Comparison of Motivic and simplicial operations in mod-l-motivic and étale cohomology

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Abstract. In this paper we explore the relationships between the motivic and simplicial cohomology defined on mod-l motivic cohomology. We also explore similar relationships in étale cohomology and conclude by considering certain operations that commute with proper push-forwards.

1. Introduction

Throughout the paper k will denote a fixed perfect field of characteristic $p \geq 0$. We will restrict to the category, Sm/k, of smooth separated schemes of finite type over k. If X denotes such a scheme, $H^n_{\mathcal{M}}(X, \mathbb{Z}(r))$ denotes the motivic cohomology with *degree* n and *weight* r and $H^n_{\mathcal{M}}(X, \mathbb{Z}/l(r))$ denotes the corresponding mod-l-variant. In [**Voev1**], Voevodsky defined a sequence of operations on motivic cohomology analogous to the operations defined by Steenrod on the singular cohomology of topological spaces. We call Voevodsky's operations *motivic operations*. They have the form

$$P^r_{\mathcal{M}} : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1))) \text{ and}$$
$$\beta P^r_{\mathcal{M}} : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1))), \tag{1.0.1}$$

where l is a prime not equal to p.

When $i = 2j$, the above motivic cohomology groups identify with the usual mod-l Chow groups. In [**Bros**], there is a construction of Steenrod operations on mod-l Chow groups, which is independent of Voevodsky's (and somewhat simpler).

Voevodsky's operations are defined by making use of a geometric model for the classifying spaces of finite groups. By using a simplicial model for these classifying spaces, one obtains another sequence of operations in mod-l motivic cohomology, which we call *simplicial* and which are defined even if $l = p$. These operations, which were originally introduced by I. Kriz and J.P. May in [**Kr-May**], have the

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form:

$$(1.0.2) \qquad \begin{aligned} P_s^r &: H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(jl)) \text{ and} \\ \beta P_s^r &: H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(jl)). \end{aligned}$$

The main goal of this paper is to explore the relationship between these two types of operations in mod-l motivic cohomology. In doing so, we elaborate on the results in an earlier preprint by the authors studying these relations after inverting the Bott-element.

The comparison of the total power operations, yields a straightforward comparison between the motivic and simplicial operations for classes with degree equal to twice the weight. By observing that both the simplical and motivic operations are stable with respect to suspension in the degree, we can then deduce the comparison between classes with degree $\leq$ twice the weight. The case when the degree is strictly greater than twice the weight is more involved and makes use of the Cartan formula. The main comparison results may be summarized as follows.

Theorem 1.1. *(See section 6 for more details.) Assume the base field has a primitive l-th root of unity and let $B \in H^0_{\mathcal{M}}(\mathrm{Spec}\, k, \mathbb{Z}/l(1))$ denote the motivic Bott element. Let F denote a pointed simplicial sheaf on $\mathrm{Sm}/k_{\mathrm{Nis}}$.*

(i) *Let $\alpha \in \tilde{H}^i_{\mathcal{M}}(F, \mathbb{Z}/(q)$ for any $i \leq 2q$. Then $P_s^r(\alpha) = B^{(q-r)(l-1)} P^r_{\mathcal{M}}(\alpha)$, $\beta P_s^r(\alpha) = B^{(q-r)(l-1)} \beta P^r_{\mathcal{M}}(\alpha)$.*

(ii) *Let $\alpha \in \tilde{H}^{2q+t}_{\mathcal{M}}(F, \mathbb{Z}/l(q))$, with $t > 0$.*

If $t = 2t'$ for some integer t', then

$$\begin{aligned} B^{t'l} P_s^r(x) &= B^{(q+t'-r)(l-1)} . B^{t'} P^r_{\mathcal{M}}(x) \quad \textit{and} \\ B^{t'l} \beta P_s^r(x) &= B^{(q+t'-r)(l-1)} . B^{t'} \beta P^r_{\mathcal{M}}(x), \quad 0 \leq r \leq q + t'. \end{aligned}$$

If $t = 2t' + 1$ for some integer t', then

$$\begin{aligned} B^{(t'+1)l} P_s^r(x) &= B^{(q+t'+1-r)(l-1)} . B^{t'+1} P^r_{\mathcal{M}}(x) \ \textit{and} \\ B^{(t'+1)l} \beta P_s^r(x) &= B^{(q+t'+1-r)(l-1)} . B^{t'+1} \beta P^r_{\mathcal{M}}(x), \quad 0 \leq r \leq q + t' + 1. \end{aligned}$$

(iii) *Both the motivic and simplicial operations extend to operations on étale cohomology with respect to the sheaf μ_l. If F denotes a pointed simplicial sheaf on $\mathrm{Sm}/k_{\acute{e}t}$ and $\alpha \in \tilde{H}^i_{\acute{e}t}(F, \mu_l(q))$, for any $i \geq 0$, then*

$$P_s^r(\alpha) = B^{(q-r)(l-1)} P^r_{\mathcal{M}}(\alpha), \quad \beta P_s^r(\alpha) = B^{(q-r)(l-1)} \beta P^r_{\mathcal{M}}(\alpha).$$

B. Guillou and C. Weibel have obtained a similar comparison of motivic and simplicial operations in section 5 of [**GW**] using results of Voevodsky on the motivic cohomology of Eilenberg-MacLane spaces: the results in [**GW**, sections 5, 6 and 7] seem to complement some of the results in this paper. Loosely speaking, our proof of Theorem 1.1 works by comparing the motivic classifying spaces of symmetric and cyclic groups with the simplicial ones. On the other hand, Guillou and Weibel use theorems of Voevodsky to show that, in appropriate bi-degrees, the group of operations on motivic cohomology injects into the group of operations on étale cohomology. (See [**GW**, Proposition 5.8].)

One may consult Examples 6.5 for various examples of the relations in Theorem 1.1.

Outline. We begin section 2 by reviewing quickly the cohomology of the classifying spaces of finite groups, using both the geometric and simplicial models for the classifying spaces. We discuss the total power operations in detail in the next two sections. First we recall the total power operations defined by Voevodsky for defining the motivic operations. We show that this may be modified to define total power operations for the simplicial operations, at least for classes whose degree is less than or equal to twice their weight. Both of these are first defined for algebraic cycles whose degree is twice their weight. Since the motivic operations are stable with respect to suspension in both the degree and the weight this suffices to define the motivic operations for all classes. However, since the simplicial operations are stable with respect to suspension in only the degree, the above total power operations do not define simplicial operations except for classes with degree less than or equal to twice their weight. Therefore we define total power operations in a different manner to be able to define simplicial operations without the above restriction and then show that these new total power operations agree with the ones defined above for classes with degree equal to twice their weight. This is carried out in detail in section 4.

At this point, the usual relations among the simplicial operations, like the Cartan formulae and Adem relations are not obvious. The quickest approach to establishing these for the simplicial operations is to show that the simplicial operations defined here identify with the operations defined operadically as in [**J1**, section 5] (making use of [**May**]), where such relations are known to hold. We prove this in section 5. The next section contains the comparison theorem relating the motivic operations with the simplicial ones.

We explore some applications of the above results in the last section. Motivic operations that commute with proper push-forward have played a major role in various degree formulae. Therefore, with a view towards such applications, we construct simplicial cohomology operations that commute with proper push-forwards and work out several examples of such push-forward formulae. The existence of mod$-p$ cohomology operations in characteristic p is not known, however, the simplicial operations exist and have expected properties even in this case. Therefore, as an application, we consider simplicial operations in mod$-p$ motivic cohomology in characteristic p, which identify with the cohomology with respect to the logarithmic de-Rham Witt complex.

Another role of the simplicial operations is the following phenomenon, which the first author first learned about in conversations with C. Weibel. Though the simplicial operations may be obtained from the motivic ones when considering the motivic cohomology of smooth schemes, this is no longer true for the motivic cohomology of general simplicial schemes. (See Examples 6.5.) Therefore, it is likely that the simplicial operations will play a non-trivial role in understanding the motivic cohomology of simplicial schemes and simplicial presheaves.

Conventions. We restrict to smooth separated schemes of finite type over a field k and l will be a fixed prime. Usually this will be assumed to be different from the characteristic of k, and k will be assumed to be provided with a primitive l-th root of unity, though such hypotheses are not required to define the simplicial operations. We will also consider simplicial schemes $X_\bullet$ over such a base field, where each X_n will be assumed to be a smooth scheme of finite type over k. The symbol $\mathrm{Sm/k_{Nis}}$ (resp. $\mathrm{Sm/k}_{\acute{e}t}$) will denote the category Sm/k provided with the big Nisnevich (resp. étale) topology. If C denotes any one of these sites, SSh(C) will denote the category of all pointed simplicial sheaves on C. The symbol $\mathcal{H}$C will denote the corresponding homotopy category obtained by inverting all $\mathbb{A}^1$-weak-equivalences. Any pointed simplicial scheme $X_\bullet$, with each $X_n \in$ (Sm/k), as well as any pointed simplicial set will be viewed as an object in each of the categories SSh(Sm/$\mathrm{k_{Nis}}$) and SSh(Sm/$\mathrm{k}_{\acute{e}t}$) in the obvious manner.

The mod-l motivic complex of weight i will be denoted $\mathbb{Z}/l(i)$. This should be distinguished from the integers mod-l, which will be denoted $\mathbb{F}_l$. The symbol $H^*_{\mathcal{M}}$ will denote cohomology with respect to the motivic or mod-l motivic complex computed on the Nisnevich site; $H^*_{\acute{e}t}$ will denote cohomology computed on the étale site. Often, when certain computations hold in any of these cases, we will simply use H^* to denote cohomology computed on any of these sites.

2. Cohomology of the classifying space for a finite group

2.1. Geometric classifying spaces. We begin by recalling briefly the construction of the *geometric classifying space of a linear algebraic group*, which is originally due to Totaro [**Tot**]. Let G be a linear algebraic group over $S = \mathrm{Spec}\, k$, i.e., a closed subgroup-scheme in $\mathbf{GL}_n$ over S for some n. For a faithful representation $i : G \to \mathbf{GL}_n$, the *geometric classifying space* $B_{gm}(G; i)$ of G with respect to i is defined as follows. For $m \geq 1$ let U_m be the open sub-scheme of $\mathbb{A}^{nm}$ where the diagonal action of G determined by i is free. Let $V_m = U_m/G$ be the quotient S-algebraic space of the action of G on U_m induced by the (diagonal) action of G on $\mathbb{A}^{nm}$. Since the action of G on U_m is free, the quotient V_m is smooth. We have closed embeddings $U_m \to U_{m+1}$ and $V_m \to V_{m+1}$ corresponding to the embeddings $(Id \times \{0\})^n : \mathbb{A}^{mn} \to (\mathbb{A}^{m+1})^n$ and we set $EG^{gm} = \lim\limits_{m\to\infty} U_m$ and $BG^{gm} = \lim\limits_{m\to\infty} V_m$ where the colimit is taken in the category of sheaves on $(\mathrm{Sm/k})_{\mathrm{Nis}}$ or on $(\mathrm{Sm/k})_{\acute{e}t}$. Observe that if $G = \Sigma_n$ (or a subgroup of it) acting on $\mathbb{A}^n$ by permuting the n-coordinates and acting on $\mathbb{A}^{nm}$ diagonally, we may take $U_m = \{(u_1, \cdots, u_n) | u_i \in \mathbb{A}^m, u_i \neq u_j, i \neq j\}$.

2.2. Affine replacement via the the Jouanolou trick. In general, U_m is only quasi-projective, but by the Jouanolou trick, one may replace U_m by an affine scheme with a free action by G, which maps G-equivariantly onto U_m and which is an affine-space bundle over U_m. (See, for example, [**Voev1**, p. 15].) Therefore, we will henceforth denote the above affine replacement of U_m by U_m itself. Then [**Nag**, Theorem 2, p.7] shows that V_m (which is the geometric quotient of U_m by G) is also affine. (These observations will be rather important for the construction of the total power operations constructed below.)

2.3. The equivariant motivic (étale cohomology) of a scheme X with an action by Σ_n will be defined to be $H^*_{\mathcal{M}}(E\Sigma_n^{gm} \underset{\Sigma_n}{\times} X, \mathbb{Z}/l(\star))$ $(H^*_{\acute{e}t}(E\Sigma_n^{gm} \underset{\Sigma_n}{\times} X, \mathbb{Z}/l(\star))$, respectively). The results in [**MV**, section 4] show that one may also define equivariant étale cohomology using the simplicial construction for $E\Sigma_n$.

2.4. Motivic cohomology of the geometric classifying space. We recall the computation of the reduced equivariant motivic cohomology of F, where F is any pointed simplicial sheaf on $\mathrm{Sm}/k)_{\mathrm{Nis}}$ and l is a prime different from $char(k) = p$. (For example, $F = X_+$ where X is a given scheme with trivial action by Σ_l). (See [**Voev1**, Section 6].) The following result is Voevodsky's [**Voev1**, Proposition 6.16].

Theorem 2.5 (Voevodsky). *Let F be a pointed simplicial sheaf. Then $\tilde{H}^*_{\mathcal{M}}(F \wedge (B\Sigma_l^{gm})_+; \mathbb{Z}/l(\star))$ is a free module over $\tilde{H}^*_{\mathcal{M}}(F; \mathbb{Z}/l(\star))$with a basis $\{c\bar{d}^i, d^i | i \geq 0\}$ where*

$$\bar{d} \text{ is a class in } \tilde{H}^{2l-2}_{\mathcal{M}}((B\Sigma_l^{gm})_+; \mathbb{Z}/l(l-1))$$

which is the mod-l reduction of a class $d \in \tilde{H}^{2l-2}_{\mathcal{M}}((B\Sigma_l^{gm})_+; \mathbb{Z}(l-1))$ and

$$c \text{ is a class in } \tilde{H}^{2l-3}_{\mathcal{M}}((B\Sigma_l^{gm})_+; \mathbb{Z}/l(l-1)) \text{ so that } \delta(c) = \bar{d}.$$

Let *cycl* denote the cycle map from motivic cohomology to étale cohomology. (By identifying the motivic cohomology with the higher Chow groups, these cycle maps identify with those defined in [**Bl**].) Now one may observe that the same computation as above holds in étale cohomology with the classes c and d replaced by their images under the above cycle map.

2.6. Motivic cohomology of the simplicial classifying space. For any discrete group G, we let EG and BG denote the standard simplicial sets (with $EG_n = G^{n+1}$ and $BG_n = G^n$). These (and other) simplicial sets give rise to simplicial sheaves in the obvious way, and we use the same notation for the simplicial set as for the corresponding simplicial sheaf.

Theorem 2.7. *Suppose F is a pointed simplicial sheaf on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$. Then $\tilde{H}^*_{\mathcal{M}}(F \wedge B\Sigma_{l,+}; \mathbb{Z}/l(\star))$ is a free module over $\tilde{H}^*_{\mathcal{M}}(F; \mathbb{Z}/l(\star))$ with a basis $\{x\bar{y}^i, \bar{y}^i | i \geq 0\}$ where $\bar{y}$ is a class in $\tilde{H}^{2l-2}_{\mathcal{M}}(B\Sigma_{l,+}; \mathbb{Z}/l(0))$ which is the mod-l reduction of a class $y \in \tilde{H}^{2l-2}_{\mathcal{M}}(B\Sigma_{l+}; \mathbb{Z}(0))$ and x is a class in $\tilde{H}^{2l-3}_{\mathcal{M}}(B\Sigma_{l+}; \mathbb{Z}/l(0))$ so that $\delta(x) = \bar{y}$.*

Proof. This follows from the Künneth formula of Dugger and Isaksen [**DI**, Theorem 8.6 and Remark 8.7] and the (easy) fact that

$$H^*_{\mathcal{M}}(B\Sigma_l; \mathbb{Z}/l(\star)) = H^*(\Sigma_l, \mathbb{Z}/l) \otimes H^*_{\mathcal{M}}(k, \mathbb{Z}/l(\star)),$$

where $H^*(\Sigma_l, \mathbb{Z}/l)$ denotes the group cohomology of Σ_l. □

Remark 2.8. One may observe that the main difference between the computations in 2.5 and in 2.7 is that the classes x, y and $\bar{y}$ have weight 0. Also it is important to observe the computations in 2.7 hold even if $l = char(k) = p$.

One may replace F above with a pointed simplicial sheaf on $(\mathrm{Sm}/k)_{ét}$ and the above cohomology with $H^*_{ét}$ to obtain:

2.9. Étale cohomology. $\tilde{H}^*_{ét}(F \wedge B\Sigma_{l,+}; \mathbb{Z}/l(\star))$ is a free module over $\tilde{H}^*_{ét}(F; \mathbb{Z}/l(\star))$ with a basis $\{x\bar{y}^i, \bar{y}^i | i \geq 0\}$ where $\bar{y}$ is a class in $\tilde{H}^{2l-2}_{ét}(B\Sigma_{l+}; \mathbb{Z}/l(0))$ and x is a class in $\tilde{H}^{2l-3}_{ét}(B\Sigma_{l+}; \mathbb{Z}/l(0))$ so that $\delta(x) = \bar{y}$.

Remarks 2.10. We add some remarks here concerning the computation of motivic cohomology in Voevodsky's Theorem 2.5. First we consider the question of stability, i.e. the effect on the above computations when $B\Sigma_l^{gm}$ ($B\Sigma_l$) is replaced by U_N/Σ_l ($E\Sigma_l \underset{\Sigma_l}{\times} U_N$) for N large. Key results here are [**Voev1**, Lemma 3.5 and Proposition 6.1]. These first of all show the following: for any smooth scheme X of finite type over k, the maps

(2.10.1) $\mathbb{H}_{Nis}(X_+ \wedge (U_{N+1}/\Sigma_l)_+, \mathbb{Z}/l(q)) \to \mathbb{H}_{Nis}(X_+ \wedge (U_N/\Sigma_l)_+, \mathbb{Z}/l(q))$ and

$$\mathbb{H}_{Nis}(X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_{N+1})_{+,n}, \mathbb{Z}/l(q)) \to \mathbb{H}_{Nis}(X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_N)_{+,n}, \mathbb{Z}/l(q))$$

are quasi-isomorphisms for all n, provided $N > q$. $\mathbb{H}_{Nis}$ denotes a hypercohomology complex, i.e. $RHom(X_+ \wedge (U_{N+1}/\Sigma_l)_+, \mathbb{Z}/l(q))$ in the first case and $RHom(X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_{N+1})_n, \mathbb{Z}/l(q))$ in the second case both computed on the Nisnevich site. If F is a pointed simplicial sheaf, and $X_{\bullet,+} \to F$ is a resolution, then the same quasi-isomorphisms hold on taking the homotopy inverse limit over the $X_{m,+}$ in the first line and on taking the homotopy inverse limit over the $X_{m,+}$ and the n in the second line. (If one prefers, one may replace the motivic complex $\mathbb{Z}/l(q)$ by a corresponding sheaf of abelian group spectra so that one may work in the category of spectra, where homotopy inverse limits are more familiar.) Therefore, one obtains the isomorphisms (observe the $\lim^1$-terms vanish):

$$(2.10.2) \qquad \tilde{H}^*_{\mathcal{M}}(F \wedge B\Sigma^{gm}_{l,+}, \mathbb{Z}/l(\star)) \cong \lim_{\infty \leftarrow N} \tilde{H}^*_{\mathcal{M}}(F \wedge (U_N/\Sigma_l)_+, \mathbb{Z}/l(\star)) \text{ and}$$
$$\tilde{H}^*_{\mathcal{M}}(F \wedge B\Sigma_{l,+}, \mathbb{Z}/l(\star)) \cong \lim_{\infty \leftarrow N} \tilde{H}^*_{\mathcal{M}}(F \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_N/\Sigma_l)_+, \mathbb{Z}/l(\star)).$$

Now the computation in 2.5 follows by first doing a similar computation for Σ_l replaced by μ_l and adopting a transfer argument as shown in [**Voev1**, Theorems 6.10 and 6.16].

3. The total power operations: I

A key step in the comparison between the motivic and simplicial cohomology operations is a thorough understanding of the total power operation. We proceed to discuss this in detail. *We will assume throughout this section that l is a prime different from $char(k) = p$.*

If one only considers the case $i = 2j$, then $\tilde{H}^{2j}(X_+, \mathbb{Z}/l(j))$, for a smooth scheme X, identifies with the usual Chow groups of X reduced mod-l. Then, as a first approximation, one could define the total power operation by simply sending a class

$$\alpha \in \tilde{H}^{2j}(X_+, \mathbb{Z}/l(j)) \mapsto \alpha^l$$

which defines a class in $\tilde{H}^{2jl}(B\Sigma^{gm}_{l,+} \wedge X_+, \mathbb{Z}/l(jl))$.

In order to be able to extend this total power operation as a natural transformation

$$\tilde{P} : \tilde{H}^{2j}(\quad ; \mathbb{Z}/l(j)) \to \tilde{H}^{2jl}(\quad \wedge B\Sigma^{gm}_{l+}, \mathbb{Z}/l(jl))$$

defined on all pointed simplicial sheaves on the big Zariski, Nisnevich or étale site over k, one needs to adopt the construction in [**Voev1**, section 5]. We will adopt this suitably modified to also define a version of total power operations when $B\Sigma^{gm}_l$ is replaced by the simplicial model $B\Sigma_l$.

Next recall the following. An augmented simplicial object $X_\bullet$ in a category C consists of a simplicial object $Y_\bullet$ in C with $Y_i = X_i, i \geq 0$ together with an object $X_{-1} \in$ C and an augmentation $\epsilon : Y_0 \to X_{-1}$ so that $d_0 \circ \epsilon = d_1 \circ \epsilon$, $i = 0, 1$.

Let $X_\bullet$ denote an augmented simplicial scheme with each X_i, $i \geq -1$, an scheme. Let $k[X_\bullet] = \{k[X_n] | n\}$ denote the corresponding co-ordinate ring. A *finitely generated $k[X_\bullet]$-module* is given by a collection $\{M_n|n\}$ where each M_n is a finitely generated $k[X_n]$-module and provided with a compatible collection of maps $\{\phi_\alpha : \alpha^*(M_n) \to M_m\}$ for each structure map $\alpha : X_m \to X_n$ of $X_\bullet$. $M_\bullet$ will be called *a finitely generated projective module (or a vector bundle on $X_\bullet$) (a finitely generated free module (or a trivial vector bundle))* if each M_m is a finitely generated projective (free, respectively) $k[X_m]$-module and each of the structure maps ϕ_α is an isomorphism of $k[X_m]$-modules.

Proposition 3.1. *Let $X_\bullet$ denote an augmented simplicial scheme so that each X_i, $i \geq -1$, is affine. If $M_\bullet = \{M_m | m\}$ is a finitely generated module on $X_\bullet$ which is the pull-back of a finitely generated $k[X_{-1}]$-module, then there exists a finitely generated free module $F_\bullet$ on $X_\bullet$ and a map $\phi : F_\bullet \to M_\bullet$ which is an epimorphism in each degree. In case $M_\bullet$ is the pull-back of a finitely generated projective $k[X_{-1}]$-module, one may also find a finitely generated projective $k[X_\bullet]$-module $N_\bullet$ so that $M_m \oplus N_n \cong F_n$ for all n and where the last isomorphism is compatible with the structure maps of the augmented simplicial scheme.*

Proof. Since $M_\bullet$ is the pull-back of a finitely generated $k[X_{-1}]$-module, it suffices to prove the first statement when the augmented simplicial scheme $X_\bullet$ has been replaced by the affine scheme X_{-1}. This is then clear since X_{-1} is affine. If N_{-1} is the kernel of the surjection, then $M_{-1} \oplus N_{-1} \cong F_{-1}$. This isomorphism pulls-back to a similar isomorphism $M_n \oplus N_n \cong F_n$ for each n and compatible with the structure maps of the augmented simplicial scheme $X_\bullet$. □

The following results relate the geometric classifying space $B\Sigma_l^{gm}$ with the simplicial classifying space $B\Sigma_l$.

Proposition 3.2. *Let U_N denote the open subscheme*

$$\{(u_1, \cdots, u_l) | u_i \in \mathbb{A}^N, u_j \neq u_k, \quad j \neq k\} \text{ of } \mathbb{A}^{Nl}.$$

For any fixed integer $N > 0$, let $gs_N : E\Sigma_l \underset{\Sigma_l}{\times} U_N \to E\Sigma_l \underset{\Sigma_l}{\times} \operatorname{Spec} k = B\Sigma_l$ denote the obvious map of simplicial schemes. This map induces a weak-equivalence in $\mathcal{H}\mathrm{Sm/k}_{\mathrm{Nis}}$ (and also in $\mathcal{H}\mathrm{Sm/k}_{\acute{e}t}$) on taking the colimit as $N \to \infty$. (The above homotopy categories are the homotopy categories of simplicial sheaves on the appropriate sites with $\mathbb{A}^1$-inverted.)

Proof. By Proposition 2.3 on page 134 of [**MV**], the colimit $U_\infty := \operatorname{colim}_N U_N$ in the category of Nisnevich (or étale) sheaves is $\mathbb{A}^1$-homotopy equivalent to $\operatorname{Spec} k$. Therefore $\operatorname{colim}_N E\Sigma_l \underset{\Sigma_l}{\times} U_N$ is $\mathbb{A}^1$-homotopy equivalent to $E\Sigma_l \underset{\Sigma_l}{\times} \operatorname{Spec} k = B\Sigma_l$. □

In view of the above proposition, we may approximate $B\Sigma_l$ by $E\Sigma_l \underset{\Sigma_l}{\times} U_N$ by taking N high enough. We will let U denote U_N and V denote V_N for a large N.

Observe that one has an obvious augmentation

$$E\Sigma_l \underset{\Sigma_l}{\times} U_N \to U_N/\Sigma_l \tag{3.2.1}$$

One may view this diagram as an augmented simplicial scheme. Observe that the schemes U_N and $V_N = U_N/\Sigma_l$ may be replaced equivariantly by affine schemes as shown in 2.2, so that Proposition 3.1 applies.

3.2.2. *A key construction used in the (motivic) total power construction.* Recall from [**Voev1**, Theorem 2.1] that the functor $X \to H^{2n}(X, \mathbb{Z}/l(n))$ is represented by the sheaf (in the $\mathbb{A}^1$-localized homotopy category of complexes of sheaves on the Nisnevich topology): $U \mapsto \mathbb{F}_{ltr}(\mathbb{A}^n)(U)/\mathbb{F}_{ltr}(\mathbb{A}^n - \{0\})(U), U \in (\mathrm{Sm/k})_{\mathrm{Nis}}$. We will denote the *presheaf* $U \mapsto \mathbb{F}_{ltr}(\mathbb{A}^n)(U)/\mathbb{F}_{ltr}(\mathbb{A}^n - \{0\})(U), U \in (\mathrm{Sm/k})_{\mathrm{Nis}}$ by $\mathrm{K}(n)^{pre}$ and the corresponding sheaf by $\mathrm{K}(n)$.

Let X denote a scheme in (Sm/k) and E, L vector bundles on X provided with an isomorphism $\phi : E \times_X L \to \mathbb{A}^N_X$ which is the N-dimensional trivial bundle on X. Given a cycle Z on E with coefficients in $\mathbb{F}_l$ and finite over X, we consider the cycle on $L \times_X E \times_X L$ whose fiber over a point (x, l) of L is (l, Z_x, l), where Z_x denotes the fiber of Z over $x \in X$. It is observed in [**Voev1**, Construction 5.1] that this is a cycle finite over L and that by identifying $E \times_X L$ with $\mathbb{A}^N_X$ using the isomorphism ϕ, one obtains a map of pointed sheaves $Th(L) \to \mathbb{F}_{ltr}(\mathbb{A}^N)/\mathbb{F}_{ltr}(\mathbb{A}^N - \{0\})$ (where $Th(L)$ denotes the Thom-space of L), i.e. a class in $\tilde{H}^{2N}(Th(L), \mathbb{Z}/l(N))$.

Remark 3.3. The main point of this construction is that the vector bundle E is usually a *non-trivial* vector bundle, and X will be an affine scheme so that one can find a complementary vector bundle L so that $E \oplus L$ is a trivial vector bundle. See 3.3.4 below where $\bar{E}$ plays the role of E above with the scheme X being $V = U/\Sigma_l$. In case the vector bundle E was trivial, one could take L to be zero-dimensional, i.e. $Th(L) = X_+$.

3.3.1. We will denote the cycle constructed above in 3.2.2 $\Delta_*(q^*(Z))$, where $q : E \underset{X}{\times} L \to E$ is the obvious projection and $\Delta : E \underset{X}{\times} L \to L \underset{X}{\times} E \underset{X}{\times} L$ is the diagonal.

Moreover, making use of the Thom-isomorphism,

$$\tilde{H}^{2rk(E)}(X_+, \mathbb{Z}/l(rk(E))) \cong \tilde{H}^{2N}(Th(L), \mathbb{Z}/l(N)),$$

one observes that this defines a class in $\tilde{H}^{2rk(E)}(X_+, \mathbb{Z}/l(dim(E)))$ which is denoted $a(Z)$ and shown to be independent of the choice of the isomorphism ϕ and the vector bundle L: see [**Voev1**, Construction 5.1]. (Here $rk(E)$ denotes the rank of E as a vector bundle.)

3.3.2. *An avatar of the motivic Thom-class for trivial bundles.* For later use, we make the following observation. The Thom class of the trivial bundle $\mathbb{A}^c$ on $\operatorname{Spec} k$ corresponds to the class of the identity in $Hom(\mathbb{F}_{ltr}(\mathbb{A}^c)/\mathbb{F}_{ltr}(\mathbb{A}^c - 0), \mathbb{F}_{ltr}(\mathbb{A}^c)/\mathbb{F}_{ltr}(\mathbb{A}^c - 0))$. In fact, this corresponds to the class of the diagonal $\Delta \subseteq \mathbb{A}^c \times \mathbb{A}^c$, which is a correspondence that is finite for the projection to the first factor. We will denote this class by $\Delta_*(1)$. Next let E denote a trivial vector bundle of rank c on the smooth scheme X. Then $X \times \Delta_{\mathbb{A}^c}$ defines a correspondence on $X \times \mathbb{A}^c \times \mathbb{A}^c = E \times \mathbb{A}^c$ whose projection to $E = X \times \mathbb{A}^c$ is finite. This defines the Thom-class, t_E, for E in $H^{2c}(Th(E), \mathbb{F}_{ltr}(\mathbb{A}^c)/\mathbb{F}_{ltr}(\mathbb{A}^c - 0))$ and will be denoted $\Delta_*(q^*(1))$ where $q : E = X \times \mathbb{A}^c \to X$ is the obvious projection.

Next let L denote a vector bundle on the smooth scheme X. Then correspondences C on $L \times \mathbb{A}^n$ whose projection to L is finite and so that C restricted to $(L - X) \times \mathbb{A}^n$ in fact is contained in $(L - X) \times (\mathbb{A}^n - 0)$ define classes in $Hom(Th(L), \mathbb{F}_{ltr}(\mathbb{A}^n)/\mathbb{F}_{ltr}(\mathbb{A}^n - 0))$ and hence in $H^{2n}(Th(L), \mathbb{Z}/l(n))$. Then cup-product with the Thom-class $t_E \cup [C]$ may be identified with the class $\Delta_*(q_E^*(C)) = \{(e, C, e) | e \in \mathbb{A}^c\}$ where $q_E : (L \oplus E) \times \mathbb{A}^n \to L \times \mathbb{A}^n$ is the obvious projection and Δ is the diagonal $L \times \mathbb{A}^c \times \mathbb{A}^n = (L \oplus E) \times \mathbb{A}^n \to (L \oplus E \oplus E) \times \mathbb{A}^n = L \times \mathbb{A}^c \times \mathbb{A}^c \times \mathbb{A}^n$.

Observe from 3.2.2, that one obtains the isomorphism:

$$\tilde{H}^{2i}(X_+, \mathbb{Z}/l(i)) \cong \tilde{H}^0(X_+, C_*(\mathrm{K}(i)))$$

3.3.3. *Construction of the total power operation.* We will start with a section of the presheaf $\mathrm{K}^{pre}(i)$, i.e. a cycle Z on $X \times \mathbb{A}^i$ finite over X. Let Z^l denote the l-th external power of Z: this is now a cycle on $(X \times \mathbb{A}^i)^l$. We will let $p^*(Z^l)$ denote its pull-back to $(X \times \mathbb{A}^i)^l \times U$, where $U = U_N$ for some suitably large N. Since Σ_l

acts freely on U, one may observe that the cycle $p^*(Z^l)$ descends to a unique cycle Z' on $((X \times \mathbb{A}^i)^l \times U)/\Sigma_l$ equi-dimensional and finite over $(X^l \times U)/\Sigma_l$. On pulling back by the diagonal, $\Delta : X \to X^l$, one obtains the cycle Z'' on $X \times (\mathbb{A}^{il} \times U)/\Sigma_l$.

3.3.4. Now $\bar{E} = (\mathbb{A}^l \times U)/\Sigma_l$ is a vector bundle on $V = U/\Sigma_l$ (where Σ_l acts diagonally) and V is affine: see 2.2. Moreover, recall that we have the augmented simplicial schemes: $E\Sigma_l \underset{\Sigma_l}{\times} (\mathbb{A}^l \times U) \to (\mathbb{A}^l \times U)/\Sigma_l$ and $E\Sigma_l \underset{\Sigma_l}{\times} U \to U/\Sigma_l$.

The pull-back of the vector bundle $\bar{E}$ to the simplicial scheme $E\Sigma_l \underset{\Sigma_l}{\times} U$ defines a vector bundle we will denote by $\hat{E}$. Since the pull-back of $\bar{E}$ to U is a trivial vector bundle, it follows that $\hat{E}$ is also a trivial bundle, i.e. trivial on restriction to each $(E\Sigma_l \underset{\Sigma_l}{\times} U)_n$. (We define a vector bundle on a simplicial scheme $X_\bullet$ to be trivial, if its restriction to each X_n is a trivial vector bundle.) By invoking Proposition 3.1, one may find a vector bundle $\bar{L}$ on $V = U/\Sigma_l$ so that $\bar{E} \times_V \bar{L} \cong \mathbb{A}^N_V$ for some N. Therefore, the pull-back $\hat{L}$ of $\bar{L}$ to $E\Sigma_l \underset{\Sigma_l}{\times} U$ also has the property that $\hat{E} \times_{E\Sigma_l \underset{\Sigma_l}{\times} U} \hat{L}$ is a trivial bundle of rank N.

Next we let $\tilde{E} = X \times \bar{E}$, $\tilde{L} = X \times \bar{L}$ denote the pull-backs of $\bar{E}$ and $\bar{L}$ to $X \times V$. We also let $E = X \times \hat{E}$, $L = X \times \hat{L}$ denote the corresponding vector bundles on the simplicial scheme $X \times (E\Sigma_l \underset{\Sigma_l}{\times} U)$. Then $E \times_{X \times (E\Sigma_l \underset{\Sigma_l}{\times} U)} L$ is a trivial bundle of rank N on the simplicial scheme $X \times (E\Sigma_l \underset{\Sigma_l}{\times} U)$ and $\tilde{E} \times_X \tilde{L}$ is a trivial bundle of rank N on $X \times V$. Moreover, $E^{\oplus i}$ $(L^{\oplus i})$ will denote the corresponding i-fold sums. Observe that there is a natural map $Th(L^{\oplus i}) = X_+ \wedge Th(\hat{L}^{\oplus i}) \to Th(\tilde{L}^{\oplus i}) = X_+ \wedge Th(\bar{L}^{\oplus i})$.

In this context, the same arguments as above show that a cycle Z on $X \times \mathbb{A}^i$ finite over X defines (pointed) maps

(3.3.5)

$$\mathcal{P}_{\mathcal{M}}(Z) : X_+ \wedge Th(\bar{L}^{\oplus i}) = Th(\tilde{L}^{\oplus i}) \to \mathbb{F}_{ltr}(\mathbb{A}^{iN})/Z_{tr}(\mathbb{A}^{iN} - 0) = \mathrm{K}(iN) \quad \textit{and}$$
$$\mathcal{P}_s(Z) : X_+ \wedge Th(\hat{L}^{\oplus i}) = Th(L^{\oplus i}) \to \mathbb{F}_{ltr}(\mathbb{A}^{iN})/Z_{tr}(\mathbb{A}^{iN} - \{0\}) = \mathrm{K}(iN)$$

with the latter being obtained by pre-composing the first map with the obvious map $Th(\hat{L}^{\oplus i}) \to Th(\bar{L}^{\oplus i})$.

The contravariant functoriality of the above constructions in X shows first of all that the above arguments suffice to define $\mathcal{P}_{\mathcal{M}}(Z)$ and $\mathcal{P}_s(Z)$ associated to section of the sheaf $\mathrm{K}(i)$). (To see this, one recalls the sheafification process starts with a presheaf P and first takes P^+ whose sections over a U are $\varinjlim ker(\Gamma(U_i, P) \overset{pr_1^* - pr_2^*}{\to} \Gamma(U_i \underset{U}{\times} U_j, P))$, where $\{U_i\}$ is an open cover of U and the colimit is over a cofinal system of open covers of U. To obtain the associated sheaf from P, one takes $(P^+)^+$.) Next consider the simplicial resolution of the sheaf $\mathrm{K}(i)$ by pointed smooth simplicial schemes defined as follows. (See [**Voev1**, p. 9] for details.) In degree n, it is given by the pointed smooth scheme

$$G_n\mathrm{K}(i) = (\sqcup_{X_0 \to X \to \cdots X_n, Z \in \Gamma(X_n, \mathrm{K}(i)_0} X_0)_+$$

Making use of the above resolution, the same contravariant functoriality of the above constructions shows that one may replace X in (3.3.5) by the sheaf $\mathrm{K}(i)$) to obtain maps of (pointed) sheaves in $(\mathrm{Sm/k})_{\mathrm{Nis}}$ (and also in $SSh\mathrm{Sm/k}_{Zar}$ and

$SSh\mathrm{Sm/k}_{ét}$):

$$\mathcal{P}_{\mathcal{M}} : \mathrm{K}(i) \wedge Th(\bar{L}^{\oplus i}) \to \mathrm{K}(iN) \quad and \tag{3.3.6}$$
$$\mathcal{P}_s : \mathrm{K}(i) \wedge Th(\hat{L}^{\oplus i}) \to \mathrm{K}(iN))$$

with the latter being obtained by composing the first map with the map induced by the map $Th(\hat{L}^{\oplus i}) \to Th(\bar{L}^{\oplus i})$.

The above natural transformations of simplicial sheaves induce the natural transformations (defined on the categories $\mathcal{H}\mathrm{Sm/k}_{\mathrm{Nis}}$ and $\mathcal{H}\mathrm{Sm/k}_{ét}$):

$$P'_{\mathcal{M}} : \tilde{H}^{2i}(\quad , \mathbb{Z}/l(i)) \to \tilde{H}^{2iN}(\quad \wedge Th(\bar{L}^{\oplus i}), \mathbb{Z}/l(iN)) \text{ and}$$
$$P'_s : \tilde{H}^{2i}(\quad , \mathbb{Z}/l(i)) \to \tilde{H}^{2iN}(\quad \wedge Th(\bar{L}^{\oplus i}), \mathbb{Z}/l(iN))$$

Making use of Thom-isomorphisms and observing also that $rk(E) = l$, these then correspond to

$$P'_{\mathcal{M}} : \tilde{H}^{2i}(\quad , \mathbb{Z}/l(i)) \to \tilde{H}^{2il}(\quad \wedge (U_N/\Sigma_l)_+, \mathbb{Z}/l(il)) \text{ and}$$
$$P'_s : \tilde{H}^{2i}(\quad , \mathbb{Z}/l(i)) \to \tilde{H}^{2il}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_N)_+, \mathbb{Z}/l(il))$$

so that the latter is obtained from the former by composing with the map induced by the augmentation $E\Sigma_l \underset{\Sigma_l}{\times} U_N \to U_N/\Sigma_l$. Moreover, the isomorphisms in (2.10.2) along with [**Voev1**, Lemma 5.7] show that these are compatible as $N \to \infty$ which provides the following result.

Proposition 3.4. *We obtain natural transformations:*

$$P_{\mathcal{M}} : \tilde{H}^{2i}(\quad , \mathbb{Z}/l(i)) \to \underset{\infty \leftarrow N}{lim} \tilde{H}^{2il}(\quad \wedge (U_N/\Sigma_l)_+, \mathbb{Z}/l(il))$$
$$\cong \tilde{H}^{2il}(\quad \wedge (B\Sigma_l)^{gm}_+, \mathbb{Z}/l(il)) \ and$$
$$P_s : \tilde{H}^{2i}(\quad , \mathbb{Z}/l(i)) \to \underset{\infty \leftarrow N}{lim} \tilde{H}^{2il}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_{N+}), \mathbb{Z}/l(i))$$
$$\cong \tilde{H}^{2il}(\quad \wedge B\Sigma_{l+}, \mathbb{Z}/l(il))$$

on $\mathcal{H}(\mathrm{Sm/k}_{\mathrm{Nis}})_+$ (and also on $\mathcal{H}(\mathrm{Sm/k}_{ét})_+$) so that the latter is obtained from the former by composing with the map induced by the augmentation $E\Sigma_l \underset{\Sigma_l}{\times} U_N \to U_N/\Sigma_l$.

Definition 3.5. The natural transformation $P_{\mathcal{M}}$ (P_s) will be called the geometric total power operation (the simplicial total power operation, respectively).

3.6. Motivic operations. Next we recall the definition of the cohomology operations of Voevodsky. Let F denote a pointed simplicial sheaf on $(\mathrm{Sm/k})_{\mathrm{Nis}}$ (or on $(\mathrm{Sm/k})_{Zar}$).

One starts with the total power operation :

$$P_{\mathcal{M}} : \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(i)) \to \tilde{H}^{2il}_{\mathcal{M}}(F \wedge (U/\Sigma_l)_+, \mathbb{Z}/l(il)) \tag{3.6.1}$$

By the results in 2.5, $\underset{i,j}{\oplus}\tilde{H}^{il}_{\mathcal{M}}(F \wedge (U/\Sigma_l)_+, \mathbb{Z}/l(jl))$ is a free module over $\tilde{H}^*_{\mathcal{M}}(F, \mathbb{Z}/(\star))$ with basis given by the elements $\bar{d}^r$ and $c\bar{d}^r$, $r \geq 0$. The operations $P^r_{\mathcal{M}}$ and $\beta P^r_{\mathcal{M}}$ are defined by the formula:

$$(3.6.2) \qquad P_{\mathcal{M}}(w) = \Sigma_{r\geq 0} P^r_{\mathcal{M}}(w)\bar{d}^{i-r} + \beta P^r_{\mathcal{M}}(w)c\bar{d}^{i-r-1}, \quad w \in \tilde{H}^{2i}(F, \mathbb{Z}/l(i))$$

Observe that, so defined $P^r_{\mathcal{M}} : \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(i)) \to \tilde{H}^{2i+2r(l-1)}_{\mathcal{M}}(F, \mathbb{Z}/l(i+r(l-1)))$ and
$\beta P^r_{\mathcal{M}} : \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(j)) \to \tilde{H}^{2i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$.

Behavior under suspension: A key observation is that, since the motivic cohomology operations are stable with respect to shifting degrees by 1, and also both degrees and weights by 1, this defines the operations $P^r_{\mathcal{M}}$ and $\beta P^r_{\mathcal{M}}$ on all $\tilde{H}^i_{\mathcal{M}}(F, \mathbb{Z}/l(j))$.

The simplicial operations are *not* stable with respect to suspension of weights, and therefore, one cannot define simplicial operations in general using the total power operations considered above. For this, we define a new total power operation when the simplicial model is used for the classifying spaces of finite groups. We also show that, when applied to classes with degree = twice their weight, with $l \neq char(k) = p$, these total power operations identify with the ones considered above. All of these are discussed in detail in the next section.

4. The total power operations: II

We proceed to define total power operations in a somewhat different manner so as to be able to define the simplicial operations on all classes. Let Σ_l denote the symmetric group on l-letters and let π denote a subgroup of Σ_l. Let $B\pi$ denote the simplicial classifying space of π with $E\pi \to B\pi$ denoting the associated principal π-fibration. We let $\mathbb{F}_l(E\pi)$ denote the chain complex obtained by taking the free $\mathbb{F}_l$-vector space in each simplicial degree and viewing that as a chain-complex in the usual manner using the alternating sums of the face maps as the differential. We let $\mathbb{F}_l(E\pi)^\vee = Hom(\mathbb{F}_l(E\pi), \mathbb{F}_l)$ which now forms a co-chain complex (i.e. with differentials of degree +1) trivial in negative degrees.

Let K denote a possibly unbounded co-chain complex. Now $K^{\otimes^l}$ is the l-fold tensor product of K:the symmetric group Σ_l acts in the obvious manner on $K^{\otimes^l}$. Therefore, one may now form the co-chain complex:

$$\mathbb{F}_l(E\pi)^\vee \underset{\mathbb{F}_l[\pi]}{\otimes} K^{\otimes^l}$$

where the differentials of the tensor-product are induced by the differentials of the two factors in the usual manner. (Since K is allowed to be an unbounded complex, one needs to exercise care in taking the above complex: strictly speaking one needs to take the homotopy inverse limit of the double co-chain complexes obtained this way: see [**J1**]. However, one may identify this with a suitable total chain-complex as is shown in [**Brow**, Appendix].) In particular, the differential, $((\mathbb{F}_l(E\pi))^\vee)^0 \underset{\mathbb{F}_l[\pi]}{\otimes} K^{\otimes^l})^n \to (\mathbb{F}_l(E\pi)^\vee \underset{\mathbb{F}_l(\pi)}{\otimes} (K^{\otimes^l})^{n+1}$ is such that if $z \in K^n$ is a cycle, then its l-th power $z^{\otimes^l}$ defines a cycle of degree nl in the above total complex we denote by

$$\tilde{\mathcal{Q}}(z) \in (\mathbb{F}_l(E\Sigma_l)^\vee \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} K^{\otimes^l})^{nl} \tag{4.0.3}$$

We will choose the complex K as follows. First we allow three distinct contexts:

(i) We work throughout on the site $(\mathrm{Sm}/k)_{Zar}$ with H^* denoting cohomology on the Zariski site.

(ii) We work throughout on the site $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ with H^* denoting cohomology on the Nisnevich site.

(iii) We work throughout on the site $(\mathrm{Sm}/k)_{ét}$ with H^* denoting cohomology on the étale site.

Next observe that the category of (possibly unbounded) co-chain complexes of abelian sheaves on any of the above two sites is a quasi-simplicial model category in the sense of [**Fausk**] and therefore it is closed under homotopy inverse limits. Let $\mathcal{H}om$ denote the internal hom in this category. Then, given co-chain complexes of abelian sheaves M, N, we let $\mathcal{RH}om(M,N) = \mathcal{H}om(M, GN)$ with G denoting the homotopy inverse limit of the cosimplicial object defined by the Godement resolution computed on the appropriate site. $RHom$ will denote the corresponding external hom, i.e. where $\mathcal{H}om$ in the above definition of $\mathcal{RH}om$ has been replaced by the external hom, Hom.

Let $X \in (\mathrm{Sm/k})$. We let $K = \Gamma(X, \mathcal{RH}om(M, \{\mathbb{F}_l(i)))$ where M is any chain complex of abelian sheaves trivial in negative degrees. Moreover, now $K = RHom(M \otimes Z(X), \{\mathbb{F}_l(i))$, where $Z(X)$ denotes the co-chain complex associated to the simplicial abelian presheaf defined by $\Gamma(U, Z(X)) = Z(\Gamma(U, X))$.

In fact we may start with a pointed simplicial sheaf F in $(\mathrm{Sm/k})_{\mathrm{Nis}}$ and let M denote the normalized co-chain complex obtained by taking the associated free simplicial sheaf $\mathbb{F}_l(F)$ of $\mathbb{F}_l$-vector spaces (with the base point of F identified with 0) and re-indexing so that we obtain a co-chain complex. Then we define $\mathcal{RH}om(F, \{\mathbb{F}_l(i)) = \mathcal{RH}om(M, \{\mathbb{F}_l(i)) = \mathcal{H}om(M, G\{\mathbb{F}_l(i))$. The above definition makes implicit use of the adjunction between the free $\mathbb{F}_l$-vector space functor and the underlying functor sending a $\mathbb{F}_l$-vector space to the underlying set. A useful observation in this context is that the natural map $\mathbb{F}_l(S)\underset{\mathbb{F}_l}{\otimes}\mathbb{F}_l(T) \to \mathbb{F}_l(S \bigwedge T)$ is a weak-equivalence for any pointed simplicial presheaves S and T. (One may prove this as follows. First this is clear if S is a presheaf of pointed sets, i.e. it is true if S is replaced by its 0-th skeleton. One may prove using ascending induction on n, that the above map is a weak-equivalence when S is replaced by its n-skeleton. Finally take the colimit as $n \to \infty$ over the n-skeleta of S.) This will enable one to obtain the weak-equivalence $\mathcal{RH}om(M' \otimes M'', \{\mathbb{F}_l(i)) \simeq \mathcal{RH}om(F' \bigwedge F'', \{\mathbb{F}_l(i))$, when $M' = \mathbb{F}_l(F')$ and $M'' = \mathbb{F}_l(F'')$.

Then $K^{\otimes^l} = \Gamma(X, \mathcal{RH}om(M, \{\mathbb{F}_l(i)))^{\otimes^l} = \Gamma(X^l, \mathcal{RH}om(M, \{\mathbb{F}_l(i))^{\boxtimes^l})$ maps to $\Gamma(X, \mathcal{RH}om(M, \{\mathbb{F}_l(il)))$ by pull-back by the diagonal $\Delta : X \to X^l$. (In fact this makes use of the diagonal map $\mathbb{F}_l(F) \to \mathbb{F}_l(F)^{\otimes^l}$ and the pairing $\{\mathbb{F}_l{}^{\otimes^l} \to \{\mathbb{F}_l$.) We proceed to show this pairing is compatible with the obvious action of Σ_l. First observe that M being the normalized chain complex obtained from the simplicial abelian sheaf $\mathbb{F}_l(F)$ (re-indexed so as to become a co-chain complex), has the structure of a co-algebra over the Barratt-Eccles operad as shown in [**B-F**, 2.1.1 Theorem]. $\{\mathbb{F}_l$ has the structure of an algebra over the same operad as shown in [**J1**, Theorem 1.1]. Therefore, one may readily show that these structures provide

$\mathcal{RH}om(M, \{\mathbb{F}_l)$ the structure of an algebra over the tensor product of the Barratt-Eccles operad and the Eilenberg-Zilber operad: see [**J1**, Proposition 6.4]. Therefore, the above pairing is compatible with the obvious action of Σ_l and one obtains the obvious map

$$\mathbb{F}_l(E\Sigma_l)^\vee \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F), \{\mathbb{F}_l(i)))^{\otimes l}$$
$$\to \mathbb{F}_l(E\Sigma_l)^\vee \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F), \{\mathbb{F}_l(il))).$$

(See for example (5.1.3), which explains such pairings in more detail.) One may identify the last term with

$$\Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F) \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \mathbb{F}_l(E\Sigma_l), \{\mathbb{F}_l(il)) = \Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F) \otimes \mathbb{F}_l(B\Sigma_l), \{\mathbb{F}_l(il)).$$

We denote the above composition

$$\begin{aligned} &\mathbb{F}_l(E\Sigma_l)^\vee \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F), \{\mathbb{F}_l(i)))^{\otimes l} \\ &\quad \to \Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F) \otimes \mathbb{F}_l(B\Sigma_l), \{\mathbb{F}_l(il))) \end{aligned} \tag{4.0.4}$$

by $\bar{Q}_s$. As observed above any cycle $z \in \Gamma(X, RHom(M, \{\mathbb{F}_l(i)))$ in degree n provides a cycle in degree nl in the source of the last map: this cycle was denoted $\tilde{\mathcal{Q}}(z)$. Therefore, $\bar{Q}_s(\tilde{\mathcal{Q}}(z))$ defines a cycle in the target of the last map in degree nl. Moreover, one may show readily that if two cycles z and $z' \in \Gamma(X, RHom(M, \{\mathbb{F}_l(i)))$ are such that their difference is a co-boundary, then the same holds for the cycles $\tilde{\mathcal{Q}}(z)$ and $\tilde{\mathcal{Q}}(z')$ as well as $\bar{Q}_s(\tilde{\mathcal{Q}}(z)$ and $\bar{Q}_s(\tilde{\mathcal{Q}}(z'))$: this may be proven as in [**St-Ep**, Chapter VII, Lemma 2.2]. Therefore, the above discussion provides the natural transformation

$$Q_s : \tilde{H}^j(\quad, \mathbb{Z}/l(i)) \to \tilde{H}^{jl}(\quad \wedge B\Sigma_{l+}, \mathbb{Z}/l(il)) \tag{4.0.5}$$

for all j and all $i \geq 0$ on the category $\mathcal{H}\mathrm{Sm/k}_{\mathrm{Nis}}$ and $\mathcal{H}\mathrm{Sm/k}_{\acute{e}t}$. (We call this the (second) *simplicial total power operation.*)

4.1. The simplicial operations. The book [**Kr-May**] first introduced operations of the form (1.0.2). A detailed construction along the lines of *loc. cit.* is given in [**J1**]. However, for the comparison with the motivic cohomology operations, it is more convenient for us to define simplicial operations using the total operation Q_s defined above in (4.0.5). We then compare the operations to those of [**Kr-May**] and [**J1**] in Section 5.

Let F denote a pointed simplicial sheaf. The computation of $\tilde{H}^*_{\mathcal{M}}(F \wedge B\Sigma_{l+}; \mathbb{Z}/l)$ in 2.7 shows that it is a free module over $\tilde{H}^*_{\mathcal{M}}(F; \mathbb{Z}/l)$ with basis $\{x\bar{y}^i, \bar{y}^i | i \geq 0\}$. The operation P^r_s (βP^r_s) is defined by the formula:

(4.1.1)
$$Q_s(w) = \Sigma_{r\geq 0} P^r_s(w)\bar{y}^{j/2-r} + \beta P^r_s(w) x\bar{y}^{j/2-r-1}, \quad w \in \tilde{H}^j(F, \mathbb{Z}/l(i)), \text{ for all j.}$$

Observe that, so defined,

$$P^r_s : \tilde{H}^j(F, \mathbb{Z}/l(i)) \to \tilde{H}^{j+2r(l-1)}(F, \mathbb{Z}/l(il)) \text{ and}$$

$$\beta P_s^r : \tilde{H}^j(F, \mathbb{Z}/l(i)) \to \tilde{H}^{j+2r(l-1)+1}(X, \mathbb{Z}/l(il)).$$

Behavior under suspension. In contrast to the motivic operations, these operations are compatible with shifting the degree alone by 1. This will follow from the comparison theorem in the next section.

Next we proceed to show that, for $l \neq char(k) = p$ and for classes with degree = twice the weight, the total power operation Q_s identifies with the total power operation P_s defined above in Proposition 3.4.

PROPOSITION 4.2. *Let $l \neq char(k) = p$. Let $\alpha \in \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(i))$ denote a class. Then $Q_s(\alpha) = P_s(\alpha)$.*

PROOF. First we observe from [**Voev1**, Theorem 2.1] that since we are only considering cycles whose degree equals twice their weight, one may replace the motivic complex $\mathbb{Z}/l(m)[2m]$ by the complex of sheaves $C_*(\mathrm{K}(m))$ defined earlier.

The next key step to is to invoke the following result proved in Proposition 3.2: the map $gs_N : E\Sigma_l \underset{\Sigma_l}{\times} U_N \to E\Sigma_l \underset{\Sigma_l}{\times} \operatorname{Spec} k = B\Sigma_l$ of simplicial schemes induces a weak-equivalence in $\mathcal{H}\mathrm{Sm/k}_{\mathrm{Nis}}$ (and also in $\mathcal{H}\mathrm{Sm/k}_{Zar}$ and $\mathcal{H}\mathrm{Sm/k}_{\acute{e}t}$) on taking the colimit as $N \to \infty$. Therefore, and in view of Proposition 3.4, one may replace $B\Sigma_l$ in the above definition of the simplicial operations by $E\Sigma_l \underset{\Sigma_l}{\times} U$, where $U = U_N$, $N >> 0$. i.e. First the map in (4.0.4) may be replaced by the map

$$(4.2.1) \quad \mathbb{F}_l(E\Sigma_l)^\vee \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F) \otimes \mathbb{F}_l(U), C_*(\mathrm{K}(i))))^{\otimes l}$$
$$\to \Gamma(X, \mathcal{RH}om(\mathbb{F}_l(F) \otimes \mathbb{F}_l(E\Sigma_l) \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \mathbb{F}_l(U), C_*(\mathrm{K}(il))))$$

Therefore, the total power operation Q_s may be defined as a map

$$(4.2.2) \qquad Q_s : \tilde{H}^j(\quad, C_*(\mathrm{K}(i))) \to \tilde{H}^{jl}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(il))).$$

Next will consider the case when $F = X$ which is a smooth scheme and with a class in $\tilde{H}^{2i}(F, \{\mathbb{F}_l(i))$ coming from a section in $\Gamma(X, C_*(\mathrm{K}(i)^{pre}))$. Since $\Gamma(X, C_n(\mathrm{K}(i)^{pre}) \subseteq \Gamma(X \times \Delta[n], \mathrm{K}(i)^{pre})$, we may assume without loss of generality that $n = 0$. Therefore, such a class is represented by a cycle Z on $X \times \mathbb{A}^i$ equi-dimensional and finite over X. One first pulls-back the cycle Z to $p^*(Z^l)$ on $X \times \mathbb{A}^{il} \times U$. This cycle is invariant under the obvious action of the symmetric group Σ_l on $\mathbb{A}^{il} \times U$ and therefore defines a cycle in

$$\mathbb{F}_l(E\Sigma_l)^\vee \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{H}om(\mathbb{F}_l(U), \mathrm{K}(il))) = \Gamma(X, \mathcal{H}om(\mathbb{F}_l(E\Sigma_l) \underset{\mathbb{F}_l(\Sigma_l)}{\otimes} \mathbb{F}_l(U), \mathrm{K}(il))).$$

Observe that this is the total complex of the double complex defined by the cosimplicial co-chain complex: $\{\Gamma(X, \mathcal{H}om(\mathbb{F}_l(\Sigma_l^{\times^n}) \otimes \mathbb{F}_l(U), \mathrm{K}(il)))|n\}$.

This cycle will be denoted $\hat{\mathcal{Q}}_s(Z)$. Since the cycle Z^l is clearly stable under the permutation action of the symmetric group Σ_l, $\hat{\mathcal{Q}}_s(Z)$ corresponds to a class in $\Gamma(X, \mathcal{H}om(\mathbb{F}_l(U), \mathrm{K}(il)))$ so that its pull-back to classes in $\Gamma(X, \mathcal{H}om(\mathbb{F}_l(\Sigma_l) \otimes \mathbb{F}_l(U), \mathrm{K}(il)))$ by the group-action $\mu : \Sigma_l \times U \to U$ and the projection $pr_2 : \Sigma_l \times U \to U$ are the same, i.e. it is an invariant cycle.

4.2.3. A *key observation* is that the assignment $Z \mapsto \hat{\mathcal{Q}}_s(Z)$ is contravariantly functorial in X and that $\hat{\mathcal{Q}}_s(Z)$ represents the class $Q_s(Z)$.

Next let $\bar{E}$ and $\bar{L}$ denote the vector bundles on $V = U/\Sigma_l$ defined as in 3.3.4. Recall $rk(\bar{E}) = l$, $l + rk(L) = N$, $\hat{E}$, $\hat{L}$ (E, L) are the corresponding pull-backs to $E\Sigma_l \underset{\Sigma_l}{\times} U$ ($X \times E\Sigma_l \underset{\Sigma_l}{\times} U$, respectively). Therefore, the same cycle as above defines the cycle

$$\begin{aligned}\Delta_*(q^*(\hat{\mathcal{Q}}_s(Z))) &\in \Gamma(X, \mathcal{H}om(\mathbb{F}_l(Th(\hat{L}_0^{\oplus i})), \mathrm{K}(iN)) \\ &= Hom(Z(X) \otimes (\mathbb{F}_l(Th(\hat{L}^{\oplus i})), \mathrm{K}(iN)).\end{aligned}$$

where $q : \hat{E}^{\oplus i} \oplus \hat{L}^{\oplus i} \to \hat{E}^{\oplus i}$ is the obvious projection and $\Delta : \hat{E}^{\oplus i} \oplus \hat{L}^{\oplus i} \to \hat{L}^{\oplus i} \oplus \hat{E}^{\oplus i} \oplus \hat{L}^{\oplus i}$ is the diagonal. Observe that the vector bundle $\bar{L}$ on V corresponds to a Σ_l-equivariant vector bundle on U, which is none other than the degree-0-term $\hat{L}_0$ of $\hat{L}$. Now the diagonal of $\hat{L}_0$ is clearly stable by the Σ_l-action. Therefore, the class denoted $\Delta_*(q^*(\hat{\mathcal{Q}}_s(Z)))$ corresponds to a class in

$$Hom(Z(X) \otimes (\mathbb{F}_l(Th(\hat{L}_0^{\oplus i})), \mathrm{K}(iN))$$

so that its pull-back by d_0 and d_1 to classes in $\Gamma(X, \mathcal{RH}om(\mathbb{F}_l(Th(\hat{L}_1^{\oplus i})), \mathrm{K}(iN)))$ identify, i.e. it again defines a Σ_l-invariant cycle in

$$Hom(Z(X) \otimes (\mathbb{F}_l(Th(\hat{L}^{\oplus i})), \mathrm{K}(iN)).$$

Moreover, the definition of the class $\mathcal{P}_s(Z)$ as in (3.3.5) shows that, $\mathcal{P}_s(Z)$ is the class in

$$\tilde{H}^{2il}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(il)))$$

that maps to the class $\Delta_*(q^*(\hat{\mathcal{Q}}_s(Z)))$, under the Thom-isomorphism:

$$\tilde{H}^{2il}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(il))) \overset{Thom-isom}{\to} \tilde{H}^{2iN}(\quad \wedge Th(\hat{L}^{\oplus i}), C_*(\mathrm{K}(iN))).$$

4.2.4. Observe that it suffices to show that the class $\Delta_*(q^*(\hat{\mathcal{Q}}_s(Z)))$ is also the image under Thom-isomorphism of the class $\hat{\mathcal{Q}}_s(Z)$ in

$$H^{2il}(X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, \mathbb{Z}/l(il)).$$

For this, we begin by observing that we have the following commutative diagram (where Hom denotes Hom in the category of simplicial sheaves)

$$\begin{array}{ccc} Hom(X_+ \wedge Th(\hat{L}^{\oplus i}), C_*(\mathrm{K}(iN))) & \xrightarrow{\Delta_* q_E^*(\quad)} & Hom(\Sigma^{2iN,iN} X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(i(N+l)))) \\ \uparrow{\scriptstyle \Delta_*(q_L^*(\quad))} & \nearrow{\scriptstyle \Delta_* q_{\mathbb{A}^{iN}}^*(\quad)} & \\ Hom(X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(il))) & & \end{array}$$

where the top horizontal map denoted $\Delta_* q_E^*$ (the left-vertical map denoted $\Delta_* q_L^*$) is the map defined by the construction in 3.3.1 where $q_L : E^{\oplus i} \oplus L^{\oplus i} \to E^{\oplus i}$ ($q_E : (L^{\oplus i} \oplus E^{\oplus i}) \times \mathbb{A}^{iN} \to L^{\oplus i} \times \mathbb{A}^{iN}$) is the obvious projection. One also obtains a similar commutative diagram using the simplicial mapping space functor Map instead of Hom.

In fact one may begin with a similar diagram involving the corresponding simplical presheaves $\mathrm{K}(s)^{pre}$ and make use of the contravariant functoriality of the construction in 3.3.1 to obtain the above diagram of simplicial sheaves. (To see this, one may again recall the sheafification process starts with a presheaf P and first takes P^+ whose sections over a U are $\lim_{\rightarrow} ker(\Gamma(U_i, P) \overset{pr_1^* - pr_2^*}{\rightarrow} \Gamma(U_i \underset{U}{\times} U_j, P))$, where $\{U_i\}$ is an open cover of U and the colimit is over a cofinal system of open covers of U. To obtain the associated sheaf from P, one takes $(P^+)^+$.)

Observe that the vector bundles $\hat{E}$ and $\hat{E} \oplus \hat{L}$ and hence $\hat{E}^{\oplus i}$ and $\hat{E}^{\oplus i} \oplus \hat{L}^{\oplus i}$ are all trivial, i.e. $\hat{E}_0$ ($\hat{E}_0 \oplus \hat{L}_0$) corresponds to $U \times \mathbb{A}^l$ ($U \times \mathbb{A}^N$, respectively) where Σ_l acts in the obvious way by permutation of the coordinates on $\mathbb{A}^l$ and the action of Σ_l on $\mathbb{A}^N$ is induced by its actions on $\hat{E}_0$ and $\hat{L}_0$ (which is some Σ_l-equivariant vector bundle on U obtained by pull-back from the vector bundle $\bar{L}$ on V.) Since E (L) is the pull-back of $\hat{E}$ ($\hat{L}$) to vector bundles over $X \times E\Sigma_l \underset{\Sigma_l}{\times} U$ it follows that E, $E \oplus L$, $E^{\oplus i}$ and $E^{\oplus i} \oplus L^{\oplus i}$ are all trivial vector bundles.

The map $q_{\mathbb{A}^{iN}}$ appearing in the slant map above is the projection map $q_{\mathbb{A}^{iN}} : \mathbb{A}^{iN} \times E^{\oplus i} = L^{\oplus i} \oplus E^{\oplus i} \oplus E^{\oplus i} \rightarrow E^{\oplus i}$ and Δ there is the diagonal $\Delta : \mathbb{A}^{iN} \times E^{\oplus i} \rightarrow \mathbb{A}^{iN} \times \mathbb{A}^{iN} \times E^{\oplus i}$. Therefore, as observed in 3.3.2, the top horizontal map and the slant map are infact concrete realizations of taking cup-product with the corresponding Thom-classes. Finally, the commutativity of the above triangle follows from the naturality of the construction 3.2.2. It may also be seen more explicitly as follows. Let $\mathcal{Y}$ denote a section of the sheaf $C_*(\mathrm{K}(il))$ over $X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+$. Then we already observed that the class $\Delta_*(q_L^*(\mathcal{Y}))$ is its image in the top left corner. Therefore, on applying the map in the top horizontal row to the class $\Delta_*(q_L^*(\mathcal{Y}))$, one obtains $\Delta_*(q_E^*(\Delta_*(q_L^*(\mathcal{Y}))))$. Let the fiber over a point $(x, u) \in X \times U$ of the cycle $\mathcal{Y}$ be denoted $\mathcal{Y}_{x,u}$. Then the corresponding fiber of $\Delta_* q_L^*(\mathcal{Y})$, will be $(l_u, \mathcal{Y}_{x,u}, l_u)$ where l_u is a point in the fiber of $L^{\oplus i}$ over u. Now the corresponding fiber of $\Delta_* q_E^*(\Delta_*(q_L^*(\mathcal{Y})))$ over (x, u) will be $(l_u, e_u, \mathcal{Y}_{x,u}, l_u, e_u)$ where e_u is a point of $E^{\oplus i}$ in the fiber over u.

Since the construction in 3.2.2 is contravariantly functorial in X, one may now let X be replaced by any smooth scheme so that we obtain the commutative diagram of simplicial presheaves:

$$\begin{array}{ccc}
Map((\ \ )_+ \wedge Th(\hat{L}^{\oplus i}), C_*(\mathrm{K}(iN))) & \xrightarrow{\Delta_* q_E^*(\ \ )} & Map(\Sigma^{2iN, iN}(\ \ )_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(i(N+l)))) \\
\uparrow {\scriptstyle \Delta_*(q_L^*(\ \ ))} & \nearrow {\scriptstyle \Delta_* q^*_{\mathbb{A}^{iN}}(\ \ )} & \\
Map((\ \ )_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(il))) & &
\end{array}$$

Since the above diagram of simplicial presheaves strictly commutes, one will obtain a similar commutative triangle, when the simplicial presheaves above have been replaced by *fibrant* simplicial presheaves. (Here one may assume an injective model structure for simplcial presheaves, where every simplcial presheaf is cofibrant and the fibrations are global fibrations.) One may now observe that this corresponds upto weak-equivalence to replacing the simplicial sheaves $C_*(\mathrm{K}(s))$ appearing above, for varying s, with globally fibrant simplicial presheaves upto weak-equivalence. These observations result in a similar commutative triangle when the simplicial sheaves $C_*(\mathrm{K}(s))$ all have been replaced by globally fibrant simplicial sheaves upto weak-equivalence and the argument ( ) can be any simplicial scheme which is smooth in all degrees. Since any pointed simplicial sheaf has a simplicial resolution by pointed smooth schemes, it follows one may put any pointed simplicial sheaf in the argument $(\quad)$. (Observe that the vector bundles $\bar{E}$ and $\bar{L}$ as in 3.2.2 on V as well as the associated vector bundles $\hat{E}$ and $\hat{L}$ on $E\Sigma_l \underset{\Sigma_l}{\times} U$ are defined independently of X, so that one may pull back these bundles to $(X_\bullet \times E\Sigma_l) \underset{\Sigma_l}{\times} U$. The pull-back of $\hat{E}$ $(\hat{L})$ will be denoted E (L, respectively): clearly $L \oplus E$ will be trivial.) These result in the commutative triangle

(4.2.5)

$$\begin{array}{ccc}
H^*((\quad)_+ \wedge Th(\hat{L}^{\oplus i}), C_*(\mathrm{K}(iN))) & \xrightarrow{\Delta_* q_E^*(\quad)} & H^*(\Sigma^{2iN,iN}(\quad)_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(i(N+l)))) \\
\uparrow {\scriptstyle \Delta_*(q_L^*(\quad))} & \nearrow {\scriptstyle \Delta_* q^*_{\mathbb{A}^{iN}}(\quad)} & \\
H^*((\quad)_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, C_*(\mathrm{K}(il))) & &
\end{array}$$

On the other hand, the naturality of Thom-isomorphisms now provides us with the following commutative triangle:

$$\begin{array}{ccc}
\tilde{H}^{2iN}((\quad)_+ \wedge Th(\hat{L}^{\oplus i}), \mathbb{Z}/l(iN)) & \longrightarrow & \tilde{H}^{2i(N+l)}(\Sigma^{2iN,iN}(\quad)_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, \mathbb{Z}/l(i(N+l))) \\
\uparrow & \nearrow {\scriptstyle \Sigma^{2i(N+l),N+l}} & \\
\tilde{H}^{2il}((\quad)_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U)_+, \mathbb{Z}/l(il)) & &
\end{array}$$

The left-vertical map is Thom-isomorphism for the vector bundle $L^{\oplus i}$, the top horizontal map is Thom-isomorphism for the vector bundle $E^{\oplus i}$ and the slant-map corresponds to Thom-isomorphism with respect to the trivial bundle $L^{\oplus i} \oplus E^{\oplus i}$. As observed above, making use of 3.3.2, the top horizontal map and the slant map in the diagram (4.2.5) are in fact Thom-isomorphism. Comparing the two commutative triangles above, it follows therefore, that the left-vertical map in (4.2.5) also identifies with the corresponding Thom-isomorphism, i.e. taking cup-product with the corresponding Thom-class. Therefore, we obtain:

$$\mathcal{P}_s(Z) = \hat{\mathcal{Q}}_s(Z)).$$

Since $\hat{\mathcal{Q}}_s(Z)$ represents the class $Q_s(Z)$, this completes the proof of the proposition first when the simplicial sheaf F is a smooth scheme and $\alpha \in \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(i))$. In the general case where F is any simplicial sheaf, the construction in 3.2.2 extended to smooth simplicial schemes defines a class $\Delta_*(q^*(\alpha)) \in H^{2iN}(F \wedge Th(\hat{L}^{\oplus i}), \mathbb{Z}/l(iN))$. The class $P_s(\alpha)$ is the class in $H^{2il}(F \wedge E\Sigma_l \underset{\Sigma_l}{\times} U_+, \mathbb{Z}/l(il))$ that corresponds to the class $\Delta_*(q^*(\alpha))$ under Thom-isomorphism. Comparison of the above two commutative triangles will show again that the map $\alpha \mapsto \Delta_* q^*(\alpha)$ identifies with the Thom-isomorphism. This completes the proof of Proposition 4.2. $\square$

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5. Comparison with the operadic definition of simplicial cohomology operations: properties of simplicial operations

An E^∞-structure on the motivic complex $\mathbb{A} = \oplus_i \mathbb{Z}/l(i))$ is shown to lead to a somewhat different definition of the *simplicial* cohomology operations on mod$-l$

motivic cohomology as discussed in [**J1**, Section 5] and based on the earlier work [**May**]. We will presently show that these operations are in fact identical to the simplicial operations defined above. Since the simplicial operations defined operadically readily inherit several well-known properties, we are thereby able to carry over such properties to the simplicial cohomological operations defined above. Some of these properties of the simplicial cohomology operations, for example, the Cartan formulae are used in an essential manner in the comparison results in the next section.

The only other way to establish such properties for the simplicial cohomology operations would be by a tedious step-by-step verification of these properties following the approach in [**St-Ep**]. Therefore we prefer the approach adopted here, which is far simpler.

Proposition 5.1. *The cohomology operations defined above coincide with the simplicial cohomology operations defined on mod-l motivic cohomology in* [**J1**, Section 5].

Proof. Throughout the proof we will denote the pairing between a vector-space over $\mathbb{F}_l$ and its dual by $< \quad , \quad >$. Recall the simplicial Barratt-Eccles operad is the operad $\{NZ(E\Sigma_n)|n\}$ where $E\Sigma_n$ denotes the simplicial bar-resolution of the finite group Σ_n and $NZ(E\Sigma_n)$ denotes the normalized chain complex associated to the simplicial abelian group $Z(E\Sigma_n)$. The operad structure obtained this way is discussed in [**J1**]. We will assume that it is an action by the simplicial Barratt-Eccles operad on the motivic complex that provides its E_∞-structure. The above action of the operad $\{NZ(E\Sigma_n)|n\}$ on the complex $\mathcal{A} = \oplus_{n\geq 0}\mathbb{Z}/l(n)$ provides us maps

$$\theta_l : NZ(E\Sigma_l) \otimes \mathcal{A}^{\otimes^l} \to \mathcal{A} \tag{5.1.1}$$

Recall that $K^\vee$ denotes $\mathcal{H}om(K, \mathbb{F}_l)$, if K is any complex of $\mathbb{F}_l$-vector spaces. From the above pairing we obtain

$$\theta_l^* : NZ(E\Sigma_l) \otimes \mathcal{A}^\vee \to (\mathcal{A}^\vee)^{\otimes^l}$$

where we define $\theta_l^*(h, a^\vee)(a_1 \otimes \cdots \otimes a_l) =< \theta_l(h \otimes a_1 \otimes \cdots \otimes a_l), a^\vee >$, $a_i \in \mathcal{A}$, $a^\vee \in \mathcal{A}^\vee$ and $h \in NZ(E\Sigma_l)$. In fact these pairings provide the dual $\mathcal{A}^\vee$ with the structure of a co-algebra over the operad $\{NZ(E\Sigma_l|l\}$. It is a standard result in this situation (i.e. for co-algebras over acyclic operads) that the map θ_l^* is a chain map and is an *approximation to the diagonal map* (i.e. homotopic to the diagonal map) $\Delta : \mathcal{A}^\vee \to (\mathcal{A}^\vee)^{\otimes^l}$. (Here, as well as elsewhere in this section, we use the observation that for any vector space V over $\mathbb{F}_l$, a vector $v \in V$ (a vector $v^\vee \in V^\vee$) is determined by its pairing $< v, w >$ with all vectors $w \in V^\vee$ (its pairing $< u, v^\vee >$ with all vectors $u \in V$, respectively.).)

We next take the dual of the pairing θ_l^* to define a chain-map:

$$((\mathcal{A}^\vee)^\vee)^{\otimes^l} \to NZ(E\Sigma_l)^\vee \otimes (\mathcal{A}^\vee)^\vee.$$

The formula defining the chain map θ_l^* shows that this map sends $\mathbb{A}^{\otimes^l} \subseteq ((\mathcal{A}^\vee)^\vee)^{\otimes^l}$ to $NZ(E\Sigma_l)^\vee \otimes \mathcal{A}$. Clearly there is a pairing $NZ(E\Sigma_l)^\vee \otimes NZ(E\Sigma_l)^\vee \to NZ(E\Sigma_l)^\vee$ induced by the diagonal $\Delta : E\Sigma_l \to E\Sigma_l \times E\Sigma_l$. Tensoring the last map with $NZ(E\Sigma_l)^\vee$ and making use of this pairing provides us with the map:

$$d : (NZ(E\Sigma_l))^\vee \otimes \mathcal{A}^{\otimes^l} \to (NZ(E\Sigma_l))^\vee \otimes \mathcal{A} \tag{5.1.2}$$

One may recall that the action of $\sigma \in \Sigma_l$ on $NZ(E\Sigma_l)$ and of σ^{-1} on $\mathcal{A}^{\otimes^l}$ cancel out. Tracing through these actions of Σ_l on the maps in the above steps, one concludes that the map d induces a map on the quotients:

$$\bar{d} : (NZ(E\Sigma_l))^\vee \underset{Z\Sigma_l}{\otimes} \mathcal{A}^{\otimes^l} \to (NZ(E\Sigma_l))^\vee \underset{Z\Sigma_l}{\otimes} \mathcal{A} \tag{5.1.3}$$

Now the cohomology of the complex $(NZ(E\Sigma_l))^\vee \underset{N(Z(\Sigma_l))}{\otimes} \mathcal{A}$ identifies with $H^*(B\Sigma_l;\mathbb{F}_l) \otimes H^*(\mathcal{A})$ whereas the cohomology of the complex $(NZ(E\Sigma_l))^\vee \underset{Z\Sigma_l}{\otimes} \mathcal{A}^{\otimes^l}$ identifies with the equivariant cohomology: $H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; \mathbb{F}_l)$. Therefore, the map $\bar{d}$ defines a map

$$\bar{d}^* : H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; \mathbb{F}_l) \to H^*(B\Sigma_l;\mathbb{F}_l) \otimes H^*(\mathcal{A}) \tag{5.1.4}$$

The formula defining d also shows that the map $\bar{d}^*$ is a map of $H^*(B\Sigma_l, \mathbb{F}_l)$-modules. One may also observe readily that the l-th power map defines a map $H^*(\mathcal{A}) \to H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; \mathbb{F}_l)$, $a \mapsto a^l$. Let $\{e_i, fe_i | i \geq 0\}$ denote a basis of the $\mathbb{F}_l$-vector space $H_*(B\Sigma_l;\mathbb{F}_l)$ dual to the basis $\{y^i, xy^i | i \geq 0\}$ for $H^*(B\Sigma_l;\mathbb{F}_l)$, i.e. $< e_i, y^j >= 0$, if $i \neq j$ and $= 1$ if $i = j$. Also $< fe_i, y^j >= 0$ for all i , j, $< fe_i, xy^j >= 0$ for $i \neq j$ and $= 1$ for $i = j$. Observe that now we have the following computation for a class $\alpha \in H^q(\mathcal{A})$:

$$< \bar{d}^*(\alpha^l), e_i \otimes (-)^\vee >=< \bar{\theta}_l^*(e_i, (-)^\vee), \alpha^l >=< (-)^\vee, \bar{\theta}_l(e_i, \alpha^l) > \text{ and}$$

$$< \bar{d}^*(\alpha^l), fe_i \otimes (-)^\vee >=< \bar{\theta}_l^*(fe_i, (-)^\vee), \alpha^l >=< (-)^\vee, \bar{\theta}_l(fe_i, \alpha^l) >$$

where $(-)^\vee \in H^*(\mathcal{A})^\vee$ and $\bar{\theta}_l^*$ is the map induced by θ_l^* on taking homology of the corresponding complexes. (One may prove the above equalities, by observing that the map $\bar{d}^*$ is essentially the dual of θ_l^*.) Since the map θ_l^* was observed to be chain homotopic to the diagonal, it follows that $\bar{d}^* = \Delta^*$ where Δ is the obvious diagonal. Therefore, the coefficient of y^i (xy^i) in the expansion of $\bar{d}^*(\alpha^l) \in H^*(B\Sigma_l;\mathbb{Z}/l) \otimes H^*(\mathcal{A})$ identifies with $\bar{\theta}_i(e_i, \alpha^l)$ ($\bar{\theta}_i(fe_i, \alpha^l)$, respectively). This completes the proof of the proposition □

The main point of the above comparison is to provide the following corollary where the corresponding results are shown to hold for the simplicial operations defined operadically in [**J1**, Theorem 5.3] invoking the results of [**May**].

Theorem 5.2. *Let F denote a pointed simplicial sheaf on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ in which case H^* will denote cohomology computed on the Nisnevich site or on $(\mathrm{Sm}/k)_{\acute{e}t}$ in which case H^* will denote cohomology computed on the étale site.*

The simplicial cohomology operations

$Q^s : \tilde{H}^q(F, \mathbb{Z}/l(t)) \to \tilde{H}^{q+2s(l-1)}(F, \mathbb{Z}/l(l.t))$ *and*

$\beta Q^s : \tilde{H}^q(F, \mathbb{Z}/l(t)) \to \tilde{H}^{q+2s(l-1)+1}(F, \mathbb{Z}/l(l.t))$.

satisfy the following properties:

(i) Contravariant functoriality: if $f : F' \to F$ is a map between simplicial sheaves, $f^* \circ Q^s = Q^s \circ f^*$

(ii) Let $x \in \tilde{H}^q(F, \mathbb{Z}/l(t))$. $Q^s(x) = 0$ if $2s > q$, $\beta Q^s(x) = 0$ if $2s \geq q$ and if $(q = 2s)$, then $Q^s(x) = x^l$.

(iii) If β is the Bockstein, $\beta \circ Q^s = \beta Q^s$.

(iv) Cartan formulae: For all primes l,

$$Q^s(x \otimes y) = \sum_{i+j=s} Q^i(x) \otimes Q^j(y) \text{ and}$$

$$\beta Q^s(x \otimes y) = \sum_{i+j=s} \beta Q^i(x) \otimes Q^j(y) + Q^i(x) \otimes \beta Q^j(y).$$

(v) Adem relations For each pair of integers $i \geq 0, j \geq 0$, we let $(i, j) = \frac{(i+j)!}{i!j!}$ with the convention that $0! = 1$. We will also let $(i, j) = 0$ if $i < 0$ or $j < 0$. (See [**May**, p. 183]*.) With this terminology we obtain:*

If ($l > 2$, $a < lb$, and $\epsilon = 0, 1$) or if ($l = 2$, $a < lb$ and $\epsilon = 0$) one has

$$\beta^\epsilon Q^a Q^b = \Sigma_i (-1)^{a+i}(a - li, (l-1)b - a + i - 1)\beta^\epsilon Q^{a+b-i} Q^i \tag{5.2.1}$$

where $\beta^0 Q^s = Q^s$ while $\beta^1 Q^s = \beta Q^s$. If $l > 2$, $a \leq lb$ and $\epsilon = 0, 1$, one also has

$$\begin{aligned} \beta^\epsilon Q^a \beta Q^b &= (1-\epsilon)\Sigma_i (-1)^{a+i}(a - li, (l-1)b - a + i - 1)\beta Q^{a+b-i} Q^i \\ &\quad - \Sigma_i (-1)^{a+i}(a - li - 1, (l-1)b - a + i)\beta^\epsilon Q^{a+b-i} \beta Q^i \end{aligned} \tag{5.2.2}$$

(vi) The operations Q^s commute with the simplicial suspension isomorphism in $H^n(F; \mathbb{Z}/l(r)) \cong H^{n+1}(S^1_s F; \mathbb{Z}/l(r))$.

(vii) The operation Q^s commutes with change of base fields and also with the higher cycle map into mod–l étale cohomology.

Remark 5.3. It is important to observe that Q^0 is not the identity. The property (ii) above shows that in general $Q^0(x) = x^l$, if $x \in H^0(X, \mathbb{Z}/l(t)) = \tilde{H}^0(X_+, \mathbb{Z}/l(t))$ for any smooth scheme X and any $t \geq 0$. This will play a major role in the comparison results in the next section.

6. Comparison between the motivic and simplicial operations

In view of the results established in the earlier sections we are able to provide a nearly complete comparison of the motivic and simplicial operations.

6.0.1. **The Motivic Bott element.** Throughout the rest of this section, we will assume that the field k has a primitive l-th root of unity. Recall that we have:

$$\begin{aligned} H^p_{\mathcal{M}}(\mathrm{Spec}, \mathbb{Z}(1)) &= 0, p \neq 1 \\ &= k^*, p = 1 \end{aligned} \tag{6.0.2}$$

Now the universal coefficient sequence associated to the short exact sequence $0 \to \mathbb{Z}(1) \stackrel{\times l}{\to} \mathbb{Z}(1) \to \mathbb{Z}/l(1) \to 0$ of motivic complexes, provides the isomorphism

$$H^0_{\mathcal{M}}(\mathrm{Spec}\, k, \mathbb{Z}/l(1)) \cong \mu_l(k) \tag{6.0.3}$$

The *Motivic Bott element* is the class in $H^0_{\mathcal{M}}(\mathrm{Spec}\, k, \mathbb{Z}/l(1))$ corresponding under the above isomorphism to the primitive l-th root of unity ζ. We will denote this element by B. Since $cycl(B) = \zeta$ in $H^*_{\acute{e}t}(\quad, \mu_l(*))$, multiplication by the class $cycl(B)$ induces an isomorphism: $H^*_{\acute{e}t}(\quad, \mu_l(r)) \to H^*_{\acute{e}t}(\quad, \mu_l(r+1))$. It follows that the cycle map *cycl* induces a map of cohomology functors:

$$cycl(B^{-1}) : H^*_{\mathcal{M}}(\quad, \mathbb{Z}/l(\star))[B^{-1}] \to H^*_{\acute{e}t}(\quad, \mu_l(\star)). \tag{6.0.4}$$

It is shown in [**Lev**] that this map is an isomorphism on smooth schemes.

As observed above, the cohomology $H^*(B\Sigma_l^{gm}; \mathbb{Z}/l)$ maps naturally to $H^*(B\Sigma_l; \mathbb{Z}/l)$ under which the total power operation $\mathcal{P}_{\mathcal{M}}$ maps to the total power operation $\mathcal{P}_s$. Therefore, a simple comparison of the degrees and weights of the classes involved provides the following proposition.

PROPOSITION 6.1. *Assume that the base field k has a primitive l-th root of unity. Let $\alpha \in H^{2q}_{\mathcal{M}}(X, \mathbb{Z}/l(q)$ for some $q \geq 0$ with $X \in (\mathrm{Sm/k})$. Then*

$$P^r_s(\alpha) = B^{(q-r).(l-1)}.P^r_{\mathcal{M}}(\alpha), \quad \beta P^r_s(\alpha) = B^{(q-r).(l-1)}.\beta P^r_{\mathcal{M}}(\alpha)$$

for $r \leq q$. For $r > q$, $P^r_s(\alpha) = 0 = P^r_{\mathcal{M}}(\alpha)$.

COROLLARY 6.2. *The same relation holds for any class $\alpha \in \tilde{H}^i_{\mathcal{M}}(F, \mathbb{Z}/l(q))$ when F is any pointed simplicial sheaf on $(\mathrm{Sm/k})_{\mathrm{Nis}}$ provided $i \leq 2q$.*

PROOF. We will first observe that the relations hold when $i = 2q$ and F is any pointed simplicial sheaf on $(\mathrm{Sm/k})_{\mathrm{Nis}}$. This follows readily in view of the observation that the two total power operations $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_s$ are compatible as natural transformations defined on the category of all pointed simplicial sheaves on $(\mathrm{Sm/k})$. Next we consider the statement when $i < 2q$. For example, if $i = 2q-1$, $\tilde{H}^{2q-1}_{\mathcal{M}}(F, \mathbb{Z}/l(q)) \cong \tilde{H}^{2q}_{\mathcal{M}}(\Sigma^1_s \wedge F, \mathbb{Z}/l(q))$. Now using the observation that both the motivic and simplicial operations are stable with respect to the suspension $\Sigma^1_s \wedge \quad$, such a degree-suspension reduces this to the case when $i = 2q$, which has been proved already. Observe also that when $i \leq 2q$, one knows that $P^r_s(\alpha) = 0 = P^r_{\mathcal{M}}(\alpha)$ for $r > q$, (see [**Voev1**, Lemma 9.9] for a proof of the last equality) so that for the classes α for which P^r_s is non-zero, the exponent $(q-r)(l-1)$ of B is ≥ 0. (If $i > 2q$ this may no longer be true apriori.)

In case F is in fact a scheme $X \in (\mathrm{Sm/k})$, the identification $H^i_{\mathcal{M}}(X, \mathbb{Z}/(q)) \cong CH^j(X, 2q-i; \mathbb{F}_l)$ shows that these groups are trivial if $i > 2q$. Therefore, it suffices to consider the case when $i \leq 2q$ in case F is in fact a scheme $X \in (\mathrm{Sm/k})$. □

Next we will consider what may be said about the case $i > 2q$. First observe that the Bott element B defines a class in $H^0_{\mathcal{M}}(X, \mathbb{Z}/l(1))$ for any smooth scheme X by pull-back. Next consider a a pointed simplicial sheaf F. Then one finds a resolution of F by pointed simplicial schemes $X_{\bullet,+}$: see [**Voev1**, section 3]. The structure map $X_1 \to \mathrm{Spec}\, k$ factors through the structure map $X_0 \to \mathrm{Spec}\, k$, so that B pulls-back to define a class (still denoted) $B \in \tilde{H}^0_{\mathcal{M}}(F, \mathbb{Z}/l(1))$.

LEMMA 6.3. *Let F denote a pointed simplicial sheaf on $(\mathrm{Smt}/k)_{\mathrm{Nis}}$. Then*

(i) $Q^0(B) = B^l$.
(ii) *if $x \in \tilde{H}^q_{\mathcal{M}}(F, \mathbb{Z}/l(t))$, then $P^r_s(B.x) = B^l P^r_s(x)$ and $\beta P^r_s(B.x) = B^l \beta P^r_s(x)$ for all $x \in \tilde{H}^q_{\mathcal{M}}(F, \mathbb{Z}/l(t))$.*

Proof. (i) Take $x = B$ in Theorem 5.2(ii). Then $q = 0 = s$ there so that $P^r_s(B) = 0$ for $r > 0$ and $Q^0(B) = B^l$. This proves (i). (ii) now follows from (i) making use of the Cartan formula in Theorem 5.2(iv). □

Our basic technique to handle the case where the degree > twice the weight (i.e. $i > 2q$) is to apply suitable weight and degree suspensions so as to reduce to the case where the degree = twice the weight. Then we handle this case by the comparison above. Both the motivic and simplicial operations commute with degree suspension, and the motivic operations commute with weight suspensions as well. The simplicial operations do not, however, commute with weight suspensions. But weight suspensions may effected by multiplying with the class B and the behavior of the simplicial operations with respect to tensoring with B is explained by the results above. Therefore, we obtain the extension of our comparison to classes of all degree and weight as explained below.

Proposition 6.4. *Suppose $x \in \tilde{H}^{2q+t}_{\mathcal{M}}(F, \mathbb{Z}/l(q))$, with $t > 0$.*
(i) If $t = 2t'$ for some integer t', then

$$B^{t'l}P^r_s(x) = B^{(q+t'-r)(l-1)}.B^{t'}P^r_{\mathcal{M}}(x) \quad \textit{and}$$
$$B^{t'l}\beta P^r_s(x) = B^{(q+t'-r)(l-1)}.B^{t'}\beta P^r_{\mathcal{M}}(x), \quad 0 \le r \le q+t'.$$

(ii) If $t = 2t'+1$,

$$B^{(t'+1)l}P^r_s(x) = B^{(q+t'+1-r)(l-1)}.B^{t'+1}P^r_{\mathcal{M}}(x) \quad \textit{and}$$
$$B^{(t'+1)l}\beta P^r_s(x) = B^{(q+t'+1-r)(l-1)}.B^{t'+1}\beta P^r_{\mathcal{M}}(x), \quad 0 \le r \le q+t'+1.$$

Proof. To obtain (i), one first applies an iterated weight suspension t'-times: this is effected by multiplying x by $B^{t'}$. Now the class $B^{t'}x \in \tilde{H}^{2q+2t'}_{\mathcal{M}}(F, \mathbb{Z}/l(q+t'))$, so that one may apply the comparison in Proposition 6.1 to it and obtain:

$$P^r_s(B^{t'}x) = B^{(q+t'-r)(l-1)}P^r_{\mathcal{M}}(B^{t'}x) \text{ and } \beta P^r_s(B^{t'}x) = B^{(q+t'-r)(l-1)}\beta P^r_{\mathcal{M}}(B^{t'}x).$$

Making use of Lemma 6.3, we see that $P^r_s(B^{t'}x)$ simplifies to $B^{t'l}P^r_s(x)$ while $\beta P^r_s(B^{t'}x)$ simplifies to $B^{t'l}\beta P^r_s(x)$. $P^r_{\mathcal{M}}(B^{t'}x) = B^{t'}P^r_{\mathcal{M}}(X)$ and $\beta P^r_{\mathcal{M}}(B^{t'}x) = B^{t'}\beta P^r_{\mathcal{M}}(x)$. These prove (i). To obtain (ii), one needs to apply an iterated weight suspension $t'+1$-times followed by a degree suspension once. This produces the class $\Sigma^1_s B^{t'+1}x \in \tilde{H}^{2q+2t'+2}_{\mathcal{M}}(\Sigma^1_s F, \mathbb{Z}/l(q+t'+1))$. Now one applies the comparison in Proposition 6.1 to it. Then one makes use of Lemma 6.3 to pull-out the B from the left-hand-side. □

Examples 6.5. (i) Take $t = 1$. In this case one obtains

$$B^lP^r_s(x) = B^{(q+1-r)(l-1)}BP^r_{\mathcal{M}}(x) \text{ and}$$
$$B^l\beta P^r_s(x) = B^{(q+1-r)(l-1)}B\beta P^r_{\mathcal{M}}(x).$$

One may now also take $r = q$ to obtain, $B^lQ^q(x) = B^lP^q(x)$ and $B^l\beta Q^q(x) = B^l\beta P^q(x)$. Since B is not invertible, multiplication by B need not be injective and therefore, one cannot conclude that therefore $Q^q(x) = P^q(X)$ or that $\beta Q^q(x) = \beta P^q(x)$.

(ii) Take $t = 2$. In this case one obtains $B^lP^r_s(x) = B^{(q+1-r)(l-1)}BP^r_{\mathcal{M}}(x)$ and $B^l\beta P^r_s(x) = B^{(q+1-r)(l-1)}B\beta P^r_{\mathcal{M}}(x)$.

(iii) Take $t = 3$. In this case one obtains $B^{2l}P_s^r(x) = B^{(q+2-r)(l-1)}B^2 P_{\mathcal{M}}^r(x)$ and $B^{2l}\beta P_s^r(x) = B^{(q+2-r)(l-1)}B^2\beta P_{\mathcal{M}}^r(x)$. If, in addition, $r = q+1$, then this becomes $B^{2l}P_s^r(x) = B^{l+1}P_{\mathcal{M}}^r(x)$ and $B^{2l}\beta P_s^r(x) = B^{l+1}\beta P_{\mathcal{M}}^r(x)$. Once again, since B in not invertible, one cannot conclude that therefore $B^{l-1}P_s^r(x) = P_{\mathcal{M}}^r(x)$ or that $B^{l-1}\beta P_s^r(x) = \beta P_{\mathcal{M}}^r(x)$.

Observe that by the multiplicative properties of the operations and the observation that $P_{\mathcal{M}}^r(B) = 0$ if $r \geq 1$ ([**Voev1**, Lemma 9.8]):

(6.5.1)
$$P_{\mathcal{M}}^r(B^j\alpha) = B^j P_{\mathcal{M}}^r(\alpha),$$
$$\beta P_{\mathcal{M}}^r(B^j\alpha) = B^j \beta P_{\mathcal{M}}^r(\alpha).$$

The above relations show that the motivic cohomology operations above induce operations on $H^*(\quad, \mathbb{Z}/l(\star))[B^{-1}]$ in the obvious manner: we define $P_{\mathcal{M}}^r(\alpha.B^{-1}) = P_{\mathcal{M}}^r(\alpha).B^{-1}$ and $\beta P_{\mathcal{M}}^r(\alpha.B^{-1}) = \beta P_{\mathcal{M}}^r(\alpha).B^{-1}$. Next we proceed to compare these induced motivic and simplicial operations on mod$-l$ étale cohomology,

Proposition 6.6. *(Comparison of operations in mod$-l$ étale cohomology.) Assume that the base field k has a primitive l-th root of unity. Let F denote a pointed simplicial sheaf on* $(\mathrm{Sm}/k)_{ét}$. *Let $\alpha \in H^i_{ét}(F, \mu_l(q))$ for some $q \geq 0$. Then*

$$P_s^r(\alpha) = B^{(q-r).(l-1)}.P_{\mathcal{M}}^r(\alpha), \quad \beta P_s^r(\alpha) = B^{(q-r).(l-1)}.\beta P_{\mathcal{M}}^r(\alpha)$$

for $r \leq i/2$ and all $i \geq 0$. $P_s^r(\alpha) = 0$ and $P_{\mathcal{M}}^r(\alpha) = 0$ for $r > i/2$.

Proof. For the case $r \leq q$ this follows from Proposition 6.1. For the other cases it follows by expanding the exponents of B on both sides of the formulae in Proposition 6.4 and canceling out all the powers of B on the left-hand-side. □

7. Cohomological operations that commute with proper push-forwards and Examples

The operations considered so far commute with pull-backs only and do not commute with push-forwards by proper maps. In this section we modify the above operations to obtain operations that commute with proper-push-forwards. The goal of this discussion is to consider the analogues of degree formulae in mod-p motivic cohomology: such degree formulae have played a major role in some of the applications of motivic cohomology operations. The key to this is the following formula, which follows by a deformation to the normal cone argument as shown in [**FL**, Chapter VI]. We state this for the convenience of the reader. Recall that motivic cohomology is a contravariant functor on smooth schemes. By identifying motivic cohomology with higher Chow groups, one may show the former is also covariant for proper maps.

Proposition 7.1. *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ X' & \xrightarrow{i'} & W' \end{array}$$

denote a cartesian square with all schemes smooth and with the vertical maps either regular closed immersions or projections from a projective space. Let the normal *bundle associated to i (i) be N (N′, respectively). Then the square commutes:*

$$\begin{array}{ccc} H^*(X', \mathbb{Z}/l(\bullet)) & \xrightarrow{i'_*} & H^*(W', \mathbb{Z}/l(\bullet)) \\ \downarrow{\scriptstyle e(N)f^*} & & \downarrow{\scriptstyle g^*} \\ H^*(X, \mathbb{Z}/l(\bullet)) & \xrightarrow{i_*} & H^*(W, \mathbb{Z}/l(\bullet)) \end{array}$$

where $N = f^*(N')/N$ is the excess normal bundle and $e(N)$ denotes the Euler-class of N. In case g and hence f are also closed immersions with normal bundles N_g and N_f, respectively, then $N \cong N_{g|X}/N_f$. Moreover if a finite constant group scheme G acts on the above schemes, the corresponding assertions holds in the G-equivariant motivic cohomology defined below.

Definition 7.2. Let G denote a finite group acting on a scheme X. Then we let $\mathbb{H}_G(X, \mathbb{Z}/(r)) = \underset{\Delta}{\text{holim}}\, R\Gamma(EG \underset{G}{\times} X, \mathbb{Z}/l(r))$ following the terminology in [**J2**, Section 6]. We let $H^n_G(X, \mathbb{Z}/l(r)) = \pi_{-n}(\mathbb{H}_G(X, \mathbb{Z}/l(r))$.

Remarks 7.3. 1. One may now verify that if $G = Z/l$, for a fixed prime l, then

$$H^*_G(\mathrm{Spec}\, k, \mathbb{Z}/l(\bullet)) \cong H^*(\mathrm{Spec}\, k, \mathbb{Z}/l(\bullet)) \otimes H^*_{sing}(BG, Z/l)$$

where $H^*(\mathrm{Spec}\, k, \mathbb{Z}/l(\bullet))$ denotes the motivic cohomology of Spec k and $H^*_{sing}(BG, Z/l)$ denotes the singular cohomology of the space BG with $\mathbb{F}_l$-coefficients. Recall that if $l = 2$, $H^*_{sing}(BG, Z/l)$ is a polynomial ring in one variable and when $l > 2$, $H^*_{sing}(BG, Z/l) = Z/l[t] \otimes \Lambda[\nu]$ where $\beta t = \nu$ and $\Lambda[\nu]$ denotes an exterior algebra in one generator ν.

2. The situation where will apply the above proposition will be the following: X will denote a given smooth scheme and X' will denote $X^{\times^l}$. W will denote another smooth scheme provided with a closed immersion $X \to W$ and W' will denote $W^{\times^l}$. In this case the normal bundle associated to the diagonal imbedding of X in $X^{\times^l}$ is $T_{X^{\times^{l-1}}}$ (the normal bundle associated to the diagonal imbedding of W in $W^{\times^l}$ is $T_{W^{\times^{l-1}}}$,respectively). As equivariant vector bundles for the obvious permutation action of $\mathbb{Z}/l$ on $X^{\times^l}$ and $W^{\times^l}$ these identify with $R \otimes_k T_X$ and $R \otimes_k T_W$ where R is the representation of $\mathbb{Z}/l$ given by $(k[x]/(x^l - 1))/k$. For a line bundle $\mathcal{E}$, let $w(\mathcal{E}, t) = 1 + c_1(L)^{l-1}t$. One extends the definition of $w(\mathcal{E}, t)$ to all vector bundles $\mathcal{E}$ by making this class take short exact sequences to products. Then the Euler-class $e(R \otimes_k T_X) = t^{dim(X)} w(T_X, 1/t)$ and $e(R \otimes_k T_W) = t^{dim(W)} w(T_W, 1/t)$.

At this point, we may adopt the arguments as in [**Bros**] to define cohomology operations that are compatible with push-forwards by proper maps between quasi-projective schemes. i.e. Let $Q^\bullet : H^*(X, \mathbb{Z}/l(\bullet)) \to H^*(X, \mathbb{Z}/l(\bullet))$ denote the total operation defined by $Q^\bullet = \Sigma_s Q^s$. Now we define the *covariantly functorial* operations Q_s by letting

$$Q_\bullet = \Sigma_s Q_s = Q^\bullet \cap w(T_X)^{-1} \tag{7.3.1}$$

(Recall that the class $w(T_X)$ is invertible.) If we re-index motivic cohomology homologically, (i.e. if X is proper and of pure dimension d, we let $H_n(X, \mathbb{Z}/l(r)) =$

$H^{2d-n}(X, \mathbb{Z}/(d-r)))$ the operations Q_s map $H_n(X, \mathbb{Z}/l(t))$ to $H_{n-2s(l-1)}(X, \mathbb{Z}/l(tl - d(l-1)))$.

Proposition 7.4. *Let $f : X \to Y$ denote a proper map between quasi-projective schemes over* $\mathrm{Spec}\, k$. *Then $Q_\bullet \circ f_* = f_* \circ Q_\bullet$.*

Proof. Since X and Y are quasi-projective, f may be factored as a closed immersion $i : X \to Y \times \mathbb{P}^n$ for some projective space $\mathbb{P}^n$ and the obvious projection $\pi : Y \times \mathbb{P}^n \to Y$. Therefore, it suffices to prove the assertion separately for $f = i$ and for $f = \pi$. The case $f = i$ is clear from the statements above. Next observe that $\mathbb{P}^n$ is a linear scheme and therefore the motivic cohomology of $X \times \mathbb{P}^n$ is given by an obvious Kunneth formula: see [**AJ**, Appendix] for example. Therefore the Cartan formula immediately implies the required assertion for the case $f = \pi$. □

We proceed to consider various examples.

7.5. Examples. The first example we consider is an operation

$$Q_s : H_q(X, \mathbb{Z}/l(t)) \to H_{q-2s(l-1)}(X, \mathbb{Z}/l(tl - d(l-1)))$$

on a projective smooth scheme X of dimension d so that the composition with the proper map $\pi_* : H_{q-2s(l-1)}(X, \mathbb{Z}/l(tl-d(l-1))) \to H_{q-2s(l-1)}(\mathrm{Spec}\, k, \mathbb{Z}/l(tl-d(l-1)))$ is in fact zero.

For example, one may take $dim(X) = 3$, $q = 2$, $t = 1$, $s = 1$ and $l = 2$. Now we have the operation

$$Q_1 : H_2(X, \mathbb{Z}/2(1)) \to H_0(X, \mathbb{Z}/2(-1)).$$

In cohomology notation this identifies with an operation $Q_1 : H^4(X, \mathbb{Z}/2(2)) \to H^6(X, \mathbb{Z}/2(4))$. The projection to $\mathrm{Spec}\, k$ sends the source to the group $H_2(\mathrm{Spec}\, k, \mathbb{Z}/2(1)) \cong H^{-2}(\mathrm{Spec}\, k, \mathbb{Z}/2(-1)) \cong CH^{-1}(Speck\, k, \mathbb{Z}/2) = 0$. It follows that $\pi_* \circ Q_1 = 0$. Recall that $H^4(X, \mathbb{Z}/2(2))$ identifies with $CH^2(X, \mathbb{Z}/2)$. Therefore any closed integral sub-scheme of X of codimension 2 defines a class in this group. If α is such a class, our conclusion is that $\pi_*(Q_1(\alpha)) = 0$.

So far we did not put any restriction on the prime l. Next we assume $l = p$. Let $\nu(r)$ be the sheaf that is kernel of $W^* - C : Z\Omega^r_{X/S} \to \Omega^r_{X^{(p)}/S}$. Here $X^{(p)}$ is the scheme obtained as the pull-back of $X \underset{S}{\times} S$ where the map $S \to S$ is the absolute Frobenius and $S = \mathrm{Spec}\, k$ is the base field. Moreover W^* is defined as the adjoint to the obvious map $\Omega^r_{X/S} \to W_* \Omega^r_{X^{(p)}/S}$ and $Z\Omega^r_{X/S}$ denotes the kernel of the differential $d : \Omega^r_{X/S} \to \Omega^{r+1}_{X/S}$. (See [**Ill**, 2.4] for more details.) It is known that $\nu(0) =$ the constant sheaf Z/p, $\nu(1) = dlog(\mathcal{O}^*_X)$ and that $\nu(r)$, viewed as a sheaf on $X_{ét}$ is generated locally by $dlog(x_1). \cdots dlog(x_r)$, $x_i \in \mathcal{O}^*_X$.

It is shown in [**GL**, Theorem 8.4] that if X is a smooth integral scheme over k and k is perfect, then one has the natural isomorphism (induced by a quasi-isomorphism $\nu(r)[-r] \simeq \mathbb{Z}/p(r)$) $H^s(X, \nu(r)) \cong H^{s+r}(X, \mathbb{Z}/p(r))$, where cohomology denotes cohomology computed either on the Zariski or étale sites.

Therefore, if we require $l = p$ and the field k is perfect, the last operation takes on the form

$$Q_1 : H^2(X, \nu(2)) \to H^2(X, \nu(4))$$

where cohomology denotes cohomology computed either on the Zariski or étale sites.

As another example, we may assume $dim(X) = 4$, $q = 3$, $t = 1$, $s = 1$ and $l = 3$. Now we obtain the operation $Q_1 : CH^3(X, \mathbb{Z}/3, 1) \cong H^5(X, \mathbb{Z}/3(3)) \to H^9(X, \mathbb{Z}/3(9))$. Re-indexing homologically this identifies with

$$Q_1 : H_3(X, \mathbb{Z}/3(1)) \to H_{-1}(X, \mathbb{Z}/3(-5)).$$

Now $\pi_* \circ Q_1 = Q_1 \circ \pi_*$ and π_* maps the group $H_3(X, \mathbb{Z}/3(1))$ to $H_3(\mathrm{Spec}\, k, \mathbb{Z}/3(1)) \cong H^{-3}(\mathrm{Spec}\, k, \mathbb{Z}/3(-1)) \cong CH^{-1}(\mathrm{Spec}\, k, \mathbb{Z}/3, 1) = 0$ since the higher Chow groups indexed by the codimension are trivial for negative codimension. Therefore, the composition $\pi_* \circ Q_1 = 0$. In case $l = p$, this operation now takes on the form

$$Q_1 : H^2(X, \nu(3)) \to H^0(X, \nu(9)).$$

As yet another example, we will presently show that the only simplicial operations that send the usual mod-l Chow groups to the usual mod-l Chow groups are the power operations. Recall that the usual mod$-l$ Chow groups are given by the mod$-l$ motivic cohomology groups $H^{2n}(X, \mathbb{Z}/l(n))$. Now let $Q^s : H^{2t}(X, \mathbb{Z}/l(t)) \to H^{2t+2s(l-1)}(X, \mathbb{Z}/l(lt))$ be given so that the $2t + 2s(l-1) = 2lt$. Then $2s(l-1) = 2t(l-1)$ so that $s = t$. Therefore we see from Theorem 5.2(ii) that the given operation is none other than the l-th power operation.

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