# BRAUER GROUPS OF ALGEBRAIC STACKS AND GIT-QUOTIENTS:I

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Abstract. In this paper we consider the Brauer groups of algebraic stacks and GIT quotients: the only algebraic stacks we consider in this paper are quotient stacks [X/G] with certain restrictions on X and G, and defined over a Dedekind domain, a discrete valuation ring, or a field. We discuss the calculation of the Brauer groups of various examples: the class of smooth toric stacks over Dedekind domains provides a large family of examples, and we show many familiar stacks fit in to this framework. This will be continued in a sequel, where we will discuss more examples, especially the Brauer groups of various moduli stacks of principal G-bundles as well as the Brauer groups of their GIT quotients.

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## 1. Introduction and the Main Results

The paper originated in an effort by the authors to study the Brauer groups quotient stacks and of GIT quotients associated to actions of reductive groups. We began by assuming the base scheme is a separably closed field, then soon extended our framework to the case where it is any field. While working on various examples, we realized that as several of the algebraic stacks one encounters often are defined over the ring of integers or Dedekind domains, it is preferable to adopt a more general framework as follows. This enables us to consider cohomological invariants of algebraic stacks defined over arbitrary Dedekind domains. The only

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cohomological invariants we consider will be the $m$ torsion part of the Brauer group, where $m$ is invertible in the given base ring.

Let B denote a regular Noetherian excellent scheme of dimension at most 1: B will serve as the *base scheme*. We also consider two basic situations here:

### 1.1. **Basic hypotheses.**

(i) $\mathrm{B} = \operatorname{Spec} \mathrm{R}$, where R is a Dedekind domain, or a DVR which is assumed to be excellent (for example, the ring of integers $\mathbb{Z}$ or its localization at a prime $p$), *or*

(ii) B is a smooth scheme of pure dimension at most 1 over a field $k$.

Let $m$ denote a fixed positive integer invertible in $\mathcal{O}_{\mathrm{B}}$ and let X denote a scheme of finite type over B. Then one begins with the *Kummer sequence*

$$1 \to \mu_m(1) \to \mathbb{G}_m \overset{m}{\to} \mathbb{G}_m \to 1, \tag{1.1}$$

which holds on the (small) étale site $\mathrm{X}_{et}$ of X, whenever $m$ is invertible in $\mathcal{O}_{\mathrm{B}}$. (See [Gr, section 3] or [Mi, p. 66].) Taking étale cohomology, we obtain corresponding long-exact sequence:

$$\to \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \overset{m}{\to} \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \cdots , \tag{1.2}$$

which holds on the étale site when $m$ is invertible in $\mathcal{O}_{\mathrm{B}}$.

**Definition 1.1.** *The cohomological Brauer group* $\mathrm{Br}(\mathrm{X})$ *is the torsion subgroup of the cohomology group* $\mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}})$*. In other words,* $\mathrm{Br}(\mathrm{X}) = \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}})_{\mathrm{tors}}$. [1]

Next assume X is smooth over the base scheme B. Then, by Hilbert's Theorem 90, we obtain the isomorphisms:

$$\mathrm{Pic}(\mathrm{X}) \cong \mathrm{CH}^1(\mathrm{X}) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}), \tag{1.3}$$

where $\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z})$ denotes motivic cohomology (in degree 2 and weight 1) whose definition for smooth schemes of finite type over B is worked out in [Geis], and we recall this in the Appendix. Then one also obtains the short-exact sequence:

$$0 \to \mathrm{Pic}(\mathrm{X})/m \cong \mathrm{NS}(\mathrm{X})/m \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \to {}_m\mathrm{Br}(\mathrm{X}) \to 0, \tag{1.4}$$

where the map $\mathrm{Pic}(\mathrm{X})/m = \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/m) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_m(1))$ is the cycle map, and therefore, ${}_m\mathrm{Br}(\mathrm{X})$ identifies with the cokernel of the cycle map. Thus it follows that for smooth schemes X over B, ${}_m\mathrm{Br}(\mathrm{X})$ is trivial if and only if the above cycle map is surjective: our approach to the Brauer group adopted in this paper is to consider the above cycle map from motivic cohomology to etale cohomology, and involves a combination of motivic and étale cohomology techniques. Moreover, in view of this, *we will always restrict to smooth schemes of finite type over the given base scheme* B. However, apart from the restriction to smooth schemes, our approach making use of both motivic and étale cohomology techniques over Dedekind domains offers considerable advantages in various computations: these will become clear in later sections of the paper.

[1] If X is a regular integral Noetherian scheme, one may observe that $\mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}})_{\mathrm{tors}} = \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}})$: see, for example, [CTS, Lemma 3.5.3]

Let G denote a not-necessarily connected smooth affine group scheme, of finite type over B, and acting on the given scheme X. Next we recall the framework of Borel-style equivariant étale cohomology, and Borel-style equivariant motivic cohomology. For this we form an ind-scheme $\{\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}|\mathrm{m}\}$ and then take its étale cohomology, and also its motivic cohomology when X is also assumed to be smooth. One may consult [Tot99], [MV99], and also section 3 of this paper for more details. Here $\mathrm{BG}^{\mathrm{gm,m}}$ is a finite dimensional approximation to the classifying space of the affine group scheme G, and $\mathrm{EG}^{\mathrm{gm,m}}$ denotes the universal principal G-bundle over $\mathrm{BG}^{\mathrm{gm,m}}$. In the terminology of Definition 2.1, $\mathrm{EG}^{\mathrm{gm,m}} = \mathrm{U_m}$ and $\mathrm{BG}^{\mathrm{gm,m}} = \mathrm{U_m/G}$. We also assume that such a $\mathrm{BG}^{\mathrm{gm,m}}$ exists for every $m \geq 0$, as a quasi-projective scheme over the given base B. There are standard arguments to prove that that the cohomology of the ind-schemes $\{\mathrm{BG}^{\mathrm{gm,m}}|\mathrm{m} \geq 0\}$, $\{\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X}|\mathrm{m} \geq 0\}$ are independent of the choice of the admissible gadgets $\{\mathrm{U_m}|\mathrm{m} \geq 0\}$ that enter into their definition: see, for example, Proposition 3.5.

*Let $m$ denote a fixed positive integer invertible in $\mathcal{O}_{\mathrm{B}}$.* Then we let $\mathrm{H}^{*,\bullet}_{\mathrm{G,M}}(\mathrm{X}, \mathbb{Z}/m)$ denote the motivic cohomology of $\{\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}|\mathrm{m}\}$ defined as the homotopy inverse limit of the motivic cohomology of the finite dimensional approximations $\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}$, that is, defined by the usual Milnor exact sequence relating $lim^1$ and $lim$ of the motivic hypercohomology of the above finite dimensional approximations. (When $* = 2i$ and $\bullet = i$, for a non-negative integer $i$, these identify with the usual (equivariant) Chow groups.) $\mathrm{H}^{*}_{\mathrm{G,et}}(\mathrm{X}, \mu_m(\bullet))$ is defined similarly and will be often denoted $\mathrm{H}^{*,\bullet}_{\mathrm{G,et}}(\mathrm{X}, \mu_m)$.

Recall that for each fixed integer $i \geq 0$, one obtains the isomorphisms (for $m$ chosen, depending on $i$):

$$\mathrm{H}^{2\mathrm{i,i}}_{\mathrm{G,M}}(\mathrm{X}, \mathbb{Z}/m) \cong \mathrm{H}^{2\mathrm{i,i}}_{\mathrm{M}}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathbb{Z}/m), m \gg 0 \text{ and X smooth, and}$$

$$\mathrm{H}^{2\mathrm{i,i}}_{\mathrm{G,et}}(\mathrm{X}, \mu_m) \cong \mathrm{H}^{2\mathrm{i}}_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}, \mu_m(\mathrm{i})), \mathrm{m} \gg 0.$$

These show that one may define the G-*equivariant Brauer group* of a G-scheme X as follows:

**Definition 1.2.** $\mathrm{Br_G(X)} = \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,u}}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathbb{G}_\mathrm{m})_{\mathrm{tors}}$*, for $u \gg 0$, where the subscript tors denotes the torsion subgroup.*

Moreover, we obtain from the Kummer-sequence the short-exact sequence:

$$(1.5)\qquad 0 \to \mathrm{Pic}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X})/m \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}, \mu_m(1)) \to {}_m\mathrm{Br}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}) = {}_m\mathrm{Br_G(X)} \to 0 \text{ and}$$

where

$\mathrm{Pic}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X})/m = \mathrm{coker}(\mathrm{Pic}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X})\xrightarrow{m}\mathrm{Pic}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}))$,
${}_m\mathrm{Br_G(X)}$ = the $m$-torsion part of $\mathrm{Br_G(X)}$.

**Definition 1.3.** *Given an Artin stack $S$ of finite type over the base scheme* B*, we define its Brauer group to be* $\mathrm{H}^2_{\mathrm{smt}}(S, \mathbb{G}_m)_{tors}$*, where* $\mathrm{H}^2_{\mathrm{smt}}(S, \mathbb{G}_m)$ *denotes cohomology computed on the smooth site (see* (3.20)*), and the subscript tors denotes its torsion subgroup. We denote this by* $\mathrm{Br}(S)$*. For a fixed positive integer $m$ invertible in $\mathcal{O}_{\mathrm{B}}$, we let ${}_m\mathrm{Br}(S)$ denote the $m$-torsion part of* $\mathrm{Br}(S)$.

Then our first result is the following, which shows the Brauer group of a quotient stack [X/G], so defined, identifies with the G-equivariant Brauer group defined in Definition 1.2.

**Theorem 1.4.** *Assume that* X *is a smooth scheme of finite type over the base scheme* B *satisfying one of the hypotheses 1.1, and provided with an action by the affine smooth group scheme* G. *Then, assuming the above terminology,*

$$_m\mathrm{Br}([\mathrm{X}/\mathrm{G}]) \cong {_m\mathrm{Br}_{\mathrm{G}}}(\mathrm{X}).$$

*Therefore,* $_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X})$ *is intrinsic to the quotient stack* [X/G].

**Remark 1.5.** The comparison theorem [J20, Theorem 1.6] shows that $\mathrm{H}^*_{\mathrm{G,et}}(\mathrm{X}, \mu_m(\bullet))$ identifies with $\mathrm{H}^*_{\mathrm{smt}}([\mathrm{X}/\mathrm{G}], \mu_m(\bullet))$, which denotes the cohomology of the quotient stack [X/G] computed on the smooth site. However, it may be important to point out that this result does *not* imply the above theorem, mainly because the Picard groups appearing in (1.5) and in the corresponding short-exact sequence for the quotient stack [X/G] need not be trivial. See the first paragraph in section 3 for a more detailed discussion on this issue.

We derive a number of results based on the above theorem, a few of which are listed below.

**Corollary 1.6.** *Assume in addition to the hypotheses of Theorem 1.4 that* $_m\mathrm{Br}(\mathrm{X}) = 0$. *Then* $_m\mathrm{Br}([\mathrm{X}/\mathrm{G}]) \cong 0$ *as well in the following cases:*

(i) G *is a split torus, or*
(ii) G *is a finite product of general linear groups.*

We define a *toric stack* $\mathcal{X}$ to be an algebraic stack of the following form: let X denote a toric variety defined over a field $k$ for the split torus $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^s$, or more generally a toric scheme over an excellent Dedekind domain R in the sense of [JL, section 4(i), (4.1)]. (Recall that the hypotheses in [JL, section 4(i), (4.1)](i.e., 10.5) require that the toric scheme X contain as an open subscheme a split torus $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^s$ and that all the T-orbits are defined and faithfully flat over B = Spec R.) We will assume such an X comes equipped with a homomorphism $\phi : \mathrm{T}_0 = \mathbb{G}_{\mathrm{m}}^r \rightarrow \mathrm{T}$, for some $r > 0$. Then we require that $\mathcal{X} = [\mathrm{X}/\mathrm{T}_0]$.

**Theorem 1.7.** *(i) The scheme* $(\mathbb{A}^2 - \{0\})^r \times_{\mathrm{T}_0} \mathrm{X}$, *where* $\mathrm{T}_0$ *acts on* X *through* $\phi$ *and it acts diagonally on* $(\mathbb{A}^2 - \{0\})^r \times \mathrm{X}$*, is a split toric scheme over* $k$*, for the split torus* $\mathrm{T}_0 \times \mathrm{T}$.

*(ii) Therefore,* $_m\mathrm{Br}([\mathrm{X}/\mathrm{T}_0])$ *is isomorphic to the* $m$*-torsion part of the Brauer group of the toric scheme* $(\mathbb{A}^2 - \{0\})^r \times_{\mathrm{T}_0} \mathrm{X}$.

**Remarks 1.8.**
(i) Making use of the determination of the Brauer groups of smooth toric schemes over any field or a Dedekind domain as in [JL, Theorems 2.1, 4.1] (i.e., Theorem 10.4), Theorem 1.7(ii) enables one to determine the Brauer groups of all toric stacks: see Theorem 5.8 and Corollary 5.9, as well as Example 5.10.
(ii) In Theorem 5.14, we also extend the above calculations to determine the Brauer groups of quotient stacks of the form $[\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]$, where $\ell$ is prime to the characteristic and $\mu_{\ell^n}$ acts on the smooth scheme X.

As an application of Theorem 1.4 for stacks defined over Dedekind domains, and Theorem 1.7 we also obtain the following computations. Let R denote any excellent Dedekind domain in which both 2 and 3 are invertible, for example, $\mathbb{Z}_{1/6}$ which denotes the localization of $\mathbb{Z}$ by inverting 6. Let $\mathcal{M}_{1,1,\mathrm{R}}$ denote the moduli stack of elliptic curves defined over R. Let $\mathrm{Y} = \mathrm{Spec}\,\mathrm{R}[\mathrm{g}_2, \mathrm{g}_3][1/\Delta] \subseteq \mathbb{A}^2_{\mathrm{R}}$, where $\Delta = g_2^3 - 27g_3^2$.

**Theorem 1.9.** *(See Corollary 6.2.) Next assume the base ring* R *is an excellent Dedekind domain or a field in which* 2 *and* 3 *are invertible and the following hold:* $\ell$ *denotes a prime not necessarily different from* 2 *or* 3*, so that* $\ell$ *is also invertible in* R*, and* R *contains a primitive* $\ell^n$*-th root of unity for some positive integer* $n$*. Then* ${}_{\ell^n}\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}}) \cong {}_{\ell^{\mathrm{n}}}\mathrm{Br}(\mathrm{Spec}\,\mathrm{R}) \oplus \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_{\ell^{\mathrm{n}}}(0))$*.*

In fact, several more related results on the Brauer group of $\mathcal{M}_{1,1,\mathrm{R}}$ under different hypotheses are discussed in section 6. For example, in Theorem 6.1, we have a more general calculation of the $m$-torsion part of the Brauer group of the stack $\mathcal{M}_{1,1,\mathrm{R}}$, where R is any Dedekind domain with the primes 2 and 3 invertible in R.

Next we discuss the following application of the torsion index of linear algebraic groups: see [Tot05, section 1].

**Theorem 1.10.** *Assume the base scheme is any field* $k$ *and that* $m$ *is a fixed positive integer invertible in* $k$*. If* H *is a connected linear algebraic group defined over* $k$ *and whose torsion index is prime to* $m$*, then* ${}_m\mathrm{Br}(\mathrm{BH}) = {}_m\mathrm{Br}(\mathrm{Spec}\,k)$*, where* BH *denotes the classifying stack of* H*, that is* $[\mathrm{Spec}\,k/\mathrm{H}]$*. In particular, the following hold:*

*(i)* ${}_m\mathrm{Br}(\mathrm{BG}) = {}_m\mathrm{Br}(\mathrm{Spec}\,k)$ *for any positive integer* $m$ *invertible in* $k$ *if* $\mathrm{G} = \mathrm{GL_n}$*,* $\mathrm{G} = \mathrm{SL_n}$ *or* $\mathrm{G} = \mathrm{Sp(2n)}$*, for any* $n$*.*

*(ii)* ${}_{\ell^{n'}}\mathrm{Br}(\mathrm{BG}) = {}_{\ell^{\mathrm{n}'}}\mathrm{Br}(\mathrm{Spec}\,k)$ *for any prime* $\ell$ *different from the characteristic of* $k$ *and* 2 *if* $\mathrm{G} = \mathrm{SO(2n)}$*,* $\mathrm{SO}(2n+1)$*, or* $\mathrm{Spin}(n)$*, for any* $n$ *and* $n'$*.*

*(iii)* ${}_{\ell^{n'}}\mathrm{Br}(\mathrm{BG}) = {}_{\ell^{\mathrm{n}'}}\mathrm{Br}(\mathrm{Spec}\,k)$ *for any simply-connected group* G*, if* $\ell$ *is different from the characteristic of* $k$ *and also different from* $2, 3$*, or* 5*.*

We conclude with the following example. Let X denote a smooth projective curve of genus $g$ over a field $k$, provided with a $k$-rational point. Then one knows the isomorphism of stacks (see for example, [Wang, Proposition 4.2.5]):

$$\mathrm{Bun}_{1,d}(\mathrm{X}) \cong \mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}} \times \mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X}), \tag{1.6}$$

where $\mathrm{B}\mathbb{G}_m^{gm} = \lim_{n\to\infty} \mathrm{B}\mathbb{G}_m^{gm,n}$, $\mathrm{Bun}_{1,d}(\mathrm{X})$ denotes the moduli stack of line bundles of degree $d$ on X and $\mathbf{Pic}^d(\mathrm{X})$ denotes the Picard scheme. In view of the above isomorphism of stacks, one may define the Brauer group of the stack $\mathrm{Bun}_{1,d}(\mathrm{X})$ to be the Brauer group of the stack $\mathrm{B}\mathbb{G}_m^{\mathrm{gm}} \times \mathbf{Pic}^d(\mathrm{X})$. Then, we obtain the following.

**Proposition 1.11.** *Assume the base field* $k$ *is separably closed, and that* $m$ *is positive integer invertible in* $k$*. Then, assuming the above situation, we obtain the isomorphism:*

$${}_m\mathrm{Br}(\mathrm{Bun_{1,d}(X)}) \cong {}_m\mathrm{Br}(\mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X})) \cong {}_m\mathrm{Br}(\mathrm{Sym}^{\mathrm{d}}(\mathrm{X})),$$

*where* $\mathrm{Sym}^d(\mathrm{X})$ *denotes the* $d$*-fold symmetric power of the curve* X*. In particular,* ${}_m\mathrm{Br}(\mathrm{Bun_{1,d}(X)}) \cong 0$ *if* X *is rational.*

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## 2. Equivariant Brauer groups

We begin by discussing the construction of geometric classifying spaces and the Borel construction making use of such gadgets. Throughout this section, the base scheme B will denote the spectrum of a regular Noetherian integral domain R of dimension at most 1 as in 1.1.

### 2.1. **Admissible gadgets.**

Let G denote a fixed smooth affine group scheme over B. We will define a pair $(\mathrm{W}, \mathrm{U})$ of smooth schemes over B to be a *good pair* for G if W is a representation of G and $\mathrm{U} \subsetneq \mathrm{W}$ is a G-invariant non-empty open subscheme on which G acts freely and so that U/G is a quasi-projective scheme over B. Moreover, one may choose $(\mathrm{W}, \mathrm{U})$ so that the complement $\mathrm{W} - \mathrm{U}$ has sufficiently high codimension. It is known (*cf.* [Tot99])that a good pair for G always exists.

**Definition 2.1.** *A sequence of pairs* $\{(\mathrm{W_m}, \mathrm{U_m}) | \mathrm{m} \geq 1\}$ *of smooth schemes over* B *is called an admissible gadget for* G*, if there exists a good pair* $(\mathrm{W}, \mathrm{U})$ *for* G *such that* $\mathrm{W_m} = \mathrm{W}^{\times^{\mathrm{m}}}$ *and* $\mathrm{U}_m \subsetneq \mathrm{W_m}$ *is a* G*-invariant open subset such that the following hold for each* $m \geq 1$.

1. $(\mathrm{U}_m \times \mathrm{W}) \cup (\mathrm{W} \times \mathrm{U_m}) \subseteq \mathrm{U}_{m+1}$ *as* G*-invariant open subschemes.*
2. $\{\mathrm{codim}_{\mathrm{U_{m+1}}}(\mathrm{U_{m+1}} \setminus (\mathrm{U_m} \times \mathrm{W})) | \mathrm{m}\}$ *is a strictly increasing sequence, that is,*
$$\mathrm{codim}_{\mathrm{U_{m+2}}}(\mathrm{U_{m+2}} \setminus (\mathrm{U_{m+1}} \times \mathrm{W})) > \mathrm{codim}_{\mathrm{U_{m+1}}}(\mathrm{U_{m+1}} \setminus (\mathrm{U_m} \times \mathrm{W})).$$
3. $\{\mathrm{codim}_{\mathrm{W_m}}(\mathrm{W_m} \setminus \mathrm{U_m}) | \mathrm{m}\}$ *is a strictly increasing sequence, that is,*
$$\mathrm{codim}_{\mathrm{W_{m+1}}}(\mathrm{W_{m+1}} \setminus \mathrm{U_{m+1}}) > \mathrm{codim}_{\mathrm{W_m}}(\mathrm{W_m} \setminus \mathrm{U_m}).$$
4. $\mathrm{U}_m$ *has a free* G*-action, the quotient* $\mathrm{U}_m/\mathrm{G}$ *is a smooth quasi-projective scheme over* B *and* $\mathrm{U}_m \to \mathrm{U}_m/\mathrm{G}$ *is a principal* G*-bundle.*
5. *In addition, we will also assume the following (see* [MV, Definition 2.1, p. 133]*):*

   *the structure map* $\mathrm{U_m} \to \mathrm{B}$ *has a section.*

**Lemma 2.2.** *Let* U *denote a smooth quasi-projective scheme over* B *with a free action by the smooth affine group scheme* G *so that the quotient* U/G *exists as a smooth quasi-projective scheme over* B*. Then if* X *is any smooth* G*-quasi-projective scheme over* B*, the quotient* $\mathrm{U} \underset{\mathrm{G}}{\times} \mathrm{X} \cong (\mathrm{U} \underset{\mathrm{Spec\,B}}{\times} \mathrm{X})/\mathrm{G}$ *(for the diagonal action of* G*) exists as a scheme over* B.

*Proof.* This follows, for example, from [MFK94, Proposition 7.1]. □

**Example 2.3.** *An example of an admissible gadget for* G *can be constructed as follows: start with a good pair* $(\mathrm{W}, \mathrm{U})$ *for* G*. The choice of such a good pair will vary depending on* G*, but may be chosen as follows. Choose a faithful representation* R *of* G *of dimension* $n$*, that is,* G *admits a closed immersion into* GL(R)*. Then* G *acts freely on an open subscheme* U *of* $\mathrm{W} = \mathrm{R}^{\oplus n} = \mathrm{End(R)}$ *so that* U/G *is a scheme. (For e.g.* $\mathrm{U} = \mathrm{GL(R)}$*.) Let* $\mathrm{Z} = \mathrm{W} \setminus \mathrm{U}$.

*Given a good pair* $(\mathrm{W}, \mathrm{U})$*, we now let*

$$\mathrm{W_m} = \mathrm{W}^{\times \mathrm{m}}, \mathrm{U_1} = \mathrm{U} \text{ and } \mathrm{U_{m+1}} = (\mathrm{U_m} \times \mathrm{W}) \cup (\mathrm{W} \times \mathrm{U_m}) \text{ for } \mathrm{m} \geq 1. \tag{2.1}$$

*Setting* $\mathrm{Z_1 = Z}$ *and* $\mathrm{Z_{m+1} = W_{m+1} \setminus U_{m+1}}$ *for* $m \geq 1$, *one checks that* $\mathrm{W_m \setminus U_m = Z^m}$ *and* $\mathrm{Z_{m+1} = Z^m \times Z}$. *In particular,* $\mathrm{codim_{W_m}(W_m \setminus U_m) = m(codim_W(Z))}$. *Moreover,* $\mathrm{U}_m \to \mathrm{U}_m/\mathrm{G}$ *is a principal* G*-bundle and the quotient* $\mathrm{V_m = U_m/G}$ *exists as a smooth quasi-projective scheme.*

2.2. **The geometric Borel construction.** Given an admissible gadget $\{(\mathrm{W_m, U_m})|\mathrm{m} \geq 0\}$ for the affine smooth group scheme G and a G-scheme X, we define

$$\mathrm{EG^{gm,m} = U_m}, \quad \mathrm{EG^{gm,m} \times_G X = U_m \times_G X}, \quad \mathrm{BG^{gm,m} = U_m \times_G B}, \text{ and} \\ \pi_m : \mathrm{EG^{gm,m} \times_G X \to BG^{gm,m}}. \tag{2.2}$$

The ind-scheme $\{\mathrm{EG^{gm,m} \times_G X}|\mathrm{m} \geq 0\}$ is called the *geometric Borel construction.* We will often denote $\lim_{m\to\infty}\{\mathrm{EG^{gm,m} \times_G X}|\mathrm{m} \geq 0\}$ by $\mathrm{EG^{gm} \times_G X}$: one may view this as a simplicial presheaf.

2.3. **Basic techniques for computing the equivariant Brauer groups.** In this section, we will discuss certain techniques that will facilitate the computation of equivariant Brauer groups. We will make use of the admissible gadgets defined in Example 2.3, as well as Corollary 3.4 and Proposition 3.5, which are all discussed in the next section.

Let $\mathrm{G = GL_n}$, for a fixed integer $n > 0$. Let $\mathrm{W = End}(\mathbb{A}^{\mathrm{n}})$ = the space of all $n \times n$-matrices with entries in $\mathcal{O}_{\mathrm{B}}$. In this case, we will let

$$\mathrm{EG^{gm,2}} = (\mathrm{GL_n} \times \mathrm{W}) \bigcup_{\mathrm{GL_n \times GL_n}} (\mathrm{W \times GL_n}). \tag{2.3}$$

The determination of the *bad set* $\mathrm{Z_m}$ as in Example 2.3 shows that the codimension of $\mathrm{EG^{gm,2}}$ in $\mathrm{W \times W}$ is 2. We also observe that the following hold, when X is a scheme provided with an action by $\mathrm{GL}_n$:

(i) $(\mathrm{GL}_n \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X}$ is open in $\mathrm{EG^{gm,2}} \times_{\mathrm{GL_n}} \mathrm{X}$ with the compliment being $((\mathrm{W - GL_n}) \times \mathrm{GL_n}) \times_{\mathrm{GL_n}} \mathrm{X}$
(ii) $((\mathrm{W - GL_n}) \times \mathrm{GL_n}) \times_{\mathrm{GL_n}} \mathrm{X}$ has codimension 1 in $(\mathrm{GL}_n \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X}$.

Therefore, [JL, Corollary 2.8] (i.e., Proposition 10.3) and Corollary 3.4 provide the short exact sequence:

$$0 \to {}_m\mathrm{Br_{GL_n}(X)} \to {}_m\mathrm{Br}((\mathrm{GL_n} \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X}) \longrightarrow \mathrm{H}^3_{((\mathrm{W-GL_n})\times \mathrm{GL_n})\times_{\mathrm{GL_n}}\mathrm{X,et}}((\mathrm{W \times GL_n}) \times_{\mathrm{GL_n}} \mathrm{X}, \mu_m(1)). \tag{2.4}$$

Clearly

$$(\mathrm{GL}_n \times \mathrm{W}) \times_{\mathrm{GL_n}} \mathrm{X} \cong \mathrm{W \times X} \text{ and } ((\mathrm{W - GL_n}) \times \mathrm{GL_n}) \times_{\mathrm{GL_n}} \mathrm{X} \cong (\mathrm{W - GL_n}) \times \mathrm{X}, \tag{2.5}$$

so that the exact sequence (2.4) identifies with

$$0 \to {}_m\mathrm{Br_{GL_n}(X)} \to {}_m\mathrm{Br(X)} \xrightarrow{\beta} \mathrm{H}^3_{(\mathrm{W-GL_n})\times \mathrm{X,et}}(\mathrm{W \times X}, \mu_m(1)). \tag{2.6}$$

We may also consider the following special cases of the above general result.

(i) Take $n = 1$, so that $\mathrm{GL}_n = \mathbb{G}_m$. In this case one may take $\mathrm{W} = \mathbb{A}^1$, so that $\mathrm{EG^{gm,2}} = \mathbb{A}^2 - \{0\} = \mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}} \bigcup \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1$. In this case the exact sequence (2.6) becomes

$$0 \to {}_m\mathrm{Br}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{X}) \to {}_m\mathrm{Br}((\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1) \times_{\mathbb{G}_{\mathrm{m}}} \mathrm{X}) \cong {}_m\mathrm{Br(X)} \xrightarrow{res} \mathrm{H}^1_{\mathrm{et}}((\mathbb{A}^1 - \mathbb{G}_{\mathrm{m}}) \times \mathbb{G}_{\mathrm{m}}) \times_{\mathbb{G}_{\mathrm{m}}} \mathrm{X}, \mu_m(0)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0)). \tag{2.7}$$

(ii) Take $n = 2$, so that in this case $\mathrm{W = End}(\mathbb{A}^2)$. Therefore, in this case the exact sequence (2.6) becomes

$$0 \to {}_m\mathrm{Br_{GL_2}(X)} \to {}_m\mathrm{Br(X)} \xrightarrow{\beta} \mathrm{H}^3_{(\mathrm{W-GL_2})\times \mathrm{X,et}}(\mathrm{W \times X}, \mu_m(1)). \tag{2.8}$$

(iii) In case G is a closed smooth affine sub-group-scheme of $\mathrm{GL}_n$, with G acting on X, one obtains an induced action by $\mathrm{GL}_n$ on $\mathrm{GL}_n \times_{\mathrm{G}} \mathrm{X}$. In view of Theorem 1.4, $\mathrm{Br}_{\mathrm{G}}(\mathrm{X}) \cong \mathrm{Br}_{\mathrm{GL_n}}(\mathrm{GL_n} \times_{\mathrm{G}} \mathrm{X})$. Therefore, in this case the exact sequence (2.6) becomes

$$0 \to {}_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{GL_n} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^3_{(\mathrm{W}-\mathrm{GL_n})\times(\mathrm{GL_n} \times_{\mathrm{G}} \mathrm{X}),\mathrm{et}}(\mathrm{W} \times (\mathrm{GL_n} \times_{\mathrm{G}} \mathrm{X}), \mu_m(1)). \tag{2.9}$$

As an *example* of this, assume X is provided by an action of $\mathrm{SL}_n$ which in fact extends to an action by $\mathrm{GL}_n$. Then $\mathrm{GL}_n \times_{\mathrm{SL}_n} \mathrm{X} \cong \mathrm{GL_n}/\mathrm{SL_n} \times \mathrm{X} \cong \mathbb{G}_{\mathrm{m}} \times \mathrm{X}$. Therefore, the exact sequence in (2.9) becomes

$$0 \to {}_m\mathrm{Br}_{\mathrm{SL_n}}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}) \to \mathrm{H}^3_{(\mathrm{W}-\mathrm{GL_n})\times(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}),\mathrm{et}}(\mathrm{W} \times (\mathbb{G}_{\mathrm{m}} \times \mathrm{X}), \mu_m(1)). \tag{2.10}$$

In particular, it follows that ${}_m\mathrm{Br}_{\mathrm{SL_n}}(\mathrm{X})$ injects into ${}_m\mathrm{Br}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}) \cong {}_m\mathrm{Br}(\mathrm{X} \oplus \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))$.

(iv) We next consider the case where $\mathrm{G} = \mathbb{G}_{\mathrm{m}}^r$, a split torus of rank $r$, or more generally a diagonalizable group scheme of the form $\mu_{n_1} \times \cdots \times \mu_{n_s} \times \mathbb{G}_m^t$, with $r = s + t$. In this case, we will always choose $\mathrm{EG}^{\mathrm{gm},2} = (\mathbb{A}^2 - \{0\})^r$. Moreover, we will also observe that when $\mathbb{G}_m^r$ is provided with an action on the scheme X with an induced action by G on X, the quotient $(\mathbb{A}^2 - \{0\})^r \times_{\mathrm{G}} \mathrm{X}$ identifies with a sum of $s$ line bundles over $(\mathbb{A}^2 - \{0\})^r \times_{\mathbb{G}_m^r} \mathrm{X}$ with their zero section removed. These observations will be very useful when we consider the Brauer groups of toric stacks.

**Example 2.4.** We will assume the base B is the spectrum of a field $k$ of characteristic different from 2. We show here as an immediate consequence of (2.8) above that if $\bar{\mathcal{H}}_g$ denotes the moduli stack of stable hyper-elliptic curves of genus $g \geq 2$ and *even*, then ${}_m\mathrm{Br}(\bar{\mathcal{H}}_{\mathrm{g}}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\, k)$, for any $\ell$ different from 2 and invertible in the base field $k$. As observed in [LP, Lemma A.3], there is an open substack $\bar{\mathcal{H}}'_g \subseteq \bar{\mathcal{H}}_g$ so that $\bar{\mathcal{H}}'_g = [\mathrm{U_g}/\mathrm{GL}_2]$, for an open subscheme $\mathrm{U_g} \subseteq \mathbb{A}^{2\mathrm{g}+3}$ so that the complement of $\mathrm{U_g}$ in $\mathbb{A}^{2g+3}$ has codimension greater than 1. Then the restriction ${}_m\mathrm{Br}(\bar{\mathcal{H}}_{\mathrm{g}}) \to {}_m\mathrm{Br}(\bar{\mathcal{H}}'_{\mathrm{g}})$ is injective. Now ${}_m\mathrm{Br}(\bar{\mathcal{H}}'_{\mathrm{g}}) = {}_m\mathrm{Br}([\mathrm{U_g}/\mathrm{GL}_2])$. By (2.8), the latter injects into ${}_m\mathrm{Br}(\mathrm{U_g}) \cong {}_m\mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\, k)$. This shows the composite map above ${}_m\mathrm{Br}(\bar{\mathcal{H}}_{\mathrm{g}}) \to {}_m\mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\, k)$ is an injection.

On the other hand, one has the pull-back $\pi^* : {}_m\mathrm{Br}(\mathrm{Spec}\, k) \to {}_m\mathrm{Br}(\bar{\mathcal{H}}_{\mathrm{g}})$: the composition of this map with the above map ${}_m\mathrm{Br}(\bar{\mathcal{H}}_{\mathrm{g}}) \to {}_m\mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\, k)$ is clearly an isomorphism. This shows that the map ${}_m\mathrm{Br}(\bar{\mathcal{H}}_{\mathrm{g}}) \to {}_m\mathrm{Br}(\mathbb{A}^{2\mathrm{g}+3}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\, k)$ is in fact an isomorphism.

## 3. Equivariant Brauer groups vs. Brauer groups of quotient stacks: Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. It may be important to point out the need for a careful proof of this result. One starts with the long exact sequence in étale cohomology obtained from the Kummer sequence:

$$\begin{aligned} &\to \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \stackrel{m}{\to} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \stackrel{\delta}{\to} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \\ &\qquad \stackrel{m}{\to} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \cdots \end{aligned} \tag{3.1}$$

Then the *cokernel*$(\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \stackrel{m}{\to} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}))$ maps to $\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1))$, by a map induced by the boundary map $\delta$: we will denote this map by $\bar{\delta}$. Then the Brauer group ${}_m\mathrm{Br}([\mathrm{X}/\mathrm{G}])$ identifies with the cokernel of the map $\bar{\delta}$. Here EG denotes the simplicial variant given in degree $n$ by $\mathrm{G}^{\times \mathrm{n}}$.

One also obtains a similar Kummer sequence where EG considered in 3.1 is replaced by a finite degree approximation $\mathrm{EG^{gm,u}}$, i.e., one obtains:

(3.2)
$$\to \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \overset{m}{\to} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \overset{\delta^{\mathrm{gm,u}}}{\to} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}})$$
$$\overset{m}{\to} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \cdots$$

Then the *cokernel*$(\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \overset{m}{\to} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}))$ maps to $\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1))$, by a map induced by the boundary map $\delta^{gm,u}$: we will denote this map by $\bar{\delta}^{gm,u}$. Then the Brauer group ${}_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X})$ identifies with the cokernel of the map $\bar{\delta}^{gm,u}$, when $u > 1$ as shown below.

The main difficulty in showing that the groups ${}_m\mathrm{Br}([\mathrm{X/G}])$ and ${}_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X})$ are isomorphic is because the terms in the corresponding Kummer sequences defining the corresponding Picard-groups,
i.e., $\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}})$ and $\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}})$ may not be trivial in general. In the special case where they are both trivial, the isomorphisms of the above Brauer groups amounts to an isomorphism $\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1)) \cong \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1))$ which may be established using the homotopy property of étale cohomology with respect to the sheaf $\mu_m$: see [J20, Proposition 5.2]. Therefore, the main effort in the proof of Theorem 1.4 is to show that one obtains an isomorphism between the Brauer groups ${}_m\mathrm{Br}([\mathrm{X/G}])$ and ${}_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X})$ without assuming the corresponding Picard groups are trivial: this really needs the use of motivic techniques (and Hilbert's Theorem 90), even when the base scheme is the spectrum of a field.

We begin discussing a simplicial variant of the Borel construction. Throughout this section, the base scheme B will again denote the spectrum of a regular Noetherian integral domain R of dimension at most 1 as in 1.1.

3.1. **The simplicial Borel construction.** We next consider $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ which is the simplicial scheme defined by $\mathrm{G^n} \times \mathrm{X}$ in degree $n$, and with the structure maps defined as follows:

(3.3)
$$\begin{aligned} d_i(g_0, \cdots, g_n, x) &= (g_1, \cdots, g_n, x), i = 0 \\ &= (g_1, \cdots, g_{i-1}.g_i, \cdots, g_n, x), 0 < i < n \\ &= (g_1, \cdots, g_{n-1}, g_n.x), i = n, \text{ and} \\ s_i(g_0, \cdots, g_{n-1}, x) &= (g_0, \cdots, g_{i-1}, e, g_i, \cdots, x) \end{aligned}$$

where $g_i \in \mathrm{G}$, $x \in \mathrm{X}$, $g_{i-1}.g_i$ denotes the product of $g_{i-1}$ and $g_i$ in G, while $g_n.x$ denotes the product of $g_n$ and $x$. $e$ denotes the unit element in G. This is the *simplicial Borel construction.* Then we obtain the following identification, which is well-known.

**Lemma 3.1.** *One obtains an isomorphism:* $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X} \cong \mathrm{cosk}_0^{[\mathrm{X/G}]}(\mathrm{X})$*, where* $cosk_0^{[\mathrm{X/G}]}(\mathrm{X})$ *is the simplicial scheme defined in degree* $n$ *by the* $(n+1)$*-fold fibered product of* X *with itself over the stack* $[\mathrm{X/G}]$*, with the structure maps of the simplicial scheme* $cosk_0^{[\mathrm{X/G}]}(\mathrm{X})$ *induced by the above fibered products.*

For each fixed $u \geq 0$, we obtain the diagram of simplicial schemes (where $p_1$ is induced by the projection $\mathrm{EG^{gm,u}} \times \mathrm{X} \to \mathrm{X}$ and $p_2$ is induced by the projection $\mathrm{EG} \times (\mathrm{EG^{gm,u}} \times \mathrm{X}) \to \mathrm{EG^{gm,u}} \times \mathrm{X}$):

$$\begin{array}{ccc} & \mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X}) & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \mathrm{EG} \times_{\mathrm{G}} \mathrm{X} & & \mathrm{EG^{gm,u}} \underset{\mathrm{G}}{\times} \mathrm{X} \end{array} \tag{3.4}$$

G acts diagonally on $\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X})$.

**Proposition 3.2.** (i) *The map*

$$\mathrm{p}_1^* : \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X}), \mathbb{G}_{\mathrm{m}}) \textit{ and the map} \tag{3.5}$$
$$\mathrm{p}_2^* : \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X}), \mathbb{G}_{\mathrm{m}}), \textit{ for } \mathrm{u} > 1,$$

*are isomorphisms.*

(ii) *The corresponding maps, for* $u > 1$ *with* $\ell$ *invertible in* $\mathcal{O}_{\mathrm{B}}$*,*

$$\mathrm{p}_1^* : \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X}), \mu_{\ell^{\mathrm{n}}}(1)), \textit{ and} \tag{3.6}$$
$$\mathrm{p}_2^* : \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mu_{\ell^{\mathrm{n}}}(1)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X}), \mu_{\ell^{\mathrm{n}}}(1))$$

*are isomorphisms.*

*Proof.* The isomorphisms in (i) are rather involved, and therefore, we discuss the proof of (i) first. A key to the proof is the observation that, over a scheme A which is a regular local ring, $\mathrm{H}^1_{\mathrm{et}}(\mathbb{A}^{\mathrm{n}}_{\mathrm{A}}, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{Pic}(\mathbb{A}^{\mathrm{n}}_{\mathrm{A}}) \cong 0$, for any $n \geq 0$. We consider the Leray spectral sequences associated to the maps $p_1$ and $p_2$:

$$\mathrm{E}_2^{\mathrm{s,t}}(1) = \mathrm{H}^{\mathrm{s}}_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathrm{R}^{\mathrm{t}}\mathrm{p}_{1*}(\mathbb{G}_{\mathrm{m}})) \Longrightarrow \mathrm{H}^{\mathrm{s+t}}_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X}), \mathbb{G}_{\mathrm{m}}) \text{ and} \tag{3.7}$$
$$\mathrm{E}_2^{\mathrm{s,t}}(2) = \mathrm{H}^{\mathrm{s}}_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathrm{R}^{\mathrm{t}}\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})) \Longrightarrow \mathrm{H}^{\mathrm{s+t}}_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG^{gm,u}} \times \mathrm{X}), \mathbb{G}_{\mathrm{m}}).$$

Since $\mathrm{s}, \mathrm{t} \geq 0$, both spectral sequences converge strongly.

The stalks of $\mathrm{R}^{\mathrm{t}}\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{\mathrm{t}}(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}})$, where A denotes a strict Hensel ring. (Strictly speaking, in order to obtain the above identification, we need to make use of the simplicial topology as in [J02] or [J20, 5.4]. But we will ignore this rather subtle point for the rest of the discussion.) Since $\mathrm{EG} \cong \mathrm{cosk}_0^{\mathrm{Spec\,B}}(\mathrm{G})$, $\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}) \cong \mathrm{cosk}_0^{\mathrm{Spec\,A}}(\mathrm{G} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A})$ is a *smooth hypercover* of Spec A. Therefore, we obtain the isomorphism:

$$\mathrm{H}^{\mathrm{t}}_{\mathrm{et}}(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{\mathrm{t}}_{\mathrm{smt}}(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^{\mathrm{t}}(\mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}). \tag{3.8}$$

These groups are trivial for $t = 1$ (see, for example, [Mi, Chapter III, Lemma 4.10]). Therefore, it follows that

$$\mathrm{R}^t\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}} \cong 0, \text{ for } \mathrm{t} = 1. \tag{3.9}$$

Next we observe the isomorphism, by taking $t = 0$ in (3.8):

$$\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}} \cong \mathrm{H}^0(\mathrm{EG} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^0(\mathrm{Spec\,A}, \mathbb{G}_{\mathrm{m}}), \tag{3.10}$$

where $\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})_{\mathrm{Spec\,A}}$ denotes the stalk of the sheaf $\mathrm{p}_{2*}(\mathbb{G}_{\mathrm{m}})$ at Spec A. Observing that $\mathbb{G}_m$ is in fact a sheaf on the flat site, and therefore also on the smooth site, it follows that there is a natural map of sheaves

$\mathbb{G}_m \to \mathrm{p}_{2*}(\mathbb{G}_\mathrm{m})$, where the $\mathbb{G}_m$ on the left (on the right) denotes the sheaf $\mathbb{G}_m$ restricted to the étale site of $\mathrm{EG}^{\mathrm{gm,u}} \times_\mathrm{G} \mathrm{X}$ (the étale site of $\mathrm{EG} \times_\mathrm{G} (\mathrm{EG}^{\mathrm{gm,u}} \times \mathrm{X})$, respectively). The isomorphism in (3.10) shows this map induces an isomorphism stalk-wise. It follows that the natural map $\mathbb{G}_m \to \mathrm{p}_{2*}(\mathbb{G}_\mathrm{m})$ of sheaves on the étale site is an isomorphism. This provides the isomorphism:

$$\mathrm{E}_2^{1,0}(2) = \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,u}} \times_\mathrm{G} \mathrm{X}, \mathrm{p}_{2*}(\mathbb{G}_\mathrm{m})) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,u}} \times_\mathrm{G} \mathrm{X}, \mathbb{G}_\mathrm{m}), \mathrm{u} > 0. \tag{3.11}$$

The stalks of $\mathrm{R}^\mathrm{t}\mathrm{p}_{1*}(\mathbb{G}_\mathrm{m}) \cong \mathrm{H}^\mathrm{t}(\mathrm{EG}^{\mathrm{gm,u}} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_\mathrm{m})$, where A denotes a strict Hensel ring, for all $t \geq 0$. Observe that this strict Hensel ring A is the stalk of the structure sheaf of $(\mathrm{EG} \times_\mathrm{G} \mathrm{X})_\mathrm{n} = \mathrm{G}^\mathrm{n} \times \mathrm{X}$, at a geometric point. Hence it is a filtered direct limit $\lim_i \mathrm{A_i}$, with each $\mathrm{A_i}$ regular.

To determine the groups $\mathrm{H}^\mathrm{t}(\mathrm{EG}^{\mathrm{gm,u}} \underset{\mathrm{Spec\,B}}{\times} (\mathrm{Spec\,A}), \mathbb{G}_\mathrm{m})$, we consider the long exact sequence (with $\mathrm{EG}^{\mathrm{gm,u}} = \mathrm{U_u}$, which is assumed to be an open subscheme of the affine space $\mathbb{A}^u$, with $\mathrm{Z_u} = \mathbb{A}^\mathrm{u} - \mathrm{U_u}$):

$$\begin{aligned} \cdots \to \mathrm{H}^0_{\mathrm{et},\mathrm{Z_u}\times_{\mathrm{Spec\,B}}\mathrm{Spec\,A}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_\mathrm{m}) \to \mathrm{H}^0_{\mathrm{et}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_\mathrm{m}) \to \\ \overset{\alpha}{\to} \mathrm{H}^0_{\mathrm{et}}(\mathrm{U_u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_\mathrm{m}) \to \mathrm{H}^1_{\mathrm{et},\mathrm{Z_u}\times_{\mathrm{Spec\,B}}\mathrm{Spec\,A}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_\mathrm{m}) \\ \to \mathrm{H}^1_{\mathrm{et}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_\mathrm{m}) \overset{\beta}{\to} \mathrm{H}^1_{\mathrm{et}}(\mathrm{U_u} \times_{\mathrm{Spec}\,k} \mathrm{Spec\,A}, \mathbb{G}_\mathrm{m}) \to \\ \to \mathrm{H}^2_{\mathrm{et},\mathrm{Z_u}\times_{\mathrm{Spec\,B}}\mathrm{Spec\,A}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}, \mathbb{G}_\mathrm{m}) \to \cdots \end{aligned} \tag{3.12}$$

Next we observe the identification of $\mathbb{G}_m$ with $\mathbb{Z}(1)[1]$ from Proposition 9.2. As a result, we obtain the following identifications, for a smooth scheme Y of finite type over the base B, which is assumed to be a Dedekind domain (which also includes the case of it being a field) and a closed smooth subscheme Z of pure codimension $c$ in Y:

$$\begin{aligned} &\mathrm{H}^1_{\mathrm{et,Z}}(\mathrm{Y}, \mathbb{G}_\mathrm{m}) \cong \mathrm{H}^1_{\mathrm{Zar,Z}}(\mathrm{Y}, \mathbb{G}_\mathrm{m}) \cong \mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{Y}), \text{ and} \\ &\mathrm{H}^0_{\mathrm{et,Z}}(\mathrm{Y}, \mathbb{G}_\mathrm{m}) \cong \mathrm{H}^0_{\mathrm{Zar,Z}}(\mathrm{Y}, \mathbb{G}_\mathrm{m}) \cong \mathrm{H}^{1,1}_{\mathrm{M,Z}}(\mathrm{Y}). \end{aligned} \tag{3.13}$$

Therefore, by Proposition 9.4, we see that if $c > 1$, then

$$\begin{aligned} &\mathrm{H}^1_{\mathrm{et,Z}}(\mathrm{Y}, \mathbb{G}_\mathrm{m}) \cong \mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{Y}) \cong 0, \text{ and} \\ &\mathrm{H}^0_{\mathrm{et,Z}}(\mathrm{Y}, \mathbb{G}_\mathrm{m}) \cong \mathrm{H}^{1,1}_{\mathrm{M,Z}}(\mathrm{Y}) \cong 0. \end{aligned} \tag{3.14}$$

The map denoted $\alpha$ ($\beta$) in the long exact sequence (10.3) identifies with the restriction

$$\mathrm{H}^{1,1}_{\mathrm{M}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{1,1}_{\mathrm{M}}(\mathrm{U_u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A})$$

$$(\mathrm{H}^{2,1}_{\mathrm{M}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{U_u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}), \text{respectively})$$

forming part of the localization sequence for the motivic cohomology groups. In fact, the corresponding localization sequence is given by:

$$\begin{aligned} \cdots \to \mathrm{H}^{1,1-c_\mathrm{u}}_{\mathrm{M}}(\mathrm{Z_u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{1,1}_{\mathrm{M}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \overset{\alpha'}{\to} \mathrm{H}^{1,1}_{\mathrm{M}}(\mathrm{U_u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \\ \to \mathrm{H}^{2,1-c_\mathrm{u}}_{\mathrm{M}}(\mathrm{Z_u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to \mathrm{H}^{2,1}_{\mathrm{M}}(\mathbb{A}^\mathrm{u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \overset{\beta'}{\to} \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{U_u} \times_{\mathrm{Spec\,B}} \mathrm{Spec\,A}) \to 0 \end{aligned} \tag{3.15}$$

where $c_u$ denotes the codimension of $\mathrm{Z_u}$ in $\mathbb{A}^u$, which we assume satisfies $c_\mathrm{u} > 1$. To see that one gets such a localization sequence, one first replaces the strict Hensel ring A by one of the $\mathrm{A_i}$, where $\mathrm{A} = \lim_\mathrm{i} \mathrm{A_i}$, with each $\mathrm{A_i}$ a regular local ring. Clearly then the corresponding localization sequence exists and the groups in (3.15) involving the $\mathrm{Z_u}$ are trivial, as $c_\mathrm{u} > 1$, by assumption. At this point, one takes the direct limit over the $\mathrm{A_i}$: since the motivic cohomology groups are contravariantly functorial for flat maps, and filtered colimits are exact, we obtain the localization sequence (3.15). Moreover, the groups appearing in (3.15) involving the $Z_u$ are all trivial, thereby proving that the maps $\alpha'$ and $\beta'$ in (3.15), and therefore, the maps $\alpha$ and $\beta$ in (10.3) are isomorphisms. This provides the isomorphisms for $t = 0, 1$:

$$\begin{aligned}(3.16)\qquad R^t p_{1*}(\mathbb{G}_m)_{\mathrm{Spec\,A}} \cong H^t_{et}(U_u \times_{\mathrm{Spec}\,k} \mathrm{Spec\,A}, \mathbb{G}_m) &\cong H^t_{et}(\mathbb{A}^u \times_{\mathrm{Spec}\,k} \mathrm{Spec\,A}, \mathbb{G}_m)\\ &\cong H^t_{et}(\mathrm{Spec\,A}, \mathbb{G}_m).\end{aligned}$$

Therefore, it follows that

$$(3.17)\qquad R^t p_{1*}(\mathbb{G}_m)_{\mathrm{Spec\,A}} \cong 0, \text{ for } t = 1.$$

Since $\mathbb{G}_m$ is a sheaf on the flat and hence on the smooth topology, there is a natural map $\mathbb{G}_m \to p_{1*}(\mathbb{G}_m)$ of sheaves where the $\mathbb{G}_m$ on the left (on the right) is a sheaf on the étale site of $EG \times_G X$ (on the étale site of $EG \times_G (EG^{gm,m} \times X)$, respectively). The stalk-wise isomorphism in (3.16) for $t = 0$ shows that the natural map $\mathbb{G}_m \to p_{1*}(\mathbb{G}_m)$ of sheaves on the étale site is an isomorphism. This provides the isomorphism:

$$(3.18)\qquad E_2^{1,0}(1) = H^1_{et}(EG \times_G X, p_{1*}(\mathbb{G}_m)) \cong H^1_{et}(EG \times_G X, \mathbb{G}_m).$$

Moreover, observing that the differentials in the spectral sequences above go from $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$, one sees (using (3.9) and (3.17)) that

$$(3.19)\qquad E_r^{0,1}(1) = E_r^{0,1}(2) = 0 \text{ for all } r \geq 2 \text{ and that } E_2^{1,0}(i) \cong E_r^{1,0}(i), \text{ for all } r \geq 2, i = 1, 2.$$

The last observation shows that $E_2^{1,0}(i)$, $i = 1, 2$ is isomorphic to the abutment in degree 1, namely, $H^1_{et}(EG \times_G (EG^{gm,u} \times X), \mathbb{G}_m)$, $u > 1$. (Observe that the assumption $u > 1$ implies the codimension $c_u$ of $Z_u$ in $\mathbb{A}^u$ is at least 2.) Therefore, the isomorphisms in (3.18) and (3.11) complete the proof of (i).

Next we consider the proof of (ii). The key point is to consider the Leray spectral sequences for the maps $p_1$ and $p_2$. In this case, one may readily compute the stalks of $R^t p_{i*}(\mu_m(1))$ to be trivial for $t = 1, 2$ and $\cong \mu_m(1)$ for $t = 0$, and for $u > 1$. In fact, one may adopt the same arguments as before, with the sheaf $\mathbb{G}_m$ on the étale site replaced by $\mu_m(1)$. Then a localization sequence corresponding to the one in (10.3) still holds, but also extends to higher degrees. The corresponding local cohomology groups there will vanish in all cohomological degrees less than 4, since the codimension of $Z_u$ in $\mathbb{A}^u$ is assumed to be at least 2. Therefore, the conclusions in (ii) follow readily. (One may also consult [J20, Theorem 1.6] for further details.) □

Let $S$ denote an algebraic stack, which we will assume is of *Artin type* and of finite type over the given base field $k$, with $x : X \to S$ an *atlas*, that is, a *smooth surjective map from an algebraic space* X. We let $B_x S = \mathrm{cosk}_0^S(X)$ denote the corresponding simplicial algebraic space. Then we let $S_{smt}$ denote the smooth site, whose objects are $y : Y \to S$, with $y$ a smooth map from an algebraic space Y to $S$, and where a morphism between two such objects $y' : Y' \to S$ and $y : Y \to S$ is given by a map $f : Y' \to Y$ making the triangle

$$(3.20)\qquad \begin{array}{ccc} Y' & \xrightarrow{f} & Y \\ & {\scriptstyle y'}\searrow \quad \swarrow{\scriptstyle y} & \\ & S & \end{array}$$

commute. The same definition defines the smooth site of any algebraic space. The smooth and étale sites of the simplicial algebraic space $B_x S$ may be defined as follows. The objects of $Smt(B_x S)$ are given by smooth maps $u_n : \mathrm{U_n} \to (\mathrm{B}_x S)_{\mathrm{n}}$ for some $n \geq 0$. Given such a $u_n : \mathrm{U_n} \to \mathrm{B}_x S_{\mathrm{n}}$ and $v_m : \mathrm{V_m} \to \mathrm{B}_x \mathrm{S_m}$, a morphism from $u_n \to v_m$ is a commutative square:

$$\begin{array}{ccc} \mathrm{U_n} & \xrightarrow{\alpha'} & \mathrm{V_m} \\ \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle v_m} \\ \mathrm{B}_x S_{\mathrm{n}} & \xrightarrow{\alpha} & \mathrm{B}_x S_{\mathrm{m}} \end{array}$$

where $\alpha$ is a structure map of the simplicial algebraic space $\mathrm{B}_x S$. The Étale site $Et(\mathrm{B}_x S)$ is defined similarly. An abelian sheaf $F$ on $Smt(\mathrm{B}_x S)$ is given by a collection of abelian sheaves $F = \{F_n | n\}$ with each $F_n$ being an abelian sheaf on $Smt(\mathrm{B}_x S_{\mathrm{n}})$, so that it comes equipped with the following data: for each structure map $\alpha : \mathrm{B}_x S_{\mathrm{n}} \to \mathrm{B}_x S_{\mathrm{m}}$, one is provided with a map of sheaves $\phi_{n,m} : \alpha^*(F_m) \to F_n$ so that the maps $\{\phi_{n,m} | n, m\}$ are compatible. Abelian sheaves on the site $Et(\mathrm{B}_x S)$ may be defined similarly. We skip the verification that the category of abelian sheaves on the above sites have enough injectives. The $n$-th cohomology group of the simplicial object $\mathrm{B}_x S$ with respect to an abelian sheaf $F$ is defined as the $n$-th right derived functor of the functor sending

$$F \mapsto kernel(\delta^0 - \delta^1 : \Gamma(\mathrm{B}_x S_0, \mathrm{F}_0) \to \Gamma(\mathrm{B}_x S_1, \mathrm{F}_1)). \tag{3.21}$$

Now we obtain the following Proposition.

**Proposition 3.3.** *Let $F$ denote an abelian sheaf on $Smt(S)$. Then we obtain the following isomorphisms:*

*(i)* $\mathrm{H}^*_{\mathrm{smt}}(\mathrm{B}_x S, x_\bullet^*(F)) \cong \mathrm{H}^*_{\mathrm{smt}}(S, F)$*, where the subscript smt denotes cohomology computed on the smooth sites and* $x_\bullet : \mathrm{B}_x S \to S$ *is the simplicial map induced by* $x : \mathrm{X} \to S$*.*

*(ii)* $\mathrm{H}^*_{\mathrm{smt}}(\mathrm{B}_x S, x_\bullet^*(F)) \cong \mathrm{H}^*_{\mathrm{et}}(\mathrm{B}_x S, \alpha_* x_\bullet^*(F))$*, where the subscript et denotes cohomology computed on the étale site and* $\alpha : Smt(\mathrm{B}_x S) \to \mathrm{Et}(\mathrm{B}_x S)$ *is the obvious morphism of sites.*

*Proof.* Observe that $x : \mathrm{X} \to S$ is a covering of the stack $S$ in the smooth topology, so that

$$kernel(\delta^0 - \delta^1 : \Gamma(\mathrm{B}_x S_0, F_0) \to \Gamma(\mathrm{B}_x S_1, F_1)) \cong \Gamma(S, F).$$

Since $\mathrm{H}^{\mathrm{n}}_{\mathrm{smt}}(S, F)$ is the $n$-th right derived functor of the above functor, in view of (3.21), we see that it identifies with $\mathrm{H}^{\mathrm{n}}_{\mathrm{smt}}(\mathrm{B}_x S, x_\bullet^*(F))$. This provides the isomorphism in (i). The isomorphism in (ii) is a straight-forward extension of a well-known result comparing the cohomology of an algebraic space computed on the smooth and étale sites. □

**Corollary 3.4.** *Assume the above context.*

*(i) Then we obtain an isomorphism*

$$\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^1_{\mathrm{smt}}([\mathrm{X/G}], \mathbb{G}_{\mathrm{m}}), \textit{ for } \mathrm{u} > 1,$$

*which is functorial in the* G*-scheme* X*.*

*(ii) Moreover, we obtain isomorphisms:*

$$\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG^{gm,u}} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1)) \cong \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1)) \cong \mathrm{H}^2_{\mathrm{smt}}([\mathrm{X/G}], \mu_m(1)) \textit{ for } \mathrm{u} > 1.$$

*which are functorial in the* G*-scheme* X*, and where $m$ is invertible in $k$.*

*Here* $\mathrm{H}^1_{\mathrm{smt}}([\mathrm{X}/\mathrm{G}], \mathbb{G}_{\mathrm{m}})$ *and* $\mathrm{H}^2_{\mathrm{smt}}([\mathrm{X}/\mathrm{G}], \mu_m(1))$ *denote the cohomology of the quotient stack* $[\mathrm{X}/\mathrm{G}]$ *computed on the smooth site.*

*(iii) One obtains an isomorphism* ${}_m\mathrm{Br}(\mathrm{EG}^{\mathrm{gm,u}} \times_{\mathrm{G}} \mathrm{X}) \cong {}_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X}) \cong {}_m\mathrm{Br}([\mathrm{X}/\mathrm{G}])$*, for any* $u > 1$*, thereby proving that* ${}_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X})$ *is an invariant of the quotient stack* $[\mathrm{X}/\mathrm{G}]$*, for any positive integer* $m$ *invertible in* $k$*.*

*Proof.* The first isomorphisms in both the statements (i) and (ii) are from Proposition 3.2. The second isomorphisms in (i) and (ii) follow from the isomorphism of the simplicial schemes: $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X} \cong \mathrm{cosk}_0^{[\mathrm{X}/\mathrm{G}]}(\mathrm{X})$ and Proposition 3.3. Next we consider the third statement.

Recall the long exact sequence in étale cohomology obtained from the Kummer sequence:

$$\begin{aligned}(3.22)\qquad &\to \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{m} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\delta} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \\ &\xrightarrow{m} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \to \cdots\end{aligned}$$

Then the *cokernel*$(\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{m} \mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}))$ maps to $\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1))$, by a map induced by the boundary map $\delta$: we will denote this map by $\bar{\delta}$. Then, in view of the isomorphisms in (i) and (ii), the Brauer group $\mathrm{Br}_{\mathrm{G}}(\mathrm{X})_{\ell^n}$ identifies with the cokernel of the map $\bar{\delta}$.

In view of Proposition 3.3, the isomorphisms in (i) and (ii) and the long exact sequence (3.22), ${}_m\mathrm{Br}([\mathrm{X}/\mathrm{G}])$ identifies with

$$kernel(\mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{m} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}})).$$

Again by Proposition 3.3, the isomorphisms in (i) and (ii) and the long-exact sequence (3.22), this identifies with

$$cokernel((\mathrm{H}^1_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mathbb{G}_{\mathrm{m}})/m \xrightarrow{\bar{\delta}} \mathrm{H}^2_{\mathrm{et}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \mu_m(1))) \cong {}_m\mathrm{Br}_{\mathrm{G}}(\mathrm{X}).$$

This proves the third assertion. □

**Proof of Theorem 1.4**. Clearly Corollary 3.4(iii) completes the proof of Theorem 1.4. □

**Proposition 3.5.** *The equivariant Brauer groups are independent of the choice of an admissible gadget defined as in Definition 2.1.*

*Proof.* Let $\{(\mathrm{W_m}, \mathrm{U_m})|\mathrm{m}\}$ and $\{(\bar{\mathrm{W}}_m, \bar{\mathrm{U}}_m)|m\}$ denote two admissible gadgets for the given linear algebraic group G. Let G act on the given scheme X. Let $\mathrm{Z_m} = \mathrm{W_m} - \mathrm{U_m}$, $\bar{\mathrm{Z}}_m = \bar{\mathrm{W}}_m - \bar{\mathrm{U}}_{\mathrm{m}}$. Then one may observe that $\{(\widetilde{\mathrm{W}}_m = \mathrm{W_m} \times \bar{\mathrm{W}}_{\mathrm{m}}, \widetilde{\mathrm{U}}_{\mathrm{m}} = \mathrm{U_m} \times \bar{\mathrm{W}}_{\mathrm{m}} \cup \bar{\mathrm{W}}_{\mathrm{m}} \times \bar{\mathrm{U}}_{\mathrm{m}})|\mathrm{m}\}$ is also an admissible gadget. Moreover,

$$codim_{\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X}}(\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X} - (\mathrm{U_m} \times \bar{\mathrm{W}}_{\mathrm{m}}) \times_{\mathrm{G}} \mathrm{X}) = codim_{\mathrm{W_m}}(\mathrm{W_m} - \mathrm{U_m}), \text{ and} \tag{3.23}$$

$$codim_{\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X}}(\widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X} - (\mathrm{W_m} \times \bar{\mathrm{U}}_{\mathrm{m}}) \times_{\mathrm{G}} \mathrm{X}) = codim_{\bar{\mathrm{W}}_m}(\bar{\mathrm{W}}_m - \bar{\mathrm{U}}_m). \tag{3.24}$$

Therefore, if $codim_{\mathrm{W_m}}(\mathrm{W_m} - \mathrm{U_m}) \geq 2$ and $codim_{\bar{\mathrm{W}}_m}(\bar{\mathrm{W}}_m - \bar{\mathrm{U}}_m) \geq 2$, then in the long exact sequence

$$\cdots \to \mathrm{H}^{2,1}_{\mathrm{Q_m}}(\widetilde{\mathrm{U}}_{\mathrm{m}} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}(\widetilde{\mathrm{U}}_{\mathrm{m}} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}((\mathrm{U_m} \times \bar{\mathrm{W}}_{\mathrm{m}}) \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{3,1}_{\mathrm{Q_m}}(\widetilde{\mathrm{U}}_{\mathrm{m}} \times_{\mathrm{G}} \mathrm{X}) \to \cdots,$$

where $\mathrm{Q_m} = \widetilde{\mathrm{U}}_{\mathrm{m}} \times_{\mathrm{G}} \mathrm{X} - (\mathrm{U_m} \times \bar{\mathrm{W}}_{\mathrm{m}}) \times_{\mathrm{G}} \mathrm{X}$, and in the long exact sequence

$$\cdots \to \mathrm{H}^{2,1}_{\bar{\mathrm{Q}}_{\mathrm{m}}}(\widetilde{\mathrm{U}}_{\mathrm{m}} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}(\widetilde{\mathrm{U}}_{\mathrm{m}} \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{2,1}((\mathrm{W_m} \times \bar{\mathrm{U}}_{\mathrm{m}}) \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{H}^{3,1}_{\bar{\mathrm{Q}}_{\mathrm{m}}}(\widetilde{\mathrm{U}}_{\mathrm{m}} \times_{\mathrm{G}} \mathrm{X}) \to \cdots,$$

where $\bar{\mathrm{Q}}_m = \widetilde{\mathrm{U}}_m \times_{\mathrm{G}} \mathrm{X} - (\mathrm{W_m} \times \bar{\mathrm{U}}_{\mathrm{m}}) \times_{\mathrm{G}} \mathrm{X}$, both the end terms are trivial, thereby showing that the middle maps in both the long exact sequences are isomorphisms. Here $\mathrm{H}^{\mathrm{i},1}$ denotes either motivic cohomology of weight 1 with $\mathbb{Z}/m$-coefficients or étale cohomology with respect to the sheaf $\mu_m(1)$. The assertion on the triviality of the local motivic cohomology terms above follows from Proposition 9.4, while the corresponding assertion for étale cohomology follows from well-known cohomological semi-purity statements. □

## 4. Proof of Corollary 1.6

**Proof of Corollary 1.6.** When $\mathrm{G} = \mathrm{GL_n}$ or $\mathbb{G}_m$, the two statements follow readily from the discussion in section 2.3: see especially (2.4) and (2.6). When $\mathrm{G} = \mathrm{GL_n}$, or $\mathrm{G} = \mathbb{G}_{\mathrm{m}}$, for $p \geq 2$, the codimension of the complement of $(\mathrm{G}^{\times \mathrm{p}} \times \mathrm{W}^{\times \mathrm{p}}) \times_{\mathrm{G}^{\times \mathrm{p}}} \mathrm{X}$ in $\mathrm{E}(\mathrm{G}^{\times \mathrm{p}})^{\mathrm{gm},2} \times_{\mathrm{G}^{\times \mathrm{p}}} \mathrm{X}$ is at least 2. Therefore in this case, ${}_m\mathrm{Br}_{\mathrm{G}^{\times \mathrm{p}}}(\mathrm{X}) \cong {}_m\mathrm{Br}((\mathrm{G}^{\times \mathrm{p}} \times \mathrm{W}^{\times \mathrm{p}}) \times_{\mathrm{G}^{\times \mathrm{p}}} \mathrm{X}) \cong {}_m\mathrm{Br}(\mathrm{X})$. □

## 5. Brauer groups of stacks of the form [X/D] where D is a smooth diagonalizable group scheme and Proof of Theorem 1.7.

The main theme of this section is the determination of the Brauer groups of quotient stacks of the form [X/D], where D is a smooth diagonalizable group scheme. We begin by observing that when D is the 1-dimensional torus $\mathbb{G}_m$, then (2.7) provides a means to compute the Brauer group of the quotient stack [X/D]. We proceed to consider some of the remaining cases.

5.1. **Toric stacks.** Let X denote a toric variety defined over a field $k$ for the split torus $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{s}}$, or more generally a toric scheme over the base B in the sense of [JL, section 4(i), (4.1)], i.e. satisfying the hypotheses as in (10.5). Recall B is assumed to be the spectrum of a Dedekind domain R which is an excellent scheme and that the hypotheses in (10.5) require that the toric scheme X contain as an open subscheme a split torus $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{s}$, and that all the T-orbits are defined and faithfully flat over B. Consider a homomorphism

$$\mathbb{G}_m^r \to \mathbb{G}_m^s$$

given by characters

$$\alpha_1, \dots, \alpha_s : \mathbb{G}_m^r \to \mathbb{G}_m.$$

Each character in turn decomposes as $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ir})$ where $\alpha_{ij} : \mathbb{G}_m \to \mathbb{G}_m$ given by $z \mapsto z^{\alpha_{ij}}$.

We are interested in the Brauer group of the quotient stack $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]$: as defined in the introduction, such stacks are what we call toric stacks. (See [GSat] and/or [JK, section 2].) We let

$$\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r) := (\mathbb{A}^2 \setminus \{0\})^r \times_{\mathbb{G}_{\mathrm{m}}^r} \mathrm{X}. \tag{5.1}$$

5.2. **Notation.** We let

$$\mathbb{A}\mathbb{G}(u) = \begin{cases} \mathbb{G}_m \times \mathbb{A}^1 & \text{if } u = 1 \\ \mathbb{A}^1 \times \mathbb{G}_m & \text{if } u = 2 \end{cases}$$

Given $v \in \{1, 2\}^r$, with $v = (v_1, \dots, v_r)$ we set

$$\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r) = (\prod_1^r \mathbb{A}\mathbb{G}(v_i)) \times_{\mathbb{G}_{\mathrm{m}}^r} \mathrm{X}. \tag{5.2}$$

Given $u \in \{1, 2\}$, we let

$$\hat{u} = \begin{cases} 1 & \text{if } u = 2 \\ 2 & \text{if } u = 1. \end{cases}$$

**Proposition 5.1.** *In the above situation, for each choice of* $v = (v_1, \cdots, v_r)$*, there is an isomorphism*

$$\phi_v : \mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r) \to \mathbb{A}^r \times \mathrm{X}$$

*given by*

$$\phi_v((x_{11}, x_{12}), \dots (x_{r1}, x_{r2}), p) = \left( \frac{x_{1\hat{v_1}}}{x_{1v_1}}, \dots, \frac{x_{r\hat{v_r}}}{x_{rv_r}}, \prod_1^r x_{iv_i}^{-1}.p \right)$$

*Proof.* One checks that the lift of $\phi_v$, defined by the same equations is equivariant for the action of $\mathbb{G}_m^r$ on $\prod_1^r \mathbb{A}^{v_i} \times \mathrm{X}$ and hence $\phi_v$ is well defined. One can write down the inverse of $\phi_v$ as

$$\phi_v^{-1}(x_1, \dots x_r, p) = [(y_{11}, y_{12}), \dots (y_{r1}, y_{r2}), p]$$

where

$$y_{i1} = \begin{cases} 1 & \text{if } v_i = 1 \\ x_i & \text{if } v_i = 2 \end{cases}$$

and

$$y_{i2} = \begin{cases} x_i & \text{if } v_i = 1 \\ 1 & \text{if } v_i = 2 \end{cases} .$$

One checks that these maps are mutually inverse. □

**Proposition 5.2.** *As $v$ varies over $\{1, 2\}^r$ the* $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$*, form an open cover of* $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$*.*

*Proof.* This is clear. □

**Proposition 5.3.** *The variety (scheme)* $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ *is toric and has dense open torus*

$$\mathbb{G}_m^r \times \mathrm{T} \xrightarrow{\lambda} \mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$$

*given by*

$$(z_1, \dots, z_r, p) \longmapsto [(1, z_1), (1, z_2), \dots, (1, z_r), p].$$

*Moreover all the* $\mathbb{G}_m^r \times \mathrm{T}$*-orbits on* $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ *are faithfully flat over* B.

*Proof.* The question that $\lambda$ is an open embedding is local on $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ so it can be checked locally on $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$. Using the isomorphism of the prior proposition, we check that it is an open embedding with dense image. To see that $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ is normal, note that normality is local for the smooth topology and the smooth cover $\mathrm{X} \times (\mathbb{A} - \{0\})^r$ is normal. One may see from the above discussion that, when $\mathrm{B} = \operatorname{Spec} \mathrm{R}$ for an excellent Dedekind domain R, the $\mathbb{G}_m^r \times \mathrm{T}$-orbits on $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ are of the form $\mathbb{G}_m^r \times \mathrm{T}'$, where $\mathrm{T}'$ is a factor of T. Therefore, they are faithfully flat over B. □

We proceed to discuss an explicit fan for the toric variety (scheme) $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$.

**Proposition 5.4.** *Choose $v, w \in \{1, 2\}^r$.*

(1) *We have* $\phi_v(\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r) \cap \mathrm{U}(w, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)) = \prod_1^r \mathbb{AG}^{v_i, w_i} \times \mathrm{X}$ *where*

$$\mathbb{AG}^{v_i, w_i} = \begin{cases} \mathbb{A}^1 & \textit{if } v_i = w_i \\ \mathbb{G}_m & \textit{otherwise.} \end{cases}$$

*(2) The composition*

$$\prod_1^r \mathbb{AG}^{v_i,w_i} \times \mathrm{X} \xrightarrow{\phi_v^{-1}} \mathrm{U}(v,\mathrm{X},\mathbb{G}_{\mathrm{m}}^r) \cap \mathrm{U}(w,\mathrm{X},\mathbb{G}_{\mathrm{m}}^r) \xrightarrow{\phi_w} \prod_1^r \mathbb{AG}^{v_i,w_i} \times \mathrm{X}$$

*is given by*

$$\phi_w \circ \phi_v^{-1}(x_1,\dots,x_r,p) = (x_1^{v_1,w_1},\dots,x_r^{v_r,w_r}, t\cdot p)$$

*where*

$$x_i^{v_i,w_i} = \begin{cases} x_i & \text{if } v_i = w_i \\ x_i^{-1} & \text{otherwise,} \end{cases}$$

*and*

$$t = \prod_{i=1,v_i\neq w_i}^r x_i^{-1}.$$

*Proof.* (1) Set

$$\widetilde{\mathbb{AG}^{v_i,w_i}} = \begin{cases} \mathbb{G}_m \times \mathbb{A}^1 & \text{if } v_i = w_i = 1 \\ \mathbb{G}_m \times \mathbb{G}_m & \text{if } v_i \neq w_i \\ \mathbb{A}^1 \times \mathbb{G}_m & \text{if } v_i = w_i = 2. \end{cases}$$

It follow from 5.2 that $\prod_1^r \widetilde{\mathbb{A}^{v_i,w_i}} \times_{\mathbb{G}_m^r} \mathrm{X} = \mathrm{U}(v,\mathrm{X},\mathbb{G}_{\mathrm{m}}^r) \cap \mathrm{U}(w,\mathrm{X},\mathbb{G}_{\mathrm{m}}^r)$.

When $v_i = w_i$ the $i$th component of $\widetilde{\mathbb{A}^{v_i,w_i}}$ is the same as that of $\mathrm{U}(v,\mathrm{X},\mathbb{G}_{\mathrm{m}}^r)$ and when $v_i \neq w_i$ then an open axis is missing. The result follows directly from the formula for $\phi_v$.

(2) Here we make use of the formula for $\phi_v^{-1}$ given in the proof of 5.1. We see that

$$\begin{aligned}\phi_w \circ \phi_v^{-1}(x_1,\dots,x_r,p) &= \phi_w((y_{11},y_{12}),\dots,(y_{r1},y_{r2}),p) \\ &= \left(\frac{y_{1,\hat{w}_1}}{y_{1w_1}},\dots,\frac{y_{r,\hat{w}_r}}{y_{rw_r}}, \prod_1^r y_{iw_i}^{-1}\cdot p\right),\end{aligned}$$

where

$$y_{i1} = \begin{cases} 1 & \text{if } v_i = 1 \\ x_i & \text{if } v_i = 2 \end{cases} \qquad y_{i2} = \begin{cases} x_i & \text{if } v_i = 1 \\ 1 & \text{if } v_i = 2. \end{cases}$$

One checks

$$y_{iw_i} = \begin{cases} x_i & \text{if } v_i = w_i \\ 1 & \text{if } v_i \neq w_i. \end{cases} \qquad y_{i\hat{w}_i} = \begin{cases} 1 & \text{if } v_i = w_i \\ x_i & \text{if } v_i \neq w_i. \end{cases}$$

It follows that

$$\frac{y_{i\hat{w}_i}}{y_{iw_i}} = \begin{cases} x_i & \text{if } v_i = w_i \\ x_i^{-1} & \text{if } v_i \neq w_i \end{cases}$$

and

$$\prod_1^r y_{i,w_i}^{-1} = \prod_{v_i\neq w_i} x_i^{-1} = t.$$

□

We will assume from now on familiarity with cones and fans and their associated toric varieties.

**Proposition 5.5.** *Consider the morphism of split tori*

$$\phi_v : \mathbb{G}_m^r \times \mathrm{T} = \mathbb{G}_{\mathrm{m}}^{r+s} \to \mathbb{G}_{\mathrm{m}}^{r+s}$$

*induced by the map* $\phi_v : \mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r) \to \mathbb{A}^r \times \mathrm{X}$ *where* $\mathbb{G}_m^r \times \mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{r+s}} \subseteq \mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ *and* $\mathbb{G}_m^{r+s} \subseteq \mathbb{A}^r \times \mathrm{X}$. *Then the induced pullback map on character lattices* $\phi_v^* : \mathbb{X}(\mathbb{G}_m^{r+s}) = \mathbb{Z}^{r+s} \to \mathbb{X}(\mathbb{G}_m^{r+s}) = \mathbb{Z}^{r+s}$ *is given by multiplication by the matrix*

$$B(v, \alpha) = \begin{pmatrix} I_v & A(v, \alpha) \\ 0_{s \times r} & I_{s \times s} \end{pmatrix},$$

*where:*

$$I_v = \begin{pmatrix} \epsilon(v_1) & 0 & \cdots & 0 \\ 0 & \epsilon(v_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \epsilon(v_r) \end{pmatrix}$$

$$A(v, \alpha) = \begin{pmatrix} -\alpha_{11}(v_1 - 1) & \cdots & -\alpha_{s1}(v_1 - 1) \\ -\alpha_{12}(v_2 - 1) & \cdots & -\alpha_{s2}(v_2 - 1) \\ \vdots & \vdots & \vdots \\ -\alpha_{1r}(v_r - 1) & \cdots & -\alpha_{sr}(v_r - 1) \end{pmatrix}$$

*Here*

$$\epsilon(v_i) = \begin{cases} 1 & \text{if } v_i = 1 \\ -1 & \text{if } v_i = 2 \end{cases}$$

*Proof.* We need to compute the morphism from the torus $\mathbb{G}_m^r \times \mathrm{T}$ to $\mathbb{A}^r \times \mathrm{X}$ via $\phi_v$. The torus $\mathbb{G}_m^r \times \mathrm{T}$ sits inside $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ as described in 5.3. Recall the embedding is

$$(z_1, \ldots, z_r, p) \longmapsto [(1, z_1), \ldots, (1, z_r), p].$$

Hence, one computes

$$\phi_v((1, z_1), (1, z_2), \ldots, (1, z_r), p_1, \ldots, p_s) = \left(z_1^{\epsilon(v_1)}, z_2^{\epsilon(v_2)}, \ldots, z_r^{\epsilon(v_r)}, tp_1, tp_2, \ldots, tp_s\right)$$

where

$$t = \prod_{i=1, v_i = 2}^{r} z_i^{-1}.$$

Now let $\chi_i : \mathbb{G}_m^{r+s} \to \mathbb{G}_m$ be the character $\chi_i(z_1, \ldots, z_{r+s}) = z_i$. We need to compute $\chi_i \circ \phi_v$. It is easy to see that

$$\chi_i \circ \phi_v = \chi_i^{\epsilon}(v_i)$$

when $i \leq r$. This proves that the left $(r+s) \times r$ block of the matrix $B(v, \alpha)$ is as stated. Now assume $i > r$. Take $1 \leq j \leq r$. Then

$$\chi_i \circ \phi_v([(1, 1), \ldots, (1, z_j), \ldots, (1, 1), 1]) = \chi_i(1, \ldots, z_j^{\epsilon(v_j)}, \ldots, 1, z_j^{\lambda}.1).$$

Now $\lambda = 0$ if $v_j = 1$ and $\lambda = -1$ if $v_j = 2$. Now recall that the torus $\mathbb{G}_m^r$ acts on $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{s}}$ via the characters $\alpha_i$ as described at the start of this section. The top right $r \times s$ block of $B(v, \alpha)$ is obtained by observing that

$$v_i - 1 = \begin{cases} 0 & \text{if } v_i = 1 \\ 1 & \text{if } v_i = 2. \end{cases}$$

It remains to check that the bottom $s \times s$ block is the identity. This is straightforward as $\mathbb{G}_m^s$ act on $\mathbb{G}_m^s$ via ordinary multiplication. □

**Proposition 5.6.** *Let* V *be a finite dimensional vector space with an automorphism* $a : \mathrm{V} \to \mathrm{V}$. *Let* $\rho$ *be a cone in* $V$ *with dual cone* $\rho^\vee$. *Then we have*

$$a(\rho)^\vee = (a^t)^{-1}(\rho^\vee).$$

*Proof.*

$$\begin{aligned} v \in (a(\rho))^\vee &\iff < a(x), v > 0 \geq \quad \forall\ x \in \rho \\ &\iff < x, a^t(v) > 0 \geq \quad \forall\ x \in \rho \\ &\iff (a^t)(v) \in \rho^\vee \\ &\iff v \in (a^t)^{-1}(\rho^\vee). \end{aligned}$$

□

**Proposition 5.7.** *Let* $\Sigma \subseteq \mathbb{Z}^s$ *be a fan for the toric variety (toric scheme)* X. *Let* $\mathrm{cone}(\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r})$ *be the standard fan for the toric variety (toric scheme)* $\mathbb{A}^r$ *so that* $\mathrm{cone}(\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}) \times \Sigma$ *is a fan for* $\mathbb{A}^r \times \mathrm{X}$. *Then a fan for* $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^r)$ *is given by taking the union of the fans*

$$(B(v, \alpha)^t)^{-1}(\mathrm{cone}(\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}) \times \Sigma)$$

*as* $v$ *varies over* $\{1, 2\}^r$.

*Proof.* Consider first the case where X is affine, given by a cone $\sigma$. We write $S_\sigma$ for the monoid consisting of lattice points in $\sigma^\vee$. We have

$$\mathrm{Spec}(k[S_{\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r})}]) = \mathrm{X}, \quad (\mathrm{Spec}(\mathrm{R}[\mathrm{S}_{\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r})}]) = \mathrm{X}).$$

As a sub-algebra of the co-ordinate ring of the torus, this is

$$k[S_{B(v,\alpha)(\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r}))}] \quad (\mathrm{R}[\mathrm{S}_{\mathrm{B}(\mathrm{v},\alpha)(\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r}))}], \text{respectively}).$$

In other words, the dual cone for $\mathrm{U}(v, \mathrm{X}, \mathbb{G}_\mathrm{m}^r)$ is $B(v, \alpha)(\sigma \times \mathrm{cone}(\mathrm{e}_1 \ldots \mathrm{e}_\mathrm{r}))$. The cone is then described via the lemma. The affine case now follows from [CLS, Exercise 3.2.11]. The same gluing procedure yields the result in general. □

**Proof of Theorem 1.7.** Clearly the above discussion proves the first statement. Now the second statement follows readily from Theorem 1.4 making use of Corollary 3.4. □

We will now recall the following basic framework from [JL, sections 2 and 4]. In the following discussion, we state the results explicitly only for toric schemes defined over the excellent Dedekind domain R: the case for toric varieties over the field $k$ should be clear by replacing R by $k$. Observe that $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^r)$ is now a smooth split toric scheme with the open orbit given by the split torus $\mathbb{G}_m^{r+s}$. We will denote the coordinates of this torus by $t_i$, $i = 1, \cdots r+s$. Let $\zeta$ denote a primitive $m$-th root of unity in $k$ and let $(t_i, t_j)_\zeta$ and $(b, t_i)_\zeta$ denote cyclic algebras with $b \in \mathrm{R}^*$. Observe that any Azumaya algebra generated by the cyclic algebras $(t_i, t_j)_\zeta$, $i < j$ will be of the form $\Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}}$, for some choice of integers $0 \leq e_{i,j} < m$, while any Azumaya algebra generated by the cyclic algebras $(b, t)_\zeta$, with $b \in \mathrm{R}^*$ and $t$ a coordinate of the torus $\mathbb{G}_m^{r+s}$ will be of the form $\Lambda = \Pi_{i=1}^r (b_i, t_i)_\zeta^{e_i}$, for some integers $0 \leq e_i < m$.

We will denote the subgroup of ${}_m\mathrm{Br}(\mathrm{T})$ generated by $\{\Pi_{1 \leq i<j \leq r+s}(t_i, t_j)_\zeta^{e_{i,j}} | e_{i,j} \geq 0\}$ by A, and the subgroup generated by $\{\Pi_{1 \leq i \leq r+s}(b, t_i)_\zeta^{e_i} | e_i \geq 0, b \in \mathrm{R}^*\}$ by B. Let $\mathbf{M}$ ($\mathbf{N}$) denote the lattice of characters (co-characters or 1-parameter subgroups) associated to the split torus $\mathbb{G}_m^{r+s}$. Let $\Delta'$ denote the fan associated to the toric scheme $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_\mathrm{m}^r)$ and let $\mathbf{N}'$ denote the subgroup generated by $\bigcup_{\sigma' \in \Delta'} \sigma' \cap \mathbf{N}$. Then $\mathbf{N}' = \mathbb{Z}a_1\mathbf{n}_1 \oplus \cdots \mathbb{Z}a_u\mathbf{n}_u$, where $\mathbf{n}_1, \cdots, \mathbf{n}_u, \mathbf{n}_{u+1}, \cdots .\mathbf{n}_{r+s}$ is a basis for $\mathbf{N}$ and $a_i \geq 0$ are integers with $a_i | a_{i+1}$,

for $i = 1, \cdots, u$. Then [JL, Theorems 2.1 and 4.1] (i.e., Theorem 10.4) readily provide the following theorem that calculates the Brauer group of the toric stack $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]$.

**Theorem 5.8.** *For a positive integer $m$ invertible in $\mathrm{R}$, let ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r])$ denote the $m$- torsion part of the Brauer group of $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]$. Then the following hold, assuming $\mathrm{R}$ contains a primitive $m$-th root of unity $\zeta$:*

(i) ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cong {}_m\mathrm{Br}(\mathrm{Spec}\,\mathrm{R}) \oplus ({}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cap \mathrm{A}) \oplus ({}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cap \mathrm{B})$*, where (following the terminology in* [JL, Theorem 3.2]*)*

(ii) ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cap \mathrm{A} =$ *the subgroup generated by* $\{\Lambda = \Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}} | m > e_{i,j} \geq 0\}$ *satisfying the following conditions: for each* $v = 1, \ldots, min\{u, r+s-1\}$, *if* $m_v = hcf\{m, e_{1,v}, e_{2,v}, \cdots, e_{v-1,v}, e_{v,v+1}, \cdots, e_{v,r+s}\}$, *then* $(\frac{m}{m_v})|a_v$. *In view of the assumption the toric scheme* $\mathrm{X}$ *is smooth, all* $a_v = 1$, *and hence hence the last condition translates to* $m_v = m$, *for all* $v$.

(iii) ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cap \mathrm{B}$ *is generated by* $\{\Lambda = \Pi_{i=1}^{r+s}(b_i, t_i)_\zeta^{e_i} |\ m > e_i \geq 0\}$, *as* $b_i \in \mathrm{R}^*$ *varies among the corresponding classes in* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(0))$ *so that the following conditions are satisfied: for each* $v = 1, \cdots, u$, *if* $m_v = hcf\{m, e_v, ord_m(b_v)\}$, *then* $(\frac{ord_m(b_v)}{m_v})|a_v$. *(For an element* $b \in \mathrm{R}^*$, *we let* $\bar{b}$ *denote the image of* $b$ *in* $\mathrm{K}^*/(\mathrm{K}^*)^m$. *Let* $ord_m(b)$ *denote the order of* $\bar{b}$, *which is the least positive integer so that* $b^{ord_m(b)} \in (\mathrm{K}^*)^m$.*)*

*In view of the assumption the toric scheme* $\mathrm{X}$ *is smooth, all* $a_v = 1$, *and hence hence the last condition translates to* $ord_m(b_v) = m_v$, *for all* $v$.

Moreover, [JL, Corollaries 3.3 and 4.2] provide the following corollary.

**Corollary 5.9.** *Assume the basic hypotheses of the last theorem. Let $\Delta$ denote the fan for the toric scheme* X. *Then the following hold:*

(i) *In case there is a cone $\sigma$ in the fan $\Delta$ with dimension$(\sigma) \geq s-1$, then* ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cap \mathrm{A}$ *is trivial.*

(ii) *In case there is a cone $\sigma$ in the fan $\Delta$ with dimension$(\sigma) \geq s$, then both* ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cap \mathrm{B}$ *and* ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cap \mathrm{A}$ *are trivial, so that* ${}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]) \cong {}_m\mathrm{Br}(\mathrm{Spec}\,\mathrm{R})$.

*Proof.* In view of Proposition 5.7, one may observe that if there is a cone $\sigma$ for the given toric scheme X with $dimension(\sigma) \geq s-1$ ($dimension(\sigma) \geq s$), then there is a cone $\sigma'$ for the toric scheme $\mathrm{X}_2(\mathrm{X}, \mathbb{G}_{\mathrm{m}}^r)$ with $dimension(\sigma') \geq r+s-1$ ($dimension(\sigma') \geq r+s$, respectively). Therefore the conclusions follow in view of [JL, Corollary 3.3 and 4.2]. □

**Example 5.10.** *Let $n \geq 1$ denote a positive integer. We will now consider the Brauer group of the weighted projective stack* $[(\mathbb{A}^n - \{0\})/\mathbb{G}_m]$ *defined over a base* B *as in* (1.1). *(Observe (see* [GSat, Example 4.14]*) that the stack* $\overline{\mathcal{M}}_{1,1,\mathrm{R}}$ *is a weighted projective stack, so that our results here apply to such stacks.)*

**Corollary 5.11.** *Assume the basic hypotheses of Theorem 5.8. Then* ${}_m\mathrm{Br}([(\mathbb{A}^{\mathrm{n}} - \{0\})/\mathbb{G}_{\mathrm{m}}]) \cong {}_m\mathrm{Br}(\mathrm{B})$.

*Proof.* From the description of the fan for the associated toric variety (scheme) $\mathrm{X}_2(\mathbb{A}^{\mathrm{n}} - \{0\}, \mathbb{G}_{\mathrm{m}})$ as in Proposition 5.7, one can see that it satisfies hypotheses in Corollary 5.9(ii). □

**Example 5.12.** *Here we consider the stack* $[\mathbb{A}^1/\mathbb{G}_m]$. *This is the stack that classifies pairs* $(\mathrm{L}, \mathrm{s})$ *with* $\mathrm{L} \to \mathrm{X}$ *a line bundle on the scheme* X *and* $s \in \mathrm{H}^0(\mathrm{X}, \mathrm{L})$. *Now* (2.7) *shows that* ${}_m\mathrm{Br}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$ *injects into* ${}_m\mathrm{Br}(\mathbb{A}^1)$. *Clearly the latter is trivial if* $m = \ell^n$ *for a prime* $\ell$ *invertible in* $k$ *(*R*) and* $k$ *(*R*) has* $\ell$*-cohomological dimension* 1 *or less. The same holds for the stack* $[\mathbb{A}^n/\mathbb{G}_m]$, *where* $\mathbb{G}_m$ *acts diagonally on* $\mathbb{A}^n$.

**Proposition 5.13.** *(The period-index problem for toric stacks)*

(i) *Assume the base field k is algebraically closed and let m denote a positive integer invertible in k. Then for* $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]$ *a toric stack as in Theorem 1.7, if* $\alpha \in {}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r])$, $Period(\alpha)|Index(\alpha)^{r+s-1}$.

(ii) *Assume the base field k is the function field of a smooth curve over an algebraically closed field and the positive integer m is invertible in k. Then for* $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r]$ *a toric stack as in Theorem 1.7, if* $\alpha \in {}_m\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}^r])$, $Period(\alpha)|Index(\alpha)^{r+s}$.

*Proof.* This follows readily from Theorem 1.7, Proposition 5.3 and [JL, Theorem 5.3 and Corollary 5.7]. □

5.3. **Quotient stacks of the form** $[\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]$**.** Next we proceed to consider quotient stacks of the form $[\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]$ where X is a smooth scheme provided with an action by the diagonalizable group scheme of the form $\mu_{\ell^n}$, all defined over the base scheme B as in 1.1, with $\ell$ a prime invertible in $\mathcal{O}_{\mathrm{B}}$. In this case, we let $\mathrm{E}\mu_{\ell^n} = \mathbb{A}^2 - \{0\}$ provided with the action of $\mu_{\ell^n}$, where it acts through the obvious injection $\mu_{\ell^n} \to \mathbb{G}_m$ and $\mathbb{G}_m$ acts on $\mathbb{A}^2$ in the obvious manner. In view of Theorem 1.4, one may identify $\mathrm{Br}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}])$ with $\mathrm{Br}(\mathrm{E}\mu_{\ell^{\mathrm{n}}} \times_{\mu_{\ell^{\mathrm{n}}}} \mathrm{X})$.

**Theorem 5.14.** *Assume the above situation. Then the following hold.*

*(i) For* $n \geq n'$*, we obtain a short exact sequence:*

$$0 \to {}_{\ell^{n'}}\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}]) \to {}_{\ell^{\mathrm{n}'}}\mathrm{Br}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]) \to \mathrm{H}^1_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(0)) \to 0.$$

*(ii) In case* $n < n'$*, we obtain a short exact sequence, where* $\sigma$ *is the first Chern classes of the line bundle* $\mathcal{O}(-1)$ *on* $\mathrm{B}\mathbb{G}_{\mathrm{m}} = \mathbb{P}^\infty$*:*

$$0 \to ({}_{\ell^{n'}}\mathrm{Br}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}]))/\ell^{\mathrm{n}}\sigma \to {}_{\ell^{\mathrm{n}'}}\mathrm{Br}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]) \to \ker(\mathrm{H}^1_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(0)) \xrightarrow{\ell^{\mathrm{n}}\bar{\sigma}} \mathrm{H}^{3,1}_{\mathrm{et}}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}])) \to 0.$$

*(iii) Next assume that* $\mathcal{O}_{\mathcal{B}}$ *has a primitive* $\ell$*-th root of unity and that the smooth scheme* X *is provided with an action by the symmetric group* $\Sigma_\ell$*, where* $\Sigma_\ell$ *denotes the symmetric group on* $\ell$*-letters. Then* $\mu_\ell$ *identifies with the constant sub-sheaf* $\mathbb{Z}/\ell$ *of* $\Sigma_\ell$ *and*

$${}_\ell\mathrm{Br}([\mathrm{X}/\Sigma_\ell]) \cong ({}_\ell\mathrm{Br}([\mathrm{X}/\mu_\ell]))^{\mathrm{Aut}(\mu_\ell)}.$$

*Proof.* We will first prove (i) and (ii) when X = B, so that $\mu_{\ell^n}$ acts trivially on X in this case. We begin with the calculations in [Voev2, section 6] on the motivic cohomology of $\mathrm{B}\mu_{\ell^{\mathrm{n}}}$. A key observation is that $\mathrm{B}\mu_{\ell^{\mathrm{n}}} = \mathcal{O}(-\ell^{\mathrm{n}}) - \mathrm{z}(\mathbb{P}^\infty)$, where $\mathcal{O}(-\ell^n)$ denotes the obvious line bundle on $\mathbb{P}^\infty$. This is clear since a model for the geometric classifying space for $\mu_{\ell^n}$ is given as the quotient $(\mathbb{A}^{n+1} - 0)/\mu_{\ell^n}$ which fibers over $\mathbb{P}^n = (\mathbb{A}^{n+1} - 0)/\mathbb{G}_m$. Therefore, the homotopy purity theorem [MV, Theorem 2.23] provides the cofiber sequence:

$$\mathrm{B}\mu_{\ell^{\mathrm{n}}+} \to (\mathcal{O}(-\ell^{\mathrm{n}})_{\mathbb{P}^\infty})_+ \to \mathrm{Th}(\mathcal{O}(-\ell^{\mathrm{n}})), \tag{5.3}$$

where $\mathrm{Th}(\mathcal{O}(-\ell^n))$ is the Thom-space of the above line bundle, which identifies with the cofiber of the first map. Next we let $\mathrm{H}^{\mathrm{even}}_{\mathrm{M}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{\mathrm{n}'}) = \oplus_{\mathrm{i}}\mathrm{H}^{2\mathrm{i},\mathrm{i}}_{\mathrm{M}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{\mathrm{n}'})$ and similarly $\mathrm{H}^{\mathrm{even}}_{\mathrm{et}}(\mathbb{P}^\infty, \mu_{\ell^{\mathrm{n}'}}) = \oplus_{\mathrm{i}}\mathrm{H}^{2\mathrm{i},\mathrm{i}}_{\mathrm{et}}(\mathbb{P}^\infty, \mu_{\ell^{\mathrm{n}'}})$. Then

$$\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathbb{P}^\infty, \mathbb{Z}/\ell^{\mathrm{n}'}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{B}, \mathbb{Z}/\ell^{\mathrm{n}'})[[\sigma]] \text{ and} \tag{5.4}$$
$$\mathrm{H}^{*,\bullet}_{\mathrm{et}}(\mathbb{P}^\infty, \mu_{\ell^{\mathrm{n}'}}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}})[[\bar{\sigma}]].$$

Thus, $\mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathbb{P}^{\infty}, \mathbb{Z}/\ell^{\mathrm{n}'})$ ($\mathrm{H}_{\mathrm{et}}^{*,\bullet}(\mathbb{P}^{\infty}, \mu_{\ell^{\mathrm{n}'}})$) is a formal power series ring in the variable $\sigma$ ($\bar{\sigma}$) over $\mathrm{H}_{\mathrm{M}}^{*,\bullet}(\mathrm{B}, \mathbb{Z}/\ell^{\mathrm{n}'})$ (over $\mathrm{H}_{\mathrm{et}}^{*,\bullet}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}})$, respectively). $\sigma$ and $\bar{\sigma}$ are the first Chern classes of the line bundle $\mathcal{O}(-1)$. (Observe that the cycle map sends $\sigma$ to $\bar{\sigma}$.)

Then the long-exact sequences in motivic and in étale cohomology associated to the above cofiber sequence provide the commutative diagram of long exact sequences

(5.5)

$$\begin{array}{ccccccccc}
\longrightarrow \mathrm{H}_{\mathrm{M}}^{0,0}(\mathrm{B}, \mathbb{Z}/\ell^{\mathrm{n}'})[[\sigma]] & \xrightarrow{\cup e} & \mathrm{H}_{\mathrm{M}}^{2,1}(\mathrm{B}, \mathbb{Z}/\ell^{\mathrm{n}'})[[\sigma]] & \longrightarrow & \mathrm{H}_{\mathrm{M}}^{2,1}(\mathrm{B}\mu_{\ell^{\mathrm{n}}}, \mathbb{Z}/\ell^{\mathrm{n}'}) & \longrightarrow & \mathrm{H}_{\mathrm{M}}^{1,0}(\mathrm{B}, \mathbb{Z}/\ell^{\mathrm{n}'})[[\sigma]] \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow \mathrm{H}_{\mathrm{et}}^{0}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}}(0))[[\bar{\sigma}]] & \xrightarrow{\cup e} & \mathrm{H}_{\mathrm{et}}^{2,1}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}}(1))[[\bar{\sigma}]] & \longrightarrow & \mathrm{H}_{\mathrm{et}}^{2}(\mathrm{B}\mu_{\ell^{\mathrm{n}}}, \mu_{\ell^{\mathrm{n}'}}) & \longrightarrow & \mathrm{H}_{\mathrm{et}}^{1}(\mathrm{B}, \mu_{\ell^{\mathrm{n}'}}(0))[[\bar{\sigma}]] \longrightarrow
\end{array}$$

where $\sigma \epsilon \mathrm{H}^{2,1}(\mathbb{P}^{\infty})$ is the first Chern class of $\mathcal{O}(-1)$.

Here $e$ denotes the Euler class of the line bundle $\mathcal{O}(-\ell^n)$. $\mathrm{H}^{*,\bullet}$ denotes either motivic or étale cohomology with $\mathbb{Z}/\ell^m$-coefficients. The map denoted $\cup e$ is the composition of the Thom isomorphism and the obvious map $\mathrm{H}^{*,\bullet}(\mathrm{Th}(\mathcal{O}(-\ell^{\mathrm{n}}))) \to \mathrm{H}^{*,\bullet}(\mathrm{E}(\mathcal{O}(-\ell^{\mathrm{n}})_{\mathbb{P}^{\infty}})) \cong \mathrm{H}^{*,\bullet}(\mathbb{P}^{\infty})$.

Next we break the remaining part of the proof into two cases, (i) when $n \geq n'$ and (ii) when $n < n'$. In the first case, observe that $\ell^n \sigma = e(\mathcal{O}(-\ell^n))$ and therefore, the above long exact sequence breaks up into short exact sequences since we are working with $\mathbb{Z}/\ell^{n'}$-coefficients for $n \geq n'$. In view of (5.4), the statement in (i) follows when $n \geq n'$ by taking the cokernels of the vertical maps, and observing that $\mathrm{H}_{\mathrm{M}}^{1,0} = 0$ (since motivic complex $\mathbb{Z}/\ell^n(0)$ identifies with the constant sheaf $\mathrm{Z}/\ell^n$). In fact one may observe that the vertical maps above are all injective, as one may see from the Kummer sequence and then apply a snake Lemma argument (see, for example: [JL, Lemma 2.6]) to obtain the required conclusion, when $\mathrm{X} = \mathrm{B}$ and the action by $\mu_{\ell^n}$ is trivial.

Next we consider the case where X is no longer the base scheme B. In this case, and for the remainder of the proof, we will denote $(\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X}$ by $[\mathrm{X}/\mu_{\ell^{\mathrm{n}}}]$ and similarly $(\mathbb{A}^2 - \{0\}) \times_{\mathbb{G}_m} \mathrm{X}$ by $[\mathrm{X}/\mathbb{G}_{\mathrm{m}}]$. A key observation now is that $(\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X} = \pi^*(\mathcal{O}(-\ell^{\mathrm{n}})) - \mathrm{z}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}])$, where $\pi : (\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X} \to \mathrm{B}\mu_{\ell^{\mathrm{n}}}$ is the projection and $\pi^*(\mathcal{O}(-\ell^n))$ denotes the pull-back of the line bundle $\mathcal{O}(-\ell^n)$ on $\mathbb{P}^{\infty} = \mathrm{B}\mathbb{G}_{\mathrm{m}}$. This is clear since a model for the geometric classifying space for $\mu_{\ell^n}$ is given as the quotient $(\mathbb{A}^{n+1} - 0)/\mu_{\ell^n}$ which fibers over $\mathbb{P}^n = (\mathbb{A}^{n+1} - 0)/\mathbb{G}_m$. Therefore, the homotopy purity theorem [MV, Theorem 2.23] provides the cofiber sequence:

$$((\mathbb{A}^2 - \{0\}) \times_{\mu_{\ell^n}} \mathrm{X})_+ \to \pi^*(\mathcal{O}(-\ell^{\mathrm{n}})_{\mathbb{P}^{\infty}})_+ \to \mathrm{Th}(\pi^*(\mathcal{O}(-\ell^{\mathrm{n}}))). \tag{5.6}$$

In this case, in place of the diagram (5.5), we obtain the diagram:

(5.7)

$$\begin{array}{ccccccc}
\longrightarrow \mathrm{H}_{\mathrm{M}}^{0,0}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mathbb{Z}/\ell^{\mathrm{n}'}) & \xrightarrow{\cup \pi^*(e)} & \mathrm{H}_{\mathrm{M}}^{2,1}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mathbb{Z}/\ell^{\mathrm{n}'}) & \longrightarrow & \mathrm{H}_{\mathrm{M}}^{2,1}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}], \mathbb{Z}/\ell^{\mathrm{n}'}) & \longrightarrow & \mathrm{H}_{\mathrm{M}}^{1,0}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mathbb{Z}/\ell^{\mathrm{n}'}) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow \mathrm{H}_{\mathrm{et}}^{0}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(0)) & \xrightarrow{\cup \pi^*(e)} & \mathrm{H}_{\mathrm{et}}^{2}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(1)) & \longrightarrow & \mathrm{H}_{\mathrm{et}}^{2}([\mathrm{X}/\mu_{\ell^{\mathrm{n}}}], \mu_{\ell^{\mathrm{n}'}}(1)) & \longrightarrow & \mathrm{H}_{\mathrm{et}}^{1}([\mathrm{X}/\mathbb{G}_{\mathrm{m}}], \mu_{\ell^{\mathrm{n}'}}(0)) \longrightarrow
\end{array}$$

where the map denoted $\cup \pi^*(e)$ is the cup product with the Euler class of the pulled-back line bundle $\pi^*(\mathcal{O}(-\ell^n)_{\mathbb{P}^{\infty}})$. Now the class $\pi^*(e) = \pi^*(\ell^n \sigma) = \ell^n \pi^*(\sigma)$ and hence is trivial. Therefore, the long-exact sequences in each row break up into short-exact sequences. One obtains the short exact sequence in (i) on taking the cokernels.

Next we consider (ii), i.e., the case when $n < n'$. Now the long exact sequence (5.5) no longer breaks up into short exact sequences so that we will argue a bit differently as follows. Here also we first consider the special case where $X = B$ and $\mu_{\ell^n}$ acts trivially. Then one obtains the following commutative diagram of short exact sequences from the long exact sequence (5.5):

$$\begin{array}{ccccccccc} 0 & \rightarrow & H_M^{2,1}(B, \mathbb{Z}/\ell^{n'})[[\sigma]]/(\ell^n\sigma) & \rightarrow & H_M^{2,1}(B\mu_{\ell^n}, \mathbb{Z}/\ell^{n'}) & \rightarrow & K & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H^2_{et}(B, \mu_{\ell^{n'}})[[\bar\sigma]]/(\ell^n\bar\sigma) & \rightarrow & H^2_{et}(B\mu_{\ell^n}, \mu_{\ell^{n'}}) & \rightarrow & \bar K & \rightarrow & 0 \end{array} \tag{5.8}$$

where

$$K = \ker(H_M^{1,0}(B, \mathbb{Z}/\ell^{n'})[[\sigma]] \xrightarrow{\ell^n\sigma} H_M^{3,1}(B, \mathbb{Z}/\ell^{n'})[[\sigma]]) \text{ and } \bar K = \ker(H^1_{et}(B, \mu_{\ell^{n'}}(0))[[\bar\sigma]] \xrightarrow{\ell^n\bar\sigma} H^3_{et}(B, \mu_{\ell^{n'}}(1))[[\bar\sigma]]).$$

Taking the cokernels of the vertical maps, and observing again that $H_M^{1,0} = 0$ now proves (ii) in this case. When X is no longer B, we will instead obtain the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \rightarrow & H_M^{2,1}([X/\mathbb{G}_m], \mathbb{Z}/\ell^{n'})/(\ell^n\sigma) & \rightarrow & H_M^{2,1}([X/\mu_{\ell^n}], \mathbb{Z}/\ell^{n'}) & \rightarrow & K & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H^2_{et}([X/\mathbb{G}_m], \mu_{\ell^{n'}})/(\ell^n\bar\sigma) & \rightarrow & H^2_{et}([X/\mu_{\ell^n}], \mu_{\ell^{n'}}) & \rightarrow & \bar K & \rightarrow & 0 \end{array} \tag{5.9}$$

where

$$K = \ker(H_M^{1,0}([X/\mathbb{G}_m], \mathbb{Z}/\ell^{n'}) \xrightarrow{\ell^n\sigma} H_M^{3,1}([X/\mathbb{G}_m], \mathbb{Z}/\ell^{n'})) \text{ and}$$

$$\bar K = ker(H^1_{et}([X/\mathbb{G}_m], \mu_{\ell^{n'}}(0)) \xrightarrow{\ell^n\bar\sigma} H^3_{et}([X/\mathbb{G}_m], \mu_{\ell^{n'}}(1))).$$

Taking the cokernels of the vertical maps, now proves (ii) in this case.

Next we will consider (iii). First observe that, under our hypotheses, one may identify the sheaf $\mu_\ell$ with the constant sheaf $\mathbb{Z}/\ell$. In this case, we will adopt the terminology from section 2.3 and let $EGL_n^{gm,2}$ denote the object defined in (2.3). Then we will let $[X/\Sigma_\ell]$ denote $EGL_n^{gm,2} \times_{GL_n} (GL_n \times_{\Sigma_\ell} X)$. Moreover we will let $[X/\mu_{\ell^n}]$ denote $EGL_n^{gm,2} \times_{GL_n} (GL_n \times_{\mathbb{Z}/\ell} X)$. Then one has a natural map $p : [X/\mu_\ell] \overset{\cong}{\rightarrow} [X/\mathbb{Z}/\ell] \rightarrow [X/\Sigma_\ell]$. The main observation now is that $|\Sigma_l/\mathbb{Z}/\ell| = (\ell-1)!$ is invertible in $\mathbb{Z}/\ell$ and $\mu_\ell$. Therefore, the induced maps

$$\begin{aligned} p^* &: H_M^{*,\bullet}([X/\Sigma_\ell], \mathbb{Z}/\ell) \rightarrow H_M^{*,\bullet}([X/\mu_\ell], \mathbb{Z}/\ell)^{Aut(\mu_\ell)} \text{ and} \\ p^* &: H^*_{et}([X/\Sigma_\ell], \mathbb{Z}/\ell) \rightarrow H^*_{et}([X/\mu_\ell], \mathbb{Z}/\ell)^{Aut(\mu_\ell)} \end{aligned} \tag{5.10}$$

are split injective. One may also observe that the action of $Aut(\mu_\ell)$ is compatible with the cycle map. Therefore the case of $[X/\mu_\ell]$ considered above completes the proof of (iii) in the Theorem. ☐

**Example 5.15.** *As an example one can consider the quotient stack* $[\mathbb{A}^1/\mu_{\ell^{n'}}]$. *Here we make use of Proposition 5.14 to compute the resulting Brauer groups. Observe that* $H^1_{et}([\mathbb{A}^1/\mathbb{G}_m], \mu_{\ell^{n'}}(0)) \cong H^1_{et}(B\mathbb{G}_m, \mu_{\ell^{n'}}(0)) \cong H^1(B, \mu_{\ell^{n'}}(0))$ *since* $B\mathbb{G}_m \cong \mathbb{P}^\infty$. *One may now complete the computation making use of the short-exact sequences in Proposition 5.14. Observe that when* $B = \operatorname{Spec} k$ *with* $k$ *a separably or algebraically closed field,* ${}_{\ell^n}Br([\mathbb{A}^1/\mu_{\ell^{n'}}])$ *is trivial.*

## 6. Brauer group of the moduli stack of elliptic curves and Proof of Theorem 1.9.

Let $\mathcal{M}_{1,1,\mathrm{R}}$ denote the moduli stack of elliptic curves over the base scheme B, which we assume is the spectrum of a Dedekind domain R, for example, the ring of integers in a number field. We proceed to compute the $\ell$-primary torsion part of the corresponding Brauer group. We will assume that the primes 2 and 3 are invertible in R. Let $\mathrm{Y} = \mathrm{Spec\,R}[\mathrm{g}_2, \mathrm{g}_3][1/\Delta] \subseteq \mathbb{A}^2_{\mathrm{R}}$, where $\Delta = g_2^3 - 27g_3^2$. We define an action of $\mathbb{G}_m$ by $g_2 \mapsto u^4 g_2, g_3 \mapsto u^6 g_3$, $u \in \mathbb{G}_m$. Let $\mathrm{B} = \mathrm{Spec\,R}$. Then we make use of the following presentation for the stack $\mathcal{M}_{1,1,\mathrm{R}}$: see [Ols, Proposition 28.6] or [Hart77, Chapter IV section 4, Theorem 4.14B]:

$$\mathcal{M}_{1,1,\mathrm{R}} = [\mathrm{Y}/\mathbb{G}_{\mathrm{m}}]. \tag{6.1}$$

Then we obtain the following result.

**Theorem 6.1.** *Assume further that the prime $\ell$ is invertible in* R. *Then the following hold:*

(i) ${}_{\ell^n}\mathrm{Br}(\mathrm{Y}) \cong {}_{\ell^n}\mathrm{Br}(\mathrm{B}) \oplus \mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(0)) \cong {}_{\ell^n}\mathrm{Br}(\mathrm{B}) \oplus \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{et}}(\mathrm{B}, *), \mathbb{Z}/\ell^{\mathrm{n}})$, *where* $\pi_1^{et}(\mathrm{B}, *)$ *denotes the étale fundamental group of* B *(pointed by a base point $*$) and* $\mathrm{Hom}_{\mathrm{cont}}(\pi_1^{et}(\mathrm{B}, *), \mathbb{Z}/\ell^{\mathrm{n}})$ *denotes the group of continuous homomorphisms into the the discrete group* $\mathbb{Z}/\ell^n$,

(ii) $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^n}(1)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(1)) \oplus \mathrm{H}^0_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(0)) \cong \mathrm{coker}(\mathrm{R}^* \xrightarrow{\ell^{\mathrm{n}}} \mathrm{R}^*) \oplus \mathbb{Z}/\ell^{\mathrm{n}}$,

(iii) ${}_{\ell^n}\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}}) = \mathrm{kernel}(\mathrm{res} : {}_{\ell^n}\mathrm{Br}((\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}) \times_{\mathbb{G}_{\mathrm{m}}} \mathrm{Y}) \cong {}_{\ell^n}\mathrm{Br}(\mathrm{Y}) \to \mathrm{H}^3_{\mathrm{et}, (\mathbb{G}_{\mathrm{m}} \times \{0\}) \times_{\mathbb{G}_{\mathrm{m}}} \mathrm{Y}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1 \times_{\mathbb{G}_{\mathrm{m}}} \mathrm{Y}, \mu_{\ell^n}(1)) \cong \mathrm{H}^1_{\mathrm{et}}((\mathbb{G}_{\mathrm{m}} \times \{0\}) \times_{\mathbb{G}_{\mathrm{m}}} \mathrm{Y}, \mu_{\ell^n}(0)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^n}(0)))$,
*where res denotes the residue map discussed in* (2.7) *and in* [JL, 2.3]. *In particular the Brauer group* ${}_{\ell^n}\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})$ *is a subgroup of* ${}_{\ell^n}\mathrm{Br}(\mathrm{Y})$. *Moreover* ${}_{\ell^n}\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}}) \cong 0$ *if the residue map in (ii) is injective.*

(iv) *In case the Brauer group* ${}_{\ell^n}\mathrm{Br}(\mathrm{B})$ *is trivial,* ${}_{\ell^n}\mathrm{Br}(\mathrm{Y}) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(0))$ *and hence* ${}_{\ell^n}\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}})$ *is generated by classes coming from* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(0))$.

(v) *In case* R *is a finite field, separably closed field, or a complete discrete valuation ring with finite or separably closed residue field,* ${}_{\ell^n}\mathrm{Br}(\mathrm{B})$ *is trivial, so that the conclusions in* (iv) *hold in these cases.*

*Proof.* Throughout the proof, we will let $x = g_2$, $y = g_3$ and $\tilde{\Delta} = Spec\,\mathrm{R}[x, y]/(x^3 - 27y^2)$. Let $\mathbb{A}^i_{\mathrm{B}}$ denote the affine space of dimension $i$ over $\mathrm{B} = \mathrm{Spec\,R}$. Observe that the curve corresponding to $\tilde{\Delta}$ has an isolated singularity at the origin, which can be resolved by taking the normalization as follows. Recall that $\tilde{\Delta}$ corresponds to the plane curve with equation : $(x/3)^3 = y^2$. Therefore, we substitute $(x/3) = t^2$ and $y = t^3$, so that $A = \mathrm{R}[x, y]/((x^3 - 27y^2) \cong \mathrm{R}[\mathrm{t}^2, \mathrm{t}^3]$ with function field K(t), where K denotes the function field of R. This is because $1/t = t^2/t^3 = (x/3)/y = x/(3y)$. Since R is assumed to be a Dedekind domain, it is integrally closed in its field of fractions K. Therefore, R[t] is integrally closed in K(t): see [StacksP, Normal rings: Lemma 10.37.8]. Clearly t is integral over $A$, and therefore the integral closure of $A$ in K(t) is $\mathrm{R}[\mathrm{t}] = \mathrm{R}[3y/x]$, which corresponds to the affine line $\mathbb{A}^1_{\mathrm{B}}$ over $\mathrm{B} = \mathrm{Spec\,R}$. This proves that the normalization of the curve $\tilde{\Delta}$ is the affine line $\mathbb{A}^1_{\mathrm{B}}$ and the normalization maps $\mathbb{A}^1_{\mathrm{B}} - \{0\}$ isomorphically to the curve $\tilde{\Delta} - \{0\}$.

Therefore, we obtain the isomorphisms:

$$\mathbb{A}^2_{\mathrm{B}} - \tilde{\Delta} \cong (\mathbb{A}^2_{\mathrm{B}} - \{0\}) - ((\tilde{\Delta}) - \{0\}) \cong \mathbb{A}^2_{\mathrm{B}} - \mathbb{A}^1_{\mathrm{B}} \cong \mathbb{A}^1_{\mathrm{B}} \times \mathbb{G}_{m,\mathrm{B}} \tag{6.2}$$

$$\mathrm{H}^3_{\mathrm{et}}(\mathbb{A}^2_{\mathrm{B}} - \tilde{\Delta}, \mu_{\ell^n}(1)) \cong \mathrm{H}^3_{\mathrm{et}}((\mathbb{A}^2_{\mathrm{B}} - \{0\}) - (\tilde{\Delta} - \{0\}), \mu_{\ell^n}(1)) \cong \mathrm{H}^3_{\mathrm{et}}((\mathbb{A}^2_{\mathrm{B}} - \mathbb{A}^1_{\mathrm{B}}), \mu_{\ell^n}(1)),$$

where the last isomorphism follows from the observation made earlier that the normalization of $\tilde{\Delta}$ is the affine line $\mathbb{A}^1$. At this point Theorem 10.1(i) applies to provide a proof of the assertion in (i) while Theorem 10.1(ii) applies to provide a proof of the assertion in (ii). The assertion in (iii) then follows from (2.7). The remaining statements are clear. □

**Corollary 6.2.** (i) *Next assume the base ring* R *is a field* $k$ *in which* 2 *and* 3 *are invertible. Let* $\ell$ *denote a prime not necessarily different from* 2 *or* 3*, so that* $\ell$ *is also invertible in* $k$*, and* $k$ *contains a primitive* $\ell^n$*-th root of unity for some positive integer* $n$*. Then*

$$ {}_{\ell^n}\mathrm{Br}(\mathcal{M}_{1,1,k}) \cong {}_{\ell^n}\mathrm{Br}(\mathrm{Spec}\, k) \oplus \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^n}(0)). \tag{6.3}$$

*Moreover the last summand corresponds to cyclic algebras of the form* $(b,t)_\zeta$*, where* $b \in k^*$ *and* $t \in \mathbb{G}_m$ *corresponds to the character that generates the lattice of characters of* $\mathbb{G}_m$*.*

(ii) *The corresponding results also hold when the base ring* R *is an excellent Dedekind domain in which* 2 *and* 3 *are invertible and where* $\ell$ *denotes a prime not necessarily different from* 2 *or* 3*, so that* $\ell$ *is also invertible in* R*, and* R *contains a primitive* $\ell^n$*-th root of unity for some positive integer* $n$*.*

*Proof.* We will first consider the proof of the first statement. The main point to observe is that, in view of (6.2), now $\mathcal{M}_{1,1,k}$ identifies with the toric stack $[(\mathbb{A}^1_k \times \mathbb{G}_{m,k})/\mathbb{G}_{m,k}]$. According to Proposition 5.7, the corresponding toric variety $\mathrm{X}_2(\mathbb{A}^1_k \times \mathbb{G}_{\mathrm{m},k}, \mathbb{G}_{\mathrm{m},k})$ has an open covering by toric varieties of the form $\mathbb{A}^1_k \times \mathbb{A}^1_k \times \mathbb{G}_{m,k}$. Now the open orbit is $\mathbb{G}^{\times 3}_{m,k}$, but there are only two codimension 1 orbits coming from the fan, namely $\{0\} \times \mathbb{G}^{\times 2}_{m,k}$ and $\mathbb{G}_{m,k} \times \{0\} \times \mathbb{G}_{m,k}$. Observe also that the fan for the toric variety $\mathbb{A}^1_k \times \mathbb{G}_{m,k}$ is 1-dimensional and hence the fan for $\mathrm{X}_2(\mathbb{A}^1_k \times \mathbb{G}_{\mathrm{m},k}, \mathbb{G}_{\mathrm{m},k})$ is 2-dimensional. Using the terminology of Theorem 5.8, $r+s=3$, and $u=2$ and since the toric schemes are all smooth, the invariant factors $a_v = 1$ for all $v$. Therefore, Corollary 5.9(i) applies to show that the term ${}_{\ell^n}\mathrm{Br}([\mathbb{A}^1_k \times \mathbb{G}_{\mathrm{m},k}/\mathbb{G}_{\mathrm{m},k}]) \cap \mathrm{A}$ is trivial. Now Theorem 5.8(iii) and Corollary 5.9(ii) apply to show that as $b \in k^*$ varies, the corresponding classes in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^n}(0))$ give rise to the cyclic algebras $(b,t)_\zeta$ contributing to the last summand $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_{\ell^n}(0))$ in (6.3). This completes the proof of (i). For the second statement, one may observe that $\mathcal{M}_{1,1,\mathrm{R}}$ identifies with the corresponding toric stack defined over $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$ and therefore a very similar calculation holds in this case. □

Now we return to the general setting where R is a Dedekind domain and consider the situation without assuming it has a primitive $\ell^n$-th root of unity. One considers the following conditions on $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$.

(i) $\mathrm{Pic}(\mathrm{B}) = 0$,
(ii) $\mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^n}(1)) = 0$. (In case $\ell$ is invertible in $\mathcal{O}_\mathrm{B}$, $\mathrm{H}^2_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(1)) = 0$.) and
(iii) $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^n}(1)) = 0$. (In case $\ell$ is invertible in $\mathcal{O}_\mathrm{B}$, $\mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(1)) = 0$.)

**Lemma 6.3.** *(i) Under the hypothesis in (i),* ${}_{\ell^n}\mathrm{Br}(\mathrm{B}) \cong \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^n}(1))$*, which denotes cohomology computed on the fppf site. If in addition to the hypothesis in (i),* $\ell$ *is invertible in* $\mathcal{O}_\mathrm{B}$*,* ${}_{\ell^n}\mathrm{Br}(\mathrm{B}) \cong \mathrm{H}^2_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(1))$*. Therefore, under the hypothesis in (i), (ii) is equivalent to* ${}_{\ell^n}\mathrm{Br}(\mathrm{B}) \cong 0$*.*

*(ii) Under the hypothesis in (i),* $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^n}(1)) \cong 0$ *if and only if the the* $\ell^n$*-th power map* $\Gamma(\mathrm{B}, \mathbb{G}_\mathrm{m}) \to \Gamma(\mathrm{B}, \mathbb{G}_\mathrm{m})$ *is surjective. If* $\ell$ *is invertible in* B *and (i) holds,* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_{\ell^n}(1)) \cong 0$ *if and only if the the* $\ell^n$*-th power map* $\Gamma(\mathrm{B}, \mathbb{G}_\mathrm{m}) \to \Gamma(\mathrm{B}, \mathbb{G}_\mathrm{m})$ *is surjective.*

*Proof.* To see (i), consider the the following part of the long exact sequence provided by the Kummer sequence:

$$\cdots \to \mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\alpha} \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}(1)) \xrightarrow{\beta} \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\gamma} \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \to \cdots \tag{6.4}$$

(If $\ell$ is invertible in $\mathcal{O}_{\mathrm{B}}$, one may also use the corresponding long-exact sequence in étale cohomology.) The map denoted $\gamma$ is the $\ell^n$-th power map, and its kernel is ${}_{\ell^n}\mathrm{Br}(\mathrm{B})$. The exactness of the long exact sequence above shows that the kernel of $\gamma$ is isomorphic to the image of $\beta$. But the map $\beta$ is clearly injective in view of the assumption that $\mathrm{Pic}(\mathrm{B}) = 0$. Thus ${}_{\ell^n}\mathrm{Br}(\mathrm{B}) \cong \mathrm{H}^2_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}(1))$. The same argument proves the corresponding statement in étale cohomology when $\ell$ is invertible in $\mathcal{O}_{\mathrm{B}}$.

For (ii) we consider the following part of the long exact sequence provided by the Kummer sequence:

$$0 \to \mathrm{H}^0_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^n}(1)) \to \Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\ell^{\mathrm{n}}} \Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\delta} \mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^n}(1)) \to \mathrm{Pic}(\mathrm{B}) \cong 0 \tag{6.5}$$

This shows that $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^{\mathrm{n}}}(1))$ is isomorphic to the cokernel of the $\ell^n$-th power map. Thus $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{B}, \mu_{\ell^n}(1)) \cong 0$ if and only if the $\ell^n$-th power map $\Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\ell^{\mathrm{n}}} \Gamma(\mathrm{B}, \mathbb{G}_{\mathrm{m}})$ is surjective. The corresponding statement for étale cohomology, when $\ell$ is invertible in $\mathcal{O}_{\mathrm{B}}$ may be proven similarly. □

Next we make the following observations. Let $\mathbb{Z}$ denote the ring of integers in $\mathbb{Q}$. Then

(i) $\mathrm{Pic}(\mathrm{Spec}\,\mathbb{Z}) \cong 0$, and
(ii) $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z}) \cong 0$. In particular ${}_{\ell^n}\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z}) \cong 0$.

One may readily see (i) is true because $\mathbb{Z}$ is a PID. The statement that $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z}) \cong 0$ may be proven using class-field-theory: see [Ay], for example. Since ${}_{\ell^n}\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z})$ denotes the $\ell^n$-torsion part of $\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z})$ the second assertion in (ii) follows.

Let $\mathbb{Z}_{1/6}$ denote the localization of the ring $\mathbb{Z}$ by inverting the number 6. Next observe that $\mathrm{Spec}\,\mathbb{Z}_{1/6} = \mathrm{Spec}\,\mathbb{Z} - \{(2), (3)\}$. Therefore, one may compute its Brauer group as follows.

**Lemma 6.4.** *${}_{\ell^n}\mathrm{Br}(\mathrm{Spec}\,\mathbb{Z}_{1/6}) \cong {}_{\ell^{\mathrm{n}}}(\mathbb{Z}/2\mathbb{Z}) \oplus {}_{\ell^{\mathrm{n}}}(\mathbb{Q}/\mathbb{Z})$, for $\ell = 2$ or 3.*

*Proof.* We skip the proof as this follows from a standard argument using the computation that $\mathrm{Br}(\mathbb{Z}) = 0$. □

Taking $\mathrm{R} = \mathbb{Z}_{1/6}$, we obtain the following corollary to Theorem 6.1.

**Corollary 6.5.** *Let $\mathrm{B} = \mathrm{Spec}\,\mathbb{Z}_{1/6}$ and let $\mathrm{Y}$ be as in (6.1). Let $\mathbb{Z}^*_{1/6}$ denote the units in the ring $\mathbb{Z}_{1/6}$. Then for $\ell = 2$, or 3,*

(i) *${}_{\ell^n}\mathrm{Br}(\mathrm{Y}) \cong {}_{\ell^{\mathrm{n}}}(\mathbb{Z}/2\mathbb{Z}) \oplus {}_{\ell^{\mathrm{n}}}(\mathbb{Q}/\mathbb{Z}) \oplus \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Z}_{1/6}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong {}_{\ell^{\mathrm{n}}}(\mathbb{Z}/2\mathbb{Z}) \oplus {}_{\ell^{\mathrm{n}}}(\mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}^*_2 \times \mathbb{Z}^*_3, \mathbb{Z}/\ell^{\mathrm{n}})$, where $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}^*_2 \times \mathbb{Z}^*_3, \mathbb{Z}/\ell^n)$ denotes the set of continuous homomorphisms from the profinite group $\mathbb{Z}^*_2 \times \mathbb{Z}^*_3$ (which denotes the group of units in $\mathbb{Z}_2 \times \mathbb{Z}_3$) to the discrete group $\mathbb{Z}/\ell^n$,*
(ii) *$\mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^n}(1)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Z}_{1/6}, \mu_{\ell^{\mathrm{n}}}(1)) \oplus \mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Z}_{1/6}, \mu_{\ell^{\mathrm{n}}}(0))$
$\cong \mathrm{coker}(\mathbb{Z}^*_{1/6} \xrightarrow{\ell^{\mathrm{n}}} \mathbb{Z}^*_{1/6}) \oplus \mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Z}_{1/6}, \mathbb{Z}/\ell^{\mathrm{n}}) \cong \mathbb{Z}^*_{1/6}/(\mathbb{Z}^*_{1/6})^{\ell^{\mathrm{n}}} \oplus \mathbb{Z}/\ell^{\mathrm{n}}$, and*
(iii) *${}_{\ell^n}\mathrm{Br}(\mathcal{M}_{1,1,\mathrm{R}}) = \ker(\mathrm{res} : {}_{\ell^n}\mathrm{Br}(\mathrm{Y}) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Y}, \mu_{\ell^{\mathrm{n}}}(1)))$ where res denotes the residue map as in Theorem 6.1.*

*Proof.* One may compute the étale fundamental group of $\mathrm{Spec}\,\mathbb{Z}_{1/6}$ to be $\mathbb{Z}^*_2 \times \mathbb{Z}^*_3$. Therefore (i) follows from the first statement in Theorem 6.1(i). The Kummer sequence in (6.5) (which holds on the étale site in this case since $\ell = 2, 3$) shows that $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Z}_{1/6}, \mu_{\ell^n}(1)) \cong \mathrm{coker}(\mathbb{Z}^*_{1/6} \xrightarrow{\ell^{\mathrm{n}}} \mathbb{Z}^*_{1/6})$. Therefore, (ii) follows from

Theorem 6.1(ii). Now (iii) follows readily from Theorem 6.1(iii) and the above observations. This completes the proof of the Corollary. □

**Remark 6.6.** We would like to point out that, though there are related results in the literature, such as [AMO], their work uses a completely different presentation of the moduli stack.We are able to obtain the results discussed above, as a direct consequence of our work on the Brauer groups of toric stacks and quotient stacks as in the earlier sections of this paper.

## 7. Brauer group and torsion index of linear algebraic groups: Proof of Theorem 1.10.

Throughout this section we will work over any base field $k$ with $m$ a positive integer invertible in $k$. All linear algebraic groups we consider will be defined over $k$. Let H denote a fixed connected linear algebraic group with a chosen Borel subgroup B and a chosen maximal torus $\mathrm{T} \subseteq \mathrm{B}$. Let N denote the dimension of H/B. For a linear algebraic group G, we will let $\mathrm{BG^{gm}}$ denote $\mathrm{BG^{gm,m}}$, for some $\mathrm{m} \gg 0$.

Next consider the diagram

$$\mathrm{H/B} \overset{\mathrm{i}}{\to} \mathrm{BB^{gm}} \cong \mathrm{EH^{gm}} \times_{\mathrm{H}} (\mathrm{H/B}) \overset{\mathrm{f}}{\to} \mathrm{BH^{gm}}, \tag{7.1}$$

where $f$ denotes the obvious map induced by the inclusion $\mathrm{B} \subseteq \mathrm{H}$ (or equivalently the projection $\mathrm{H/B} \to \operatorname{Spec} k$). Observe that $\mathrm{BB^{gm}} \simeq \mathrm{BT^{gm}}$, where $\simeq$ denotes an isomorphism in the motivic homotopy category. We next recall the definition of the *torsion index* of connected linear algebraic groups from [Tot05, section 1].

**Definition 7.1.** *(See* [Tot05, section 1]*.) Let* $\mathrm{N} = \dim(\mathrm{H/B})$*, so that* $\mathrm{CH^N(H/B)} \cong \mathbb{Z}$*. Then the torsion index of* H *(denoted* $t(\mathrm{H})$*) is the least positive integer* $t(\mathrm{H})$ *so that*

$$image(i^* : \mathrm{CH^N}(\mathrm{BB^{gm}}) \to \mathrm{CH^N(H/B)})$$

*equals* $t(\mathrm{H}).\mathrm{CH^N(H/B)}$*.*

Observe that there exists a class $a \in \mathrm{CH^N}(\mathrm{BB^{gm}}, \mathrm{Z}/m) (\cong \mathrm{H_M^{2N,N}}(\mathrm{BB^{gm}}, \mathbb{Z}/m))$ so that

$$f_*(a) = \overline{t(\mathrm{H})} \in \mathrm{CH^0}(\mathrm{BH^{gm}}, \mathrm{Z}/m) \cong \mathrm{Z}/m, \tag{7.2}$$

where $\overline{t(\mathrm{H})}$ is the image of the torsion index $t(\mathrm{H})$.

**Remark 7.2.** *For linear algebraic groups defined over the complex numbers, the torsion index can also be defined using singular cohomology with integral coefficients, as for such groups the singular cohomology of* H/B *with integral coefficients is isomorphic to the corresponding Chow group.*

Next we consider the following squares that commute:

$$\begin{array}{ccc} \mathrm{H_M^{*,\bullet}}(\mathrm{BB^{gm}}, \mathbb{Z}/m) & \xrightarrow{f_*} & \mathrm{H_M^{*-2N,\bullet-N}}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \\ \downarrow{\scriptstyle cycl} & & \downarrow{\scriptstyle cycl} \\ \mathrm{H^*_{et}}(\mathrm{BB^{gm}}, \mu_m(\bullet)) & \xrightarrow{\bar{f}_*} & \mathrm{H^{*-2N}_{et}}(\mathrm{BH^{gm}}, \mu_m(\bullet - \mathrm{N})) \end{array} \tag{7.3}$$

$$\begin{array}{ccc} \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/m) & \xleftarrow{f^*} & \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BH}^{\mathrm{gm}}, \mathbb{Z}/m) \\ \downarrow{\scriptstyle cycl} & & \downarrow{\scriptstyle cycl} \\ \mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_m(\bullet)) & \xleftarrow{\bar{f}^*} & \mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BH}^{\mathrm{gm}}, \mu_m(\bullet)) \end{array} \tag{7.4}$$

To see that these squares commute, one may observe that $\mathrm{BB}^{\mathrm{gm,m}}$ identifies with $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} (\mathrm{H}/\mathrm{B})$ while $\mathrm{BH}^{\mathrm{gm,m}}$ identifies with $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} (\mathrm{H}/\mathrm{H}) \cong \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} (\mathrm{Spec}\, k)$. Now observe that, since B is a Borel subgroup in H, H/B is a flag variety and therefore admits an H-equivariant closed immersion into a projective space $\mathbb{P}^n$ on which H acts. Therefore, the map $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{H}/\mathrm{B} \to \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{H}/\mathrm{H} \cong \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{Spec}\, k$ factors as the composition $\mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{H}/\mathrm{B} \overset{i}{\to} \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathbb{P}^{\mathrm{n}} \overset{\pi}{\to} \mathrm{EH}^{\mathrm{gm,m}} \times_{\mathrm{H}} \mathrm{Spec}\, k$. Therefore it suffices to show that the cycle map commutes with $i_*$ and $i^*$ as well as with $\pi_*$ and $\pi^*$. One may prove $i_*$ commutes with the cycle map by using a deformation to the normal cone argument. To prove $\pi_*$ commutes with the cycle map, one may use the projective space bundle formula for motivic and étale cohomology. The commutativity of $i^*$ and $\pi^*$ with the cycle map may be proven similarly.

**Lemma 7.3.** *(i) The cycle map* $cycl : \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/m) \to \mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_m(\bullet))$ *is an isomorphism when the base field* $k$ *is separably closed.*

*(ii) For any base field* $k$, $\mathrm{H}^{u}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_m(v)) \cong \oplus_{i+2m=u, j+m=v} \mathrm{H}^{i}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_m(j)) \otimes_{\mathbb{Z}/m} \mathrm{H}^{2m}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/m(m))$. *In particular,*

$$\begin{aligned} \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_m(1)) &\cong \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_m(1)) \otimes_{\mathbb{Z}/m} \mathrm{H}^{0}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/m(0)) \oplus \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_m(0)) \otimes_{\mathbb{Z}/m} \mathrm{H}^{2}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/m(1)) \\ &\cong \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_m(1)) \oplus \mathrm{H}^{2}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/m(1)). \end{aligned}$$

*(iii) For any base field* $k$*, the cycle maps in the left column in the commutative squares* (7.3) *and* (7.4) *are both injective.*

*(iv)* $\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{Spec}\, k, \mathbb{Z}/m)$ *is a split summand of both* $\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm}}, \mathbb{Z}/m)$ *and* $\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BH}^{\mathrm{gm}}, \mathbb{Z}/m)$. $\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_m(\bullet))$ *is a split summand of both* $\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm}}, \mu_m(\bullet))$ *and* $\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BH}^{\mathrm{gm}}, \mu_m(\bullet))$*. It follows that* ${}_m\mathrm{Br}(\mathrm{Spec}\, k)$ *is a split summand of both* ${}_m\mathrm{Br}(\mathrm{BB}^{\mathrm{gm}})$ *and* ${}_m\mathrm{Br}(\mathrm{BH}^{\mathrm{gm}})$.

*Proof.* Observe that $\mathrm{BB}^{\mathrm{gm,m}} \simeq \mathrm{BT}^{\mathrm{gm,m}} = \times^r \mathbb{P}^{\mathrm{m}}$, where T is a split maximal torus in H of rank $r$. Therefore, the first two statements follow readily from the calculation of the motivic and étale cohomology of a projective space $\mathbb{P}^n$. In fact, we will presently provide the following details on this. One starts with the isomorphisms provided by the projective space bundle formula:

$$\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm,m}}, \mathbb{Z}/m) \cong \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{Spec}\, k, \mathbb{Z}/m) \underset{\mathbb{Z}/m}{\otimes} (\otimes^r_{\mathbb{Z}/m, \mathrm{i}=1} \mathbb{Z}/m[\mathrm{t_i}]/(\mathrm{t_i^{m+1}})), \tag{7.5}$$

where each $t_i$ has bi-degree $(2,1)$. Similarly one see that

$$\mathrm{H}^{*}_{\mathrm{et}}(\mathrm{BB}^{\mathrm{gm,m}}, \mu_m(\bullet)) \cong \mathrm{H}^{*,\bullet}_{\mathrm{et}}(\mathrm{Spec}\, k, \mathbb{Z}/m) \underset{\mathbb{Z}/m}{\otimes} (\otimes^r_{\mathbb{Z}/m, \mathrm{i}=1} \mathbb{Z}/m[\mathrm{t_i}]/(\mathrm{t_i^{m+1}})), \tag{7.6}$$

where each $t_i$ has bi-degree $(2,1)$. Next one may observe that

$$\oplus_{j=0}^{m} \mathrm{H}^{\mathrm{2j,j}}_{\mathrm{M}}(\mathrm{BB}^{\mathrm{gm,m}}, \mathbb{Z}/m) \cong \otimes^r_{\mathbb{Z}/m, \mathrm{i}=1} \mathbb{Z}/m[\mathrm{t_i}]/(\mathrm{t_i^{m+1}}).$$

On letting $m \to \infty$, these observations prove the statement in (i) and the first statement in (ii). The second statement in (ii) is now an immediate consequence of the first statement.

The third statement is an immediate consequence of the first two. Observe from the definition of admissible gadgets as in (2.1), we require that the $\mathrm{U_m}$ there always has a $k$-rational point. It follows that the finite degree approximations $\mathrm{BB^{gm,m}}$ and $\mathrm{BH^{gm,m}}$ also have $k$-rational points. The statements in (iv) are immediate consequences of this. □

Moreover, one may observe that the cycle map $\mathrm{H^{0,0}_M}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \to \mathrm{H^0_{et}}(\mathrm{BH^{gm}}, \mu_m(0)) \cong \mathrm{Z}/m$ is also an isomorphism. In view of these observations, one may define the torsion index $\overline{t(\mathrm{H})}$ as the class in $\mathbb{Z}/m$ so that if $\bar{a} = cycl(a)$, with the class $a$ as in (7.2)

$$\bar{f}_*(\bar{a}) = \overline{t(\mathrm{H})} \in \mathrm{H}^0_{et}(\mathrm{BH^{gm}}, \mu_m) \cong \mathrm{Z}/m, \tag{7.7}$$

**Proposition 7.4.** *(See* [Tot05, section 1]*.) (i) The kernel of the cycle map*

$$cycl : \mathrm{H^{*,\bullet}_M}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \to \mathrm{H^*_{et}}(\mathrm{BH^{gm}}, \mu_m),$$

*as well as the kernel of the restriction map*

$$f^* : \mathrm{H^{*,\bullet}_M}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \to \mathrm{H^{*,\bullet}_M}(\mathrm{BB^{gm}}, \mathbb{Z}/m)$$

*are killed by* $t(\mathrm{H})$.

*(ii)* ${}_m\mathrm{Br}(\mathrm{BH^{gm}})/({}_m\mathrm{Br}(\mathrm{Spec}\, k)) = \mathrm{H^2_{et}}(\mathrm{BH^{gm}}, \mu_m(1))/(\mathrm{Im(cycl)} + \mathrm{H^2_{et}}(\mathrm{Spec}\, k, \mu_m(1)))$ *is killed by* $t(\mathrm{H})$, *where* $\mathrm{Im}(cycl)$ *denotes the image of the cycle map cycl* : $\mathrm{H^{2,1}_M}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \to \mathrm{H^2_{et}}(\mathrm{BH^{gm}}, \mu_m(1))$.

*Proof.* Define a map $\alpha : \mathrm{CH^i}(\mathrm{BB^{gm}}, \mathbb{Z}/m) \cong \mathrm{H^{2i,i}_M}(\mathrm{BB^{gm}}, \mathbb{Z}/m) \to \mathrm{CH^i}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \cong \mathrm{H^{2i,i}_M}(\mathrm{BH^{gm}}, \mathbb{Z}/m)$ by $\alpha(x) = f_*(a.x)$. Then, $\alpha(f^*(x)) = f_*(a.f^*(x)) = f_*(a).x = t(\mathrm{H}).x$. As $\mathrm{BT^{gm}}$ identifies with $\mathrm{BB^{gm}}$, the map $f^*$ identifies with the restriction homomorphism $res : \mathrm{H^{*,\bullet}_M}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \to \mathrm{H^{*,\bullet}_M}(\mathrm{BT^{gm}}, \mathbb{Z}/m)$, thereby proving that its kernel is killed by the class $t(\mathrm{H})$. In view of the fact that cycle map forming the left vertical map in (7.4) is injective, it follows that the kernel of the cycle map

$$cycl : \mathrm{H^{*,\bullet}_M}(\mathrm{BH^{gm}}, \mathbb{Z}/m) \to \mathrm{H^*_{et}}(\mathrm{BH^{gm}}, \mu_m(\bullet)) \tag{7.8}$$

is contained in the kernel of $f^*$, and hence is killed by the class $t(\mathrm{H})$. This completes the proof of (i).

We next consider the statement in (ii). Therefore, let $\bar{x} \in \mathrm{H^2_{et}}(\mathrm{BH^{gm}}, \mu_m(1)$ denote a class. Then

$$t(\mathrm{H}).\bar{x} = \bar{\mathrm{f}}_*(\mathrm{cycl(a)}.\bar{\mathrm{f}}^*(\bar{x})). \tag{7.9}$$

In view of statements (ii) and (iv) in Lemma 7.3, one may write

$$\bar{f}^*(\bar{x}) = cycl(y) + z = cycl(y) + \bar{f}^*(\bar{z}), y \in \mathrm{H^{2,1}_M}(\mathrm{BB^{gm}}, \mathbb{Z}/m), \bar{z} \in \mathrm{H^2_{et}}(\mathrm{Spec}\, k, \mu_m(1)) \tag{7.10}$$

Therefore,

$$t(\mathrm{H}).\bar{x} = \bar{f}_*(\mathrm{cycl}(\mathrm{a}.y) + \mathrm{cycl(a)}.\bar{f}^*(\bar{z})) = \mathrm{cycl}(\bar{f}_*(\mathrm{a}.y)) + \bar{f}_*(\mathrm{cycl(a)}.\bar{f}^*\bar{z}) = \mathrm{cycl}(\bar{f}_*(\mathrm{a}.y)) + \mathrm{t(H)}\bar{z}. \tag{7.11}$$

This proves the statement in (ii), thereby completing the proof of the Proposition. □

7.1. **Information on the torsion index.** Using the fact that connected reductive groups over any algebraically closed field of positive characteristic admit liftings to characteristic 0, one may make use of the determination of the torsion index for compact Lie groups. The following are known:

(i) The only primes that divide the torsion index of simply-connected groups are $2, 3$ and $5$.
(ii) The torsion index for $\mathrm{GL}_n$, $\mathrm{SL}_n$ and $\mathrm{Sp}(2n)$, for any $n$ is 1.
(iii) The torsion index for $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n+1)$, for any $n$, are powers of 2.
(iv) The torsion index of $\mathrm{Spin}(n)$ is a power of 2: see [Tot05] for more details.
(v) The torsion index of E6 is 6 and of E8 is $2^6 3^2 5$: see [Tot05.2].

**Corollary 7.5.** *(i)* ${}_m\mathrm{Br}(\mathrm{BG}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\, k)$ *for any positive integer* $m$ *invertible in* $k$ *if* $\mathrm{G} = \mathrm{GL_n}$, $\mathrm{G} = \mathrm{SL_n}$ *or* $\mathrm{G} = \mathrm{Sp(2n)}$, *for any* $n$.

*(ii)* ${}_{\ell^{n'}}\mathrm{Br}(\mathrm{BG}) \cong {}_{\ell^{n'}}\mathrm{Br}(\mathrm{Spec}\, k)$ *for any prime* $\ell$ *different from the characteristic of* $k$ *and* $2$ *if* $\mathrm{G} = \mathrm{SO(2n)}$, $\mathrm{SO}(2n+1)$, *or* $\mathrm{Spin}(n)$, *for any* $n$ *and* $n'$.

*(iii)* ${}_{\ell^n}\mathrm{Br}(\mathrm{BG}) \cong {}_{\ell^n}\mathrm{Br}(\mathrm{Spec}\, k)$ *for any prime* $\ell$ *different from the characteristic of* $k$ *for any simply-connected group* $\mathrm{G}$, *if* $\ell$ *is also different from* $2, 3$, *or* $5$.

**Proof of Theorem 1.10**. Clearly the above discussion completes the proof of the theorem. □

## 8. Proof of Proposition 1.11.

We first observe that $\mathrm{B}\mathbb{G}_m^{\mathrm{gm}} \cong \lim_{n\to\infty} \mathbb{P}^n$. Since each $\mathbb{P}^n$ is a linear scheme which is projective and smooth, it follows from [J01, Theorem 4.5, Corollary 4.6] that one obtains isomorphisms for any smooth scheme Y:

$$(8.1) \qquad \begin{aligned} &\oplus_i \mathrm{H}_{\mathrm{M}}^{2\mathrm{i},\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}} \times \mathrm{Y}, \mathbb{Z}/m) \cong (\oplus_i \mathrm{H}_{\mathrm{M}}^{2\mathrm{i},\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}, \mathbb{Z}/m)) \otimes (\oplus_{\mathrm{i}} \mathrm{H}_{\mathrm{M}}^{2\mathrm{i},\mathrm{i}}(\mathrm{Y}, \mathbb{Z}/m)) \text{ and} \\ &\oplus_i \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}} \times \mathrm{Y}, \mu_m(\mathrm{i})) \cong (\oplus_i \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}, \mu_m(\mathrm{i}))) \otimes (\oplus_{\mathrm{i}} \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{Y}, \mu_m(\mathrm{i}))). \end{aligned}$$

Since the cycle map *cycl* : $\oplus_i \mathrm{H}_{\mathrm{M}}^{2\mathrm{i},\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}}, \mathbb{Z}/m) \to \oplus_{\mathrm{i}} \mathrm{H}_{\mathrm{et}}^{2\mathrm{i}}(\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm}}, \mu_m(\mathrm{i}))$ is an isomorphism, the Brauer group ${}_m\mathrm{Br}(\mathrm{Bun_{1,d}}(\mathrm{X}))$, which is the cokernel of cycle map, identifies with ${}_m\mathrm{Br}(\mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X}))$. Finally the isomorphism ${}_m\mathrm{Br}(\mathbf{P}\mathrm{ic}^{\mathrm{d}}(\mathrm{X})) \cong {}_m\mathrm{Br}(\mathrm{Sym}^{\mathrm{d}}(\mathrm{X}))$ is proven in [IJ20, Theorem 1.2].

Recall that $\mathrm{Br}(\mathrm{Y}) = 0$ if Y is a connected projective smooth variety that is *rational*: this follows from the well-known fact that the Brauer group is a stable birational invariant for connected projective smooth varieties. The last statement follows from this observation. □

## 9. Appendix A: Motivic cohomology over regular Noetherian base schemes

First one may observe that the higher cycle complex may be defined over any base scheme B: if X is a scheme of finite type over B, and $c \geq 0$ is a fixed integer, one defines $Z^c(\mathrm{X}, .)$ to be the chain complex defined in degree $n$, by

$$(9.1) \qquad \{\mathrm{Z} = \text{ a pure codim c cycle on } \mathrm{X} \times_{\mathrm{B}} \Delta_{\mathrm{B}}[\mathrm{n}] \,|\, \mathrm{Z} \text{ intersects the faces of } \mathrm{X} \times \Delta_{\mathrm{B}}[\mathrm{n}] \text{ properly}\}.$$

**Definition 9.1.** *We let* $\mathbb{Z}(c)$ *denote the co-chain complex* $Z^c(\mathrm{X}, .)[-2\mathrm{c}]$ *in cohomological degree* $m$, *that is,* $\mathbb{Z}(c) = Z^c(\mathrm{X}, 2\mathrm{c} - \mathrm{m})$. *(Observe that* $\mathbb{Z}(c)$ *is contravariantly functorial for flat maps.) If* X *denotes a smooth scheme of finite type over* B, *we will let* $\mathbb{Z}^{\mathrm{X}}(c)$ *denote the restriction of the complex* $\mathbb{Z}(c)$ *to the small Zariski or Nisnevich site of* X. *We call the complex* $\mathbb{Z}(c)$ *the motivic complex of weight* $c$. *If* $c < 0$ *is an integer, we define* $\mathbb{Z}(c)$ *to be* $\{0\}$.

Next we will assume that $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$, where R is a Dedekind domain.

**Proposition 9.2.** *Assume in addition that* X *denotes a smooth scheme of finite type over* B*. Then* $\mathbb{Z}^{\mathrm{X}}(1)[1]$ *identifies with* $\mathbb{G}_m^{\mathrm{X}}$*, which denotes the restriction of the sheaf* $\mathbb{G}_m$ *to the small Nisnevich site of* X*.*

*Proof.* This is discussed in [Bl, section 6], where the discussion does not assume the base scheme is a field. ☐

**Definition 9.3.** *(Motivic cohomology) Let* X *denote a scheme of finite type over* B*. We let* $\mathrm{H}_{\mathrm{M}}^{\mathrm{i,j}}(\mathrm{X}) = \mathrm{H}_{\mathrm{Zar}}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}(\mathrm{j}))$*, where* $\mathrm{H}_{\mathrm{Zar}}$ *denotes the hypercohomology on the Zariski site.*

It is observed in [Geis, Corollary 3.3], that one obtains the identification $\mathrm{H}_{\mathrm{Zar}}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}(\mathrm{j})) \cong \mathrm{H}_{\mathrm{Zar}}(\mathrm{B}, \pi_*(\mathbb{Z}(\mathrm{j})))$, where $\pi : \mathrm{X} \to \mathrm{B}$ denotes the structure map.

Let X denote a scheme of finite type over B and let Z denote a closed subscheme of X of pure codimension $c$ with open complement U. Let $i : \mathrm{Z} \to \mathrm{X}$ and $j : \mathrm{U} \to \mathrm{X}$ denote the corresponding immersions. Then it is shown in [Geis, Corollary 3.3] that one obtains the distinguished triangle

$$0 \to i_*\mathbb{Z}^{\mathrm{Z}}(n-c)[-2c] \to \mathbb{Z}^{\mathrm{X}}(n) \to j_*\mathbb{Z}^{\mathrm{U}}(n) \tag{9.2}$$

in the derived category of Zariski sheaves on X. In particular, this provides the identification of the terms in the long-exact sequence forming the top row in the diagram:

$$\begin{array}{ccccc} \mathrm{H}_{\mathrm{Z}}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) & \longrightarrow & \mathrm{H}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) & \longrightarrow & \mathrm{H}^{\mathrm{i}}(\mathrm{U}, \mathbb{Z}(\mathrm{n})) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathrm{H}_{\mathrm{Zar}}^{\mathrm{i}}(\mathrm{X}, \mathrm{i}_*\mathbb{Z}^{\mathrm{Z}}(\mathrm{n}-\mathrm{c})[-2\mathrm{c}]) & \longrightarrow & \mathrm{H}_{\mathrm{Zar}}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}^{\mathrm{X}}(\mathrm{n})) & \longrightarrow & \mathrm{H}_{\mathrm{Zar}}^{\mathrm{i}}(\mathrm{X}, \mathrm{j}_*\mathbb{Z}^{\mathrm{X}}(\mathrm{n})) \end{array} \tag{9.3}$$

**Proposition 9.4.** *Assume the above situation. If* $n < c = codim_{\mathrm{X}}(\mathrm{Z})$*, where* $codim_{\mathrm{X}}(\mathrm{Z})$ *denotes the codimension of* Z *in* X*, then* $\mathrm{H}_{\mathrm{Z}}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) = 0$*. A corresponding result also holds when* $\mathbb{Z}(n)$ *is replaced by* $\mathbb{Z}/m(n)$*, when* $m$ *is invertible in* R*.*

*Proof.* The first statement is clear from the identification of the first terms in the commutative diagram (9.3), since when $n < c$, the complex $\mathbb{Z}^{\mathrm{Z}}(n-c)$ is trivial. This proves the first statement.

To obtain the second statement, one first tensors the localization sequence in (9.2) with $\mathbb{Z}/m$ (i.e., the ring of integers modulo $m$) to obtain the corresponding localization sequence involving the motivic complexes $\mathbb{Z}/m(n)$. This then provides the commutative diagram corresponding to the one in (9.3) where the integral motivic complex $\mathbb{Z}(n)$ is replaced by $\mathbb{Z}/m(n)$. ☐

## 10. Appendix B: Brauer groups of smooth split toric schemes

We would like to point out that most of the current paper is self-contained and the only place we invoke results of [JL] are in Theorems 1.9 and 5.8 as well as Corollary 5.9. For the convenience of the reader, here we will provide a quick review of some of the key results and techniques of [JL] on the Brauer groups of toric schemes. One of the main results in [JL] is the following theorem, describing the Brauer group of a split torus. Throughout $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$, for an excellent Dedekind domain R.

**Theorem 10.1.** *(See* [JL, Theorem 2, Corollary 2.3(iv) and (iii)]*.) Let* $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^r$ *denote a split torus defined over the base* B *and let* X *denote a smooth scheme of finite type over* B*. Then:*

*(1)*

$$ {}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}^r) \cong {}_m\mathrm{Br}(\mathrm{X}) \oplus (\oplus^r \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))) \oplus (\oplus^{\binom{r}{2}} \mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mu_m(-1))), \tag{10.1}$$

*with the understanding that* $\binom{r}{2} = 0$ *for* $r = 1$*, where* ${}_m\mathrm{A}$ *denotes the* $m$*-torsion part of the abelian group* A*. Here* $\mu_m(0)$ *is identified with the constant sheaf* $\mathbb{Z}/m$*.*

*(2) One also obtains the following isomorphisms for étale cohomology with respect to* $\mu_m(i)$*, where* $\mu_m(0)$ *is identified with the constant sheaf* $\mathbb{Z}/m$*:*

$$\mathrm{H}^1_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times r}, \mu_m(1)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \oplus (\oplus_{i=1}^r \mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mu_m(0))).$$

The proof makes intrinsic use of the results of Appendix A, on the motivic cohomology of schemes over Dedekind domains. We also provide an interpretation of the summands on the right in Theorem 10.1 in terms of cyclic algebras. We will assume that X and Z are smooth schemes over B; in addition to assuming the integer $m$ is invertible in $\mathcal{O}_{\mathcal{B}}$, we will further assume $\mathcal{O}_{\mathrm{B}}$ has a primitive $m$-th root of unity. We first consider the following external product pairing, where $\mu_m(0)$ is identified with the constant sheaf $\mathbb{Z}/m$:

$$\mathbf{x} : \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{Z} \times \mathrm{X}, \mu_m(1)). \tag{10.2}$$

One can also interpret the above pairing in terms of the following well-known construction of cyclic algebras. Observe that the boundary map

$$\delta : cokernel(\Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}}) \overset{m}{\to} \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(1))$$

(obtained from the Kummer sequence) is always injective. (In case Pic(Z) $\cong 0$, for example, if Z is the spectrum of a local ring or a Noetherian unique factorization domain, then the above map is an isomorphism, but we do not need to assume this.)

Let $a \in \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})$ and let $\mathrm{Y} \to \mathrm{X}$ denote a $\mathbb{Z}/m$-torsor corresponding to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))$. Let $\sigma$ denote the generator of $Aut_{\mathrm{X}}(\mathrm{Y})) \cong \mathbb{Z}/\mathrm{m}$. Associated to Y and the class $a$ (identified with $a \otimes 1 \in \mathcal{O}_{\mathrm{Z}} \otimes \mathcal{O}_{\mathrm{Y}} \cong \mathcal{O}_{\mathrm{Z} \times \mathrm{Y}}$), one defines the *cyclic algebra* $\mathcal{O}_{\mathrm{Z} \times \mathrm{Y}}[x]_\sigma/(x^m - a)$, where $x.y' = \sigma(y').x$, for all $y' \in \mathcal{O}_{\mathrm{Y} \times \mathrm{Z}}$. This defines a class in ${}_m\mathrm{Br}(\mathrm{Z} \times \mathrm{X})$ and identifies with the class defined as the image of $\delta(a) \in \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(1))$ and Y under the external product pairing in (10.2).

Next we take $\mathrm{Z} = \mathbb{G}_{\mathrm{m}}$, the multiplicative group scheme defined over B. Now $\mathcal{O}_{\mathbb{G}_m} = \mathcal{O}_{\mathrm{B}}[t, t^{-1}]$. Let $\mathrm{Y} \to \mathrm{X}$ denote a $\mathbb{Z}/m$-torsor corresponding to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))$ as in the last paragraph. Then, one may verify that the mapping $\mathrm{Y} \mapsto \mathcal{O}_{\mathrm{Y} \times \mathbb{G}_{\mathrm{m}}}[x]_\sigma/(x^m - t)$, is an injection $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \to {}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}})$, with inverse defined by the residue map associated to the divisor obtained by setting $t = 1$ in $\mathbb{G}_m$: see [CTS, p. 32]. (To be able to invoke [CTS, p. 32], one needs to first pull back classes in $\mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_m(1))$ and in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))$ to classes in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}), \mu_m(1))$ and $\mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}), \mu_m(0))$. Observe that the composite map $p_2^* : \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0)) \to \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}, \mu_m(0)) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}), \mu_m(0))$ is an injection.)

Next one may take $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$ to obtain the external product pairing:

$$\mathbf{x} : \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_m(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_m(0)) \to \mathrm{H}^2_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}, \mu_m(1)). \tag{10.3}$$

We proceed to interpret this pairing also in terms of cyclic algebras, under the assumption the base scheme B has the property that $m$ is invertible in $\mathcal{O}_{\mathrm{B}} = \mathrm{R}$ and that it has a primitive $m$-th root of unity $\zeta$. Therefore, the sheaf $\mu_m(1)$ identifies the with the constant sheaf $\mathbb{Z}/m$. Given a unit $b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$, let $\mathrm{Y} \to \mathbb{G}_{\mathrm{m}}$ denote the $\mathbb{Z}/m$-torsor given by $\mathrm{Spec}\,(\mathcal{O}_{\mathbb{G}_m}[x]/(x^m - b)) \to \mathbb{G}_m$: we equip this torsor with the automorphism $\sigma$ given by sending $x \mapsto x\zeta$. Therefore, given two units $a, b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$, one may define a cyclic algebra $(a, b)_\zeta$, by applying the construction in the last paragraph with $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$, and the torsor $\mathrm{Y} \to \mathrm{X}$ given by the torsor $\mathrm{Spec}\,(\mathcal{O}_{\mathrm{X}}[x]/(x^m - b)) \to \mathrm{X} = \mathbb{G}_{\mathrm{m}}$.

At this point if X is any smooth scheme of finite type over B, pre-composing the external product pairing in (10.3) with the cup-product with $\mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\mathrm{m})$ defines classes in $\mathrm{H}^2_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}}; \mu_m(1))$, and hence classes in ${}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}})$. In terms of cyclic algebras this corresponds to letting $a = t_1$ and $b = t_2$ in the discussion in the last paragraph, and where $\mathcal{O}_{\mathbb{G}^2_m} = \mathrm{R}[\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_1^{-1}, \mathrm{t}_2^{-1}]$. This defines the summand $(\oplus^2 \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))) \oplus \mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mu_m(-1))$ in ${}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}})$ appearing on the right-hand-side of Theorem 10.1 with $r = 2$.

Now we take $\mathrm{Z} = \mathbb{G}_{\mathrm{m}}^{\times r}$, where $\mathbb{G}_m^{\times r} = \mathrm{Spec}\,\mathrm{R}[t_1, t_1^{-1}, \cdots, t_r, t_r^{-1}]$. We may take $t_i \in \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}}) = \mathrm{Hom}(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})$: observe that the latter identifies with the ring of characters of Z. We will also take $\mathrm{X} = \mathrm{Spec}\,\mathrm{R}$. If $b \in \Gamma(\mathrm{Spec}\,\mathrm{K}, \mathbb{G}_m) = \mathrm{R}^*$, $b$ corresponds to a $\mathbb{Z}/m$-torsor $\mathrm{Y} = \mathrm{Spec}\,(\mathrm{R}[\mathrm{x}]/(\mathrm{x}^{\mathrm{m}} - \mathrm{b}))$ over $\mathrm{Spec}\,\mathrm{R}$. Thus $b$ corresponds to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{K}, \mu_m(0))$. Now we may form the cyclic algebra $(b, t)_\zeta$. These cyclic algebras as $b$ varies over the units in R and $t$ varies among the characters $t_i$, $i = 1, \cdots, r$ of the split torus $\mathrm{T} = \mathbb{G}^r_{\mathrm{m}}$ correspond to the sum $\oplus^r \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(0))$ in the right hand side of (10.1).

Next we take $\mathrm{Z} = \mathbb{G}_{\mathrm{m}}$ and $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$, so that we may take $a = t_1$, and $b = t_2$, where $\mathbb{G}^2_m = \mathrm{Spec}\,\mathrm{R}[t_1, t_2, t_1^{-1}, t_2^{-1}]$. We will view $b$ as corresponding to the $\mathbb{Z}/m$-torsor $\mathrm{Spec}\,(\mathcal{O}_{\mathbb{G}_{\mathrm{m}}}[\mathrm{x}]/(\mathrm{x}^{\mathrm{m}} - \mathrm{b}))$ over $\mathbb{G}_m$. Now, corresponding to the pair of coordinates $t_1, t_2$, we form the cyclic algebra $(t_1, t_2)_\zeta$. One may now repeat this construction taking two factors of $\mathbb{G}_m$ corresponding to the coordinates $t_i, t_j$, $i < j$ and form the cyclic algebras $(t_i, t_j)_\zeta$. As we vary $t_i, t_j$ over all ordered pairs of coordinates, we obtain $\binom{r}{2}$ such cyclic algebras. These will account for each of the $\binom{r}{2}$ summands $\mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(-1)))$. Consequently, we obtain the following Corollary to Theorem 10.1:

**Corollary 10.2.**

$$
{}_m\mathrm{Br}(\mathbb{G}^r_{\mathrm{m}}) = {}_m\mathrm{Br}(\mathrm{B}) \bigoplus \overbrace{\left( \bigoplus_{i \le i < j \le r} \mathbb{Z}/m\mathbb{Z} \cdot (t_i, t_j)_\zeta \right)}^{\mathrm{A}} \bigoplus \overbrace{\left( \bigoplus_{i=1}^{r} \sum_{b_i \in k^*} \mathbb{Z}/m\mathbb{Z} \cdot (b_i, t_i)_\zeta \right)}^{\mathrm{B}} \tag{10.4}
$$

In order to proceed with the computation of the Brauer group of a toric variety or toric scheme, we also need localization sequences.

**Proposition 10.3.** *(See* [JL, Corollary 2.8 and Proposition 2.9]*.) Assume* X *is a smooth scheme of pure dimension over* B*, with* Z *a closed subscheme of pure codimension in* X *with open complement* U*.*

(i) *Then, we obtain the exact sequence:*

$$0 \to {}_m\mathrm{Br}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{U}) \longrightarrow \mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)),$$

*and in case* Z *has pure codimension* 1 *in* X *and is also regular, one has the identification* $\mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(0))$.

(ii) *In case* Z *is of pure codimension 1, but only generically smooth, we obtain the exact sequence:*

$$0 \to {}_m\mathrm{Br}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{U}) \longrightarrow \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z} - \mathrm{Z_s}, \mu_m(0)),$$

*where* $\mathrm{Z_s}$ *denotes the singular locus of* Z.

(iii) *In case the* Z *has pure codimension* $> 1$ *in* X*, we obtain:*

$$ {}_m\mathrm{Br}(\mathrm{X}) \overset{\cong}{\to} {}_m\mathrm{Br}(\mathrm{U}). $$

(iv) *The above localization sequence is functorial in* X, U *and* Z *in the following sense: if* $\mathrm{X}' \to \mathrm{X}$ *is a map of smooth schemes and* $\mathrm{U}' = \mathrm{U} \times_{\mathrm{X}} \mathrm{X}'$, $\mathrm{Z}' = \mathrm{Z} \times_{\mathrm{X}} \mathrm{X}'$*, then one obtains a commutative diagram of localization sequences:*

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & {}_m\mathrm{Br}(\mathrm{X}) & \longrightarrow & {}_m\mathrm{Br}(\mathrm{U}) & \longrightarrow & \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(0)) & \longrightarrow & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}_m\mathrm{Br}(\mathrm{X}') & \longrightarrow & {}_m\mathrm{Br}(\mathrm{U}') & \longrightarrow & \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}', \mu_m(0)) & \longrightarrow & . \end{array} $$

(v) *Assume that* X *is a smooth scheme of finite type over the given base scheme* B *provided with the action of a smooth affine group scheme* G *with finitely many orbits. Let* U *denote the open* G *stable subscheme which is the (disjoint) union of the open orbits and let* $\{\mathcal{O}_i | i = 1, \cdots, n\}$ *denote the codimension* 1*-orbits. Let* $\mathrm{Z} = \cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{\mathcal{O}}_{\mathrm{i}}$. *Then there exists a localization sequence:*

$$ 0 \to {}_m\mathrm{Br}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{U}) \to \oplus_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{H}^1_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_m(0)). $$

*Moreover, the last map may then be identified with the residue map.*

In view of Proposition 10.3(v), taking X to be the given toric scheme, and $\mathrm{U} = \mathrm{T} = \mathbb{G}_{\mathrm{m}}^r$, it suffices to determine the kernel of the map ${}_m\mathrm{Br}(\mathrm{U}) \to \oplus_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{H}^1_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_m(0))$.

We will make the following simplifying assumption to deal with toric schemes over Dedekind domains R:

(10.5)
the toric scheme X is smooth, contains as an open dense subscheme the split torus $\mathrm{T} \cong \mathbb{G}_{\mathrm{m}}^{\times \mathrm{r}}$ (defined over Spec R), and that all the orbits of T on X are schemes that are smooth and faithfully flat over Spec R.

Making use of the localization sequence above, we then obtain the following key result.

**Theorem 10.4.** *(See* [JL, Theorem 4.1]*.) Let* R *denote a Dedekind domain, which we assume is an excellent ring. We will also assume that the positive integer* $m$ *is a unit in* R *and that* R *contains a primitive* $m$*-th root of unity* $\zeta$. *Then, under the assumption* (10.5) *the following hold:*

(i) ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}$ = *the subgroup generated by* $\{\Lambda = \Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}} | m > e_{i,j} \geq 0\}$ *satisfying the following conditions: for each* $s = 1, \ldots, min\{u, r-1\}$, *if* $m_s = hcf\{m, e_{1,s}, e_{2,s}, \cdots, e_{s-1,s}, e_{s,s+1}, \cdots, e_{s,r}\}$, *then* $(\frac{m}{m_s})|a_s$. *In view of the assumption the toric scheme* X *is smooth, all* $a_s = 1$, *and hence hence the last condition translates to* $m_s = m$ *for all* $s$.

(ii) ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B}$ *is the subgroup generated by* $\{\Lambda = \Pi_{i=1}^r (b_i, t_i)_\zeta^{e_i} | \ m > e_i \geq 0\}$, *as* $b_i \in \mathrm{R}^*$ *varies among the corresponding classes in* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,R}, \mu_m(0))$ *so that the following conditions are satisfied: for each* $s = 1, \cdots, u$, *if* $m_s = hcf\{m, e_s, ord_m(b_s)\}$, *then* $(\frac{ord_m(b_s)}{m_s})|a_s$. *In view of the assumption the toric scheme* X *is smooth, all* $a_s = 1$, *and hence hence the last condition translates to* $ord_m(b_s) = m_s$, *for all* $s$.

(iii) *Moreover,* ${}_m\mathrm{Br}(\mathrm{X}) \cong {}_m\mathrm{Br}(\mathrm{Spec\,R}) \oplus ({}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}) \oplus ({}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B})$.

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