MOTIVIC E_∞-ALGEBRAS AND THE MOTIVIC DGA

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ABSTRACT. In this paper we define an E_∞-structure, i.e. a coherently homotopy associative and commutative product on chain complexes defining (integral and mod-l) motivic cohomology as well as mod -l étale cohomology. We also discuss several applications.

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1. Introduction

The main result of this paper is the existence of explicit E_∞-structures on complexes defining mod-l motivic and étale cohomology of all smooth schemes. Such results have often been implicitly assumed in the past: we make them explicit by constructing explicit operad actions on these complexes.

Our constructions already show that there are important differences between the E_∞-structure on the mod$-l$ singular complex of a topological space and the E_∞-structure we provide on the mod-l motivic complex. In the first case the existence of cohomology operations is a consequence of the fact that the passage from simplicial abelian groups to associated chain complexes is only homotopy commutative. In this case, this pairing involves the Alexander-Whitney map (from the chain complex associated to the tensor product of two simplicial abelian groups to the tensor product of the corresponding chain complexes) which is only homotopy commutative. In the motivic case, *the intersection pairing* on the motivic complexes is obtained from an external pairing of chain complexes followed by a pairing that involves only the shuffle maps which are strictly commutative. Therefore the source of the lack of strict commutativity for the product on the motivic complex is the lack of strict commutativity of the corresponding external pairing. In other words, our results show that the E_∞-structure on the singular co-chain complex and on complexes defining étale cohomology are strictly chain-theoretic phenomena whereas the E_∞-structure on the motivic complex is not. The E_∞-structure on the motivic complex also allows the construction of an E_∞-dga structure on the integral motivic complex and on the motivic complex with finite coefficients as well as a dga-structure on the rational motivic complex associated to any smooth scheme over a field. The latter applies to provide a construction of categories of mixed Tate motives associated to a large class of schemes.

The following is an outline of the paper. The second section is devoted to setting up the basic framework for the rest of the paper. There we also discuss the simplicial Barratt-Eccles operad. The third section is devoted to defining in detail an action of the above operad on teh integral motivic complexes. This leads to the construction of the motivic dga and motivic E_∞-dga. We summarize some of these results in following theorem.

Theorem 1.1. *Let* (Sm/k) *denote the category of all smooth separated schemes of finite type over a fixed field k with morphisms being maps of finite type. Let l be a prime (not necessarily different from $char(k) = p$) and let $\nu \geq 0$.*

1. Then there exists a functor

$$\underline{\mathbb{Q}}^{mot} : (\mathrm{Sm}/k)^{op} \to (DG - algebras \quad over \quad \mathbb{Q})$$

so that $\underline{\mathbb{Q}}^{mot} = \underset{r \geq 0, q}{\oplus} \underline{\mathbb{Q}}^{mot,q}(r)$. *If X is any smooth separated scheme of finite type over k and $\mathbb{Q}_X^{mot} = \Gamma(X, \underline{\mathbb{Q}}^{mot})$, then $H^q(\mathbb{Q}_X^{mot}(r)) \cong H^q_{\mathcal{M}}(X, \mathbb{Q}(r)) =$ the (rational) motivic cohomology of X in degree q and weight r.*

2. More generally there exist functors

$$\underline{\mathbb{Z}}^{mot} : (\mathrm{Sm}/k)^{op} \to (E^\infty \quad DG - algebras), \quad \underline{\mathbb{Z}}^{et}/l^\nu : (\mathrm{Sm}/k)^{op} \to (E^\infty \quad DG - algebras)$$

so that $\underline{\mathbb{Z}}^{mot} = \underset{r \geq 0, q}{\oplus} \underline{\mathbb{Z}}^{mot,q}(r)$ *and* $\underline{\mathbb{Z}}^{et}/l^\nu = \underset{r \geq 0, q}{\oplus} \underline{\mathbb{Z}}/l^{\nu et,q}(r)$. *If X is any smooth separated scheme of finite type over k and*

$$\mathbb{Z}_X^{mot} = R\Gamma(X, \underline{\mathbb{Z}}^{mot})$$

(computed on the Zariski site of X),

$$\mathbb{Z}/l^{\nu et}_X = R\Gamma(X, \underline{\mathbb{Z}}/l^{\nu et})$$

(computed on the étale site of X) then, $H^q(\mathbb{Z}_X^{mot}(r)) \cong H^q_{\mathcal{M}}(X, \mathbb{Z}(r)) =$ the motivic cohomology of X in degree q and weight r while $H^q(\mathbb{Z}/l^{\nu et}_X(r)) \cong H^q_{et}(X, \mathbb{Z}/l^\nu(r)) =$ the étale cohomology of X in degree q and weight r. $\mathbb{Z}_X^{mot}$ and $\mathbb{Z}/l^{\nu et}_X$ are E_∞-algebras over the operad $NZ(E\Sigma) \otimes \mathcal{E}nd_{\mathrm{C}_*(\Delta)}$ where $NZ(E\Sigma)$ is the simplicial Barratt-Eccles operad considered in Definition 2.3 and $\mathcal{E}nd_{\mathrm{C}_*(\Delta)}$ is the classical Eilenberg-Zilber operad.

Moreover, the functors $\underline{\mathbb{Z}}^{mot}$ and $\underline{\mathbb{Q}}^{mot}$ are additive presheaves satisfying cohomological descent (as in definition 6.5) on the big Zariski and Nisnevich sites of smooth quasi-projective schemes over k.

The rest of the paper is devoted to applications. In the fourth section, making use of the existence of the motivic dga, we construct a category of mixed Tate motives for a large class of varieties that includes all quasi-projective smooth linear varieties over a field which are the complements of closed projective smooth linear subvarieties in a bigger projective smooth linear variety. The main result here is Theorem 4.9 which we quote here (see section four for more details):

Theorem 1.2. *If the DGA A is connected (in the sense of 4.0.8), $\mathcal{MTF}(X)$ is equivalent to the category $\mathcal{FH}_A$. In particular, this holds for the following classes of smooth quasi-projective varieties assuming the Beilinson-Soulé conjecture (see Corollary 4.6) holds for the motivic cohomology of $Spec\, k$, for example if k is a number field:*

(i) all smooth (connected) projective linear varieties over k

(ii) any of the varieties appearing in Example 4.4 which are also connected, projective and smooth

(iii) any quasi-projective variety U of the form $X - Y$, where X and Y are smooth projective varieties as in (i) or (ii) and Y is closed in X.

So far the only construction of a category of mixed Tate motives is for a field (see [Bl-2], [Bl-K]) and also for the ring of integers in a number field (see [DG]) (where they apply it also to study the motivic fundamental group of $\mathbb{P}^1$ - several points.)

In the fifth section, we show that the operad actions defined earlier lead to *classical* cohomology operations on both mod$-l$ and mod-p motivic cohomology. These operations differ from the operations

constructed by Voevodsky (see [VV]) in the way they behave with respect to the weights. A follow-up to this paper, worked out jointly with Patrick Brosnan, (See [BroJ]) explores the relation between the classical cohomology operations as constructed in this paper and the motivic cohomology operations of Voevodsky.

In an earlier draft of the paper, we had provided a different E_∞-structure on the complex defining mod$-l$ étale cohomology as well as on the motives of smooth schemes. This E_∞-structure was provided by an algebraic variant of the Eilenberg-Zilber operad. However, this made it necessary to first set up and discuss in detail a model structure on the category of chain-complexes of sheaves of Z/l-modules on the big Nisnevich or étale site of smooth schemes and seemed to take the focus away from the main results on the action of the simplicial Barratt-Eccles operad on the motivic complexes. Therefore, we have removed all discussions on this second construction which will be discussed elsewhere separately.

Acknowledgments. This has been a rather long project for us, partly because the area of operads had been new to us when we embarked on this project and partly because we had been busy with several other more pressing projects. Over the years, we have benefitted from discussions with several mathematicians working in this and related areas: we thank Arvind Asok, Spencer Bloch, Patrick Brosnan, Zig Fiedorowicz, Herbert Gangl, Eric Friedlander, Mark Hovey, Mark Levine, Peter May, Bertrand Toen and Chuck Weibel for several helpful discussions/correspondence.

2. **The basic framework**

Throughout the paper k will denote a fixed field of arbitrary characteristic $p \geq 0$. (Sm/k) will denote the category of separated smooth schemes of finite type over k. $(\mathrm{Sm}/k)_{Zar}$, $(\mathrm{Sm}/k)_{Nis}$ and $(\mathrm{Sm}/k)_{et}$ will denote this category provided with the big Zariski, Nisnevich or étale topologies. If X denotes a separated smooth scheme of finite type over k, X_{Zar} (X_{Nis}, X_{et}) will denote the Zariski site (the (small) Nisnevich site, the (small) étale site, respectively) of X. We will let $\mathcal{S}$ denote any one of these sites: since the schemes we consider are of finite type over k, it follows that these sites are all skeletally small. $\mathrm{C}(\mathcal{S})$ will denote the category of unbounded co-chain complexes of abelian sheaves on $\mathcal{S}$ with differentials of degree $+1$; complexes of abelian sheaves with differentials of degree -1 will be referred to as chain complexes. By default, a complex will usually mean a co-chain complex, i.e. one whose differentials are of degree $+1$. The category of all pointed simplicial sheaves on a site $\mathcal{S}$ will be denoted $SSh(\mathcal{S})$: observe that this category contains as a full sub-category, the category of all sheaves of pointed sets. The base point in this category will be denoted $*$.

For the most part $\mathrm{C}_-(\mathcal{S})$ $(\mathrm{C}_0(\mathcal{S}))$ will denote the full sub-category of bounded above complexes (that are also trivial in positive degrees, respectively). For a fixed sheaf of commutative rings $\mathcal{R}$ with unit, $\mathrm{C}(\mathcal{S}, \mathcal{R})$ $(\mathrm{C}_-(\mathcal{S}, \mathcal{R}), \mathrm{C}_0(\mathcal{S}, \mathcal{R}))$ will denote the corresponding categories of complexes of sheaves of $\mathcal{R}$-modules. If R is a fixed commutative ring with unit, we will let R also denote the obvious associated constant sheaf. Observe that $\mathrm{C}(\mathcal{S})$ and $\mathrm{C}(\mathcal{S}, \mathcal{R})$ are symmetric monoidal with product $\otimes$ and an internal Hom functor we denote by $\mathcal{H}om$. Observe that $\mathrm{C}_-(\mathcal{S}, \mathcal{R})$ and $\mathrm{C}_0(\mathcal{S}, \mathcal{R})$ are not closed under the formation of the internal hom: this is the main reason for considering the category $\mathrm{C}(\mathcal{S}, \mathcal{R})$ of unbounded complexes in this paper. While it is convenient for us to have a model category structure on $\mathrm{C}(\mathcal{S}, \mathcal{R})$ that is compatible with the above tensor structure, such a model structure does not play any key role for the constructions in the rest of this paper. Therefore, we have chosen not to consider such structures in this paper. (This is discussed, for example, in [?] for the case where the site $\mathcal{S}$ is the usual site associated to a topological space and may be readily extended to any Grothendieck site.) Throughout the rest of the paper, we will assume that $\mathcal{R} = R$ the constant sheaf associated to a commutative ring R with unit. We conclude this section by recalling the definition of the motivic complexes and of the simplicial Barratt-Eccles operad.

Definition 2.1. Given a fixed $X \, \varepsilon \, (smt.schemes/k)$, one defines presheaves with transfers, $\mathbb{Z}_{eq}(X)$, $\mathbb{Z}_{tr}(X)$ by $\Gamma(V, \mathbb{Z}_{eq}(X)) = \mathrm{Cor}_{q.f}(V, X)$ and $\Gamma(V, \mathbb{Z}_{tr}(X)) = \mathrm{Cor}_f(V, X)$. It is observed in [F-S-V], that these define *sheaves with transfers* on $(\mathrm{Sm}/k)_{Nis}$ and $(\mathrm{Sm}/k)_{Zar}$.

Definition 2.2. We let $\underline{\mathbb{Z}}(n)$ =the sheaf $C^*(\mathbb{Z}_{eq}(\mathbb{A}^n))[-2n]$ on either of the sites $(\mathrm{Sm}/k)_{Nis}$ or $(\mathrm{Sm}/k)_{Zar}$. Alternatively we may also use the sheaf $C^*(\mathbb{Z}_{tr}((\mathbb{P}^1)^{\wedge^n}))[-2n]$ as defined in (3.2.5). Therefore, we will use whichever definition is more convenient. If X is a smooth scheme over k, we also let $\underline{\mathbb{Z}}_X(n)$ denote the restriction of $\underline{\mathbb{Z}}(n)$ to the Zariski or Nisnevich site of X. $\mathbb{Z}_X(n) = R\Gamma(X, \underline{\mathbb{Z}}_X(n))$. We define $\underline{\mathbb{Z}} = \underset{n}{\oplus}\underline{\mathbb{Z}}(n)$ and $\mathbb{Z}_X = R\Gamma(X, \underline{\mathbb{Z}}_X)$. $\underline{\mathbb{Z}}/l^\nu(n)$ will denote $\underline{\mathbb{Z}}(n)\underset{Z}{\otimes}Z/l^\nu$ for each fixed prime l and $\nu \geq 0$. $\underline{\mathbb{Z}}^{et}/l^\nu(n)$ denotes the same complex on the site $(Sm/k)_{et}$.

2.1. **The Barratt-Eccles operads.** One of the standard examples of E_∞-operads are the ones commonly called the Barratt-Eccles operads. We will only consider the simplicial variants of these in this section. (The geometric variant considered in an earlier version of the paper will discussed elsewhere as it does not play any role in this paper.)

Given a discrete group G, one forms the simplicial group EG given in degree n by $EG_n = G^{n+1}$. One may verify readily that if G and H are two groups, $E(G \times H)$ is naturally isomorphic to $EG \times EH$. One may also observe the isomorphism $EG \cong cosk_0(G)$ of simplicial objects: the advantage of $cosk_0(G)$ for us is that it does not involve the group structure. (Recall $cosk_0(G)_m = G^{\times^{m+1}}$. The face map $d_i : cosk_0(G)_m \to cosk_0(G)_{m-1}$ just drops the i-th factor while the degeneracy $s_i : cosk_0(G)_m \to cosk_0(G)_{m+1}$ just sends the i-th factor, G, diagonally into $G \times G$ forming the i-th and $i+1$-st factors.) Therefore, taking $cosk_0$, we see that the map $\gamma_k : \Sigma_k \times \Sigma_{n_1} \times ... \times \Sigma_{n_k} \to \Sigma_{\Sigma_i n_i}$ defined by $(\sigma_k, \sigma_{n_1}, \cdots, \sigma_{n_k}) \mapsto (\sigma_{n_{\sigma_k(1)}}, \cdots, \sigma_{n_{\sigma_k(k)}})$ induces the structure map $\gamma_k : E(\Sigma_k) \times E(\Sigma_{n_1}) \times E(\Sigma_{n_k}) \to E(\Sigma_{\Sigma_i n_i})$.

It may be important to point that since each $E\Sigma_m$ is identified with $cosk_0(\Sigma_m)$, the operad as defined above is an operad of simplicial sets only and not of simplicial groups. One may also want to observe that each $E\Sigma_m$ is acyclic and has a free action by Σ_m. Given a simplicial set X, one lets $X \otimes Spec\,k$ denote the simplicial scheme defined by $(X \otimes Spec\,k)_n = \underset{x_n\,\varepsilon\,X_n}{\sqcup} Spec\,k$ and with the structure maps induced by the structure maps of the simplicial set X. Thus $E\Sigma_m \otimes Spec\,k$ is a simplicial scheme over $Spec\,k$ for each m.

Definition 2.3. (The simplicial Barratt-Eccles operad) Henceforth we will let $E\Sigma_m$ denote the sheaf on the site $(\mathrm{Sm}/k)_{Nis}$ represented by this simplicial scheme. We will apply the free abelian group functor Z followed by the normalization functor N (that produces the Moore-complex: see (6.0.8)) to produce an operad in $C((\mathrm{Sm}/k)_{Nis})$ and in $C((\mathrm{Sm}/k)_{et})$ from the collection $\{E\Sigma_m | m \geq 0\}$. This defines the simplicial Barratt-Eccles operad on the above site: this will be denoted $NZ(E\Sigma) = \{N(Z(E\Sigma_n))|n\}$. Since we do not consider any other variant of this operad in this paper, we will refer to this operad henceforth as the *Barratt-Eccles operad*. We will often denote this operad by BE with $BE(n) = N(Z(E\Sigma_n))$.

For later use, it will be important to observe that the pairing induced by γ_k on this operad takes the following form. Given a q-simplex, $s_q = (\tau_1, \cdots, \tau_q)\,\varepsilon\,N(Z(E\Sigma_k))_q$ and p_i-simplices, $s_{p_i} = (\sigma_1(i), \cdots, \sigma_{p_i}(i))\,\varepsilon\,N(Z(E\Sigma_{j_i}))$, with $j = \Sigma_{i=1}^k j_i$, one first produces the $(q, p_1, \cdots, p_k)$-multi-simplex of $N(Z(E\Sigma_j))$ given by $(\tau_1 \circ \sigma_1(1), \cdots, \tau_1 \circ \sigma_{p_1}(1), \cdots, \tau_1 \circ \sigma_1(k), \cdots \tau_1 \circ \sigma_{p_k}(k), \cdots, \tau_q \circ \sigma_1(1), \cdots, \tau_q \circ \sigma_{p_1}(1), \cdots, \tau_q \circ \sigma_1(k), \cdots, \tau_q \circ \sigma_{p_k}(k))$. (Here we are viewing Σ_k acting as the subgroup of Σ_j where it permutes the k-blocks, $(1, \cdots, j_1), (j_1 + 1, \cdots, j_1 + j_2), \cdots, (\Sigma_{i=1}^{k-1}j_i + 1, \cdots, \Sigma_{i=1}^k j_i)$ and each Σ_{j_i} is viewed as the subgroup of Σ_j where it permutes the entries of the i-th block, $(\Sigma_{l=1}^{i-1}j_l + 1, \cdots, \Sigma_{l=1}^i j_l)$.) Then one uses the shuffle maps to produce a $q + \Sigma_i p_i$-simplex of $N(Z(E\Sigma_j))$.

Remark 2.4. Recall $M(Spec\,k) = Z[0]$ so that $M(E\Sigma_n \otimes Spec\,k) \cong N(Z(E\Sigma_n))$.

Proposition 2.5. *The Barratt-Eccles operad is an acyclic operad in the category* $C((\mathrm{Sm}/k)_{Nis})$ *and* $C((\mathrm{Sm}/k)_{et})$.

Proof. The acyclicity of the Barratt-Eccles operad follows readily in view of the observation that each $EG \cong cosk_0(G)$ is acyclic: this simplicial object has an extra degeneracy map s_{-1} induced by the map sending the trivial group $\{e\}$ into G as the identity element of G. $\qquad \square$

3. **Action of the Barratt-Eccles operads on the motivic complex**

In this section, we will define explicitly an action by the simplicial Barratt-Eccles operad $\{N(Z(E\Sigma_n))|n\}$ on the motivic complexes.

To motivate and clarify our construction, we will consider first explicitly how one proves (first order) homotopy commutativity of the product on the motivic complexes. We will once again assume the basic framework of section 1 throughout this section. In addition, l will denote a fixed prime, not necessarily different from p.

We will also first consider a construction using motivic complex of weight n defined as $\underline{\mathbb{Z}}(n) = C^*\text{Cor}_{q.f}(\ ,\mathbb{A}^n)[-2n]$. We will show that while one may construct the higher order homotopies explicitly this way, the structure that one obtains on the graded motivic complex $\oplus_{n\geq 0}\mathbb{Z}(n)$ will be only one of an *algebra over a colored operad.* Nevertheless this construction clarifies our basic ideas and therefore we will begin by discussing it briefly. Afterwards we show that a variant of this construction using $(\mathbb{P}^1)^{\wedge^n}$ in the place of $\mathbb{A}^n$ does indeed provide the structure of an algebra on $\oplus_{n\geq 0}\mathbb{Z}(n)$ over the Barratt-Eccles operad.

Since $\underline{\mathbb{Z}}(n) = C^*\text{Cor}_{q.f}(\ ,\mathbb{A}^n)[-2n]$, the product is defined by the pairing

$$(3.0.1) \qquad \text{Cor}_{q.f}(\ \times\Delta[.],\mathbb{A}^l)\otimes\text{Cor}_{q.f}(\ \times\Delta[.],\mathbb{A}^m)\to\text{Cor}_{q.f}(\ \times\Delta[.],\mathbb{A}^{l+m})$$

of simplicial abelian sheaves, which may be described as follows. Here $\Delta[.]$ denotes the cosimplicial scheme $\{\Delta[n]|n\}$ with the obvious structure maps. We start with correspondences Γ_1 on $U\times\Delta[n]\times\mathbb{A}^l$ and Γ_2 on $U\times\Delta[n]\times\mathbb{A}^m$ and take their external product to define a correspondence $\Gamma_1\times\Gamma_2$ on $U\times U\times\Delta[n]\times\Delta[n]\times\mathbb{A}^{l+m}$. Next we pull-back by the diagonal $\Delta:U\times\Delta[n]\to U\times\Delta[n]\times U\times\Delta[n]$. This defines the pairing of simplicial abelian groups given above. Finally we may apply the normalization functor to pass from a simplicial abelian group to the associated chain complex.

Key observation: Alternatively one may first take the chain complexes associated to the simplicial abelian groups on the left-side of (3.0.1) and define a pairing on them. This will involve passing to the corresponding simplicial abelian groups using shuffle maps, making use of the above pairing at the level of simplicial abelian groups and passing to the chain complex associated to the simplicial abelian group on the right-side of (3.0.1). Since the shuffle maps from the product of the resulting chain complexes to the chain complex associated to the product of the simplicial abelian groups strictly commute with the action of the symmetric group Σ_2 permuting the two factors, the need for homotopy commutativity arises only from the switching of the two weight-factors $\mathbb{A}^l$ and $\mathbb{A}^m$. (In particular, the required homotopy may be constructed from an $\mathbb{A}^1$-homotopy in $\mathbb{A}^{l+m}$: this observation should shed some perspective on the detailed constructions below.)

Remark 3.1. One may contrast the above pairing with the pairing on singular cohomology. There one starts with a pairing of cosimplicial abelian groups and on passage to the associated co-chain complexes, the resulting pairing involves the Alexander-Whitney maps which do *not* commute strictly with the action of the symmetric group. The failure of this strict commutativity leads to the cohomology operations in singular cohomology. Thus, already one can see there are *important differences* between the pairing on the motivic complexes and the one on the singular co-chain complex of a topological space.

3.1. Assume that we are given 2 cycles $Z_1\ \varepsilon\ \text{Cor}_{q.f}(U\times\Delta[n],\mathbb{A}^l)$ and $Z_2\ \varepsilon\ \text{Cor}_{q.f}(U\times\Delta[n],\mathbb{A}^m)$. After pull-back by the diagonal $\Delta:U\times\Delta[n]\to U\times\Delta[n]\times U\times\Delta[n]$, $Z_1\times Z_2$ ($Z_2\times Z_1$) defines the cycle $\Delta^*(Z_1\times Z_2)\ \varepsilon\ \text{Cor}_{q.f}(U\times\Delta[n],\mathbb{A}^l\times\mathbb{A}^m)$ (the cycle $\Delta^*(Z_2\times Z_1)\ \varepsilon\ \text{Cor}_{q.f}(U\times\Delta[n],\mathbb{A}^m\times\mathbb{A}^l)$, respectively). *At this point we may assume that the cycles Z_i are in fact closed sub-schemes of $U\times\Delta[n]\times\mathbb{A}^l$ and $U\times\Delta[n]\times\mathbb{A}^m$. By replacing $\Delta^*(Z_1\times Z_2)$ with an irreducible component, if necessary, we may assume $\Delta^*(Z_1\times Z_2)$ is a closed sub-scheme of $U\times\Delta[n]\times\mathbb{A}^{l+m}$.*

The basic idea is to define a subscheme of $U\times\Delta[n]\times\mathbb{A}^1\times\mathbb{A}^{l+m}$ that joins a point on the scheme $\Delta^*(Z_1\times Z_2)$ with the corresponding point on $\Delta^*(Z_2\times Z_1)$ by a line and show that this defines a cycle in $\text{Cor}_{q.f}(U\times\Delta[n]\times\Delta[1],\mathbb{A}^{l+m})$. This will provide the first order homotopy relating the classes

$\Delta^*(Z_1 \times Z_2)$ *and* $\Delta^*(Z_2 \times Z_1)$. (Since there are technical difficulties for doing this in $U \times \Delta[n] \times \mathbb{A}^1 \times \mathbb{A}^{l+m}$ as explained below in Examples 3.2, we carry out this construction in the scheme, $U \times \Delta[n] \times \Gamma_{\phi_{s_1}}$ (as defined below) which is naturally isomorphic to the above scheme.)

3.2. The graph construction. The detailed construction is as follows. There is an obvious action by the group Σ_2 on $\sqcup_{l,m} \mathbb{A}^{l+m} = \sqcup_{l,m} \mathbb{A}^l \times \mathbb{A}^m$ switching the two factors $\mathbb{A}^l$ and $\mathbb{A}^m$. If $\sigma \, \varepsilon \, \Sigma_2$ is the non-identity element, $\sigma(x_1, \cdots x_l, y_1, \cdots, y_m) = (y_1, \cdots, y_m, x_1, \cdots, x_l)$. Let $s_1 = (id, \sigma)$ denote the obvious 1-simplex of $E\Sigma_2$. We define a map of schemes $\phi_{s_1} : \mathbb{A}^1 \times (\sqcup_{l,m \geq 1} \mathbb{A}^l \times \mathbb{A}^m) \to \sqcup_{l,m \geq 1}(\mathbb{A}^{m+l})$ by

$$\phi_{s_1}(t, x_1, \cdots, x_l, y_1, \cdots y_m) = (1-t).(x_1, \cdots, x_l, y_1, \cdots, y_m) + t(\sigma(x_1, \cdots, x_l, y_1, \cdots, y_m)).$$

The restriction of ϕ_{s_1} to $\mathbb{A}^1 \times \mathbb{A}^l \times \mathbb{A}^m$ will be denoted $\phi_{s_1,l,m}$: if l, m are fixed throughout our discussion, as below, we will abbreviate this to ϕ_{s_1}. Now the *graph* of $\phi_{s_1,l,m}$, $\Gamma_{\phi_{s_1,l,m}}$, is the closed subscheme of $\mathbb{A}^1 \times (\mathbb{A}^{l+m} \times \mathbb{A}^{l+m})$ defined by the ideal

$$\{(t, x_1, \cdots, x_l, y_1, \cdots, y_m, \bar{x}_1, \cdots \bar{x}_l, \bar{x}_{l+1}, \cdots, \bar{x}_{l+m})|$$
$$\bar{x}_i - p_i((1-t)(x_1, \cdots x_l, y_1, \cdots y_m) + t.\sigma(x_1, \cdots, x_l, y_1, \cdots, y_m))|i\}.$$

(Here p_i denotes projection to the i-th factor.)

3.2.1. Moreover, the graph of $\phi_{s_1,l,m}$ is isomorphic (as a scheme) by projection to the domain of $\phi_{s_1,l,m}$, i.e. to $\mathbb{A}^1 \times \mathbb{A}^{l+m}$. In view of this, it may be important to observe that $U \times \Delta[n] \times \Gamma_{\phi_{s_1,l,m}}$ is isomorphic to the product of $U \times \Delta[n]$ and $\mathbb{A}^1 \times \mathbb{A}^{l+m}$ in the category of schemes though it is imbedded in $U \times \Delta[n] \times \mathbb{A}^1 \times \mathbb{A}^{l+m} \times \mathbb{A}^{l+m}$ not as a co-ordinate hyperplane, i.e. not by putting some of the co-ordinates in $\mathbb{A}^{l+m} \times \mathbb{A}^{l+m}$ zero.

We consider a closed integral subscheme $Z \subseteq U \times \Delta[n] \times \mathbb{A}^{l+m}$ so that its projection to $U \times \Delta[n]$ is quasi-finite and dominant. For the most part we may assume $Z = \Delta^*(Z_1 \times Z_2)$ chosen as in 3.1, but the graph-construction does not require this. Its inverse image under the obvious projection $p : U \times \Delta[n] \times \mathbb{A}^1 \times \mathbb{A}^{l+m} \to U \times \Delta[n] \times \mathbb{A}^{l+m}$ will define the closed subscheme $p^{-1}(Z)$ (which $= p^{-1}(\Delta^*(Z_1 \times Z_2))$, if $Z = \Delta^*(Z_1 \times Z_2)$). Now the graph of the restriction of $id \times \phi_{s_1}$ to this subscheme defines the closed subscheme $\Gamma_{id \times \phi_{s_1}|p^{-1}(Z)}$ of $\Gamma_{id_{U \times \Delta[n]}} \times \Gamma_{\phi_{s_1}}$ which is contained in $U \times \Delta[n] \times U \times \Delta[n] \times \mathbb{A}^1 \times \mathbb{A}^{l+m} \times \mathbb{A}^{l+m}$. Since the composition $U \times \Delta[n] \overset{\Delta}{\to} U \times \Delta[n] \times U \times \Delta[n] \overset{pr_1}{\to} U \times \Delta[n]$ is the identity, and hence in particular proper, the image of $\Gamma_{id \times \phi_{s_1}|p^{-1}(Z)}$ under the projection $pr_1 \times id$ to $U \times \Delta[n] \times \mathbb{A}^1 \times \mathbb{A}^{l+m} \times \mathbb{A}^{l+m}$ is closed. Since it is the image of an irreducible scheme, it is also irreducible. *We will denote this by* $s_1(Z)$. We summarize some of the main properties of this construction here:

3.2.2.

- Let $\sigma(Z)$ denote the image of Z under the permutation of the two factors $\mathbb{A}^l$ and $\mathbb{A}^m$ in $U \times \Delta[n] \times \mathbb{A}^{l+m}$. (When $Z = \Delta^*(Z_1 \times Z_2)$, clearly $\sigma(Z) = \Delta^*(Z_2 \times Z_1)$.) It is clear from the above definition that the scheme $s_1(Z)$ consists of triples, consisting of a point of Z, a point $t \, \varepsilon \, \mathbb{A}^1$ together with a point on the line joining this point of Z with the corresponding point of $\sigma(Z)$ parametrized by $t \, \varepsilon \, \mathbb{A}^1$. Therefore, the projection of $s_1(Z)$ to $U \times \Delta[n] \times \mathbb{A}^1$ is in fact *quasi-finite*. It is also *dominant* since the projection of $Z \to U \times \Delta[n]$ is dominant.
- $p^{-1}(Z)$ being a product of $\mathbb{A}^1$ and Z is evidently irreducible. Since the graph $\Gamma_{id \times \phi_{s_1}|p^{-1}(Z)}$ is isomorphic to $p^{-1}(Z)$), it is also *irreducible*. Therefore so is $s_1(Z)$).
- The isomorphism of the graph $\Gamma_{\phi_{s_1,l,m}}$ with $\mathbb{A}^1 \times \mathbb{A}^{l+m}$ shows the following: let $Cor_{q.f}(U \times \Delta[n] \times \Gamma_{\phi_{s_1}}) =$ all correspondences on $U \times \Delta[n] \times \Gamma_{\phi_{s_1}}$ whose projection to $U \times \Delta[n] \times \mathbb{A}^1$ (where $\mathbb{A}^1$ is contained in the domain of ϕ_{s_1}) is quasi-finite and dominant. Then this abelian group identifies with $Cor_{q.f}(U \times \Delta[n] \times \mathbb{A}^1, \mathbb{A}^{l+m}) =$ the correspondences on $U \times \Delta[n] \times \mathbb{A}^1 \times \mathbb{A}^{l+m}$

whose projections to $U \times \Delta[n] \times \mathbb{A}^1$ are quasi-finite and dominant. *Since this isomorphism is natural in $U \times \Delta[n]$, by varying $U \times \Delta[n]$, we obtain the identification*

(3.2.3) $$Cor_{q.f}(\ \times \Delta[.] \times \mathbb{A}^1, \mathbb{A}^{l+m}) \cong Cor_{q.f}(\ \times \Delta[.] \times \Gamma_{\phi_{s_1}})$$

of the corresponding simplicial abelian sheaves. In view of this isomorphism, we will identify the associated (co-)chain complex of the term on the right with the associated (co-)chain complex of the term on the left, i.e. $\Gamma(U \times \mathbb{A}^1, \mathbb{Z}(l+m))$. *Henceforth we will denote the term $\mathbb{A}^1$ appearing above by $\Delta[1]$ to signify that it is not a weight-factor.*

- Next assume $Z = \Delta^*(Z_1 \times Z_2)$ as in 3.1. Then $s_1(\Delta^*(Z_1 \times Z_2))$ defines a class in $Cor_{q.f}(U \times \Delta[n] \times \Gamma_{\phi_{s_1}}) \cong Cor_{q.f}(U \times \Delta[n] \times \mathbb{A}^1, \mathbb{A}^{l+m})$. Restricting to the two faces of $\Delta[1] \subseteq \Gamma_{\phi_{s_1}}$ provides the two classes in $Cor_{q.f}(U \times \Delta[n] \times \Gamma_{id})$ and $Cor_{q.f}(U \times \Delta[n] \times \Gamma_{\sigma})$: *the restriction maps*

$$Cor_{q.f}(\ \times \Delta[.] \times \Gamma_{\phi_{s_1}}) \to Cor_{q.f}(\ \times \Delta[.] \times \Gamma_{\tau})$$

are both quasi-isomorphisms in view of the homotopy property of the motivic complexes and the isomorphism in (3.2.3). (For each permutation $\tau \, \varepsilon \, \Sigma_2$, we let Γ_{τ} denote the graph of the induced map $\tau^* : \mathbb{A}^{l+m} \to \mathbb{A}^{l+m}$. $Cor_{q.f}(U \times \Delta[n] \times \Gamma_{\tau})$ denotes the correspondences on $U \times \Delta[n] \times \Gamma_{\tau}$ whose projections to $U \times \Delta[n]$ are quasi-finite and dominant.) One may further take the image of these classes under the composite map $\Gamma_{\tau} \subseteq \mathbb{A}^{l+m} \times \mathbb{A}^{l+m} \overset{pr_2}{\to} \mathbb{A}^{l+m}$ (where the last map is the projection to the second factor) to obtain classes that identify with $\Delta^*(Z_1 \times Z_2)$) and $\sigma^*(\Delta^*(Z_1 \times Z_2))$, respectively. (One may also see readily that the above classes in $Cor_{q.f}(U \times \Delta[n] \times \Gamma_{\tau})$ map isomorphically to their images in $Cor_{q.f}(U \times \Delta[n], \mathbb{A}^{l+m})$ under the above projections.)

Observe that, in the description of the first order homotopy as above, we only considered the 1-simplices of $E\Sigma_2$ of the form $s_1 = (id, \sigma)$, $\sigma \, \varepsilon \, \Sigma_2$. We may extend the above construction to define an action by all 1-simplices of $E\Sigma_2$ which are of the form $s_1 = (\sigma_0, \sigma_1)$, $\sigma_i \, \varepsilon \, \Sigma_2$ as follows: we simply replace the subscheme $\Delta^*(Z_1 \times Z_2)$ by $\sigma_0^*(\Delta^*(Z_1 \times Z_2))$ where σ_0^* is the map induced by the permutation action $\sigma_0 : \sqcup_{l,m} \mathbb{A}^l \times \mathbb{A}^m \to \sqcup_{l,m} \mathbb{A}^l \times \mathbb{A}^m$ and apply the same constructions as before with σ_1 playing the role of σ. Therefore, the above construction provides a pairing:

$$Z(\{1 - simplices\, s_1 = (\sigma_0, \sigma_1) \, \varepsilon \, E\Sigma_2\}) \otimes Cor_{q.f}(U \times \Delta[n], \mathbb{A}^l) \otimes Cor_{q.f}(U \times \Delta[n], \mathbb{A}^m) \to Cor_{q.f}(U \times \Delta[n] \times \Gamma_{\phi_{s_1}}) \cong Cor_{q.f}(U \times \mathbb{A}^1 \times \Delta[n], \mathbb{A}^{l+m})$$

where $Z(\{1 - simplices\, s_1 = (\sigma_0, \sigma_1) \, \varepsilon \, E\Sigma_2\})$ is the free abelian group on the 1-simplices of $E\Sigma_2$. This pairing is clearly compatible with restriction to the faces of the n-simplex $\Delta[n]$. By defining the graph $\Gamma_{\phi_{\sigma_0}}$ associated to a 0-simplex $\sigma_0 \, \varepsilon \, E\Sigma_2$ as the graph of the permutation σ_0 applied to $\sqcup_{l,m} \mathbb{A}^l \times \mathbb{A}^m$, one extends the above pairing to a pairing

(3.2.4) $$\underset{s}{\oplus} Z(s) \otimes \{Cor_{q.f}(U \times \Delta[n], \mathbb{A}^l)|n\} \otimes \{Cor_{q.f}(U \times \Delta[n], \mathbb{A}^m)|n\} \to \{Cor_{q.f}(U \times \Delta[n] \times \Gamma_{\phi_s})|n\}$$

where the direct sum is taken over all the 1-simplices in $E\Sigma_2$ and $Z(s)$ denotes the chain complex obtained by first taking the (free) simplicial abelian group on the 1-simplex s and then by normalizing it. Moreover, restricting to the two faces of $\Delta[1] \subseteq \Gamma_{\phi_{s_1}}$ provides the two classes corresponding to $\sigma_i^*(\Delta^*(Z_1 \times Z_2))$, $i = 0, 1$ as observed above.

Observe that both the left and right-sides of the above pairing are double-complexes whose bi-degrees are indexed by n and the degree of terms in the complex $Z(s)$. The pairing is compatible with restrictions to the faces of $\Delta[n]$ as well as to the faces of the 1-simplices $s_1 \, \varepsilon \, E\Sigma_2$ as the above descriptions show. The descriptions above show that this is compatible with the differentials of the complexes on either side. *Therefore, this is indeed a pairing of double complexes.* Moreover, if $\tau \, \varepsilon \, \Sigma_2$, it acts on a simplex $s_1 = (\sigma_0, \sigma_1)$ by $\tau \circ s_1 = (\tau \circ \sigma_0, \tau \circ \sigma_1)$. The resulting pairing lands in $Cor_{q.f}(U \times \Delta[.] \times \Gamma_{\phi_{\tau \circ s_1}})$: one may see that this is the same as the cycle obtained by applying τ first

to (Z_1, Z_2) and then applying the pairing with $\Gamma_{\phi_{s_1}}$. This completes the construction of an *explicit first order homotopy for the pairing of the motivic complexes considered in* (3.0.1).

It is important to observe that the target of this pairing is the complex $\{Cor_{q.f}(\quad \times \Delta[.] \times \Gamma_{\phi_s})|n\}$, *i.e. it varies depending on the* 1-*simplex* $s \, \varepsilon \, E\Sigma_2$. It is possible to make an identification

$$\{Cor_{q.f}(\quad \times \Delta[.] \times \Delta[1], \mathbb{A}^{l+m})|n\} \overset{\cong}{\to} \{Cor_{q.f}(\quad \times \Delta[.] \times \Gamma_{\phi_s})|n\}$$

of complexes of abelian sheaves, so that the above pairing may be viewed as an *avatar* of a pairing:

$$NZ(sk_1 E(\Sigma_2)) \otimes \underline{\mathbb{Z}}(l) \otimes \underline{\mathbb{Z}}(m) \to \underline{\mathbb{Z}}(l+m).$$

(Recall that the complex $NZ(sk_1 E(\Sigma_2))$ is trivial in degrees greater than 1.) However, the above identification clearly looses the extra information in the complex $\{Cor_{q.f}(\quad \times \Delta[.] \times \Gamma_{\phi_s})|n\}$, coming in particular from the graph Γ_{ϕ_s}. Keeping this complex as the target of the above pairing (and on considering the iterated pairing involving higher dimensional simplices of $E\Sigma_k$), it becomes necessary to replace the operad $\{Z(E\Sigma_k)|k\}$ with a related *colored operad* (where the n-simplices (and the $(n_1, \cdots, n_k)$-multi-simplices) are the colors) and *the last pairing as one between such a colored operad and a chain complex*. (Colored operads and algebras over such operads have just begun to appear in the literature: see [BM2] and [Lein].) Rather than adopt this approach, we modify the graph construction below using projective spaces in the place of the affine space $\mathbb{A}^n$ and obtain an action of the Barratt-Eccles operad itself on the motivic complexes. The above discussion is put in here mainly to motivate the constructions below and to point out the intricacies of our constructions.

Since the isomorphism $Cor_{q.f}(\quad \times \Delta[.] \times \mathbb{A}^1, \mathbb{A}^{l+m}) \cong Cor_{q.f}(\quad \times \Delta[.] \times \Gamma_{\phi_{s_1}})$ is compatible with restriction to the faces of $\Delta[n]$ and also to $\{0, 1\} \subseteq \mathbb{A}^1 = \Delta[1]$, this construction *does provide an explicit first order homotopy for the pairing of motivic complexes considered in* (3.0.1).

Examples 3.2. In the above construction, one may be tempted to replace the graph $\Gamma_{\phi_{s_1}}$ by the scheme $\mathbb{A}^1 \times \mathbb{A}^{l+m}$, where the $\mathbb{A}^{l+m}$ is the target of the map ϕ_{s_1}. This would mean one will need to replace the scheme $s_1(\Delta^*(Z_1 \times Z_2))$ constructed above by its image under the composite map $s_1(\Delta^*(Z_1 \times Z_2)) \subseteq U \times \Delta[n] \times \Gamma_{\phi_{s_1}} \subseteq U \times \Delta[n] \times \Delta[1] \times \mathbb{A}^{l+m} \times \mathbb{A}^{l+m} \to U \times \Delta[n] \times \Delta[1] \times \mathbb{A}^{l+m}$, where the last map is dropping the first-factor of $\mathbb{A}^{l+m}$. This would be essentially projecting to the image of $\phi_{s_1}(p^{-1}(\Delta^*(Z_1 \times Z_2)))$. However, the following counter-examples show this image (which will be denoted Y in the following examples) may not be closed in $U \times \Delta[n] \times \Delta[1] \times \mathbb{A}^{l+m}$ and its closure may not be quasi-finite over $U \times \Delta[n] \times \Delta[1]$.

(1) Let $U = Spec\, k$, $n = l = m = 1$. Then let $Z = \{(x, x, y) \, \varepsilon \, \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 | xy = 1\}$. Now the projection $Z \to U \times \mathbb{A}^1$ (where $\mathbb{A}^1$ denotes the first $\mathbb{A}^1$) is quasi-finite. (In fact, this projection is *not finite*, but only quasi-finite.) Now consider

$$Y = \{(x, t, tx + (1-t)y, (1-t)x + ty) | xy = 1\} :$$

this is the image of $\phi_{s_1}(p^{-1}(Z))$ contained in $U \times \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$. For $t = 1/2$, the fiber of Y over t will be denoted $Y_{1/2}$. Now $Y_{1/2} = \{(x, 1/2, (1/2)x + (1/2)y, (1/2)y + (1/2)x) | xy = 1\} = \{(x, 1/2, (1/2)x + (1/2x), (1/2x) + (1/2)x) | x \neq 0\}$. There is no limit as $x \to 0$, so that in this example, $Y_{1/2}$ and Y are closed in $U \times \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$.

However, one can modify this example a bit to get another example, where the corresponding Y will not be closed. Here we take $Z = \{(x, x, y, x, z) | xy = 1, xz = -1\}$ viewed as a closed subscheme of $\mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^2$, i.e. $U = Spec\, k$ and $n = 1, l = m = 2$. Now the projection $Z \to U \times \mathbb{A}^1$ is quasi-finite. Then

$$Y = \{(x, t, t(x, y) + (1-t)(x, z), (1-t)(x, y) + t(x, z)) | xy = 1, xz = -1\}$$

which is the image of $\phi_{s_1}(p^{-1}(Z))$ contained in $U \times \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^2$. Therefore, $Y_{1/2} = \{(x, 1/2, ((1/2)x + (1/2)x), ((1/2)y + (1/2)z), ((1/2)x + (1/2)x), ((1/2)z + (1/2)y)) | xy = 1, xz = -1\} = \{(x, 1/2, x, 0, x, 0) | x \neq 0\}$. Clearly this has $(0, 1/2, 0, 0, 0, 0)$ as a limit point which is outside of Y. Therefore, $Y_{1/2}$ and Y are not closed. In this case though, the closure of Y just

adds the point $(0, 1/2, 0, 0, 0, 0)$ so that this closure is still quasi-finite over the product of the first two $\mathbb{A}^1$.

(2) Next we will construct an example, where Y is not closed and the closure of Y is not quasi-finite over $U \times \Delta[n] \times \Delta[1]$. Here again, $U = Spec\,k$ and $n = 2$ and $l = m = 1$. This example is obtained from the blow-up of $\mathbb{A}^2$ at the origin with a divisor at infinity removed, so that we obtain an affine scheme over $\mathbb{A}^2$. Let $Z = \{(x_1, x_2, y_1, z_1)|x_1y_1 = 1, x_1z_1 = x_2\}$. (Recall the blow-up of $\mathbb{A}^2$ at the origin is given by equations $x_1u_2 - x_2u_1 = 0$ in $\mathbb{A}^2 \times \mathbb{P}^1$, where (x_1, x_2) are parameters for $\mathbb{A}^2$ and $[u_1 : u_2]$ are the homogeneous coordinates for $\mathbb{P}^1$. So we are letting $z_1 = u_2/u_1$ and throwing out the part $u_1 = 0$.) Now Z is clearly closed in $\mathbb{A}^4$, as it is given by the equations $x_1y_1 = 1, x_1z_1 = x_2$. But the projection of Z into the three factors dropping y_1 is not closed in $\mathbb{A}^3$: call this Z'. In fact the projection of Z' to the (x_1, x_2)-coordinates is not quasi-finite as the fiber over $(0, 0)$ will be a whole $\mathbb{A}^1$.

Now one can modify the above example to obtain a counter example where Y (which is the image of $\phi_{s_1}(p^{-1}(Z))$) is not closed and its closure is not quasi-finite for the projection to the first two factors. Let $Z = \{(x_1, x_2, (y_1, z_1), (y_2, z_2))|x_1y_1 = 1, x_1y_2 = -1, x_1z_1 = x_2, x_1z_2 = x_2\}$ viewed as a closed sub-scheme of $\mathbb{A}^2 \times \mathbb{A}^2 \times \mathbb{A}^2$, i.e. $U = Spec\,k$, $n = 2, l = m = 2$. Clearly the projection of Z to the (x_1, x_2) coordinates is quasi-finite. Now

$$Y = \{(x_1, x_2, t, t(y_1, z_1) + (1 - t)(y_2, z_2), (1 - t)(y_1, z_1) + t(y_2, z_2))|$$

$$x_1y_1 = 1, x_1y_2 = -1, x_1z_1 = x_2, x_1z_2 = x_2\}$$

so that $Y_{1/2} = \{(x_1, x_2, 1/2, 1/2(y_1+y_2), 1/2(z_1+z_2), 1/2(y_1+y_2), 1/2(z_1+z_2))|x_1y_1 = 1, x_1y_2 = -1, x_1z_1 = x_2, x_1z_2 = x_2\}$. Clearly this equals $\{(x_1, x_2, 1/2, 0, x_2/x_1, 0, x_2/x_1)|x_1 \neq 0\}$. Clearly this is not closed and the fiber of the closure over $x_1 = 0, x_2 = 0$ has a whole $\mathbb{A}^1$ in the fifth and last coordinate, so that the projection of the closure of $Y_{1/2}$ to the first factor $\mathbb{A}^2$ is not quasi-finite.

The above examples make it necessary to make use of the graph $\Gamma_{\phi_{s_1}}$ and adopt the construction we have used above, i.e. if we use the definition of the motivic complex as $\underline{\mathbb{Z}}(n) = C^*\mathrm{Cor}_{q.f}(\quad, \mathbb{A}^n)[-2n]$. One way to avoid using the graph $\Gamma_{\phi_{s_1}}$ is to replace it by its projection to the co-domain: however, for this to work, one needs to replace the affine spaces $\mathbb{A}^n$ appearing with $(\mathbb{P}^1)^n$ and with the motivic complex $\underline{\mathbb{Z}}(n)[2n]$ defined as the complex of quotient sheaves

(3.2.5) $$C^*\mathrm{Cor}_f(\quad, (\mathbb{P}^1)^n)/C^*\mathrm{Cor}_f(\quad, (\mathbb{P}^1)^n - \mathbb{A}^n)$$

This identification should be clear if we realize that the above complex is nothing but $C^*\mathrm{Cor}_f(\quad, \mathbb{P}^1 \wedge \ldots \wedge \mathbb{P}^1)$: recall $\mathbb{P}^1 \simeq \mathbb{G}_m \wedge S^1$ in the $\mathbb{A}^1$-motivic homotopy category. Therefore, we will denote the complex in (3.2.5) by $C^*\mathrm{Cor}_f(\quad, (\mathbb{P}^1)^{\wedge^n})$, where $\mathbb{P}^1$ is pointed by ∞.

We proceed to define an action of the Barratt-Eccles operad on $\oplus_{n \geq 0}\mathbb{Z}(n)$ with $\mathbb{Z}(n)$ defined as above. First we extend the graph construction in 3.2 as follows.

3.3. The graph construction revisited. There is an obvious action by the group Σ_2 on

$$\sqcup_{l,m}(\mathbb{P}^1)^{l+m} = \sqcup_{l,m}(\mathbb{P}^1)^l \times (\mathbb{P}^1)^m$$

switching the two factors $(\mathbb{P}^1)^l$ and $(\mathbb{P}^1)^m$. If $\sigma \,\varepsilon\, \Sigma_2$ is the non-identity element, $\sigma([x_{1,0} : x_{1,1}], \cdots, [x_{l,0} : x_{l,1}], [y_{1,0} : y_{1,1}], \cdots, [y_{m,0} : y_{m,1}]) = ([y_{1,0} : y_{1,1}], \cdots, [y_{m,0} : y_{m,1}], [x_{1,0} : x_{1,1}], \cdots, [x_{l,0} : x_{l,1}])$. Let $s_1 = (id, \sigma)$ denote the obvious 1-simplex of $E\Sigma_2$.

The basic strategy is the following. Let j_1, j_1' and j_2, j_2' denote the integers 0 or 1 so that $j_1 + j_1' = 1$ and $j_2 + j_2' = 1$. Observe that the $\mathbb{P}^1$ forming the i_1-th factor in $(\mathbb{P}^l)^l$ has homogeneous coordinates $x_{i_1,0}$ and $x_{i_1,1}$ and the $\mathbb{P}^1$ forming the i_2-th factor in $(\mathbb{P}^l)^m$ has homogeneous coordinates $y_{i_2,0}$ and $y_{i_2,1}$. We will begin by defining a map of schemes

$$\bar{\phi}_{s_1} : \mathbb{A}^1 \times (\sqcup_{l,m \geq 1}(\mathbb{P}^1)^l \times (\mathbb{P}^1)^m) \to \sqcup_{l,m \geq 1}((\mathbb{P}^1)^{m+l})$$

by defining its restriction to be the map ϕ_{s_1} (as in 3.2) on the affine space $\mathbb{A}^{l+m} = (\mathbb{A}^1)^l \times (\mathbb{A}^1)^m$ defined by $x_{i_1,j_1'}.y_{i_2,j_2'} \neq 0$ where i_1 ranges over $1 \leq i_1 \leq l$ and i_2 ranges over $1 \leq i_2 \leq m$ with j_1' and j_2' either 0 or 1 but depending on i_1 and i_2, respectively. Then we show that these restrictions are compatible under the gluing used to produce the $\mathbb{P}^1$ forming the various factors in the domain. One may in fact define $\bar{\phi}_{s_1}$ in homogeneous coordinates as follows.

$\bar{\phi}_{s_1}(t, [x_{1,0} : x_{1,1}], \cdots, [x_{l,0} : x_{l,1}], [y_{1,0} : y_{1,1}], \cdots [y_{m,0} : y_{m,1}]) = $ the $l + m$-tuple whose k-th entry is the point with homogeneous coordinates

(3.3.1)
$$= [tx_{k,j_1}.y_{k,j_2'} + (1-t)y_{k,j2}.x_{k,j_1'} : x_{k,j_1'}.y_{k,j_2'}], k \leq min\{l,m\},$$
$$= [tx_{k,j_1}.x_{k-m,j_2'} + (1-t)x_{k-m,j2}.x_{k,j_1'} : x_{k,j_1'}.x_{k-m,j_2'}], m = min\{l,m\} < k \leq l, \text{ if } m = min\{l,m\},$$
$$= [ty_{k-l,j_1}.x_{k-m,j_2'} + (1-t)x_{k-m,j2}.y_{k-l,j_1'} : x_{k-m,j_2'}y_{k-l,j_1'}], l < k \leq l+m, \text{ if } m = min\{l,m\},$$
$$= [ty_{k-l,j_1}.y_{k,j_2'} + (1-t)y_{k,j2}.y_{k-l,j_1'} : y_{k-l,j_1'}.y_{k,j_2'}], l = min\{l,m\} < k \leq m, \text{ if } l = min\{l,m\}, \text{ and}$$
$$= [ty_{k-l,j_1}.x_{k-m,j_2'} + (1-t)x_{k-m,j2}.y_{k-l,j_1'} : x_{k-m,j_2'}.y_{k-l,j_1'}], m < k \leq l+m, \text{ if } l = min\{l,m\}$$

Lemma 3.3. $\bar{\phi}_{s_1}$ *defines a map* $\mathbb{A}^1 \times (\sqcup_{l,m \geq 1}(\mathbb{P}^1)^l \times (\mathbb{P}^1)^m) \to \sqcup_{l,m \geq 1}((\mathbb{P}^1)^{m+l})$ *as claimed.*

Proof. i.e. The restrictions of the map $\bar{\phi}_{s_1}$ to the affine spaces $\mathbb{A}^1$ which glue together to form the factors $\mathbb{P}^1$ in the *domain* are compatible. We will consider the case $k \leq min\{l,m\}$. In this case, it suffices to show for example, the following: let $[x_{k,0} : x_{k,1}] \, \varepsilon \, \mathbb{P}^1$ forming the k-th factor in $(\mathbb{P}^1)^l$ and let $[y_{k,0} : y_{k,1}] \, \varepsilon \, \mathbb{P}^1$ forming the k-th factor in $(\mathbb{P}^1)^m$ so that $x_{k,1}.y_{k,1} \neq 0$ and $x_{k,0} \neq 0$. Then the point $[x_{k,0} : x_{k,1}]$ is identified with $\frac{x_{k,0}}{x_{k,1}}$ in the affine space where $x_{k,1} \neq 0$ and is identified with $\frac{x_{k,1}}{x_{k,0}}$ in the affine space where $x_{k,0} \neq 0$. The gluing needed to produce $\mathbb{P}^1$ from these two affine spaces sends

$$\frac{x_{k,0}}{x_{k,1}} \mapsto \frac{x_{k,1}}{x_{k,0}}.$$

We will presently check from the definition that the function $\bar{\phi}_{s_1}$ is defined so as to be compatible with this identification. There are two affine pieces $\mathbb{A}^1$ covering the $\mathbb{P}^1$ forming the k-th factor in $(\mathbb{P}^1)^l$: one where $x_{k,0} \neq 0$ and the other where $x_{k,1} \neq 0$. We are fixing the one of the affine pieces $\mathbb{A}^1$ covering the $\mathbb{P}^1$ forming the k-th factor of $(\mathbb{P}^1)^m$: we may assume for simplicity that this corresponds to where $y_{k,1} \neq 0$. On the affine piece where $x_{k,1}.y_{k,1} \neq 0$, the k-th entry of

$\bar{\phi}_{s_1}(t, [x_{1,0} : x_{1,1}], \cdots, [x_{l,0} : x_{l,1}], [y_{1,0} : y_{1,1}], \cdots [y_{m,0} : y_{m,1}]) = [tx_{k,0}.y_{k,1} + (1-t)y_{k,0}.x_{k,1} : x_{k,1}.y_{k,1}] = t\frac{x_{k,0}}{x_{k,1}} + (1-t)\frac{y_{k,0}}{y_{k,1}}$

and on the affine piece where $x_{k,0}.y_{k,1} \neq 0$, the k-th entry of

$\bar{\phi}_{s_1}(t, [x_{1,0} : x_{1,1}], \cdots, [x_{l,0} : x_{l,1}], [y_{1,0} : y_{1,1}], \cdots [y_{m,0} : y_{m,1}]) = [tx_{k,1}.y_{k,1} + (1-t)y_{k,0}.x_{k,0} : x_{k,0}.y_{k,1}] = t\frac{x_{k,1}}{x_{k,0}} + (1-t)\frac{y_{k,0}}{y_{k,1}}.$

Therefore this is compatible with the identification $\frac{x_{k,0}}{x_{k,1}} \mapsto \frac{x_{k,1}}{x_{k,0}}$ used in the gluing to obtain $\mathbb{P}^1$. There are several remaining cases to consider, which are dealt with similarly. $\square$

Example 3.4. To further clarify this definition, one may consider explicitly the case $l = m = 1$. In this case $\bar{\phi}_{s_1} : \mathbb{A}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Since $l = 1$ and $m = 1$ we will completely omit the first subscript of the coordinates. Now the map $\bar{\phi}_{s_1}$ is defined on the following four affine spaces $(\cong \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1)$

as follows:

(3.3.2)
$$\bar{\phi}_{s_1}(t, [x_0 : x_1], [y_0 : y_1]) = ([tx_1.y_0 + (1-t)y_1.x_0 : x_0.y_0], [(1-t)x_1.y_0 + ty_1.x_0 : x_0.y_0]), \text{ if } x_0y_0 \neq 0,$$
$$= ([tx_1.y_1 + (1-t)x_0.y_0 : x_0.y_1], [(1-t)x_1.y_1 + tx_0.y_0 : x_0.y_1]), \text{ if } x_0y_1 \neq 0,$$
$$= ([tx_0.y_0 + (1-t)x_1.y_1 : x_1.y_0], [(1-t)x_0.y_0 + tx_1.y_1 : x_1.y_0]), \text{ if } x_1y_0 \neq 0, \text{ and}$$
$$= ([tx_0.y_1 + (1-t)x_1.y_0 : x_1.y_1], [(1-t)x_0.y_1 + tx_1.y_0 : x_1.y_1]), \text{ if } x_1y_1 \neq 0.$$

In this case the first (second) factor of $\mathbb{P}^1$ in the domain is obtained by gluing the two affine spaces $\mathbb{A}^1$ corresponding to where $x_0 \neq 0$ and where $x_1 \neq 0$ (where $y_0 \neq 0$ and where $y_1 \neq 0$, respectively). If $\eta : \mathbb{P}^1_{y_0.y_1 \neq 0} \to \mathbb{P}^1_{y_1.y_0 \neq 0}$ is the map sending $\frac{y_0}{y_1} \mapsto \frac{y_1}{y_0}$ used in the gluing to produce $\mathbb{P}^1$, then one verifies that this is compatible with definition of $\bar{\phi}_{s_1}$ on the first two affine pieces. One verifies the compatibility of $\bar{\phi}_{s_1}$ with the other gluings used in the domain of $\bar{\phi}_{s_1}$ similarly.

At this point one may apply a variant of the constructions in 3.2.1. Let l, m be two fixed positive integers. We begin by defining a map of presheaves on $(\mathrm{Sm}/k)^{op}$

(3.3.3)
$$s_1(= s_1^{l,m}) : Cor_f(U, (\mathbb{P}^1)^{l+m}) \to Cor_f(U \times \Delta[1], (\mathbb{P}^1)^{l+m})$$

(Strictly speaking the map above should be denoted $s_1^{(l,m)}$ as it depends not only on the 1-simplex s_1, but also on the pair (l, m). But we will usually omit the (l, m) for the sake of brevity.) This will be contravariant in U, so that it is a map of presheaves. Let Z denote a closed integral subscheme $\subseteq U \times (\mathbb{P}^1)^{l+m}$ for which the projection $Z \to U$ is finite. Its inverse image under the obvious projection $p : U \times \mathbb{A}^1 \times (\mathbb{P}^1)^{l+m} \to U \times (\mathbb{P}^1)^{l+m}$ will define the closed subscheme $p^{-1}(Z))$. Now the graph of the restriction of $id \times \bar{\phi}_{s_1}$ to this subscheme defines the closed subscheme $\Gamma_{id \times \bar{\phi}_{s_1}|p^{-1}(Z)}$ of $\Gamma_{id_U} \times \Gamma_{\bar{\phi}_{s_1}}$ which is contained in $U \times U \times \mathbb{A}^1 \times (\mathbb{P}^1)^{l+m} \times (\mathbb{P}^1)^{l+m}$. Since the composition $U \xrightarrow{\Delta} U \times U \xrightarrow{pr_1} U$ is the identity, and hence in particular proper, the image of $\Gamma_{id \times \bar{\phi}_{s_1}|p^{-1}(Z))}$ under the projection $pr_1 \times id$ to $U \times \mathbb{A}^1 \times (\mathbb{P}^1)^{l+m} \times (\mathbb{P}^1)^{l+m}$ is closed. Since it is the image of an irreducible scheme, it is also irreducible. *We will denote this by* $s_1'(Z)$. Since the projection $(\mathbb{P}^1)^{l+m} \times (\mathbb{P}^1)^{l+m} \to (\mathbb{P}^1)^{l+m}$ (which is the projection to the second factor) is clearly proper, we will project $s_1'(Z)$ into $U \times \mathbb{A}^1 \times (\mathbb{P}^1)^{l+m}$ using this map and denote the image by $s_1(Z)$. We summarize some of the main properties of this construction here:

3.3.4.

- Let $\sigma : (\mathbb{P}^1)^l \times (\mathbb{P}^1)^m \to (\mathbb{P}^1)^m \times (\mathbb{P}^1)^l$ denote the obvious map interchanging two factors. Then, it is clear from the above definition that the scheme $s_1(Z)$ consists of pairs, consisting of a point $t \, \varepsilon \, \mathbb{A}^1$ together with a point on the line joining a point of Z with the corresponding point of $\sigma(Z)$ parametrized by $t \, \varepsilon \, \mathbb{A}^1$. Therefore, the projection of $s_1(Z))$ to $U \times \mathbb{A}^1$ is in fact *finite*. It is also *surjective* since the projection of $Z \to U$ is surjective.
- $p^{-1}(Z)$ being a product of $\mathbb{A}^1$ and Z is evidently irreducible. Since the graph $\Gamma_{id \times \phi_{s_1}|p^{-1}(Z))}$ is isomorphic to $p^{-1}(Z)$, it is also *irreducible*. Therefore so is $s_1(Z)$.
- Therefore
$$s_1(Z) \, \varepsilon \, Cor_f(U \times \mathbb{A}^1, (\mathbb{P}^1)^{l+m}).$$

  We will henceforth denote the factor $\mathbb{A}^1$ by $\Delta[1]$. By extending the map s_1 by linearity to all cycles in $Cor_f(U \times \mathbb{A}^1, (\mathbb{P}^1)^{l+m})$, we obtain the required map.
- The above construction is contravariantly functorial in U.
- Let $Z_i \, \varepsilon \, Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{l_i})$, $i = 1, \cdots, 2$ be given. We will assume, as before, that each Z_i is, in fact, a closed irreducible subscheme of $U \times \Delta[n] \times (\mathbb{P}^1)^{l_i}$ for which the projection $Z_i \to U \times \Delta[n]$ is finite. Let $Z = \Delta^*(Z_1 \times Z_2)$. Then, $s_1(\Delta^*(Z_1 \times Z_2))_{|t=1} = \Delta^*(Z_1 \times Z_2)$ while, $s_1(\Delta^*(Z_1 \times Z_2))_{|t=0} = \Delta^*(Z_2 \times Z_1)$.

Next we proceed to use the map s_1 to define a map

$$(3.3.5) \qquad \bar{s}_1 (= \bar{s}_1^{l,m}) : Cor_f(U, (\mathbb{P}^1)^{\wedge l+m}) \to Cor_f(U \times \Delta[1], (\mathbb{P}^1)^{\wedge l+m})$$

Starting with the map s_1 and passing to the quotient defines a map $s_1' : Cor_f(U, (\mathbb{P}^1)^{l+m}) \to Cor_f(U \times \Delta[1], (\mathbb{P}^1)^{\wedge l+m})$. We will modify the definition of s_1' so that it will descend to the obvious quotient of the domain. We let $\pi_j : U \times (\mathbb{P}^1)^{l+m} \to U \times (\mathbb{P}^1)^{l+m-1}$ denote the projection which omits the j-th factor, $j = 1, \cdots, l+m$ and let $\eta_j : U \times (\mathbb{P}^1)^{l+m} \to (\mathbb{P}^1)^{l+m} \to \mathbb{P}^1$ denote the composition of the projection to the second factor followed by projection to the j-th factor. Next let $i_j : (\mathbb{P}^1)^{l+m-1} \to (\mathbb{P}^1)^{l+m}$, $j = 1, \cdots, l = m$ denote the imbedding with the j-th factor being the base-point ∞. (Observe that for any cycle Z in $Cor_f(U, (\mathbb{P}^1)^{l+m})$, the projection $U \times (\mathbb{P}^1)^{l+m} \to U$ is finite. Therefore, π_j defines a push-forward $\pi_{j*} : Cor_f(U, (\mathbb{P}^1)^{l+m}) \to Cor_f(U, (\mathbb{P}^1)^{l+m-1})$. i_j clearly defines a pushforward $Cor_f(U, (\mathbb{P}^1)^{l+m-1}) \to Cor_f(U, (\mathbb{P}^1)^{l+m})$.) Given a closed irreducible and reduced subscheme $Z \subseteq U \times (\mathbb{P}^1)^{l+m}$, we let $D_Z = \{j = 1, \cdots, l+m | \eta_j(Z) = \infty\}$. Then we let

$$(3.3.6) \qquad \delta(Z) = \Sigma_{j=1}^{l+m} i_{j*} \pi_{j*}(Z), \text{ if } D_Z = \phi$$

$$(3.3.7) \qquad \qquad = Z + \Sigma_{j \notin D_Z} i_{j*} \pi_{j*}(Z), \text{ if } D_Z \neq \phi$$

Now we let

$$(3.3.8) \qquad \bar{s}_1(Z) = s_1(Z) - s_1(\delta(Z))$$

Lemma 3.5. *Any class in $Cor_f(U, (\mathbb{P}^1)^{l+m} - \mathbb{A}^{l+m})$ is send by $\bar{s}_1$ to zero in the target. The map $\bar{s}_1$ is well-defined and is compatible with pull-backs in the argument U.*

Proof. First let $Z = \infty^{l+m}$ denote the cycle U imbedded in $U \times (\mathbb{P}^1)^{l+m}$ at the point ∞ in each factor of $\mathbb{P}^1$. Clearly $D_Z = \{1, \cdots, l+m\}$, so that $\bar{s}_1(Z) = s_1(Z) - s_1(Z) = 0$. Next let Z denote a cycle in $Cor_f(U, (\mathbb{P}^1)^{l+m} - \mathbb{A}^{l+m})$ represented by a closed irreducible and reduced subscheme. Then $D_Z \neq \phi$ and therefore, $\bar{s}_1(Z) = s_1(Z) - s_1(\delta(Z)) = s_1(Z) - s_1(Z) - \Sigma_{j \notin D_Z} s_1(i_{j*} \pi_{j*}(Z))$.

For any fixed $j_0 \notin D_Z$,

$$s_1(i_{j_0*} \pi_{j_0*}(Z)) = s_1(i_{j_0*} \pi_{j_0*}(Z)) - s_1(i_{j_0*} \pi_{j_0*}(Z)) - \Sigma_{j \notin D_Z, j \neq j_0} s_1(i_{j*} \pi_{j*} i_{j_0*} \pi_{j_0*}(Z)).$$

For any j appearing in the last sum, $|\{l | \eta_{l*}(i_{j*} \pi_{j*} i_{j_0*} \pi_{j_0*}(Z)) = \infty\}| > |D_Z|$. Therefore, one may use ascending induction on $l + m - |D_Z|$ to complete the proof: recall the case $l + m = |D_Z|$ is when $Z = \infty^{l+m}$ and this was considered already in the beginning of this proof.

For a fixed U, recall $Cor_f(U, (\mathbb{P}^1)^{l+m})$ is the free abelian group generated by closed irreducible and reduced subschemes $Z \subseteq U \times (\mathbb{P}^1)^{l+m}$ whose projection to U is finite and surjective. The well-definedness of the map $\bar{s}_1$ follows from this observation: the above definition shows that $\bar{s}_1$ simply sends the $\mathbb{Z}$-basis elements Z with $D_Z \neq \phi$ to 0. If $f : U' \to U$ is a map of smooth schemes of finite type over k, any irreducible and reduced closed subscheme $Z \subseteq U \times (\mathbb{P}^1)^{l+m}$ with $D_Z \neq \phi$ is pulled back to a cycle $f^*(Z) \subseteq U' \times (\mathbb{P}^1)^{l+m}$ with $D_{Z'} \neq \phi$ for each irreducible component Z' of $f^*(Z)$. Therfore if Z is sent to 0 by $\bar{s}_1$, then so is $f^*(Z)$. This completes the proof of the lemma. $\qquad \square$

Next recall that one has a natural pairing:

$$Cor_f(U, (\mathbb{P}^1)^l) \otimes Cor_f(U, (\mathbb{P}^1)^m) \to Cor_f(U, (\mathbb{P}^1)^{l+m})$$

This induces a pairing:

$$(3.3.9) \qquad Cor_f(U, (\mathbb{P}^1)^{\wedge l}) \otimes Cor_f(U, (\mathbb{P}^1)^{\wedge m}) \to Cor_f(U, (\mathbb{P}^1)^{\wedge l+m}).$$

Therefore, composing with the map $\bar{s}_1$ considered in (3.3.5), one obtains a pairing

$$(3.3.10) \qquad \mu'(s_1, \quad) : Cor_f(U, (\mathbb{P}^1)^{\wedge l}) \otimes Cor_f(U, (\mathbb{P}^1)^{\wedge m}) \to Cor_f(U \times \Delta[1], (\mathbb{P}^1)^{\wedge l+m}).$$

Recall the pairing s_1 is contravariantly functorial in U as observed in 3.3.4. Therefore, so is $\bar{s}_1$ and the above pairing μ'_{s_1}. In particular, it follows that one may replace U by $U \times \Delta[n]$, for any n and that the induced pairing would then be compatible with the structure maps of the cosimplicial scheme $\{\Delta[n]|n\}$. Therefore, we obtain a well-defined pairing of complexes of quotient presheaves. Clearly this induces a corresponding map of the associated complexes of quotient sheaves.

In view of the property, $s_1(\Delta^*(Z_1 \times Z_2))_{|t=1} = \Delta^*(Z_1 \times Z_2)$ while, $s_1(\Delta^*(Z_1 \times Z_2))_{|t=0} = \Delta^*(Z_2 \times Z_1)$, the above construction provides the first order homotopy for the pairing of the motivic complexes.

Observe that, in the description of the first order homotopy as above, we only considered the 1-simplices of $E\Sigma_2$ of the form (id, σ), $\sigma \, \varepsilon \, \Sigma_2$. We may extend the above construction to define an action by all 1-simplices of $E\Sigma_2$ which are of the form (σ_1, σ_2), $\sigma_i \, \varepsilon \, \Sigma_2$ as follows: we simply replace the subscheme $\Delta^*(Z_1 \times Z_2)$ by $\sigma_1^*(\Delta^*(Z_1 \times Z_2))$ where σ_1^* is the map induced by the permutation action $\sigma_1 : \sqcup_{l,m}\mathbb{A}^l \times \mathbb{A}^m \to \sqcup_{l,m}\mathbb{A}^l \times \mathbb{A}^m$ and apply the same constructions as before with σ_2 playing the role of σ. We may also define an action of the 0-simplices of $E\Sigma_2$ which are given by $\sigma \, \varepsilon \, \Sigma_2$ on the motivic complex by sending the cycle $\Delta^*(Z_1 \times Z_2)$ to $\sigma^*(\Delta^*(Z_1 \times Z_2))$. Therefore, the above construction provides a pairing:

$$(3.3.11) \quad \mu' : \underset{s}{\oplus} Z(s) \otimes \{Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{\wedge^l})|n\} \otimes \{Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{\wedge^m})|n\}$$

$$\to \{Cor_f(U \times \Delta[n] \times \Delta[1], (\mathbb{P}^1)^{\wedge^{l+m}})|n\}$$

where the direct sum is taken over all the 0- and 1-simplices in $E\Sigma_2$ and $Z(s)$ denotes the chain complex obtained by first taking the (free) simplicial abelian group on the 1-simplex s and then by normalizing it.

This pairing is clearly compatible with restriction to the faces of the n-simplex $\Delta[n]$. Moreover, restricting to the two faces of $\Delta[1]$ in $\Delta[n] \times \Delta[1]$ provides the two classes corresponding to $\sigma_i^*(\Delta^*(Z_1 \times Z_2))$, $i = 0, 1$ as observed above. Observe that both the left and right-sides of the above pairing are double-complexes whose bi-degrees are indexed by n and the degree of terms in the complex $Z(s)$. The pairing is compatible with restrictions to the faces of $\Delta[n]$ as well as to the faces of the 1-simplices $s_1 \, \varepsilon \, E\Sigma_2$ as the above descriptions show. Therefore, on taking the associated total complexes, the induced pairing is compatible with the differentials of the complexes on either side. The constructions so far may now be viewed as a pairing:

$$\mu' : NZ(sk_1E(\Sigma_2)) \otimes \underline{\mathbb{Z}}(l) \otimes \underline{\mathbb{Z}}(m) \to \underline{\mathbb{Z}}(l + m)$$

(Here $Z(sk_1E(\Sigma_2))$ denotes the simplicial abelian group obtained by applying the free abelian group functor to the simplicial set $sk_1E(\Sigma_2)$. Recall once again that the complex $N(Z(sk_1E(\Sigma_2)))$ is trivial in degress greater than 1.)

3.3.12. *Action by higher dimensional simplices.* We will presently extend the above construction to provide higher order homotopies for the product structure on the motivic complexes. *Here we consider a k-fold product of the motivic complexes and obtain higher order homotopies for the various resulting products extending the constructions above.* Observe that for $q \geq 1$, a q-simplex, s_q, of $E\Sigma_k$ is given by a sequence $(\sigma_0, \cdots, \sigma_q)$ with each $\sigma_i \, \varepsilon \, \Sigma_k$.

3.3.13. There is an obvious action by the symmetric group Σ_k on $\sqcup_{l_1,\cdots,l_k}\mathbb{A}^{l_1+\cdots+l_k} = \sqcup_{l_1,\cdots,l_k}\mathbb{A}^{l_1+\cdots+l_k}$ permuting the k-factors $\mathbb{A}^{l_1}, \cdots, \mathbb{A}^{l_k}$. If $s_q = (\sigma_0, \cdots, \sigma_q)$ denotes the above q-simplex of $E\Sigma_k$, σ_i will also denote the corresponding self-map of $\mathbb{A}^{l_1+\cdots+l_k}$ henceforth. We define a map of schemes $\phi_{s_q} : \mathbb{A}^q \times \mathbb{A}^{\Sigma_{i=1}^k l_i} \to \mathbb{A}^{\Sigma_{i=1}^k l_i}$ by

$$(3.3.14) \qquad \phi_{s_q} = t_0\sigma_0 + t_1\sigma_1 + \cdots + t_{q-1}\sigma_{q-1} + (1 - t_0 - \cdots - t_{q-1})\sigma_q$$

where $(t_0, \cdots, t_{q-1}) \, \varepsilon \, \mathbb{A}^q$. We will identify $\mathbb{A}^q$ with $\Delta[q]$ sending $(t_0, \cdots, t_{q-1})$ to $(t_0, \cdots, t_{q-1}, (1 - t_0 - \cdots - t_{q-1}))$ and view ϕ_{s_q} as a map $\Delta[q] \times \mathbb{A}^{\Sigma_i l_i} \to \mathbb{A}^{\Sigma_i l_i}$.

Next we proceed to adapt the above construction to define a map

$$(3.3.15) \qquad \bar{\phi}_{s_q} = \mathbb{A}^q \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i} \to (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$$

as follows. First we let each permutation σ_i act on $(\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$ by letting the blocks $(\mathbb{P}^1)^{l_1}, \cdots, (\mathbb{P}^1)^{l_k}$ be permuted by σ_i. Let $\alpha(\sigma_i(j), p)$ and $\beta(\sigma_i(j), p)$, for $0 \leq i \leq q$, $1 \leq j \leq k$ and $1 \leq p \leq l_{\sigma_i(j)}$ be functions that take values either 0 or 1, so that $\alpha(\sigma_i(j), p) + \beta(\sigma_i(j), p) = 1$. The homogeneous coordinates of a point in the first-block of $\sigma_i(\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$ are now defined by

$$([x_{1, \alpha(\sigma_i^{-1}(1), 1)}^{\sigma_i^{-1}(1)} : x_{1, \beta(\sigma_i^{-1}(1), 1)}^{\sigma_i^{-1}(1)}], \cdots, [x_{l_{\sigma_i^{-1}(1)}, \alpha(\sigma_i^{-1}(1), l_{\sigma_i^{-1}(1)})}^{\sigma_i^{-1}(1)} : x_{l_{\sigma_i^{-1}(1)}, \beta(\sigma_i^{-1}(1), l_{\sigma_i^{-1}(1)})}^{\sigma_i^{-1}(1)}])$$

Therefore, on the affine space defined by $\beta(\sigma_i^{-1}(1), p) \neq 0$ for all $1 \leq p \leq l_{\sigma_i^{-1}(1)}$, $1 \leq i \leq q$, this point is defined by the coordinates:

$$(\frac{x_{1, \alpha(\sigma_i^{-1}(1), 1)}^{\sigma_i^{-1}(1)}}{x_{1, \beta(\sigma_i^{-1}(1), 1)}^{\sigma_i^{-1}(1)}}, \cdots, \frac{x_{l_{\sigma_i^{-1}(1)}, \alpha(\sigma_i^{-1}(1), l_{\sigma_i^{-1}(1)})}^{\sigma_i^{-1}(1)}}{x_{l_{\sigma_i^{-1}(1)}, \beta(\sigma_i^{-1}(1), l_{\sigma_i^{-1}(1)})}^{\sigma_i^{-1}(1)}})$$

Therefore, we define $\bar{\phi}_{s_q}$ on the affine piece that is defined by the conditions $\beta(\sigma_i^{-1}(j), p) \neq 0$ for all $1 \leq p \leq l_{\sigma_i^{-1}(1)}$, $1 \leq i \leq q$, $1 \leq j \leq k$ to be given by

$$\Sigma_{i=0}^{q-1} t_i (\frac{x_{1, \alpha(\sigma_i^{-1}(1), 1)}^{\sigma_i^{-1}(1)}}{x_{1, \beta(\sigma_i^{-1}(1), 1)}^{\sigma_i^{-1}(1)}}, \cdots, \frac{x_{l_{\sigma_i^{-1}(1)}, \alpha(\sigma_i^{-1}(1), l_{\sigma_i^{-1}(1)})}^{\sigma_i^{-1}(1)}}{x_{l_{\sigma_i^{-1}(1)}, \beta(\sigma_i^{-1}(1), l_{\sigma_i^{-1}(1)})}^{\sigma_i^{-1}(1)}}, \cdots, \frac{x_{1, \alpha(\sigma_i^{-1}(k), 1)}^{\sigma_i^{-1}(k)}}{x_{1, \beta(\sigma_i^{-1}(k), 1)}^{\sigma_i^{-1}(k)}}, \cdots, \frac{x_{l_{\sigma_i^{-1}(k)}, \alpha(\sigma_i^{-1}(k), l_{\sigma_i^{-1}(k)})}^{\sigma_i^{-1}(k)}}{x_{l_{\sigma_i^{-1}(k)}, \beta(\sigma_i^{-1}(k), l_{\sigma_i^{-1}(k)})}^{\sigma_i^{-1}(k)}})$$

$$+ (1 - \Sigma_{i=0}^{q-1} t_i)(\frac{x_{1, \alpha(\sigma_q^{-1}(1), 1)}^{\sigma_q^{-1}(k)}}{x_{1, \beta(\sigma_q^{-1}(k), 1)}^{\sigma_q^{-1}(k)}}, \cdots, \frac{x_{l_{\sigma_q^{-1}(k)}, \alpha(\sigma_q^{-1}(k), l_{\sigma_q^{-1}(k)})}^{\sigma_q^{-1}(k)}}{x_{l_{\sigma_q^{-1}(k)}, \beta(\sigma_q^{-1}(k), l_{\sigma_q^{-1}(k)})}^{\sigma_q^{-1}(k)}}, \frac{x_{1, \alpha(\sigma_q^{-1}(k), 1)}^{\sigma_q^{-1}(k)}}{x_{1, \beta(\sigma_q^{-1}(k), 1)}^{\sigma_q^{-1}(k)}}, \cdots, \frac{x_{l_{\sigma_q^{-1}(k)}, \alpha(\sigma_q^{-1}(k), l_{\sigma_q^{-1}(k)})}^{\sigma_q^{-1}(k)}}{x_{l_{\sigma_q^{-1}(k)}, \beta(\sigma_q^{-1}(k), l_{\sigma_q^{-1}(k)})}^{\sigma_q^{-1}(k)}})$$

It suffices to show that this definition is compatible with the identifications used when one glues together two copies of $\mathbb{A}^1$ to produce $\mathbb{P}^1$. This may be verified as in the case $k = 2$ and $q = 1$ that we considered above.

For any scheme U, the graph of the map, $\Gamma_{id \times \bar{\phi}_{s_q}}$, is a closed subscheme of $U \times \Delta[q] \times U \times \Delta[q] \times (\mathbb{P}^1)^{\Sigma_i l_i} \times (\mathbb{P}^1)^{\Sigma_i l_i}$ naturally isomorphic to $U \times \Delta[q] \times (\mathbb{P}^1)^{\Sigma_i l_i}$. Let $Z \subseteq U \times \Pi_{i=1}^k (\mathbb{P}^1)^{l_i}$ be a closed irreducible and reduced sub-scheme so that the projection to U is finite and surjective. (For example, one may begin with $Z_i \, \epsilon \, Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{l_i})$, $i = 1, \cdots, k$. We will assume, as before, that each Z_i is, in fact, a closed irreducible and reduced subscheme of $U \times \Delta[n] \times (\mathbb{P}^1)^{l_i}$ for which the projection $Z_i \to U \times \Delta[n]$ is finite. Let Z denote an irreducible component of $\Delta^*(\Pi_{i=1}^k Z_i)$.) Its inverse image under the obvious projection $p : U \times \Delta[n] \times \Delta[q] \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i} \to U \times \Delta[n] \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$ will define the closed subscheme $p^{-1}(\Delta^*(\Pi_{i=1}^k Z_i))$. Now the graph of the restriction of $id \times \bar{\phi}_{s_q}$ to this subscheme defines the closed subscheme $\Gamma_{id \times \bar{\phi}_{s_q} | p^{-1}(Z))}$ of $\Gamma_{id_{U \times \Delta[n]}} \times \Gamma_{\bar{\phi}_{s_q}}$ which is contained in $U \times \Delta[n] \times U \times \Delta[n] \times \Delta[q] \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i} \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$.

Since the composition $U \times \Delta[n] \xrightarrow{\Delta} U \times \Delta[n] \times U \times \Delta[n] \xrightarrow{pr_1} U \times \Delta[n]$ is the identity, and hence in particular proper, the image of $\Gamma_{id \times \bar{\phi}_{s_1} | p^{-1}(Z)}$ under the projection $pr_1 \times id$ to $U \times \Delta[n] \times \Delta[q] \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i} \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$ is closed. Since it is the image of an irreducible scheme, it is also irreducible. *We will denote this by $s_q'(Z) = s_q'(Z))$. We will next project this into $U \times \Delta[n] \times \Delta[q] \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$*

using the projection $(\mathbb{P}^1)^{\Sigma_{i=1}^k l_i} \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i} \to (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$ to the second factor and denote the image of $s'_q(Z)$ by $s_q(Z)$. It is clear from the above definition that the projection of $s'_q(Z)$ to $U \times \Delta[q]$ is in fact *finite*. It is also *surjective* since the projection of $Z \to U$ is surjective. Therefore, we obtain the map:

$$s_q(= s_q^{l_1,\cdots,l_k}) : \underset{s_q \,\varepsilon\, sk_q E(\Sigma_k)}{\oplus} Z(s_q)_q \otimes Cor_f(U, (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}) \to Cor_f(U \times \Delta[q], (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i})$$

where $Z(s_q)$ denotes the sub-complex of $ZNE\Sigma_k$ generated by the q-simplex s_q. One may observe that above pairing defines in fact a map of complexes:

$$s_q : \underset{s_q \,\varepsilon\, sk_q E(\Sigma_k)}{\oplus} Z(s_q) \otimes Cor_f(U, (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}) \to Cor_f(U \times \Delta[q], (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i})$$

As in (3.3.5), we may use this to define a map

$$\bar{s}_q(= \bar{s}_q^{l_1,\cdots,l_k}) : \underset{s_q \,\varepsilon\, sk_q E(\Sigma_k)}{\oplus} Z(s_q) \otimes Cor_f(U, (\mathbb{P}^1)^{\wedge \Sigma_{i=1}^k l_i}) \to Cor_f(U \times \Delta[q], (\mathbb{P}^1)^{\wedge \Sigma_{i=1}^k l_i})$$

Composing with the pairing $Cor_f(U, (\mathbb{P}^1)^{\wedge l_1}) \otimes \cdots Cor_f(U, (\mathbb{P}^1)^{\wedge l_k}) \to Cor_f(U, (\mathbb{P}^1)^{\wedge \Sigma_{i=1}^k l_i})$, this provides a pairing

(3.3.16) $$\bar{s}_q : \underset{s_q \,\varepsilon\, sk_q E(\Sigma_k)}{\oplus} Z(s_q) \otimes \otimes_{i=1}^k Cor_f(U, (\mathbb{P}^1)^{\wedge l_i}) \to Cor_f(U \times \Delta[q], (\mathbb{P}^1)^{\wedge \Sigma_{i=1}^k l_i})$$

As before, one may verify this pairing is contravariantly functorial in U, and is compatible with the face maps sending the simplex s_q to its faces, so that one obtains the pairing of complexes:

(3.3.17)
$$\mu'(s_q, \quad) : \underset{s_q \,\varepsilon\, sk_q E(\Sigma_k)}{\oplus} Z(s_q) \otimes \{Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{l_1})|n\} \otimes \cdots \otimes \{Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{l_k})|n\}$$
$$\to \{Cor_f(U \times \Delta[n] \times \Delta[q], (\mathbb{P}^1)^{\Sigma_{i=1}^k l_k})|n\}$$

Making use of the isomorphisms defined using shuffle maps as in (6.0.9) and taking the sum over all such shuffles, one may similarly define a pairing

(3.3.18)
$$\mu'(s_q, \quad) : Z(s_q) \otimes Cor_f(U \times \Delta[n] \times \Delta[p_1], (\mathbb{P}^1)^{\wedge l_1}) \otimes \cdots \otimes Cor_f(U \times \Delta[n] \times \Delta[p_k], (\mathbb{P}^1)^{\wedge l_k})$$
$$\to Cor_f(U \times \Delta[n + \Sigma_i p_i] \times \Delta[q], (\mathbb{P}^1)^{\wedge \Sigma_{i=1}^k l_i})$$

where $Z(s_q)$ denotes the sub-complex of $ZNE\Sigma_k$ generated by the q-simplex s_q.

3.4. Clearly the action of the symmetric group Σ_k on the simplex s_q, $(\sigma', \sigma_1, \cdots \sigma_q) \mapsto (\sigma_1 \circ \sigma', \cdots, \circ \sigma_q \circ \sigma')$ corresponds to the action of $\sigma' \varepsilon \Sigma_k$ on $Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{\Sigma_i l_i})$ permuting the weight-factors $(\mathbb{P}^1)^{l_1}, \cdots, (\mathbb{P}^1)^{l_k}$.

Though the above constructions provide higher order homotopies, *we need to extend these to multi-simplices so as to obtain the associativity of the operad action as proved in* Theorem 3.6. We begin by extending the above construction to bi-simplices.

If $s_q = (\sigma_0, \cdots, \sigma_q)$, $s_p = (\tau_0, \cdots, \tau_p)$ are q and p simplices of $E\Sigma_k$, let $\phi_{(s_q, s_p)} : \Delta[q] \times \Delta[p] \times \mathbb{A}^{\Sigma_{i=1}^k l_i} \to \mathbb{A}^{\Sigma_{i=1}^k l_i}$ be the map defined by

$$(t_0\sigma_0 + t_1\sigma_1 + \cdots + t_{q-1}\sigma_{q-1} + (1 - t_0 - \cdots - t_{q-1})\sigma_q) \circ (s_0\tau_0 + s_1\tau_1 + \cdots + s_{p-1}\tau_{p-1} + (1 - s_0 - \cdots - s_{p-1})\tau_p)$$

$$= \Sigma_{i=0,j=0}^{i=q-1,j=p-1} t_i.s_j\sigma_i \circ \tau_j + (1 - t_0 - \cdots - t_{q-1}).\Sigma_{j=0}^{p-1} s_j\sigma_q \circ \tau_j + (1 - t_0 - \cdots - t_{q-1}).(1 - s_0 - \cdots s_{p-1})\sigma_q \circ \tau_p$$

with $(t_0, \cdots, t_q) \,\varepsilon\, \mathbb{A}^q$, $(s_0, \cdots, s_p) \,\varepsilon\, \mathbb{A}^p$.

One then obtains a corresponding map $\bar{\phi}_{(s_q,s_p)} : \Delta[q] \times \Delta[p] \times (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i} \to (\mathbb{P}^1)^{\Sigma_{i=1}^k l_i}$. Applying the graph constructions as above, one obtains a pairing:

(3.4.1)

$$\mu'(s_q, s_p, \quad) : (Z(s_q) \otimes Z(s_p)) \otimes \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[p_1], (\mathbb{P}^1)^{\wedge^{l_1}}) \otimes \cdots \otimes \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[p_k], (\mathbb{P}^1)^{\wedge^{l_k}})$$

$$\to \mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i] \times \Delta[q] \times \Delta[p], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}})$$

Since $\phi_{(s_q,s_p)} : \Delta[q] \times \Delta[p] \times \mathbb{A}^{\Sigma_{i=1}^k l_i} \to \mathbb{A}^{\Sigma_{i=1}^k l_i}$ is the composition $\phi_{(s_q)} \circ (id_{\Delta[q]} \times \phi_{(s_p)})$, one may see readily that $\bar{\phi}_{(s_q,s_p)} = \bar{\phi}_{(s_q)} \circ (id_{\Delta[q]} \times \bar{\phi}_{(s_p)})$. Observe that the action σ_i and τ_j on $\mathbb{A}^{\Sigma_{j=1}^k l_j}$ is linear and hence τ_j, σ_i commute with the variables $t_0, \cdots, t_{q-1}, s_0, \cdots, s_{p-1}$. Moreover, for each increasing map $(\phi, \psi) : [p+q] \to [p] \times [q]$, one may see readily from the above definition that

(3.4.2) $$\mu'((\phi, \psi)^*(s_q \times s_p), \quad) = (\phi, \psi)^* \circ \mu'(s_q, s_p, \quad) \circ \eta$$

Here $(\phi, \psi)^*(s_q \times s_p) \, \varepsilon \, E\Sigma_k$ denotes the $q+p$-simplex defined as in (6.0.11). $((\phi, \psi)^* : \mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i] \times \Delta[q] \times \Delta[p], (\mathbb{P}^1)^{\wedge^{\Sigma_i l_i}}) \to \mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i] \times \Delta[q+p], (\mathbb{P}^1)^{\wedge^{\Sigma_i l_i}})$ is defined using the shuffle map (ϕ, ψ). $\eta : Z((\phi, \psi)^*(s_q \times s_p)) \to Z(s_q) \otimes Z(s_p)$ is the obvious map induced by the isomorphism $Z((\phi, \psi)^*(s_q \times s_p))_{p+q} \overset{\cong}{\to} Z(s_q)_q \otimes Z(s_p)_p$ and the fact that $Z((\phi, \psi)^*(s_q \times s_p))$ is generated by the single simplex in degree $p + q$.

One may extend the above pairings to products of several simplices: i.e. if $s_{p_1}, \cdots, s_{p_l}$ are simplices of $E\Sigma_k$ of dimensions $p_1, \cdots, p_l$ and $s_q \, \varepsilon \, E\Sigma_k$ is a q-simplex, then one may define a pairing:

(3.4.3)

$$\mu'(s_q, s_{p_1}, \cdots, s_{p_l}, \quad) : (Z(s_q) \otimes Z(s_{p_1})) \otimes \cdots \otimes Z(s_{p_l})) \otimes \mathrm{Cor}_f(U \times \Delta[n], (\mathbb{P}^1)^{\wedge^{l_1}}) \otimes \cdots \otimes \mathrm{Cor}_f(U \times \Delta[n], (\mathbb{P}^1)^{\wedge^{l_k}})$$

$$\to \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[q] \times \Delta[p_1] \times \cdots \Delta[p_l], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}})$$

Moreover if $\alpha : [q + p_1 + \cdots + p_l] \to [q] \times [p_1] \times \cdots \times [p_l]$ is any increasing map, then composing the last pairing with $\alpha^* : \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[q] \times \Delta[p_1] \times \cdots \Delta[p_l], (\mathbb{P}^1)^{\wedge^{\Sigma_i l_i}}) \to \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[q + p_1 + \cdots + p_l], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}})$ and precomposing this with the (obvious) map $Z(\alpha^*(s_q \times s_{p_1} \times \cdots \times s_{p_l})) \to Z(s_q) \otimes Z(s_{p_1}) \otimes \cdots \otimes Z(s_{p_l})$ identifies with the pairing $\mu'(\alpha^*(s_q, s_{p_1}, \cdots, s_{p_l}, \quad))$.

3.4.4. One may observe that the above construction of the higher order homotopies takes place in the weight-factor $(\mathbb{P}^1)^{\Sigma_j l_j}$ only and is (contravariantly) functorial in the remaining arguments: it follows readily that the above pairing is compatible with pull-backs in the remaining arguments. This, in turn, implies that the above pairing commutes with shuffle maps (which are induced by pull-backs on the non-weight-factors) in the following sense so that the composition (where s_q denotes a q-simplex of $NZ(E\Sigma_k)$)

$$Z(s_q) \otimes \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[p_1], (\mathbb{P}^1)^{\wedge^{l_1}}) \otimes \cdots \otimes \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[p_k], (\mathbb{P}^1)^{\wedge^{l_k}}) \overset{\mu'}{\to} \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[p_1] \times \cdots \times \Delta[p_k] \times \Delta[q], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}}) \overset{\mathrm{shuffle}}{\to} \mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i] \times \Delta[q], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}})$$

factors as

$$Z(s_q) \otimes \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[p_1], (\mathbb{P}^1)^{\wedge^{l_1}}) \otimes \cdots \otimes \mathrm{Cor}_f(U \times \Delta[n] \times \Delta[p_k], (\mathbb{P}^1)^{\wedge^{l_k}}) \overset{id \otimes \mathrm{shuffle}}{\to} Z(s_q) \otimes \mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i], (\mathbb{P}^1)^{\wedge^{l_1}}) \otimes \cdots \otimes \mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i], (\mathbb{P}^1)^{\wedge^{l_k}}) \overset{\Delta^* \circ \mu'}{\to} \mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i] \times \Delta[q], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}}).$$

Composing the above pairing with another (obvious) shuffle map, one maps the last term above to $\mathrm{Cor}_f(U \times \Delta[n + \Sigma_i p_i + q], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}})$.

We may also start with cycles $Z_i \, \varepsilon \, \mathrm{Cor}_f(U \times \Delta[n_i], (\mathbb{P}^1)^{\wedge^{l_i}})$, $i = 1, \cdots, k$; using shuffle maps one first produces corresponding cycles in $\mathrm{Cor}_f(U \times \Delta[n], (\mathbb{P}^1)^{\wedge^{\Sigma_{i=1}^k l_i}})$ where $n = \Sigma_i n_i$ and then one applies the

above construction. Moreover, the above description shows the action is compatible with restriction to the faces of the simplex s_q, i.e. the $\Delta[q]$ appearing above. It is also compatible with restriction to the faces of $\Delta[n]$ and $\Delta[p_i]$ appearing above, so that the above construction defines a pairing of complexes:

$$(3.4.5) \qquad NZ(s_q) \otimes \underline{\mathbb{Z}}(l_1) \otimes \cdots \otimes \underline{\mathbb{Z}}(l_k) \to \underline{\mathbb{Z}}(\Sigma_{i=1}^k l_i)$$

where $\underline{\mathbb{Z}}(m)$ *here* denotes the motivic complex of weight m. $Z(s_q)$ denotes the sub-chain complex of $NZ(E\Sigma_k)$ generated by a *chosen* q simplex s_q. Let $\Delta[n]_{ss}$ denote the usual simplicial set given by $\Delta[n]_{ss,k} = Hom_\Delta([k],[n])$. By composing with the obvious map $\Delta[s_q] \to E\Sigma_k$ sending the generator i_{s_q} to the simplex s_q, we also obtain a pairing

$$(3.4.6) \qquad NZ(\Delta[q]_{ss})) \otimes \mathbb{Z}(l_1) \otimes \cdots \otimes \mathbb{Z}(l_k) \to \mathbb{Z}(\Sigma_{i=1}^k l_i)$$

where $NZ(\Delta[q]_{ss})$ is the obvious chain complex obtained from the simplicial abelian group $Z(\Delta[q]_{ss})$.

By ascending induction on $q \geq 1$, we may now assume that we have already established a pairing

$$NZ(sk_{q-1}(E\Sigma_k)) \otimes \underline{\mathbb{Z}} \otimes \cdots \otimes \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}, \text{ where } \underline{\mathbb{Z}} = \oplus_r \underline{\mathbb{Z}}(r)$$

denotes the motivic complex and there are k-factors of this on the left. $NZsk_{q-1}(E\Sigma_k)$ denotes the chain complex obtained by normalizing the simplicial abelian group obtained by applying the free abelian group functor Z dimension-wise to $sk_{q-1}E\Sigma_n$. Observe the q-skeleton of $E\Sigma_k$ is the filtered colimit of all its m-cells, $m \leq q$. Now observe the co-cartesian square (where the sum ranges over all q-simplices of $NZ(E\Sigma_k)$):

$$
\begin{array}{ccc}
\oplus NZ(\overset{\circ}{\Delta}[q]_{ss}) & \longrightarrow & \oplus NZ(\Delta[q]_{ss}) \\
\downarrow & & \downarrow \\
NZ(sk_{q-1}(E\Sigma_k)) & \longrightarrow & NZ(sk_q(E\Sigma_k))
\end{array}
$$

Clearly one may start the above induction when $q = 1$; then one uses the above co-cartesian square and the pairing (3.4.6) to define the pairing

$$(3.4.7) \qquad NZ(sk_q(E\Sigma_k)) \otimes \underline{\mathbb{Z}}(l_1) \otimes \cdots \otimes \underline{\mathbb{Z}}(l_k) \to \underline{\mathbb{Z}}(\Sigma_{i=1}^k l_i)$$

where there are k-factors of $\mathbb{Z}$ on the left. Finally take the colimit over $q \to \infty$ to get the pairing

$$(3.4.8) \qquad \mu_k : NZ(E\Sigma_k) \otimes \underline{\mathbb{Z}}(l_1) \otimes \cdots \otimes \underline{\mathbb{Z}}(l_k) \to \underline{\mathbb{Z}}(\Sigma_{i=1}^k l_i)$$

3.5. The observation in 3.4 shows that the action of the symmetric group Σ_k by an element $\sigma \, \varepsilon \, \Sigma_k$ on $NZ(E\Sigma_k)$ cancels out with the action by the element σ^{-1} on the k-factors $\mathbb{Z}(l_1), \cdots, \mathbb{Z}(l_k)$. Abbreviating $\underset{l}{\oplus} \underline{\mathbb{Z}}(l)$ to $\underline{\mathbb{Z}}$, the above pairing may be shortened to $\mu_k : NZ(E(\Sigma_k)) \otimes \underline{\mathbb{Z}}^{\otimes k} \to \underline{\mathbb{Z}}$.

3.5.1. Given cycles $Z_i \, \varepsilon \, Cor_{q.f}(U \times \Delta[n], \mathbb{A}^{l_i})$, $i = 1, \cdots, s$, we let $\overset{s}{\underset{i=1}{\bullet}} Z_i = \Delta^*(Z_1 \times \cdots Z_s) \, \varepsilon \, Cor_{q.f}(U \times \Delta[n], \mathbb{A}^{\Sigma_i l_i})$.

Next one starts with cycles $Z(j_i)_1, \cdots Z(j_i)_{j_i}$ in $Cor_{q.f}(U \times \Delta[n], \mathbb{A}^{l(j_i)_1}), \cdots, Cor_{q.f}(U \times \Delta[n], \mathbb{A}^{l(j_i)_{j_i}})$, $i = 1, \cdots, k$. Let $j = \Sigma_{i=1}^k j_i$ and $s_{j_1}, \cdots, s_{j_k}$ denote simplices of dimension $p_1, \cdots, p_k$ in $E\Sigma_{j_1}, \cdots, E\Sigma_{j_k}$, respectively. Now one may observe readily that

$(3.5.2)$

$$\mu'(s_{j_1}, \cdots, s_{j_k}, \overset{j_1}{\underset{i=1}{\bullet}} Z(1)_i, \cdots, \overset{j_k}{\underset{i=1}{\bullet}} Z(k)_i) = \mu'(s_{j_1}, (\overset{j_1}{\underset{i=1}{\bullet}} Z(1)_i)) \bullet \cdots \bullet \mu'(s_{j_k}, (\overset{j_1}{\underset{i=1}{\bullet}} Z(k)_i))$$

$(3.5.3)$

$$= \mu'(s_{j_1}, Z(j_1)_1, \cdots, Z(j_1)_{j_1}) \bullet \cdots \bullet \mu'(s_{j_k}, Z(j_k)_1, \cdots, Z(j_k)_{j_k})$$

The left-hand-side is defined as in (3.4.3) by using the obvious diagonal imbedding $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ in Σ_j so that all the simplices s_{j_i} are viewed as simplices in $E\Sigma_j$. In view of this, the actions of the simplices s_{j_i} commute with each other, with s_{j_i} acting trivially on all the terms $\overset{j_1}{\underset{i=1}{\bullet}} Z(j_l)_i, l \neq i$, so that one obtains the identifications stated above. The right-hand-side defines the image of the product $\mu'(s_{j_1}, \overset{j_1}{\underset{i=1}{\bullet}} Z(j_1)_i) \bullet \cdots \bullet \mu'(s_{j_k}, \overset{j_k}{\underset{i=1}{\bullet}} Z(j_k)_i)$ in $Cor_{q.f}(U \times \Delta[n], \mathbb{A}^{l(j_1)+\cdots+l(j_k)})$.

Theorem 3.6. *(Associativity of μ) Let $\gamma_k : NZ(E(\Sigma_k)) \otimes NZ(E(\Sigma_{n_1})) \otimes \cdots \otimes NZ(E(\Sigma_{n_k})) \to NZ(E(\Sigma_{\Sigma_i n_i}))$ denote the pairing defined by the operad-structure on $\{NZ(E(\Sigma_k))|k\}$. Then the following diagram commutes strictly:*

(3.5.4)

$$NZ(E(\Sigma_k)) \otimes NZ(E(\Sigma_{j_1})) \otimes ... \otimes NZ(E(\Sigma_{j_k})) \otimes \underline{\mathbb{Z}}^{\overset{j}{\otimes}} \overset{\gamma \otimes id}{\longrightarrow} \quad NZ(E(\Sigma_j)) \otimes \underline{\mathbb{Z}}^{\overset{j}{\otimes}}$$

with vertical map $regroup$ on the left and maps μ on the right to $\underline{\mathbb{Z}}$,

$$NZ(E(\Sigma_k)) \otimes NZ(E(\Sigma_{j_1})) \otimes \underline{\mathbb{Z}}^{\overset{j_1}{\otimes}} \otimes ... \otimes NZ(E(\Sigma_{j_k})) \otimes \underline{\mathbb{Z}}^{\overset{j_k}{\otimes}} \overset{id \otimes \mu^k}{\longrightarrow} \quad NZ(E(\Sigma_k)) \otimes \underline{\mathbb{Z}}^{\overset{k}{\otimes}}$$

Proof. The key observation here is that the pairing μ' as in (3.4.1), and (3.4.3) defined above commutes with shuffle maps. The commutativity of the above diagram follows essentially from this observation. However, we provide some details below, mainly for the sake of completeness.

Let $s_{j_1}, \cdots s_{j_k}$ denote simplices in $NZ(E(\Sigma_{j_1})), \cdots, NZ(E(\Sigma_{j_k}))$ of dimensions $p_1, \cdots p_k$ respectively. Let t_k denote a q-simplex of $N(ZE(\Sigma_k))$.

We will first consider the $q + p_1 + \cdots + p_k$-simplex of $NZ(E(\Sigma_{j_1+\cdots j_k}))$ defined by $\gamma(t_k, s_{j_1}, \cdots, s_{j_k})$: see Definition 2.3. One first produces a $(q, p_1, \cdots, p_k)$-multisimplex by pairing these as in Definition 2.3. Next one applies a shuffle map to produce a $q + \Sigma_i p_i$ simplex from these. The associativity of the shuffle maps entering into its definition shows that this may be obtained as follows. One first applies a $(p_1, \cdots, p_k)$-shuffle $(\mu_{j_1}, \cdots \mu_{j_k})$ to the simplex $(s_{j_1}, \cdots, s_{j_k})$ to obtain the $p_1 + \cdots + p_k$-simplex $(\mu_{j_1}(s_{j_1}), \cdots, \mu_{j_k}(s_{j_k}))$ in the image of $Z(E(\Sigma_{j_1})) \otimes \cdots \otimes Z(E(\Sigma_{j_k})) \to Z(E(\Sigma_{\Sigma_i j_i}))$. Next one applies a $(q, \Sigma_i p_i)$ -shuffle (μ_k, μ_j) to $(t_k, (\mu_{j_1}(s_{j_1}), \cdots, \mu_{j_k}(s_{j_k})))$ to obtain a $q + \Sigma_i p_i$-simplex in the image of $Z(E(\Sigma_k)) \otimes Z(E(\Sigma_{j_i})) \to Z(E(\Sigma_{\Sigma_i j_i}))$.

Next one starts with cycles $Z(j_i)_1, \cdots Z(j_i)_{j_i}$ in $\Gamma(U, \underline{\mathbb{Z}}_n(l(j_i)_1)), \cdots, \Gamma(U, \underline{\mathbb{Z}}_n(l(j_i)_{j_i})), i = 1, \cdots, k.$ (i.e. $Z(j_i)_m \, \varepsilon \, Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{\wedge^{l(j_i)m}})$.) The cycle

$$\mu(\gamma(t_k, (s_{j_1}, \cdots, s_{j_k})), Z(j_1)_1, \cdots, Z(j_1)_{j_1}, \cdots, Z(j_k)_1, \cdots, Z(j_k)_{j_k})$$

is obtained by applying the pairing of (3.4.7). Since the shuffle maps commute with the pairing μ' (see 3.4.4, (3.4.2) and (3.4.3)), it follows that the last pairing may be obtained as

(3.5.4)   shuffle $\circ \mu'(t_k, s_{j_1}, \cdots, s_{j_k}, Z(j_1)_1, \cdots, Z(j_1)_{j_1}, \cdots, Z(j_k)_1, \cdots Z(j_k)_{j_k})$

Here *shuffle* denotes the composition of all the shuffle maps for passage from $\Delta[n] \times \Delta[p_1] \times \cdots \Delta[p_k] \times \Delta[q] \to \Delta[n + \Sigma_i p_i + q]$.

Next consider the pairing defined by the composition of maps in the left-most column, the bottom row and the bottom part of the right column applied to the same cycles as above. One may see that the resulting cycle may be obtained from the cycles $Z(j_i) = \mu'(s_{j_i}, Z(j_i)_1, \cdots, Z(j_i)_{j_i}) \, \varepsilon \, Cor_f(U \times \Delta[n] \times \Delta[p_i], (\mathbb{P}^1)^{\wedge^{l_{j_i}}})$ as follows. (Here $l_{j_i} = l(j_i)_1 + \cdots + l(j_i)_{j_i}$ where $Z(j_i)_m \, \varepsilon \, Cor_f(U \times \Delta[n], (\mathbb{P}^1)^{\wedge^{l(j_i)m}})$.)

One composes this with a suitable shuffle map to obtain the class

$$\mu(s_{j_i}, Z(j_i)_1, \cdots, Z(j_i)_{j_i}) \, \varepsilon \, \mathrm{Cor}_f(U \times \Delta[n + p_i], (\mathbb{P}^1)^{\wedge^{l_{j_i}}}).$$

Next one applies shuffle maps corresponding to $(\mu_{j_1}, \cdots, \mu_{j_k})$ to obtain classes in

$$\mathrm{Cor}_f(U \times \Delta[n + p], (\mathbb{P}^1)^{\wedge^{l_{j_i}}}). \ (\text{Recall } p = \Sigma_i p_i.)$$

These will still be denoted $\mu(s_{j_i}, Z(j_i)_1, \cdots, Z(j_i)_{j_i})$ for the sake of notational simplicity. Next one applies the pairing in (3.4.7) to these $k+1$ simplices as σ varies among the vertices of the simplex t_k to obtain the class

$$\mu(t_k, \mu(s_{j_1}, Z(j_1)_1, \cdots, Z(j_1)_{j_1}), \cdots, \mu(s_{j_k}, Z(j_k)_1, \cdots, Z(j_k)_{j_k}))$$

$$= \mathrm{shuffle}_3 \circ \mu'(t_k, \mathrm{shuffle}_2 \circ \mathrm{shuffle}_1 \circ \mu'(s_{j_1}, Z(j_1)_1, \cdots, Z(j_1)_{j_1}),$$

$$\cdots, \mathrm{shuffle}_2 \circ \mathrm{shuffle}_1 \mu'(s_{j_k}, Z(j_k)_1, \cdots, Z(j_k)_{j_k})).$$

Here the shuffle map marked $\mathrm{shuffle}_1$ ($\mathrm{shuffle}_2$, $\mathrm{shuffle}_3$) corresponds to the passage from $\Delta[n] \times \Delta[p_i]$ to $\Delta[n+p_i]$ (the passage from $\Pi_i \Delta[n+p_i]$ to $\Delta[n+p]$, the passage from $\Delta[q] \times \Delta[n+p]$ to $\Delta[n+p+q]$, respectively.)

In view of the observation in 3.4.4, one may postpone applying the shuffle maps so that one may identify the last product with

$$(3.5.5) \qquad \mathrm{shuffle}\mu'(t_k, \mathrm{shuffle}_4\mu'(s_{j_1}, Z(j_1)_1, \cdots, Z(j_1)_{j_1}), \cdots, \mathrm{shuffle}_4\mu'(s_{j_k}, Z(j_k)_1, \cdots, Z(j_k)_{j_k}))$$

Here $shuffle_4$ denotes the passage from $\Pi_i \Delta[p_i]$ to $\Delta[p]$, the last shuffle maps are the obvious remaining shuffle maps and

$$\mu'(t_k, \mathrm{shuffle}_4\mu'(s_{j_1}, Z(j_1)_1, \cdots, Z(j_1)_{j_1}), \cdots, \mathrm{shuffle}_4\mu'(s_{j_k}, Z(j_k)_1, \cdots, Z(j_k)_{j_k})) \, \varepsilon \, \mathrm{Cor}_{q.f}(U \times \Delta[n] \times \Delta[p] \times \Delta[q], \mathbb{A}^{\Sigma_i l_{j_i}}).$$

The p-simplex $\Delta[p]$ denotes the p simplex of $NZ(E(\Sigma_{j_1})) \otimes \cdots \otimes NZ(E(\Sigma_{j_k}))$ given by the product $\sigma_{j_1}(s_{j_1}) \otimes \cdots \otimes \sigma_{j_k}(s_{j_k})$.

3.5.6. In view of the above observations the shuffle maps all commute with μ'. Therefore, in view of the identification in (3.4.3) and the observation in (3.5.2), the cycle in (3.5.5) may be identified with

$$(3.5.7) \quad \mu'(t_k, s_{j_1}, \cdots, s_{j_k}, \overset{j_1}{\underset{i=1}{\bullet}} Z(j_1)_i, \cdots, \overset{j_k}{\underset{i=1}{\bullet}} Z(j_k)_i)$$
$$= \mu'(t_k, s_{j_1}, \cdots, s_{j_k}, Z(j_1)_1, \cdots, Z(j_1)_{j_1}, \cdots Z(j_k)_1, \cdots, Z(j_k)_{j_k})$$

followed by appropriate shuffle maps. An inspection of the definition of (3.5.4) shows this clearly identifies with the product in (3.5.4) thereby proving the theorem. $\square$

Theorem 3.7. *The motivic complex $\underline{\mathbb{Z}} = \underset{r}{\oplus}\underline{\mathbb{Z}}(r)$ is an E^∞-algebra over the E^∞-operad $\{NZ(E(\Sigma_k))|k\}$ in the category $\mathrm{C}((\mathrm{Sm}/k)_{Nis})$. Similarly the motivic complex $\underline{\mathbb{Z}}^{et}/l^\nu = \underset{r}{\oplus}\underline{\mathbb{Z}}^{et}/l^\nu(r)$ is an E^∞-algebra over the E^∞-operad $\{NZ(E(\Sigma_k))|k\}$ in the category $\mathrm{C}((\mathrm{Sm}/k)_{et})$ for each fixed prime l and $\nu \geq 0$.*

Proof. The theorem above proves that the pairing μ satisfies the condition (6.1.5) in the appendix. The fact that the pairing μ also satisfies the condition (6.1.6) should be clear. The observations in 3.5 show that it also satisfies the condition (6.1.7). $\square$

Definition 3.8. (i) $\underline{\mathbb{Z}}^{mot}$ will denote the motivic complex viewed as an algebra over the operad $\{NZ(E(\Sigma_k))|k\}$.

(ii) Recall from [K-M, Theorem 1.4, Chapter II], that there exists a functor W that converts any E^∞-algebra tensored with $\mathbb{Q}$ to a quasi-isomorphic strictly commutative differential graded algebra. We will let $\underline{\mathbb{Q}}^{mot}$ denote $W(\underline{\mathbb{Z}}^{mot}\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$. We also let $\underline{\mathbb{Q}}^{mot}(n)$ the corresponding piece of weight n. If X

is a smooth separated scheme of finite type over k, $\mathbb{Q}_X^{mot} = R\Gamma(X, \underline{\mathbb{Q}}_X^{mot})$ will be called the *motivic DGA* associated to X.

Next we obtain the following corollary.

Corollary 3.9. *$\underline{\mathbb{Q}}_X^{mot}$ is quasi-isomorphic to $\underline{\mathbb{Z}}_X \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as a sheaf of differential graded algebras. Moreover, $\underline{\mathbb{Q}}_X^{mot}(n)$ is quasi-isomorphic to $\underline{\mathbb{Z}}_X(n) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and if X is a smooth quasi-projective variety, $\mathbb{Q}_X^{mot}(n)$ is quasi-isomorphic to $\Gamma(X, \underline{\mathbb{Z}}_X^{mot}(n)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ for each $n \geq 0$.*

Proof. It follows from the proof [K-M, Corollary (1.5), Part II] that there is a natural map $\underline{\mathbb{Z}}_X(n) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \underline{\mathbb{Q}}_X^{mot}(n)$ which is a quasi-isomorphism compatible with the pairings. The quasi-isomorphism in the last statement now follows from the observation that the integral motivic complex, being quasi-isomorphic to the higher cycle complex, satisfies the localization property and hence has cohomological descent on the Zariski site of smooth quasi-projective schemes. $\qquad\square$

Corollary 3.10. *$\mathbb{Z}_X = R\Gamma(X, \underline{\mathbb{Z}})$ and $\mathbb{Z}/l^{\nu et}_X = R\Gamma(X, \underline{\mathbb{Z}/l^{\nu et}})$ are algebras over $NZ(E\Sigma) \otimes \mathcal{E}nd_{C_*(\Delta)}$.*

Proof. This follows immediately from Proposition 6.4 in view of the following observations. Observe that the Nisnevich site of a scheme of finite type over k has finite cohomological dimension. Therefore the derived functor $R\Gamma(X, \underline{\mathbb{Z}})$ may be computed as $TotN(\{\Gamma(X, G^n(\underline{\mathbb{Z}}))|n\})$. This is the total complex of the double complex obtained by normalizing in the cosimplicial direction applied to the cosimplicial object $\{\Gamma(X, G^n(\underline{\mathbb{Z}}))|n\}$. Observe also that $\underline{\mathbb{Z}/l^{\nu et}}(r) \simeq \mu_l^\nu(r)[0]$ so that once again one may compute $R\Gamma(X_{et}, \underline{\mathbb{Z}/l^{\nu et}}(r))$ by using the total complex construction as above. $\qquad\square$

Remark 3.11. It will be convenient and often necessary to obtain actions by other operads, (for example, the geometric Barratt-Eccles operad) on the motivic complex. These actions will be induced by the above action of the (simplicial) Barratt-Eccles operad and will be produced by defining a map from the new operad to the above Barratt-Eccles operad. It is convenient to invoke model structures on the category of operads (provided, for example, by Hinich and improved upon by Spitzweck: see [H] and [Sp]) to define such maps. These will be explored elsewhere.

4. Mixed Tate motives for smooth linear schemes over a field k

The results of this section generalize the constructions of [Bl-2], [Bl-K] and [K-M] for the category of mixed Tate motives over a field. The existence of the motivic dga extends these constructions to any smooth quasi-projective scheme if one assumes the Beilinson-Soulé vanishing conjecture holds for the motivic cohomology of that scheme. *In particular we verify this for a large class of quasi-projective smooth varieties including all projective smooth toric and spherical varieties over number fields. The main result is Theorem 4.9.*

We fix a smooth quasi-projective scheme X over k. We let $A = \mathbb{Q}_X^{mot}$. We may assume therefore that A is bi-graded via k-modules $A^q(r)$, where $q \,\varepsilon\, \mathbb{Z}$ and $r \geq 0$. (Recall that A has an augmentation $A \to \mathbb{Q}[0]$.) Let $\mathbb{D}_A$ denote the derived category of cohomologically bounded below A-modules, i.e. differential bi-graded A-modules $M = \underset{q,r}{\oplus} M^q(r)$ where $M^q(r)$ may be non-zero for any pair of integers so that $\mathcal{H}^q(M)(r) = 0$ for all sufficiently small q. (One may first show that this derived category is equivalent to that of cohomologically bounded below *cell A-modules* in the sense of [K-M, Part III]. By construction, every cell A-module is *flat* over A, in the sense that the tensor product $-\underset{A}{\otimes}M$ preserves distinguished triangles in the first argument for every cell A-module M. Then the following derived tensor product may be replaced by a tensor product.) Now one may define a functor $Q : \mathbb{D}_A \to D(\mathbb{Q} - vector \quad spaces)$ by $Q(M) = \mathbb{Q} \overset{L}{\underset{A}{\otimes}} M = $ the $\mathbb{Q}$-vector space of *indecomposable elements* of M: see, for example, [K-M, Part IV]. Here $D(\mathbb{Q} - vector \quad spaces)$ denotes the derived category of bounded below complexes of $\mathbb{Q}$-vector spaces. Observe that this category has a natural t-structure, the heart of which is given by the complexes that have cohomology trivial in all degrees except 0. We let $\mathcal{H}_A$ denote the full sub-category of $\mathbb{D}_A$ consisting of complexes K so that $\mathcal{H}^q(Q(K)) = 0$ for all

$q \neq 0$. Let $\mathcal{FH}_A$ denote the full sub-category of $\mathcal{H}_A$ consisting of complexes K so that $\mathcal{H}^0(Q(K))$ is a finite dimensional $\mathbb{Q}$-vector space. We will make the following assumption throughout:

4.0.8. *the DGA A is connected in the following sense: $H^i(A)(r) = 0$ for $i < 0$, $H^0(A)(r) = 0$ if $r \neq 0$ and $H^0(A)(0) = \mathbb{Q}$.*

Now we obtain the following theorem as in [K-M].

Theorem 4.1. *The triangulated category $\mathbb{D}_A$ admits a t-structure whose heart is $\mathcal{H}_A$. Moreover $\mathcal{FH}_A$ is a graded neutral Tannakian category over $\mathbb{Q}$ with fiber functor $w = \mathcal{H}^0 \circ Q$.*

Proof. The proof is essentially in [K-M, Theorem 1.1, Part IV] . (The key idea here is to use the theory of minimal models.) $\qquad\square$

One may apply the bar construction (see [K-M, Part IV, section 1]) to the algebra A: we will denote this by $\bar{B}A$. Let IA denote the augmentation ideal of A. We let $\chi_A = H^0(\bar{B}A)$. This is a commutative Hopf-algebra and, as in [K-M, Part IV, section 1] , is a polynomial algebra with its k-module of indecomposable elements a co-Lie algebra which is denoted γ_A. Now we obtain the following result.

Theorem 4.2. *(See [K-M, Pat IV, Theorem 1.2].) Assume the hypothesis (4.0.8). Then the following categories are equivalent:*

(i) The heart $\mathcal{H}_A$ of $\mathbb{D}_A$

(ii) The category of generalized nilpotent representations of the co-Lie algebra γ_A

(iii) The category of co-modules over the Hopf-algebra χ_A

(iv) The category $\mathcal{T}_A$ of generalized nilpotent twisting matrices in A

The full sub-categories of finite dimensional objects in the categories (i), (ii) and (iii) and of finite matrices in the category (iv) are also equivalent.

Definition 4.3. (Linear schemes over k) (i) A scheme over $Spec\, k$ is 0-linear if it is either empty or isomorphic to any affine space $\mathbb{A}^n_{Spec\, k}$.

(ii) Let $n > 0$ be an integer. A scheme Z, over $Spec\, k$, is n-linear, if there exists a triple (U, X, Y) of schemes over $Spec\, k$ so that $Y \subseteq X$ is an S-closed immersion with U its complement, Y and one of the schemes U or X is $(n-1)$-linear and Z is the other member in $\{U, X\}$. We say Z is linear if it is n-linear for some $n \geq 0$.

(iii) Recall any reduced scheme X of finite type over $Spec\, k$ is called a variety. Linear varieties over k are varieties over $Spec\, k$ that are linear schemes.

Example 4.4. The following are common examples of linear varieties. In these examples we fix a separably closed base field k and consider only varieties over k.

- All toric varieties
- All spherical varieties (A variety X is spherical if there exists a reductive group G acting on X so that there exists a Borel subgroup having a dense orbit.)
- Any variety on which a connected solvable group acts with finitely many orbits. (For example projective spaces and flag varieties.)
- Any variety that has a stratification into strata each of which is the product of a torus with an affine space.

Remark 4.5. If the field is not separably closed, not all tori are split; therefore the varieties appearing above need not be linear in the sense of the definition 4.3. Over non-separably closed fields, any of the examples above will be linear if and only if the tori appearing in the strata are all split.

Corollary 4.6. *Let X denote a smooth (connected) projective linear variety over a field k (not necessarily separably closed), or any one of the schemes appearing in the examples above which are also connected, projective and smooth. Assume that the Beilinson-Soulé conjecture holds for the rational motivic cohomology of $Spec\,k$, i.e. $H^i_\mathcal{M}(Spec\,k; \mathbb{Q}(r)) = 0$ if $i < 0$, $H^0_\mathcal{M}(Spec\,k; \mathbb{Q}(r)) = 0$ if $r \neq 0$ and $H^0_\mathcal{M}(Spec\,k; \mathbb{Q}(0)) \cong \mathbb{Q}$. (For example, k is a number field). Then the conclusions of theorem (4.2) hold for X (i.e. with $A = \mathbb{Q}_X^{mot}$). Let $U = X - Y$, where X and Y are either projective smooth linear varieties or any of the projective smooth schemes appearing in the above list and that, in either case, Y is closed in X. Then the conclusions of Theorem 4.2 also hold for U.*

Proof. It suffices to show that the DGA, A appearing in the theorem is connected. If the field k is not separably closed, one may find a finite separable extension k' of k so that all the tori in the stratification of X split. By a transfer argument, one may therefore readily reduce to the case where k is separably closed. Next we will consider the case where the scheme is *projective*. In this case, the variety in question is also linear; therefore we may invoke the strong Kunneth decomposition for the class of the diagonal Δ in $CH^*(X \times X)$. (See [J-1] or [Tot].) i.e.

$$(4.0.9) \qquad\qquad \Delta = \Sigma_i \alpha_i \times \beta_i = \Sigma_i p_1^*(\alpha_i) \circ p_2^*(\beta_i)$$

where $p_i : X \times X \to X$ is the projection to the i-factor, $\circ$ denotes the intersection product and $\alpha_i, \beta_i \, \varepsilon \, CH^*(X)$. Now we proceed to show that any class $x \, \varepsilon \, CH^*(X, n)$ may be written as a linear combination

$$(4.0.10) \qquad\qquad x = \Sigma_i \alpha_i \circ p_{1*}(p_2^*(\beta_i \circ x)) = \Sigma_i \alpha_i \circ p_2'^*(p_{1*}'(\beta_i \circ x))$$

Here $p_i' : X \to Spec\,k$ is the obvious projection. To obtain (4.0.10), first observe that $x = p_{1*}([\Delta] \circ p_2^*(x))$. By the projection formula and the observation that the class $[\Delta] = \Delta_*(1)$, $1 = [X] \, \varepsilon \, CH^*(X;\,)$, we obtain equality of the classes $[\Delta] \circ p_2^*(x) = \Delta_*(\Delta^*(p_2^*(x))$. Therefore $p_{1*}([\Delta] \circ p_2^*(x)) = p_{1*}(\Delta_*(\Delta^*(p_2^*(x)))) = (p_1 \circ \Delta)_*((p_2 \circ \Delta)^*(x)) = x$. Now substitute the formula for $[\Delta]$ from (4.0.9) and use the projection formula to obtain the first equality in (4.0.10). The equality of this with the right-hand-side follows by flat-base-change. Observe that $\alpha_i \, \varepsilon \, CH^*(X, 0)$. Therefore, the hypothesis that $H^i_\mathcal{M}(Spec\,k; \mathbb{Q}(r)) = 0$ for $i < 0$ shows readily that $H^q_\mathcal{M}(X; \mathbb{Q}(r)) = CH^r(X, 2r - q; \mathbb{Q}) = 0$ for $q < 0$. (In more detail: suppose $x \, \varepsilon \, H^q_\mathcal{M}(X; \mathbb{Q}(r))$ for $q < 0$. Let $\alpha_i \, \varepsilon \, CH^s(X, 0; \mathbb{Q}) = H^{2s}_\mathcal{M}(X; \mathbb{Q}(s))$ and let $p_{1*}'(\beta_i \circ x) \, \varepsilon \, CH^r(Spec\,k, 2r - q; \mathbb{Q}) = H^q_\mathcal{M}(Spec\,k; \mathbb{Q}(r))$ so that $\alpha_i \circ p_2'^*(p_{1*}'(\beta_i \circ x)) \, \varepsilon \, CH^{s+r}(X, 2r - q; \mathbb{Q}) = H^{2s+q}_\mathcal{M}(X, \mathbb{Q}(s + r))$. Since $s \geq 0$, if $2s + q < 0$, then q must be also negative, which implies, by the hypotheses that $p_{1*}'(\beta_i \circ x) = 0$. It follows that $x = 0$.)

The hypothesis that $H^0_\mathcal{M}(Spec\,k; \mathbb{Q}(r)) = 0$ for $r \neq 0$ implies similarly that $H^0_\mathcal{M}(X; \mathbb{Q}(r)) = 0$ also for $r \neq 0$. Since X is connected, the hypothesis that $H^0_\mathcal{M}(Spec\,k; \mathbb{Q}(0)) = \mathbb{Q}$ now implies $H^0_\mathcal{M}(X; \mathbb{Q}(0)) = \mathbb{Q}(0)$. i.e. We have verified that the DGA $A = A_X = Q_X^{mot}$ associated to the motivic complex of X is connected in the sense of 4.0.8. This proves the first statement. The last statement follows by making use of the localization sequence in motivic cohomology:

$$... \to H^i_\mathcal{M}(Y, \mathbb{Q}(j)) \cong H^{i+2c}_{\mathcal{M}, Y}(X, \mathbb{Q}(j+c)) \to H^{i+2c}_\mathcal{M}(X, \mathbb{Q}(j+c)) \to H^{i+2c}_\mathcal{M}(U, \mathbb{Q}(j+c)) \to H^{i+1}_\mathcal{M}(Y, \mathbb{Q}(j)) \to$$
...

where c is the codimension of Y in X. In case Y is a k-rational point, clearly its motivic cohomology is trivial in negative degrees, in view of our hypotheses. Observe that, in general, Y is a projective scheme satisfying the hypotheses above. Therefore, we observe that $H^i_\mathcal{M}(Y, \mathbb{Q}(j)) = 0$ for $i < 0$ and hence that, for $i + 2c < 0$, the map $H^{i+2c}_\mathcal{M}(X, \mathbb{Q}(j + c)) \to H^{i+2c}_\mathcal{M}(U, \mathbb{Q}(j + c))$ is injective. Moreover since $c > 0$, $i + 1 < 0$ if $i + 2c < 0$ and therefore, $H^{i+1}_\mathcal{M}(Y, \mathbb{Q}(j)) = 0$ as well. These prove that $H^{i+2c}_\mathcal{M}(U, \mathbb{Q}(j)) = 0$ if $i < -2c$. The same observations again show that if $i = -2c$, the map $H^{i+2c}_\mathcal{M}(X, \mathbb{Q}(0)) \to H^{i+2c}_\mathcal{M}(U, \mathbb{Q}(0))$ is an isomorphism. The required conclusion now follows. $\square$

The DGA A has a 1-minimal model, $i : A < 1 > \to A$. (Recall (see [K-M, Part IV, section 2]) a connected DGA B over the rationals $\mathbb{Q}$ and provided with an augmentation $B \to \mathbb{Q}$ is said to be minimal if it is a free graded Q-module (with decomposable differential : $d(B) \subseteq (I(B))^2$ where $I(B)$ is the augmentation ideal of B. $B < 1 >$ is the sub-DGA of B generated by the elements of degree

≤ 1 and their differentials. The 1-minimal model of a DGA A is a composite map $B < 1 > \subseteq B \to A$ with the last map a quasi-isomorphism and with B minimal.) The map i induces an isomorphism on H^1 and is injective on H^2. We *say A is a $K(\pi, 1)$* if i is a quasi-isomorphism.

Theorem 4.7. *(See* [K-M, Part IV, Theorem 1.3]*.) The derived category of bounded below chain complexes in $\mathcal{H}_A$ is equivalent to the derived category $\mathbb{D}_{A<1>}$.*

Definition 4.8. (The category of mixed Tate motives over X.) Let χ_X^{mot} denote the Hopf algebra $H^0(\bar{B}A)$. The category of (rational) mixed Tate motives over X, denoted $\mathcal{MTF}(X)$, will be defined to be the category of finite dimensional co-modules over χ_X^{mot}.

Theorem 4.9. *If the DGA A is connected (in the sense of 4.0.8), $\mathcal{MTF}(X)$ is equivalent to the category $\mathcal{FH}_A$. In particular, this holds for the following classes of smooth quasi-projective varieties assuming the Beilinson-Soulé conjecture (see above) holds for the motivic cohomology of $\operatorname{Spec} k$, for example if k is a number field:*

(i) all smooth (connected) projective linear varieties over k

(ii) any of the varieties over k appearing in the list 4.4 which are also connected, projective and smooth

(iii) any quasi-projective variety U (over k) of the form $X - Y$, where X and Y are smooth projective varieties both as in (i) or (ii) and Y is closed in X.

Proof. The proof is clear in view of Theorem 4.2 and Corollary 4.6 above. $\qquad\square$

Let $\mathbb{Q}(r)$ be the copy of $\mathbb{Q}$ concentrated in bi-degree $(0, r)$ and regarded as a representation of γ_A in the obvious manner.

Corollary 4.10. *(See* [K-M, Part IV, Corollary 1.4]*.) If $A = A_X = \mathbb{Q}_X^{mot}$ is a $K(\pi, 1)$, then*
$$Ext^q_{\mathcal{MTF}(X)}(\mathbb{Q}, \mathbb{Q}(r)) = Ext^q_{\mathcal{H}_A}(\mathbb{Q}, \mathbb{Q}(r)) \cong H^q(A(r)) = H^q_{\mathcal{M}}(X, r) = CH^r(X, 2r - q; \mathbb{Q})$$

4.1. Étale realization: A DGA structure for étale cohomology. Let $\{NZ(E\Sigma_n)_{et} = \epsilon^*(NZ(E\Sigma_n)) | n \geq 0\}$ denote the simplicial Barratt-Eccles operad sheafified on the big étale site; this is a sheaf of E^∞-operads on the big étale site. Clearly $\underline{\mathbb{Z}/l}_{et}^{\nu \, mot}$ is an algebra over this operad.

Moreover (see [MVW, Chapter 10]) by the rigidity theorem, one knows that $\underline{\mathbb{Z}/l}_{et}^{\nu \, mot}$ identifies with the complex $\mu_{l^\nu}[0]$ and that the (étale) hyper-cohomology of a smooth scheme X over k with respect to the former complex identifies with the étale cohomology with respect to μ_{l^ν}. Therefore $\underline{\mathbb{Z}/l}_{et}^{\nu \, mot}$ provides an E^∞-algebra structure for étale cohomology. Taking $R\Gamma(X, \quad)$ over a given smooth scheme X, then the homotopy inverse limit over $\nu \to \infty$, tensoring with $\mathbb{Q}$ and applying a functor W as in 3.8, one obtains a strict DGA-structure for continuous l-adic étale cohomology of X.

Observe as a consequence of the above arguments that the cycle map from the motivic complex to mod-l^ν-étale cohomology may now be identified with the map induced by the obvious map: $\underline{\mathbb{Z}}^{mot} \to R\epsilon_*(\underline{\mathbb{Z}/l}_{et}^{\nu \, mot})$ where ϵ is the obvious map of sites from the big étale site to the big Nisnevich site of smooth schemes of finite type over k. $R\epsilon_*(\underline{\mathbb{Z}/l}_{et}^{\nu \, mot}) = Tot\{NG^n(\underline{\mathbb{Z}/l}_{et}^{\nu \, mot})\}$. Therefore, by Proposition 6.4, $R\epsilon_*(\underline{\mathbb{Z}/l}_{et}^{\nu \, mot})$ gets the structure of an E_∞-algebra over the operad $\mathcal{E}nd_{\mathrm{M}_*(\Delta)} \otimes \mathcal{E}nd_{\mathrm{C}_*(\Delta)}$. It follows that the above cycle map is compatible with E^∞-DGA structures.

5. Classical cohomology operations

The results of this section follow readily by invoking standard results (see for example, [May] or [H-Sch]) which deduce the existence of cohomological operations from the existence of an E^∞-structure on complexes defining cohomology. However, several nice features of these operations (and hence our constructions) need to be clarified.

- The operad $\{N\mathbb{Z}(E\Sigma_n)|n\}$ is a *classical operad* in the sense that the homology of the complexes $\{N\mathbb{Z}(B\Sigma_n) = N\mathbb{Z}(E\Sigma_n/\Sigma_n)|n\}$ is classical, i.e. in particular there are not enough classes to define the motivic operations of Voevodsky. (In fact all the classes have weight 0.)
- However, in [BroJ] we will pursue the relation between these operations and the motivic operations in great detail. We will see there, that the motivic operations of Voevodsky and the classical operations considered here differ by multiplication by suitable powers of the Bott element.
- Another interesting feature of our construction is that it provides cohomology operations even when $l = p = char(k)$: these are also explored in detail in [BroJ]. (The existence of such operations was left open in [Voev2, section 3, p. 73].)
- It also needs to be pointed out that our operations are *not* bi-stable, i.e. do not commute with weight-suspension, but only with respect to degree-suspension in the sense made precise in the theorem below.

Remark 5.1. In [Ep] a purely homological-algebraic technique to defining cohomology operations is considered. However, this requires that the cohomology be with respect to a *sheaf of strictly associative and commutative algebras*. Therefore, while this approach readily applies to produce cohomology operations in étale cohomology with respect to the sheaves $\{\mu_l|(r)|r\}$, $l \neq char(k)$, it does not apply to motivic cohomology (computed on the Zariski or Nisnevich sites). Using this approach, Gillou and Weibel have come come up with a construction of cohomology operations in étale cohomology: see [GW]. Then making use of the Beilinson-Lichtenbaum conjecture, which is a corollary to the Bloch-Kato conjecture they are able to push this to the motivic setting thereby obtaining a different construction of the classical operations in motivic cohomology. However, constructions use nothing more than the existence of an E_∞-structure on the motivic complex.

Let X denote a smooth separated scheme of finite type over the base field k. (We may assume this is quasi-projective for the sake of simplicity.) Let l denote a fixed prime (not necessarily different from the characteristic of k) and let $\underline{\mathbb{Z}}_X^{mot}$ denote the motivic complex associated to X.

5.1. Let $\mathcal{A}_X^\bullet = \underline{\mathbb{Z}}_X^{mot} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/l.\mathbb{Z}$ and let $A_X^\bullet = R\Gamma(X, \mathcal{A}_X)$ where the derived functor is taken on the Zariski (or Nisnevich site). We let $\mathcal{A}_{Spec\,k}$ and $A_{Spec\,k}$ denote the corresponding objects when $X = Spec\,k$. Recall that $\mathcal{A}_X$ and $\mathcal{A}_{Spec\,k}$ are sheaves of bi-graded E^∞-algebras over the E^∞-operad defined in the last section. Therefore, A_X and $A_{Spec\,k}$ are now bi-graded E^∞-differential graded algebras, so that $A_X = \underset{r}{\oplus} A_X^\bullet(r)$. Moreover $H_\mathcal{M}^i(X; \mathbb{Z}/l(r)) = H^i(A_X^\bullet(r))$ which will be isomorphic to $CH^r(X, 2r - i; \mathbb{Z}/l)$ when X is assumed to be quasi-projective.

5.2. Let $\mathcal{A}_{X,et}^\bullet = \underline{\mathbb{Z}}_X^{et} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/l.\mathbb{Z}$ and let $A_{X,et}^\bullet = R\Gamma(X, \mathcal{A}_{X,et})$ where the derived functor is taken on the étale site. Now $H_{et}^i(X; \mathbb{Z}/l(r)) = H^i(A_{X,et}^\bullet(r))$.

5.3. The motivic and étale derived categories associated to a scheme. For the purposes of this section we will define this as follows. Recall $(Sm/k)_{Nis}$ $((Sm/k)_{et})$ denote the big Nisnevich (étale, respectively) site of all smooth separated schemes of finite type over the given field k. For the most part, we will restrict to a fixed smooth scheme X. We consider for each smooth separated scheme X of finite type over k, the small Nisnevich site X_{Nis} and the corresponding small étale site X_{et}: we may denote either of these generically by X_{st}. Unless the distinction is important we will continue to denote both the complexes $\mathcal{A}_X$ and $\mathcal{A}_{X_{et}}$ by $\mathcal{A}_X$ itself. We consider unbounded (co-chain) complexes of sheaves M of $\mathcal{A}_X$-modules on the site X_{st}: this forms a category where the morphisms are module maps. We consider the corresponding homotopy category and the mod$-l$ motivic derived category will be the localization of this homotopy category by inverting maps that are quasi-isomorphisms. This category will be denoted $D(X)$. (Making use of the theory of *cell-A-modules* as in [K-M, Part III] one may show that this derived category has small Hom-sets: we skip these details about the derived category $D(X)$ as they are available in the literature.) The external hom (internal hom) in this category will be denoted $Ext(\quad, \quad)$ ($\mathcal{RHom}(\quad, \quad)$, respectively). (Observe that $Ext(M, N) =$

$H^0(R\Gamma(X, \mathcal{R}Hom(M, N))).$) We define the mod$-l$ cohomology of an object $M \, \varepsilon \, D(X)$ with weight r to be $Ext^*(M; \mathcal{A}_X(r))$. This will be denoted $H^*(M; \mathbb{Z}/l(r)) = H^{*,r}(M; \mathbb{Z}/l)$.

Let $D_0(X)$ denote the full sub-category of $D(X)$ consisting of (co-chain) complexes K that are trivial in positive degrees. By identifying such co-chain complexes with chain complexes that are trivial in negative degrees, one may see that that the derived category $D_0(X)$ is equivalent to the derived category of simplicial Abelian sheaves on X_{st}. Recall that for any simplicial Abelian sheaf F there is a diagonal map $\Delta : F \to F \otimes F$; this (together with the Alexander-Whitney map and the equivalence between simplicial Abelian sheaves and complexes of sheaves trivial in negative degrees) induces a diagonal map $\Delta : F \to F \otimes F$, $F \, \varepsilon \, D_0(X)$.

5.3.1. We will denote the operad $\{BE(n)|n\} \otimes \mathcal{E}nd_{\mathrm{C}_*(\Delta)}$ henceforth by $\{\mathcal{O}(n)|n\}$. The corresponding operad on the site $(Sm/k)_{et}$ will be denoted $\{\mathcal{O}(n)_{et}|n\}$. Observe (making use of the above diagonal map) that if $M \, \varepsilon \, D_0(X)$, $\mathcal{R}Hom(M, \mathcal{A}_X)$ has the obvious induced structure of a sheaf of E^∞-algebras over the operad $\{\mathcal{O}(n)|n \geq 0\}$ (the operad $\{\mathcal{O}(n)_{et}|n \geq 0\}$ in the étale case, respectively). (The required pairings are defined as the composition

$$\mathcal{O}(n) \otimes \mathcal{R}Hom(M, \mathcal{A}_X)^{\overset{n}{\otimes}} \overset{id \otimes eval}{\to} \mathcal{O}(n) \otimes \mathcal{R}Hom(M^{\overset{n}{\otimes}}, \mathcal{A}_X^{\overset{n}{\otimes}}) \overset{id \otimes \Delta^*}{\to} \mathcal{O}(n) \otimes \mathcal{R}Hom(M, \mathcal{A}_X^{\overset{n}{\otimes}})$$

$$\overset{\mathcal{R}Hom(M, \theta_n)}{\to} \mathcal{R}Hom(M, \mathcal{A}_X).$$

The last map is defined by its adjoint: $\mathcal{O}(n) \otimes M \otimes \mathcal{R}Hom(M, \mathcal{A}_X^{\overset{n}{\otimes}}) \overset{id \otimes eval}{\to} \mathcal{O}(n) \otimes \mathcal{A}_X^{\overset{n}{\otimes}} \overset{\theta_n}{\to} \mathcal{A}_X$.) Hence one obtains a graded ring structure on $\oplus_r H^*(M; \mathbb{Z}/l(r))$. Moreover, if $\mathbb{Z}/l(0)$ denotes the mod-l motivic complex of weight 0, $\mathcal{R}Hom(\mathbb{Z}/l(0), \mathcal{A}_X) \simeq \mathcal{A}_X$ and there is a natural pairing $\mathcal{R}Hom(M, \mathcal{A}_X) \otimes \mathcal{R}Hom(\mathbb{Z}/l(0), \mathcal{A}_X) \to \mathcal{R}Hom(M, \mathcal{A}_X)$ that is compatible with the above algebra structure on

$\mathcal{R}Hom(M, \mathcal{A}_X)$.

Recall the complex $\mathbb{Z}/l(0)$ is the complex with the constant sheaf $\underline{Z}/l$ in degree 0 and trivial elsewhere in both the motivic and the étale cases. Therefore, $\underline{Z}/l(0)[i]$ is the complex concentrated in degree $-i$ where it is the constant sheaf $\underline{Z}/l$: tensoring with this complex defines the degree-suspension S_{deg}^i. One may also obtain the following characterization of the degree-suspension (or the simplicial suspension): $Ext^*(S_s^1 M, K) \cong Ext^*(M, K[1])$, for $M, K \, \varepsilon \, D(X)$. We define the Tate suspension (in the motivic case), $S_t^1 M$ by $\mathbb{Z}_{tr}(\mathbb{A}^1 - 0) \otimes M$. More precisely, we may make use of the pairing $\mathcal{R}Hom(\mathbb{Z}_{tr}(\mathbb{A}^1 - 0), \mathbb{Z}(1)) \otimes \mathcal{R}Hom(M, \mathcal{A}_X(r)) \to \mathcal{R}Hom(\mathbb{Z}_{tr}(\mathbb{A}^1 - 0) \otimes M, \mathcal{A}_X(r+1))$ to define the Tate suspension of the motivic cohomology of M. The composition of these two suspensions may be effected by tensoring with the canonical class $T \, \varepsilon \, H^2(\mathbb{P}^1; \mathbb{Z}/l(1))$. We denote the composite suspension of M by $S_T^1 M$. Now we obtain the natural isomorphisms for any $M \, \varepsilon \, D_0(X)$:

$$H^n(M; \mathbb{Z}/l(r)) \cong H^{n+1}(S_s^1 M; \mathbb{Z}/l(r))$$

(5.3.2) $$H^n(M; \mathbb{Z}/l(r)) \cong H^{n+1}(S_t^1 M; \mathbb{Z}/l(r+1)) \quad and$$

$$H^n(M; \mathbb{Z}/l(r)) \cong H^{n+2}(S_T^1 M; \mathbb{Z}/l(r+1))$$

Remarks 5.2. 1. Observe that the second isomorphism in the motivic case shows $H^1(\mathcal{R}Hom(\mathbb{Z}_{tr}(\mathbb{A}^1 - 0), \mathbb{Z}(1))) \cong Z$. Let τ denote the canonical class corresponding to $1 \, \varepsilon \, Z$: clearly the Tate suspension in the motivic case may be effected by tensoring with this class.

2. All of the above discussion in the motivic case applies equally well when the Nisnevich site is replaced by the Zariski site.

Throughout the following discussion H^* will denote either motivic or étale cohomology. We define *bistable mod-l cohomology operations of bi-degree (i, j)* to be sequences of natural transformations $\{H^n(\quad, \underline{\mathbb{Z}}/l(r)) \to H^{n+i}(\quad, \underline{\mathbb{Z}}/l(r+j))|n, r\}$ on $D_0(X)$ and which are contravariantly functorial in

$X \varepsilon (\mathrm{Sm}/k)$. In view of the suspension-isomorphisms above, these are determined by their restrictions to $\{H^{2n}(\quad, \underline{\mathbb{Z}}/l(n))|n\}$.

Recall there are *Bockstein* homomorphisms $\beta : H^n(M; \underline{\mathbb{Z}}/l(r)) \to H^{n+1}(M; \underline{\mathbb{Z}}/l(r))$ which are defined in the usual manner as the boundary homomorphism associated to the short-exact sequence: $0 \to \underline{\mathbb{Z}}/l(r) \to \underline{\mathbb{Z}}/l^2(r) \to \underline{\mathbb{Z}}/l(r) \to 0$. These are clearly bi-stable cohomology operations.

One of the main results in this section is the following theorem, which shows the existence of classical motivic and étale cohomology operations for all primes l.

Theorem 5.3. *There exist operations $Q^s : H^q(X, \mathbb{Z}/l(t)) \to H^{q+2s(l-1)}(X, \mathbb{Z}/l(l.t))$ and*

$\beta Q^s : H^q(X, \mathbb{Z}/l(t)) \to H^{q+2s(l-1)+1}(X, \mathbb{Z}/l(l.t))$.

These operations satisfy the following properties:

(i) Contravariant functoriality: if $f : X \to Y$ is a map between smooth separated schemes of finite type over k, $f^* \circ Q^s = Q^s \circ f^*$

(ii) Let $x \varepsilon H^q(X, \mathbb{Z}/l(t))$. $Q^s(x) = 0$ if $2s > q$, $\beta Q^s(x) = 0$ if $2s \geq q$ and if $(q = 2s)$, then $Q^s(x) = x^l$.

(iii) If β is the Bockstein, $\beta \circ Q^s = \beta Q^s$.

(iv) Cartan formulae: For all primes l, $Q^s(x \otimes y) = \underset{i+j=s}{\Sigma} Q^i(x) \otimes Q^j(y)$ and

$\beta Q^s(x \otimes y) = \underset{i+j=s}{\Sigma} \beta Q^i(x) \otimes Q^j(y) + Q^i(x) \otimes \beta Q^j(y)$

(v) Adem relations For each pair of integers $i \geq 0, j \geq 0$, we let $(i,j) = \frac{(i+j)!}{i!j!}$ with the convention that $0! = 1$. We will also let $(i,j) = 0$ if $i < 0$ or $j < 0$. (See [May, p. 183].) With this terminology we obtain:

If $(l > 2$, $a < lb$, and $\epsilon = 0, 1)$ or if $(l = 2$, $a < lb$ and $\epsilon = 0)$ one has

$$(5.3.2) \qquad \beta^\epsilon Q^a Q^b = \Sigma_i (-1)^{a+i}(a - li, (l-1)b - a + i - 1)\beta^\epsilon Q^{a+b-i} Q^i$$

where $\beta^0 Q^s = Q^s$ while $\beta^1 Q^s = \beta Q^s$. If $l > 2$, $a \leq lb$ and $\epsilon = 0, 1$, one also has

$$(5.3.3) \qquad \begin{aligned} \beta^\epsilon Q^a \beta Q^b = (1 - \epsilon)\Sigma_i (-1)^{a+i}(a - li, (l-1)b - a + i - 1)\beta Q^{a+b-i} Q^i \\ - \Sigma_i (-1)^{a+i}(a - li - 1, (l-1)b - a + i)\beta^\epsilon Q^{a+b-i} \beta Q^i \end{aligned}$$

(vi) More generally, for any $M \varepsilon D_0(X)$, there exist cohomology operations

$Q^s : H^q(M; \mathbb{Z}/l(t)) \to H^{q+2s(l-1)}(M; \mathbb{Z}/l(l.t))$ *and*

$\beta Q^s : H^q(M; \mathbb{Z}/l(t)) \to H^{q+2s(l-1)+1}(M; \mathbb{Z}/l(l.t))$

satisfying the properties (ii) through (v) as above. Moreover, if $f : M' \to M$ is a map in $D_0(X)$, the operations Q^s commute with pull-back by f. They also commute with the simplicial suspension isomorphism in (5.3.2).

(vii) The operation Q^s commutes with change of base fields and also with the higher cycle map into $\mathrm{mod}-l$ étale cohomology.

Proof. Let π denote the cyclic group Z/l and let $\{e_i|i\}$ form a Z/l-basis for $H^*(B\pi; Z/l)$. We will let $M \varepsilon D_0(X)$ and define cohomology operations on $H^*(M; Z/l(t))$. For a smooth scheme X, we obtain cohomology operations on $H^q(X, \mathbb{Z}/l(t))$ by taking $M = \mathbf{Z}/l_X$, the constant sheaf on X with stalks

isomorphic to the integers. Recall that $\mathcal{A}_X$ is an E^∞-dga over the E^∞-operad $NZ(E\Sigma) \otimes \mathcal{E}nd_{C_*(\Delta)}$ (denoted $\{\mathcal{O}(n)|n\}$ above). Since the cohomology operations are assumed to be stable under simplicial suspension as in (ii), it suffices to define these on classes $x \, \varepsilon \, H^{2q}(M; \mathbb{Z}/l(t))$. Therefore, one obtains the existence of cohomology operations Q^s which are defined as follows (see [May, p. 161]): if $l = 2$, we let

$$(5.3.4) \qquad \begin{aligned} Q^s(x) &= \bar{\theta}_*(e_{(2s-2q)} \otimes x^l), \\ \beta Q^s(x) &= \bar{\theta}_*(e_{(2s-2q)+1} \otimes x^l) \end{aligned}$$

and if $l > 2$, we let:

$$(5.3.5) \qquad \begin{aligned} Q^s(x) &= (-1)^{q-s}\bar{\theta}_*(e_{(2s-2q)(l-1)} \otimes x^l), \\ \beta Q^s(x) &= (-1)^{q-s}\bar{\theta}_*(e_{(2s-2q)(l-1)+1} \otimes x^l) \end{aligned}$$

In [May, p. 161], an extra sign is introduced. We have gotten rid of this by including this sign into the choice of the basis elements $\{e_i|i\}$ which form a basis for $H_*(B\pi; \mathbb{Z}/l)$. Observe that $e_{(2s-2q)(l-1)}$ has degree $(2s - 2q)(l - 1)$ so that the total degree of $e_{(2s-2q)(l-1)} \otimes x^l$ is $2q + 2s(l - 1)$. Since the weight of x^l is $t.l$ and $\theta_*(e_{(2s-2q)(l-1)} \otimes x^l)$ leaves the weight unchanged, $Q^s(x)$ has weight $t.l$. The map $\bar{\theta}_*$ is the map

$$(5.3.6) \qquad \bar{\theta}_* : H_*(B\pi; \mathbb{Z}/l) \otimes H^*(M; \mathbb{Z}/l(r))^{\overset{l}{\otimes}} \to H^*(M; \mathbb{Z}/l(r)), \quad \bar{\theta}_*(\bar{e} \otimes \bar{x}^{\overset{l}{\otimes}}) = [\theta(e \otimes x^{\overset{l}{\otimes}})]$$

where e (x) denotes a cycle representing the cohomology class $\bar{e}$ ($\bar{x}$, respectively) and $[z]$ denotes the cohomology class represented by the cycle z.

Now all assertions except for the last assertion are immediate consequences of standard results on cohomology operations on algebras over E_∞-operads: see [May] or [H-Sch]. (Recall that e_i is defined only for $i \leq 0$; therefore we let $e_i = 0$ for $i > 0$ so that $Q^s(x) = 0$ if $q < 2s$.) The action of the operad $\{\mathcal{O}(p)|p\}$ on the complex $\mathcal{A}_X$ was shown to be functorial in the base field k; therefore the operation Q^s commutes with respect to change of base fields. Moreover, as observed in Remark 5.2, the action of the operads $\{\mathcal{O}(p)|p\}$ and $\{\mathcal{O}(p)_{et}|p\}$ are compatible on the complexes $\mathcal{A}_X$ and $\mathcal{A}_{X_{et}}$; therefore, the operation Q^s is also compatible with respect to the higher cycle map as defined above. $\qquad \square$

Remark 5.4. The above operations cannot commute with the Tate suspension as one may see from elementary weight considerations. Therefore they are *not* bi-stable cohomology operations. Moreover, observe that Q^0 is not necessarily the identity and therefore βQ^0 is not necessarily β. Observe also that some of the above cohomology operations may be trivial owing to the fact that the weight may be high enough. (Recall: $H^i_{\mathcal{M}}(X; \mathbb{Z}/(j)) \cong CH^j(X, \mathbb{Z}/l; 2j - i) = 0$ if $j > dim(X) + 2j - i$.) This shows that the motivic cohomology operations of Voevodsky *cannot* be deduced from the classical operations considered above. However, the classical operations on étale cohomology indeed can be deduced from the motivic cohomology operations : this is discussed in detail in the forthcoming paper [BroJ].

We will denote the operations Q^s considered above on étale cohomology with l *different the characteristic p* as Q^s_{et}. When the base field is separably closed, one may identify $\mu_l(r)$ with the constant sheaf Z/l; in this case, therefore, the weights are irrelevant, and we obtain cohomology operations in the usual sense, once it is shown that $Q^0_{et} = id$. (This is proved below.) For example, if the base field is the field of complex numbers, the operations we obtain identify with the usual cohomology operations in mod-l singular cohomology (once the latter is identified with mod$-l$ étale cohomology).

Proposition 5.5. *If the base field is separably closed, the operation $Q^0_{et} = id$.*

Proof. Since the Tate suspension is irrelevant now, Q^0 commutes with the simplicial suspension and is contravariantly functorial we may reduce to checking this when M = the constant sheaf Z/l. In this case, $Q^0_{et}(\alpha) = \alpha^l = \alpha$, for any $\alpha \, \varepsilon \, \mathbb{Z}/l$. This proves the proposition. $\qquad \square$

6. Appendix: chain complexes and operads

Let $\mathbf{A}$ denote an Abelian category; a chain complex K in $\mathbf{A}$ will denote a sequence $K_i \, \varepsilon \, \mathbf{A}$ provided with maps $d : K_i \to K_{i-1}$ so that $d^2 = 0$. Let $\mathrm{C}_0(\mathbf{A})$ denote the category of chain complexes in $\mathbf{A}$ that are trivial in negative degrees. One defines the de-normalizing functor:

$$(6.0.7) \qquad DN : \mathrm{C}_0(\mathbf{A}) \to (simplicial\ objects\ in\ \mathbf{A})]$$

as in [Ill, pp.8-9]. DN will be inverse to the functor

$$(6.0.8) \qquad N : (simplicial\ objects\ in\ \mathbf{A}) \to \mathrm{C}_0(\mathbf{A})$$

defined by $(NK)_n = \underset{i \neq 0}{\cap} ker(d_i : K_n \to K_{n-1})$ with $\delta : (NK)_n \to (NK)_{n-1}$ induced by d_0. We will also often consider the functor $C : (simplicial\ objects\ in\ \mathbf{A}) \to \mathrm{C}_0(\mathbf{A})$ defined by $K \mapsto$ the chain complex which in degree n is K_n and where the differential $d : C(K)_n \to C(K)_{n-1}$ is given by $d = \Sigma_i (-1)^i d_i$. Given a chain complex K, trivial in negative degrees with differentials of degree -1, one may form the corresponding (co-)chain complex trivial in positive degrees with differentials of degree $+1$ by re-indexing K in the obvious manner. This functor composed with the functor C above will be denoted $C^\bullet : (simplicial\ objects\ in\ \mathbf{A}) \to \mathrm{C}(\mathbf{A})$ where the last is the category of (co-)chain complexes in the Abelian category $\mathbf{A}$.

Given two positive integers p and q, a (q, p)-shuffle π is a permutation of $(1, ..., p + q)$ so that $\pi(i) < \pi(j)$ for $1 \le i < j \le q$ and for $q + 1 \le i < j \le q + p$. We let $\mu =$ the restriction of π to $(1, ..., q)$ and $\nu(j) = \pi(j + q)$, $1 \le j \le p$. Clearly π is determined by (μ, ν) and therefore, we will identify π with the pair (μ, ν). The set of all (q, p)-shuffles is in one-one correspondence with the set of all strictly increasing maps $(\phi, \psi) : [p + q] \to [q] \times [p]$: the correspondence is given by sending a shuffle (μ, ν) to (ϕ, ψ) where $\phi(x) =$ the cardinality of the set $\{1 \le i \le q | (\mu, \nu)(i) \le x\}$ and $\psi(x) =$ the cardinality of the set $\{q + 1 \le i \le q + p | (\mu, \nu)(i) \le x\}$. Each such map (ϕ, ψ) (and hence each shuffle map) defines an isomorphism of schemes

$$(6.0.9) \qquad \Delta[p + q] \to \Delta[q] \times \Delta[p]$$

by the formula: $(t_0, \cdots, t_{p+q}) \mapsto t_0(v_{\phi(0)}, v_{\psi(0)}) + \cdots + t_{p+q}(v_{\phi(p+q)}, v_{\psi(p+q)})$.

6.0.10. Let Σ_k denote the symmetric group on k-letters and let $E\Sigma_k$ denote the simplicial group defined by the bar construction. Observe that a p-simplex of $E\Sigma_k$ is given by a sequence $(\sigma_0, \cdots, \sigma_p)$, $\sigma_i \, \varepsilon \, \Sigma_k$. Let $s_p = (\sigma_0, \cdots, \sigma_p)$ and $s_q = (\tau_0, \cdots, \tau_q)$ denote a p-simplex and a q-simplex of $E\Sigma_k$. Then, viewing s_p as a map $\Delta[p] \to E\Sigma_k$, s_q as a map $\Delta[q] \to E\Sigma_k$ of simplicial sets, ϕ as an element of $\Delta[q]_{p+q}$ and ψ as an element of $\Delta[p]_{p+q}$, each map (ϕ, ψ) (as above) associates a $p + q$-simplex $(s_q(\phi(0)) \circ s_p(\psi(0)), \cdots, s_q(\phi(p + q)) \circ s_p(\psi(p + q)))$ of $E\Sigma_k$ to the (q, p)-simplex (s_q, s_p). Here $\circ$ denotes composition in Σ_k. We denote this product by

$$(6.0.11) \qquad (\phi, \psi)^*(s_q \times s_p)$$

Given two simplicial objects K and K', one obtains a map called the shuffle map (where C denotes the functor considered above)

$$(6.0.12) \qquad s = \text{shuffle} : K_q \otimes K'_p = C(K)_q \otimes C(K')_p \to C(K \otimes K')_{p+q}$$

which is defined by

$$(6.0.13) \qquad \text{shuffle}(k_q \otimes k'_p) = \Sigma_{(\mu, \nu)}(-1)^{\sigma(\nu)}(s_{\mu_p} \circ ... \circ s_{\mu_1}(k_q) \otimes s_{\nu_q} \circ ... \circ s_{\nu_1}(k'_p))$$

where the sum is over all (q, p)-shuffles (μ, ν) and where $\sigma(\nu)$ is the signature of the permutation ν.

Proposition 6.1. *Assume the above situation. Then the functor C is compatible with pairings. Moreover, the shuffle map above is strictly associative and strictly commutative.*

6.1. Operads. Basic definitions of operads and algebras over operads for topological spaces may be found in [May] or [H-Sch]. The same definitions apply with minor modifications to the unital symmetric monoidal category $C(\mathcal{S}, \mathcal{R})$ as in section 2. We include these definitions here mainly for the sake of the reader unfamiliar with operads.

An *associative operad* (or simply operad) $\mathcal{O}$ in $C(\mathcal{S}, \mathcal{R})$ is given by a sequence $\{\mathcal{O}(k)|k \geq 0\}$ of objects in $C(\mathcal{S}, \mathcal{R})$ along with the following data:

for every integer $k \geq 1$ and every sequence $(j_1, ..., j_k)$ of non-negative integers so that $\Sigma_{l=1}^{l=k} j_l = j$ there is given a map $\gamma_k : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \mathcal{O}(j_k) \to \mathcal{O}(j)$ so that the following associativity diagrams commute, where $\Sigma_{l=1}^{l=k} j_l = j$ and $\Sigma_{t=1}^{t=j} i_t = i$; we set $g_s = j_1 + ... + j_s$ and $h_s = i_{g_{s-1}+1} + ... + i_{g_s}$ for $1 \leq s \leq k$:

(6.1.1)

$$
\begin{array}{ccc}
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) & \xrightarrow{\gamma \otimes id} & \mathcal{O}(j) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) \\
\downarrow{\scriptstyle shuffle} & & \downarrow{\scriptstyle \gamma} \\
& & \mathcal{O}(i) \\
& & \uparrow{\scriptstyle \gamma} \\
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j_s}{\underset{q=1}{\otimes}} \mathcal{O}(i_{g_{s-1}+q})) & \xrightarrow{id \otimes (\otimes_s \gamma)} & \mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(h_s))
\end{array}
$$

In addition one is provided with a unit map $\eta : u \to \mathcal{O}(1)$ so that the diagrams

$$
\begin{array}{ccc}
\mathcal{O}(k) \otimes (u^{\overset{k}{\otimes}}) & \xrightarrow{\cong} & \mathcal{O}(k) \\
\downarrow{\scriptstyle id \otimes \eta^k} & \nearrow{\scriptstyle \gamma} & \\
\mathcal{O}(k) \otimes \mathcal{O}(1)^{\overset{k}{\otimes}} & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
u \otimes \mathcal{O}(j) & \xrightarrow{\cong} & \mathcal{O}(j) \\
\downarrow{\scriptstyle \eta \otimes id} & \nearrow{\scriptstyle \gamma} & \\
\mathcal{O}(1) \otimes \mathcal{O}(j) & &
\end{array}
$$

commute.

A *commutative operad* is an operad as above provided with an action by the symmetric group Σ_k on each $\mathcal{O}(k)$ so that the diagrams

(6.1.2)

$$
\begin{array}{ccc}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k) & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{O}(k) \otimes \mathcal{O}(j_{\sigma(1)}) \cdots \mathcal{O}(j_{\sigma(k)}) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
\mathcal{O}(j) & \xrightarrow{\sigma(j_{\sigma(1)}, \cdots, j_{\sigma(k)})} & \mathcal{O}(j)
\end{array}
$$

and

(6.1.3)

$$
\begin{array}{ccc}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k) & \xrightarrow{id \otimes \tau_1 \otimes \cdots \otimes \tau_k} & \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
\mathcal{O}(j) & \xrightarrow{\tau_1 \oplus \cdots \oplus \tau_k} & \mathcal{O}(j)
\end{array}
$$

commute. Here $\sigma \, \varepsilon \, \Sigma_k$, $\tau_s \, \varepsilon \, \Sigma_{j_s}$, the permutation $\sigma(j_{\sigma(1)}, ..., j_{\sigma(k)})$ permutes the k-blocks of letters of length $j_{\sigma(1)} \cdots j_{\sigma(k)}$ as σ permutes k letters and $\tau_1 \oplus \cdots \tau_k \, \varepsilon \, \Sigma_j$ is the block sum.

For the remaining statements, we will provide the category $C(\mathcal{S}, \mathcal{R})$ with the flat model structure discussed in section 2. An operad is an A^∞-operad (or *acyclic* operad) if *each $\mathcal{O}(k)$ is acyclic*. It is an E^∞-operad, if in addition, it is commutative and *the given action of Σ_k on $\mathcal{O}(k)$ is free*.
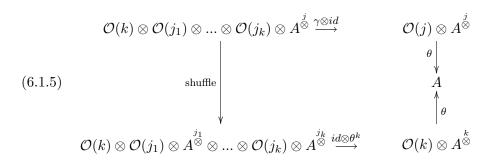
Example 6.2. The classical Eilenberg-Zilber operad. We will recall the definition of this operad briefly starting with endomorphism operads. Consider the functor $\Delta \to C(\mathcal{S}, R)$, defined by $n \mapsto C_*(\Delta[n]_{ss}, R)$, where $\Delta[n]_{ss}$ denotes the simplicial set $\{Hom_\Delta([k], [n]) | [k] \, \varepsilon \, \Delta\}$. This will be denoted $\Lambda = C_*(\Delta, R)$. (The subscript $_{ss}$ is used in $\Delta[n]_{ss}$ to distinguish this object from the scheme $\Delta[n]$ considered below.)

We have the diagonal power functor $\Lambda^{\otimes^n}$. It sends d to the n-fold $\otimes$-power $(\Lambda_d)^{\otimes^n}$. By convention, the 0-fold power of Λ is the constant functor at R. We define the *endomorphism operad $\mathcal{E}nd_\Lambda$* of the functor Λ by letting

$$(6.1.4) \qquad\qquad \mathcal{E}nd_\Lambda(n) = \mathcal{H}om_\Delta(\Lambda, \Lambda^{\otimes^n})$$

with $\mathcal{E}nd_\Lambda(0) = R$. The structure morphisms are defined as follows. $\theta_n : \mathcal{E}nd_{C_*(\Delta,R)}(n) \otimes \mathcal{E}nd_{C_*(\Delta,R)}(k_1) \otimes \cdots \otimes \mathcal{E}nd_{C_*(\Delta,R)}(k_n) \to \mathcal{E}nd_{C_*(\Delta,R)}(\Sigma_i k_i)$ is defined as the composition of $\mathcal{H}om(\Lambda, \Lambda^{\otimes^n}) \otimes \mathcal{H}om(\Lambda, \Lambda^{\otimes^{k_1}}) \otimes \cdots \otimes \mathcal{H}om(\Lambda, \Lambda^{\otimes^{k_n}}) \to \mathcal{H}om(\Lambda, \Lambda^{\otimes^n}) \otimes \mathcal{H}om(\Lambda^{\otimes^n}, \Lambda^{\otimes^{\Sigma_i k_i}}) \to \mathcal{H}om(\Lambda, \Lambda^{\otimes^{\Sigma_i k_i}})$ where both maps are the obvious ones. This operad is known to be an acyclic operad: see [H-Sch] for more details.

Definition 6.3. An algebra A over an operad $\mathcal{O}$ is an object in $C(\mathcal{S}, \mathcal{R})$ provided with maps $\theta : \mathcal{O}(j) \otimes A^{\overset{j}{\otimes}} \to A$ for all $j \geq 0$ that are associative and unital in the sense that the following diagrams commute:

$$(6.1.5)$$

$$
\begin{array}{ccc}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) \otimes A^{\overset{j}{\otimes}} \xrightarrow{\gamma \otimes id} & & \mathcal{O}(j) \otimes A^{\overset{j}{\otimes}} \\
\Big\downarrow \text{shuffle} & & \Big\downarrow \theta \\
& & A \\
& & \Big\uparrow \theta \\
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes A^{\overset{j_1}{\otimes}} \otimes ... \otimes \mathcal{O}(j_k) \otimes A^{\overset{j_k}{\otimes}} \xrightarrow{id \otimes \theta^k} & & \mathcal{O}(k) \otimes A^{\overset{k}{\otimes}}
\end{array}
$$

and

$$(6.1.6)$$

$$
\begin{array}{ccc}
u \otimes A & \xrightarrow{\simeq} & A \\
{\scriptstyle \eta \otimes id} \Big\downarrow & \nearrow {\scriptstyle \theta} & \\
\mathcal{O}(1) \otimes A & &
\end{array}
$$

If the operad is A^∞, we will refer to the algebra A as an A^∞-algebra. If A is an algebra over an operad $\mathcal{O}$ as above one defines a left A-module M to be an object in $C(\mathcal{S}, \mathcal{R})$ provided with maps $\lambda : \mathcal{O}(j) \otimes A^{j-1} \otimes M \to M$ satisfying similar associativity and unital conditions. Right-modules are defined similarly.

A commutative algebra over a commutative operad $\mathcal{O}$ is an A^∞ algebra over the operad $\mathcal{O}$ so that the following diagrams commute:

$$(6.1.7) \qquad \begin{array}{ccc} \mathcal{O}(j) \otimes A^{\overset{j}{\otimes}} & \xrightarrow{\;\;\sigma \otimes \sigma^{-1}\;\;} & \mathcal{O}(j) \otimes A^{\overset{j}{\otimes}} \\ & \searrow{\scriptstyle\theta} \qquad {\scriptstyle\theta}\swarrow & \\ & A & \end{array}$$

If, in addition, the operad is E^∞, we will refer to the algebra A as an E^∞-algebra.

A *cosimplicial algebra* $A^\bullet$ over a commutative operad $\{\mathcal{O}(n)|n\}$ is a functor $\Delta \to$ (algebras over the operad $\{\mathcal{O}(n)|n\}$), i.e. a cosimplicial object in the category of algebras over the operad $\{\mathcal{O}(n)|n\}$. Given a cosimplicial algebra $A^\bullet$, one may take its normalization $N(A^\bullet)$ to be the total complex of the double complex obtained by first normalizing $A^\bullet$ in the cosimplicial direction. (The total complex of a double complex $K = \{K^{n,m}|n,m\}$ will denote the complex $Tot(K)$ given by $Tot(K)^n = \underset{n=i+j}{\oplus} K^{i,j}$.)

Proposition 6.4. *If $A^\bullet$ is a cosimplicial algebra over the commutative operad $\{\mathcal{O}(n)|n\}$, its normalization $N(A^\bullet)$ is a an algebra over the operad $\{\mathcal{O}(n)|n\} \otimes \mathcal{E}nd_{C_*(\Delta)}$.*

Proof. This is a straightforward extension of [H-Sch, (2.3) Theorem]. One first shows that the double complex obtained by normalizing $A^\bullet$ in the cosimplicial direction is an algebra over the operad $\{\{\mathcal{O}^i(n) \otimes \mathcal{E}nd^j_{C_*(\Delta)}(n)|i,j\}|n\}$ in the category of double complexes. Now one takes the associated double complexes. (See [H-Sch, proof of (2.3) Theorem] for further details.) $\qquad\square$

Definition 6.5. Recall we have assumed that the site $\mathcal{S}$ has *enough points*. Therefore one may define *the Godement resolution* as a cosimplicial resolution of a any object $K \,\varepsilon\, C(\mathcal{S}, \mathcal{R})$: see [J-2], for example. (i) We say $K \,\varepsilon\, C(\mathcal{S}, \mathcal{R})$ has cohomological descent on the site $\mathcal{S}$, if for each V in $\mathcal{S}$, the natural map $\Gamma(V, K) \to \underset{\Delta}{\mathrm{holim}}\{\Gamma(V, G^n K)|n\}$ is a weak-equivalence.

(ii) Let $\mathcal{S}$ denote the Zariski site of a Noetherian scheme X. We say that a $K \,\varepsilon\, C(X_{Zar})$ has the Mayer-Vietoris property on X, if for any two Zariski open sub-schemes U, V of X, the diagram

$$\Gamma(U \cup V, K) \to \Gamma(U, K) \oplus \Gamma(V, K) \to \Gamma(U \cap V, K) \to \Gamma(U \cup V, K[1])$$

is a distinguished triangle of complexes of abelian groups.

The following is a well-known result: see for example, [T].

Proposition 6.6. *Suppose X is a Noetherian scheme and $K \,\varepsilon\, C(X_{Zar})$ having the Mayer-Vietoris property. Then K has cohomological descent on the Zariski site of X.*

In general, for a site $\mathcal{S}$ with enough points and $K \,\varepsilon\, C(\mathcal{S}, \mathcal{R})$ we let $R\Gamma(V, K) = \underset{\Delta}{\mathrm{holim}}\{\Gamma(V, G^n K)|n\}$.

6.2. Next assume that one of the following hypotheses holds:

(i) the site $\mathcal{S}$ has finite uniform cohomological dimension for all sheaves of the form $\mathcal{H}^i(K)$ for each fixed $K \,\varepsilon\, C(\mathcal{S}, \mathcal{R})$, i.e. there exists an integer $N >> 0$, so that $H^n(V, \mathcal{H}^i(K)) = 0$ for all $n > N$, all V in the site $\mathcal{S}$ and all i.

or (ii) K is a bounded complex.

In this case one may also define $R\Gamma(V, K) = Tot(N(\{\Gamma(V, G^n K)|n\}))$, where N denotes normalizing in the cosimplicial direction and $Tot(K^{\bullet,\bullet})^n = \underset{i+j=n}{\oplus} K^{i,j}$ for a double complex $K^{\bullet,\bullet} \,\varepsilon\, C(\mathcal{S})$).

(Observe that so defined, there is a strongly convergent spectral sequence $\mathrm{E}_2^{s,t} = R^s\Gamma(V, \mathcal{H}^t(K)) \Rightarrow R^{s+t}\Gamma(V, K)$.)

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