the ρ_i, we need a version of Liebermann's trick (cf. [7, Proof of Theorem 3.1]); first we prove the following simple lemma:

Lemma 2.8. *For every* $g, h \in G$, $\rho_j \bullet (g \times h)^*[\Delta_A] = \rho_j$.

Proof. Certainly the lemma is true if $g = h = 1$. In the general case,

$$\rho_j \bullet (g \times h)^*[\Delta_A] = p_{13*}\left(p_{12}^*(g \times h)^*[\Delta_A] \cdot p_{23}^*\rho_j\right)$$

$$= p_{13*}\left((g \times h \times 1)^* p_{12}^*[\Delta_A] \cdot p_{23}^*\rho_j\right)$$

$$= p_{13*}(g \times h \times 1)^*\left(p_{12}^*[\Delta_A] \cdot (g^{-1} \times h^{-1} \times 1)^* p_{23}^*\rho_j\right)$$

$$= p_{13*}\left(g^{-1} \times h^{-1} \times 1\right)_*\left(p_{12}^*[\Delta_A] \cdot (g^{-1} \times h^{-1} \times 1)^* p_{23}^*\rho_j\right)$$

$$(2.11) \qquad = \left(g^{-1} \times 1\right)_* p_{13*}\left(p_{12}^*[\Delta_A] \cdot p_{23}^*(h^{-1} \times 1)^*\rho_j\right).$$

Since ρ_j is $G \times G$-invariant the last term equals

$$(g \times 1)^* p_{13*}\left(p_{12}^*[\Delta_A] \cdot p_{23}^*\rho_j\right) = (g \times 1)^*\left(\rho_j \bullet [\Delta_A]\right) = (g \times 1)^*(\rho_j) = \rho_j. \qquad \square$$

Proposition 2.9. (Liebermann's trick). *For every* $i, j, i \neq j, \rho_i \bullet \rho_j = 0$.

Proof. Suppose $n \in E$. By Proposition 2.7,

$$n^j \rho_j = (1 \times \mathbf{n})^*\rho_j = (1 \times \mathbf{n})^*\left(\rho_j \bullet [\Delta_A]\right).$$

By Lemma 2.8, the last term equals

$$\frac{1}{|G|^2}(1 \times \mathbf{n})^*\left(\rho_j \bullet \sum_{g,h}(g \times h)^*[\Delta_A]\right)$$

$$= \frac{1}{|G|^2}(1 \times \mathbf{n})^*\left(\rho_j \bullet \sum_{i=0}^{2d}\rho_i\right)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2d} (1 \times \mathbf{n})^* p_{13*} \big(p_{12}^* \rho_j \cdot p_{23}^* \rho_i \big)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2d} p_{13*} (1 \times 1 \times \mathbf{n})^* \big(p_{12}^* \rho_j \cdot p_{23}^* \rho_i \big)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2d} p_{13*} \big(p_{12}^* \rho_j \cdot p_{23}^* (1 \times \mathbf{n})^* \rho_i \big)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2d} n^i (\rho_j \bullet \rho_i).$$

Hence

$$n^j \big((\rho_j \bullet \rho_j) - |G|^2 \rho_j \big) + \sum_{i \neq j} n^i (\rho_i \bullet \rho_j) = 0$$

for all $n \in E$. Since E is infinite, this forces $\rho_i \bullet \rho_j = 0$ for all $i \neq j$, and also $\rho_j \bullet \rho_j = |G|^2 \rho_j$. $\qquad \square$

This final step in the proof of Theorem 2.3 is to show that the images of the η_i under the cycle map $cl_{A/G \times_k A/G} : CH^*(A/G \times_k A/G) \to H^*(X/G \times_k X/G)$ to any Weil cohomology theory are in fact the Künneth components of the class of the diagonal. This follows easily from the analogous fact for the variety A and commutativity of the following diagram:

$$
\begin{array}{ccc}
CH^*_{\mathbf{Q}}(A \times_k A) & \xrightarrow{\;cl_{A \times_k A}\;} & H^*(A \times_k A) \\
\Big\downarrow{\scriptstyle (f \times_k f)_*} & & \Big\downarrow{\scriptstyle (f \times_k f)_*} \\
CH^*_{\mathbf{Q}}(A/G \times_k A/G) & \xrightarrow{\;cl_{A/G \times_k A/G}\;} & H^*(A/G \times_k A/G)
\end{array}
$$

Here we will show that any Weil cohomology theory, H^*, extends to pseudo-smooth schemes and show that the above square commutes. First observe that if G is a finite group acting on a smooth scheme X, each $g \in G$ acts on X as an automorphism: therefore, the action of G on X induces an action on the given Weil cohomology theory applied to X, i.e. on $H^*(X)$. Since $H^n(X)$ are all vector spaces over a field of characteristic 0, one obtains a decomposition of $H^*(X)$ into irreducible representations of G. One defines $H^*(X/G)$ to be $H^*(X)^G$. Corresponding assertions also hold for the rational Chow groups.

Observe that if $f : X \to X/G$ is the quotient map, one may identify $f_* : CH^*_{\mathbf{Q}}(X) \to CH^*_{\mathbf{Q}}(X/G)$ ($f_* : H^*(X) \to H^*(X/G)$) with the projection $CH^*_{\mathbf{Q}}(X) \to CH^*_{\mathbf{Q}}(X)^G$ (the projection $H^*(X) \to H^*(X)^G$, respectively). Since the cycle map commutes with group action, one can now see that it commutes with f_*: we obtain the commutativity of the square above.

This concludes all but the last statement in the proof of Theorem 2.3. We proceed to establish this presently. Therefore, we will assume the hypotheses there. Now observe that, by [3], we have a decomposition:

$$(2.12) \qquad CH_{\mathbf{Q}}^j(A) = \bigoplus_{s=j-d}^{j} CH_{\mathbf{Q}}^j(A)_s$$

where $CH_{\mathbf{Q}}^j(A)_s = \{\alpha \in (CH_{\mathbf{Q}}^j(A))^G \mid n^*(\alpha) = n^{2j-s}\alpha, n \in E\}$. When each $g(0)$ is torsion, one may show as in Proposition 2.6 that the group action commutes with multiplication by n, for n belonging to the infinite subset E. Therefore, the above decomposition (at least when n is restricted to belong to the infinite set E) respects the group action. We take G-invariants to obtain

$$(2.13) \qquad \left(CH_{\mathbf{Q}}^j(A)\right)^G = \bigoplus_{s=j-d}^{j} \left(CH_{\mathbf{Q}}^j(A)\right)_s^G$$

where $(CH_{\mathbf{Q}}^j(A))_s^G = \{\alpha \in (CH_{\mathbf{Q}}^j(A))^G \mid n^*(\alpha) = n^{2j-s}\alpha, n \in E\}$. (Clearly the sum of the G-invariants of each of the summands of the right-hand side of (2.12) belongs to the G-invariant part of the left-hand side of (2.12). To see the G-invariant part of the left-hand side of (2.12) belongs to the sum of the G-invariants of the summands on the right-hand side, one may use an argument as in the proof of Proposition 2.9.) Now we have the following lemma:

Lemma 2.10. *The correspondence ρ_i acts as zero on $(CH_{\mathbf{Q}}^j(A))_s^G$ for $s \neq 2j - i$ and acts as the identity if $s = 2j - i$. It follows that ρ_i acts as zero on $(CH_{\mathbf{Q}}^j(A))^G$ if $i < j$ and also if $i > j + d$. Moreover, if $j = 0, 1, d-2, d-2, d$, ρ_i acts as zero on $CH_{\mathbf{Q}}^j(A/G)$ for $i > 2j$. In particular, if $d \leqslant 4$, then ρ_i acts as zero on $CH_{\mathbf{Q}}^j(A/G)$ for both $i < j$ and $i > 2j$.*

Proof. The key observation is that the correspondence ρ_i is $G \times G$-invariant, so that it induces the map $\rho_i : (CH_{\mathbf{Q}}^j(A))^G \to (CH_{\mathbf{Q}}^j(A))^G$ for all j. Now the proof of the first statement is exactly the same as in [22, Lemma 2.5.1]: simply take the G-invariant parts of the commutative diagram there. The second statement now follow by exactly the same argument as in [22, Corollary 2.5.2]. We skip the remaining details. $\square$

Proof of the last statement of Theorem 2.3. It suffices to show that $\eta_i = 1/|G|^3 (f \times f)_*(\rho_i)$ acts as zero on $CH_{\mathbf{Q}}^j(A/G)$ in the following cases: (i) if $i < j$ and also if $i > j + d$ or (ii) if $j = 0, 1, d-2, d-1, d$ and $i > 2j$. This follows from the following computation: Let $\alpha \in CH_{\mathbf{Q}}^j(A/G)$ for j as above. Now

$$(f \times f)_*(\rho_i) \bullet \alpha = f_*\left(\rho_i \bullet f^*(\alpha)\right).$$

Observe $f^*(\alpha) \in CH_{\mathbf{Q}}^j(A)^G$ and therefore the last term is trivial for the values of j considered above. This completes the proof of Theorem 2.3. $\square$

Among the formulae proved by Künnemann is the so-called Poincaré duality for abelian varieties [17, Theorem 3.1.1(iii)]; in our notation, this reads $\pi_{2d-i} = {}^t\pi_i$ for each i. This fact immediately implies the analogue for quotients:

Corollary 2.11. (Poincaré duality for quotients). *The Chow–Künneth decomposition for A/G of Theorem 2.3 satisfies Poincaré duality: that is, for any i, $\eta_{2d-i} = {}^t\eta_i$.*

Next we proceed to show that the torsion hypothesis in Theorem 2.3 may be removed. This will follow from the sequence of results considered below.

Let G denote a finite group and let M denote an abelian group that is a G-module. Recall that a derivation $d : G \to M$ is a function d so that $d(g_1.g_2) = dg_1 + g_1 \circ dg_2$ where $\circ$ denotes the G-action on M. Given a fixed element $m \in M$, an inner derivation associated to m is the map $d : G \to M$ defined by $d(g) = g \circ m - m$, $g \in G$. Clearly every inner derivation is a derivation. We will denote the set of all derivations of G in M (the set of all inner derivations of G in M) by $Der(G, M)$ ($IDer(G, M)$, respectively). It is well known that one has the isomorphism (see, for example, [10]):

$$(2.14) \quad H^1(G, M) = Der(G, M)/IDer(G, M).$$

Next recall that if A is an abelian variety defined over the field k and G acts on A, there is an induced action $\mu : G \times A(k) \to A(k)$ of G on $A(k)$. Now define $\mu_h : G \times A(k) \to A(k)$ by $\mu_h(g, a) = \mu(g, a) - \mu(g, 0)$. It is easily checked that μ_h is also an action of G on $A(k)$; that is, μ_h defines a G-module structure on $A(k)$.

Lemma 2.12. *Assume the above situation. Then the map $d : G \to A(k)$ defined by $d(g) = \mu(g, 0)$ is a derivation of G in $A(k)$, considered as a G-module via the action μ_h.*

Proof. One simply calculates:

$$
\begin{aligned}
d(g) + gd(h) &= \mu(g, 0) + \mu_h\big(g, \mu(h, 0)\big) \\
&= \mu(g, 0) + \mu\big(g, \mu(h, 0)\big) - \mu(g, 0) \\
&= \mu\big(g, \mu(h, 0)\big) = \mu(gh, 0) = d(gh). \qquad \square
\end{aligned}
$$

Proposition 2.13. *Assume the above situation. Then $H^1(G, A(k))$ is annihilated by $|G|$, where G acts on $A(k)$ through μ_h.*

Proof. This follows from the general fact that for any finite group H and any H-module M, $H^n(H, M)$ is annihilated by $|H|$ for $n > 0$; see [31, Theorem 6.5.8]. $\square$

Proposition 2.14. *Let $\bullet : G \times A \to A$ denote the action of a finite group on an abelian variety A defined over a field k. Then there exists a finite extension L of k and a point $a \in A(L)$ so that the new action of G on $A_L = A \times_{\operatorname{Spec} k} \operatorname{Spec} L$ defined*

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by $(g, c) \mapsto g \bullet_a c = T_a \circ g \circ T_{-a}(c)$ satisfies the torsion hypothesis in Theorem 2.3 (i.e. after translating the origin of the abelian variety to the new point a, the new action satisfies the torsion hypothesis). Here $c \in A_L(R)$, where R is any L-algebra, T_x denotes the translation by $x \in A(L)$ and $\circ$ denotes composition. Moreover the geometric quotients of A_L by G for the actions $\bullet$ and $\bullet_a$ are isomorphic.

Proof. Observe that $d(g) = \mu(g, 0)$ is a 1-cocycle. This is torsion as a cohomology class, i.e. some multiple of it is an inner derivation.) In view of the Proposition 2.13, there exists a point $a \in A(k)$ so that $|G|\mu(g, 0) = \mu_h(g, a) - a$ for all $g \in G$. If k is algebraically closed we may write $a = |G|b$, for some $b \in A(k)$. In general, we may do the same for some $b \in A(L)$ where L is a finite extension of k. Then

(2.15) $\mu(g, 0) - \mu_h(g, b) + b$ is $|G|$-torsion for all $g \in G$.

Now for every k-algebra R, we may define a new action of G on $A_L(R)$ by $\nu(g, c) = \mu(g, 0) + b + \mu_h(g, c - b)$. Then for any $g \in G$, $\nu(g, 0) = \mu(g, 0) + b + \mu_h(g, -b) = \mu(g, 0) - \mu_h(g, b) + b$ is annihilated by $|G|$, so ν satisfies the torsion hypothesis of Theorem 2.3.

The last statement of the proposition follows from the commutative square:

$$
\begin{array}{ccc}
G \times A_L & \xrightarrow{\ \bullet\ } & A_L \\
{\scriptstyle id \times T_{-a}} \downarrow & & \downarrow {\scriptstyle T_{-a}} \\
G \times A_L & \xrightarrow{\ \bullet_a\ } & A_L
\end{array}
$$ $\quad\square$

2.1. Proof of the main theorem

In view of the last proposition, if the torsion hypothesis in Theorem 2.3 is not satisfied by the given G-action, one may replace that action by the above modified action. Then the quotient schemes are isomorphic and the new action satisfies the torsion hypothesis of Theorem 2.3. This completes the proof of all but the last statement in Theorem 1.1. The last statement there was already proven as part of Theorem 2.3.

2.2. Examples

1. *Symmetric products of abelian varieties.* Let X denote an abelian variety and X^n/Σ_n the n-fold symmetric power of X. Observe that for every $\sigma \in \Sigma_n$, $\sigma(0, \ldots, 0) = (0, \ldots, 0)$. Therefore the hypotheses of Theorem 2.3 are satisfied irrespective of the base field k. Therefore, we obtain a Chow–Künneth decomposition for X^n/Σ_n. (Observe that the action of Σ_n is not in general free so that the quotient X^n/Σ_n is only pseudo-smooth and *not* smooth.)

2. *Example of Igusa.* (See [11]). Let X be an elliptic curve over k, with $char(k) \neq 2$. Let t denote a point of order 2 on X. Define an action of $\mathbb{Z}/2\mathbb{Z}$ on $X \times X$ by $: (x, y) \mapsto (x + t, -y)$, and let Y denote the quotient variety for this action. The

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resulting surface is a so-called *bielliptic surface*; see [3, VI, pp. 19–20]. (This example may be generalized by taking X to be an abelian variety.) Now one sees easily that the action is free so that Y is smooth. Nevertheless, in positive characteristic, Y need not be an abelian variety as shown in [11]. Theorem 2.3 provides a Chow–Künneth decomposition for Y.

Observe that in both the above examples, the group action satisfies the torsion hypotheses. When the schemes considered in these examples have dimension $\leqslant 4$, our main result implies that the Chow–Künneth projectors we construct satisfy the full set of conjectured properties, so that the resulting filtration on their Chow groups satisfies the conjectured properties of the Bloch–Beilinson filtration.

Comparison with the construction of Chow–Künneth projectors using the method of finite dimensionality of motives

For purposes of comparison, we will construct, at least in outline, Chow–Künneth projectors for the last two examples using finite dimensionality of the motives of abelian varieties and their finite quotients considered above.

First one observes that the motives of all abelian varieties and hence their finite quotients are finite dimensional: see [15, Section 9; 9, Theorem 11]. Moreover the Künneth components of the diagonal for these schemes are algebraic. It is observed in [9, Corollary 9] that therefore such schemes have a Chow–Künneth decomposition. However, the construction of these Chow–Künneth projectors this way is quite laborious and proceeds as follows.

Let A denote an abelian variety of dimension d provided with the action of a finite constant group scheme G. To simplify the discussion, we will furthermore assume that A/G is smooth. As observed above, the motive of A/G is finite dimensional, of (Kimura) dimension $= \dim H^*(A/G) = \dim H^*(A)^G$ where H^* is a fixed Weil cohomology theory. Here dim denotes the dimension as a vector space over the coefficient field. This is bounded above by $\dim H^*(A)$ and using a lifting of A to characteristic 0 (see [24]), one may compute this to be 2^{2d}.

Next one shows from [15, Sections 6, 7] that every correspondence on A/G that is homologically trivial is *nilpotent* and that there is a uniform bound on the order of nilpotence for such correspondences as a function of the (Kimura) dimension of A. We will denote the Kimura dimension of A/G by $\mathrm{kdim}(A/G)$. This function may be computed from [15, Sections 6, 7] which shows this is $\mathrm{kdim}(A/G)^3$. In view of the above observations, all we can say is that this is bounded above by 2^{6d}.

Let $n = \mathrm{kdim}(A/G)$. Now the Nagata–Higman Lemma [1, Lemma 7.2.8] shows that the ideal of homologically trivial correspondences on A/G is itself nilpotent with the order of nilpotence $2^n - 1$.

At this point [12, Lemma 5.4] shows that one may inductively lift the projectors from $H^*(A/G \times A/G)$ (where H^* is the fixed Weil cohomology theory) to Chow–Künneth projectors. The argument given in [12, Lemma 5.4] is when the index of nilpotency is 2 and he uses an induction on the order of nilpotency to consider the general case. As observed above, the index of nilpotency in the examples above will be $2^n - 1$ where $n = kdim(A/G)$ so that one needs to repeat this construction

n-times to complete the lifting. *Recall n itself is, in general, 2^{6d} so that one needs to repeat the inductive steps in this construction, in general, 2^{6d}-times to be able to write down explicitly all the Chow–Künneth projectors, by this method.*

Moreover, since these projectors do not have any close relation to the Chow–Künneth projectors for the original abelian variety, results like the last statement of Theorem 1.1, seem difficult to obtain by this method.

3. THE STRONG KÜNNETH DECOMPOSITION FOR FINITE QUOTIENTS

Now suppose X is a pseudo-smooth, projective, equidimensional scheme over a field k and G a finite group of automorphisms of X. As in [20], we may form the quotient variety $Y = X/G$ and ask whether an explicit strong Künneth decomposition for X may be used to construct a strong Künneth decomposition for Y. We answer this question in the affirmative below.

First we consider an elementary calculation showing that strong Künneth decompositions are preserved under finite maps.

Proposition 3.1. *Let X and Y be pseudo-smooth proper varieties and $f : X \to Y$ a finite surjective map. If X has a strong Künneth decomposition, then Y also has a strong Künneth decomposition.*

Proof. Let $d = \dim X$, $m = \deg f$. The hypothesis that X has a strong Künneth decomposition allows us to write

$$[\Delta_X] = \sum_i \sum_j a_{i,j} \times b_{d-i,j}$$

where as before $a_{i,j}, b_{i,j} \in CH^i_{\mathbf{Q}}(X)$. Furthermore, $(f \times f)_*[\Delta_X] = m[\Delta_Y]$, so it suffices to prove that $(f \times f)_*(a_{i,j} \times b_{d-i,j}) = f_*(a_{i,j}) \times f_*(b_{d-i,j})$. This is accomplished by the next lemma, whose proof is immediate. $\qquad \square$

Lemma 3.2. *Let $f : X \to Y$ be a morphism of pseudo-smooth varieties.*

(1) *If f is proper, then for all $\alpha, \beta \in CH^*_{\mathbf{Q}}(X)$, $f_*(\alpha \times \beta) = f_*(\alpha) \times f_*(\beta)$.*
(2) *For all $\gamma, \delta \in CH^*_{\mathbf{Q}}(Y)$, $f^*(\gamma \times \delta) = f^*(\gamma) \times f^*(\delta)$.*

We note the following as a special case:

Corollary 3.3. *Let X be a pseudo-smooth quasi-projective variety, G a finite group of automorphisms of X. If X possesses a strong Künneth decomposition, so does $Y = X/G$.*

The utility of the previous statements becomes evident from the following easy result:

Proposition 3.4. *Let X be a pseudo-smooth projective variety possessing a strong Künneth decomposition. Then X has a Chow–Künneth decomposition.*

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Proof. Suppose $[\Delta_X] = \sum_{i=0}^{d} \sum_{j} a_{i,j} \times b_{d-i,j}$, where $a_{i,j}, b_{d-i,j} \in CH_{\mathbf{Q}}^i(X)$. For $0 \leqslant r \leqslant d$, set $\pi_r = \sum_j a_{r,j} \times b_{d-r,j}$ and for $d + 1 \leqslant r \leqslant 2d$, set $\pi_r = 0$. Then $[\Delta_X] = \sum_{r=0}^{2d} \pi_r$. We decorate p with subscripts and superscripts to denote the various projections to sub-factors of $X \times_k X \times_k X$; for example, $p_{13}^{123} : X \times_k X \times_k X \to X \times_k X$ sends (x, y, z) to (x, z), etc. Finally, we let $\sigma : X \to \operatorname{Spec} k$ and $\tau : X \times_k X \to \operatorname{Spec} k$ denote the respective structure maps. We claim that $\pi_r \bullet \pi_s = 0$ when $r \neq s$ and $\pi_r \bullet \pi_r = \pi_r$ for all $r, 0 \leqslant r \leqslant 2d$. The first equality is a consequence of the following more general fact proved in Lemma 3.5 (below). To conclude the proof of Proposition 3.4, we calculate:

$$\pi_r \bullet \pi_r = \left([\Delta_X] - \sum_{s \neq r} \pi_s \right) \bullet \pi_r = [\Delta_X] \bullet \pi_r = \pi_r. \qquad \square$$

Lemma 3.5. *With notation as above, suppose $a_r \in CH^r(X)$, $b_{d-r} \in CH^{d-r}(X)$, $a_s \in CH^s(X)$, $b_s \in CH^{d-s}(X)$, and set $\gamma_r = a_r \times b_{d-r}$, $\gamma_s = a_s \times b_{d-s}$. If $r \neq s$, then $\gamma_s \bullet \gamma_r = 0$.*

Proof. The proof is a straightforward computation and is therefore skipped. $\square$

As an application, we compute the strong Künneth decomposition for the nth symmetric product of projective space $\mathbb{P}_k^m$. Let $\ell \in CH_{\mathbf{Q}}^1(\mathbb{P}_k^m)$ be the class of a generic hyperplane in $\mathbb{P}_k^m$. It is well-known (cf. [18, p. 455]) that $\mathbb{P}_k^m$ has a strong Künneth decomposition:

$$\Delta_{\mathbb{P}_k^m} = \sum_{i=0}^{m} \ell^i \times \ell^{m-i}.$$

Let $X = (\mathbb{P}_k^m)^n$. By the Künneth formula for motives, we have

$$[\Delta_X] = \sum_{0 \leqslant i_1, \dots, i_n \leqslant m} f_{i_1, \dots, i_n}$$

where $f_{i_1, \dots, i_n} = \ell^{i_1} \times \cdots \times \ell^{i_n} \times \ell^{m-i_1} \times \cdots \times \ell^{m-i_n} \in CH_{\mathbf{Q}}^{mn}(X \times_k X)$.

Now consider the action of the symmetric group on n letters (denoted Σ_n) on $X = (\mathbb{P}_k^m)^n$ by interchanging of factors. Let $Y = X/S_n$ and $q : X \to Y$ the quotient map. Note also that for any $\sigma \in \Sigma_n$, $(q \times q)_* f_{i_1, \dots, i_n} = (q \times q)_* f_{\sigma(i_1), \dots, \sigma(i_n)}$.

Applying $(q \times q)_*$ to the strong Künneth decomposition for $[\Delta_X]$ given above, and noting that $\deg q = n!$, we obtain

$$(3.1) \qquad [\Delta_Y] = \sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant m} (q \times q)_* f_{i_1, \dots, i_n}$$

$$= \sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant m} \bar{\ell}^{i_1} \times \cdots \times \bar{\ell}^{i_n} \times \bar{\ell}^{m-i_1} \times \cdots \times \bar{\ell}^{m-i_n}$$

where $\bar{\ell}^i = q_*(\ell^i)$. This provides a strong Künneth decomposition for Y.

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Corollary 3.6. *Let Y denote the nth symmetric product of $\mathbb{P}_k^m$. Then*

$$CH^*(Y, \mathbb{Q}, r) \cong CH^*(Y, \mathbb{Q}, 0) \otimes CH^*(\operatorname{Spec} k, \mathbb{Q}, r)$$

where $CH^*(Z, \mathbb{Q}, r) = \pi_r(z^*(Z, .) \otimes \mathbb{Q})$ and $z^*(Z, .)$ denotes the higher cycle complex of the scheme Z.

Proof. This follows readily from the above strong Künneth decomposition for the class Δ_Y and Theorem A.1. $\square$

APPENDIX: FUNCTORIALITY FOR THE COHOMOLOGY OF PSEUDO-SMOOTH SCHEMES AND THE STRONG RELATIVE KÜNNETH DECOMPOSITION OF COHOMOLOGY

In this appendix, we first discuss various functoriality properties of cohomology theories for pseudo-smooth schemes with rational coefficients. We conclude, with a view to applications in the Corollary 3.6, a relative form of the strong Künneth decomposition in arbitrary cohomology theories satisfying certain mild conditions. It should be remarked that in this section, it suffices to assume the cohomology theory is, at least in principle, part of a twisted duality theory in the sense of Bloch–Ogus, but extended to the category of pseudo-smooth schemes. (See [4].)

Recall that a pseudo-smooth scheme X is the quotient of a smooth scheme by a finite group G; the associated Deligne–Mumford stack $[X/G]$ is smooth although the quotient X/G is not in general. Nevertheless, one may readily identify any cohomology of the quotient stack $[X/G]$ with that of the geometric quotient X/G provided one works with rational coefficients. This provides a convenient mechanism for extending the formalism of Bloch–Ogus style cohomology theories to pseudo-smooth schemes as is done below. One may first make the following definitions (mainly for conveniently stating the results below). Let $\tilde{X}$ ($\tilde{Y}$) be a smooth scheme provided with the action of a finite group G (H, respectively). Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ denote a map compatible with the group actions, i.e. we are given a homomorphism $G \to H$ such that $\tilde{f}$ is equivariant for the given action of G on $\tilde{X}$ and the induced action of G on $\tilde{Y}$. Let $f : X = \tilde{X}/G \to Y = \tilde{Y}/H$ be the induced map. We say f is *pseudo-flat* (*pseudo-smooth*, respectively) if the map $\tilde{f}$ is flat (smooth, respectively). Since the groups G and H are finite, one may verify that f is proper if and only if $\tilde{f}$ is proper.

Observation A.1. In this situation, we may replace $\tilde{f}$ with the map $\hat{f} : H \times_G X \to Y$ which is now H-equivariant.

(H.0) The cohomology theory will be denoted by $H^s(X, r)$, where r is the twist or weight and will be defined on the category of all pseudo-smooth schemes of finite type over a given field k: *this will always be a vector space over $\mathbb{Q}$*. One may assume that $H^s(X, r) = H^s(EG \times_G \tilde{X}, r) \cong H^s(\tilde{X}, r)^G$ where $EG \to BG$ is a principal G-bundle with BG denoting a suitable model for the *classifying space of the group G*: one may assume this is the simplicial scheme defined by the usual bar construction (or the Totaro–Edidin–Graham approximation as in [30]) and that

the given cohomology theories extend to all simplicial schemes with smooth face maps. (Using the observation that $EG \times \tilde{X} \to EG \times_G \tilde{X}$ is a principal G-bundle and since we are working with rational coefficients, one may establish the isomorphisms $H^*(X, r) \cong H^*(EG \times_G \tilde{X}, r) \cong H^*(\tilde{X}, r)^G$ readily.) Throughout this section we will make the following additional hypotheses on our cohomology theories: these should be easy to establish by viewing them as equivariant cohomology theories in the above sense and by making use of the observation A.1. Observe that, in this situation, $H^*(EH \times_H (H \times_G X); .) \cong H^*(EG \times_G X; .)$.

(H.1) for every pseudo-flat map $f : X \to Y$, there is an induced map $f^* : H^s(Y, r) \to H^s(X, r)$ and this is natural in f.

(H.2) for every proper and pseudo-smooth map $f : X \to Y$ of relative dimension d, there is a push-forward $f_* : H^i(X; j) \to H^{i-2d}(Y; j - d)$ so that if $g : Y \to Z$ is another proper map of relative dimension d', one obtains $g_* \circ f_* = (g \circ f)_*$. In case f is proper and pseudo-flat, the obvious projection formula $f_*(x \circ f^*(y)) = f_*(x) \circ y$, $x \in H^s(X, r)$, $y \in H^s(Y, r)$ holds.

(H.3) for each pseudo-smooth scheme $X = \tilde{X}/G$ and closed pseudo-smooth sub-scheme $Y = \tilde{Y}/H$ (with $\tilde{Y}$ a smooth closed sub-scheme of pure codimension c in $\tilde{X}$ and H a subgroup of G), there exists a canonical class $[Y] \in H^{2c}(X; c)$. Moreover the last class lifts to a canonical class $[Y] \in H^{2c}_Y(X; c)$. (The latter has the obvious meaning in the setting Bloch–Ogus twisted duality theories. In case the cohomology theory is defined as hyper-cohomology with respect to a complex, we let $\mathbb{H}_Y(X; c) =$ the canonical homotopy fiber of the obvious map $\mathbb{H}(X; c) \to \mathbb{H}(X - Y; c)$; now $H^{2c}_Y(X, c) = H^{2c}(\mathbb{H}_Y(X; \Gamma c))$.) The cycle classes are required to pull-back under flat pull-back and push-forward under proper push-forward.

(H.4) if X is a pseudo-smooth scheme, there exists the structure of a *graded commutative ring* on $H^*(X; .) = \bigoplus_{r,s} H^r(X; s)$, i.e. $\circ : H^r(X; s) \otimes H^{r'}(X; s') \to H^{r+r'}(X; s + s')$. In addition to this, there exists an external product $H^r(X; s) \otimes H^{r'}(X; s') \to H^{r+r'}(X \times X; s + s')$ so that the internal product is obtained from the latter by pull-back with the diagonal.

(H.5) if X is a pseudo-smooth projective scheme $H^*(X \times X; .)$ has the structure of a ring under the composition of correspondences defined as in Section 1. The class of the diagonal acts as the unit for this operation. The ring structure by composition of correspondences is easy to establish. In order to show that the class of the diagonal Δ_X is the unit for this operation, one may proceed as follows. Assume $X = \tilde{X}/G$ for some smooth scheme $\tilde{X}$ and finite group G. The proof as in Lemma 2.8 shows that for any class $\tilde{\alpha} \in H^*(\tilde{X} \times \tilde{X}; .)^G$, $\tilde{\alpha} \bullet (g \times h)^* \Delta_X = \tilde{\alpha}$. Therefore, $\tilde{\alpha} \bullet \sum_{(g,h) \in G^2} (g \times h)^* \Delta_{\tilde{X}} = |G|^2 \tilde{\alpha}$. If $\alpha \in H^*(X \times X; .)$, $\tilde{\alpha} = (f \times f)^*(\alpha) \in H^*(\tilde{X} \times \tilde{X}; .)^G$ and so $(f \times f)_*(\tilde{\alpha}) = |G|^2 \alpha$; moreover $(f \times f)_*(\sum_{(g,h) \in G^2} (g \times h)^* \Delta_{\tilde{X}}) = |G|^3 \Delta_X$. Now the arguments in Lemma 2.4 show that if $f : \tilde{X} \to \tilde{X}/G = X$ is the quotient map, $(f \times f)_*(\tilde{\alpha}) \bullet (f \times f)_*(\sum_{(g,h) \in G^2} (g \times h)^* \Delta_X) = |G|(f \times f)_*(\tilde{\alpha} \bullet (\sum_{(g,h) \in G^2} (g \times h)^* \Delta_{\tilde{X}})) = |G|.|G|^2 (f \times f)_*(\tilde{\alpha}) = |G|^5 \alpha$, i.e. $(|G|^2 \alpha) \bullet (|G|^3 \Delta_X) = |G|^5 \alpha$, so that $\alpha \bullet \Delta_X = \alpha$.

We will next consider an application of the strong relative Künneth decomposition of cohomology for pseudo-smooth schemes. For the purposes of this discussion

it is also convenient to consider only cohomology theories that are singly graded or non-weighted. Given a bi-graded cohomology theory $H^s(X; r)$, we will re-index it as follows: we let

(A.1) $h^r(X; 2r - s) = H^s(X; r)$ and $h^*(X; n) = \bigoplus_r h^r(X; n)$.

We view $\{h^*(X; n) \mid n\}$ as a singly graded cohomology theory. Observe that if $f : X \to Y$ is a proper smooth map of relative dimension d, the induced map f_* sends $h^*(X; n)$ to $h^*(Y; n)$. Similarly if $f : X \to Y$ is a flat map, the induced map f^* sends $h^*(Y; n)$ to $h^*(X; n)$.

Theorem A.1. *Let $\tilde{f} : \tilde{X} \to \tilde{Y} = Y$ denote a proper smooth map of smooth schemes of relative dimension d. We will assume that the scheme $\tilde{X}$ is provided with the action of a finite group G so that with the trivial action of G on Y, the map $\tilde{f}$ is G-equivariant. Let $X = \tilde{X}/G$, $[\Delta] \in H^{2d}(X \times_Y X; d)$ denote the class of the diagonal. Assume that $[\Delta] = \sum_{i,j} a_{i,j} \times b_{d-i,j}$, with each $a_{i,j} \in H^{2i}(X; i)$, $b_{d-i,j} \in H^{2d-2i}(X; d - i)$. Then for every fixed integer n one obtains the isomorphism:*

(A.2) $h^*(X; n) \cong h^*(X; 0) \underset{h^*(Y;0)}{\bigotimes} h^*(Y; n)$.

Proof. We will first prove that the classes $\{a_{i,j} \mid i\}$ generate $h^*(X; n)$ as a module over $h^*(Y; .)$, i.e. the obvious map from the right-hand side to the left-hand side of (A.2) (which we will denote by ρ) is *surjective*.

Let $p_i : X \times_Y X \to X$ denote the projection to the ith factor. For each $x \in h^*(X; n)$ we will first observe the equality:
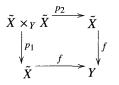
(A.3) $x = p_{1*}\big(\Delta.p_{2*}(x)\big)$

where . denotes the intersection pairing. To see this observe that $[\Delta] = \Delta_*(1)$, $1 \in H^*(X; \Gamma(.))$. Therefore, $\Delta.p_{2*}(x) = \Delta_*(\Delta^* p_2^*(x))$ and hence $p_{1*}(\Delta.p_2^*(x)) = p_{1*}\Delta_*(\Delta^* p_2^*(x)) = (p_1 \circ \Delta)_*((p_2 \circ \Delta)^*(x)) = x$. The corresponding formulae hold in the cohomology of $\tilde{X}$ and $\tilde{X} \times_Y \tilde{X}$; by taking the G and $G \times G$-invariants one may establish these formulae in the cohomology of X and $X \times_Y X$.

Now we substitute $[\Delta] = \sum_{i,j} p_1^*(a_{i,j}).p_2^*(b_{d-i,j})$ into the above formula to obtain:

(A.4) $x = p_{1*}\left(\sum_{i,j} p_1^*(a_{i,j}).p_2^*\big(b_{d-i,j}.p_2^*(x)\big)\right)$

$= p_{1*}\left(\sum_{i,j} p_1^*(a_{i,j}).p_2^*(b_{d-i,j}.x)\right)$

$= \sum_{i,j} a_{i,j}.p_{1*}p_2^*(b_{d-i,j}.x)$

$= \sum_{i,j} a_{i,j}.f^*\big(f_*(b_{d-i,j}.x)\big)$.

(The base-change formula is justified by observing that in the Cartesian square

$$
\begin{array}{ccc}
\tilde{X} \times_Y \tilde{X} & \xrightarrow{\ p_2\ } & \tilde{X} \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\tilde{X} & \xrightarrow{\ f\ } & Y
\end{array}
$$

all maps may be made $G \times G$-equivariant by letting $G \times G$ act by the appropriate factor on the two copies of $\tilde{X}$. Clearly the base-change formula holds in $G \times G$-equivariant cohomology applied to the vertices of the above diagram. By our hypothesis, the $G \times G$-equivariant cohomology identifies with the cohomology of the appropriate quotient variety.)

This proves the assertion that the classes $\{a_{i,j} \mid i\}$ generate $h^*(X; .)$ i.e. the map ρ is surjective.

The rest of the proof is to show that the map ρ is *injective*. The key is the following diagram:

$$
\begin{array}{ccc}
h^*(X; n) & \xleftarrow{\ \ \rho\ \ } & h^*(X, 0) \underset{h^*(Y,0)}{\otimes} h^*(Y; n) \\
& {\scriptstyle \mu}\searrow & \big\downarrow{\scriptstyle \alpha} \\
& & Hom_{h^*(Y;0)}(h^*(X, 0), h^*(Y; n))
\end{array}
$$

where the map α $(\mu(x),\ x \in h^*(X, 0))$ is defined by $\alpha(x \otimes y) =$ the map $x' \mapsto f_*(x' \bullet x) \bullet y$ (the map $x' \mapsto f_*(x' \bullet x)$, respectively). The commutativity of the above diagram is an immediate consequence of the projection formula: observe $\rho(x \otimes y) = x \bullet f^*(y)$. Therefore, to show the map ρ is injective, it suffices to show the map α is injective. For this we define a map β to be a splitting for α as follows: if $\phi \in Hom_{h^*(Y;0)}(h^*(X, 0), h^*(Y; n))$, we let $\beta(\phi) = \sum_{i,j} a_{i,j} \otimes (\phi(b_{d-i,j}))$. Observe that $\beta(\alpha(x \otimes y)) = \beta$ (the map $x' \to f_*(x' \bullet x) \otimes y) = (\sum_{i,j} a_{i,j} \otimes f_*(b_{d-i,j}.x)) \otimes y$. Now observe that $f_*(b_{d-i,j}.x) \in h^*(Y; 0)$ so that we may write the last term as $= (\sum_{i,j} a_{i,j}.f^* f_*(b_{d-i,j}.x)) \otimes y$. By (A.4), the last term $= x \otimes y$. This proves that α is injective and hence that so is ρ. $\square$

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