BRAUER GROUPS OF SMOOTH SPLIT TORIC SCHEMES AND THEIR FANS

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Abstract. In this paper we consider the Brauer groups of toric schemes, with the open dense torus assumed to be split. We first discuss the basic theory when the base scheme is an arbitrary field. We also consider toric schemes for a split torus over discrete valuation rings in the sense of Mumford *et al.*, and Dedekind domains. We show that fans of such toric varieties and schemes have considerable bearing on the m-torsion part of their Brauer groups, where m is a positive integer invertible in the base ring.

We also consider the period-index problem for classes in the Brauer groups of toric varieties for large classes of toric varieties. Our results also find immediate application in determining the Brauer groups of toric stacks as in the work of Dhillon, Iyer and the first author.

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1. Introduction

This paper revisits and greatly expands upon the computation of Brauer groups of toric varieties started in the 1993 paper of DeMeyer and Ford (see [DF]) where they computed the Brauer groups of toric varieties over the complex numbers. In section 2, we begin with a rather slick computation of the Brauer groups of split tori over any field. One may see a related computation of such Brauer groups in [SGS]: but our approach is bit different. We then show that the same computation works also over discrete valuation rings and Dedekind domains. The third section discusses the computation of the Brauer groups of smooth split toric varieties over any field, under the assumption that the field contains certain primitive roots of unity. When the field is algebraically closed and of characteristic 0, this computation reduces to the one in [DF]. The fourth section discusses in detail the computation of the Brauer groups of toric schemes over Dedekind

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domains and also discrete valuation rings in the sense of [KKMSD]. These results also find immediate application in determining the Brauer groups of toric stacks as in the work of Dhillon, Iyer and the first author: see [DIJ].

The fifth section discusses the period-index problem for smooth split toric varieties: surprisingly we are able to verify that the expected relation between the period and index holds for the classes in the Brauer groups of all smooth toric varieties over algebraically closed fields.[1] In particular, we show that the counter-examples such as those obtained by Hotchkiss for complex analytic tori do not appear. For smooth split toric varieties over fields that are quasi-algebraically closed, such as the function field of a curve over an algebraically closed field, we also show the expected relationship holds for the classes in the Brauer groups of all smooth toric varieties. For smooth-split toric varieties over a finite field $\mathbf{F}_q$, the corresponding results also hold (in a slightly stronger form) but only for m-torsion classes in the Brauer group of the corresponding toric varieties, where m divides $q-1$. We conclude the paper with some relatively straight-forward examples. It is a pleasure to thank Aji Dhillon for helpful discussions.

Since we also consider toric schemes over Dedekind domains and discrete valuation rings, we will work in the following slightly more general framework. Let B denote a regular Noetherian excellent scheme of dimension at most 1: B will serve as the *base scheme*. We also consider two basic situations here:

1.1. **Basic hypotheses.** $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$, where R is a Dedekind domain, or a DVR which is assumed to be excellent (for example, the ring of integers $\mathbb{Z}$ or its localization at a prime p).

Let m denote a fixed positive integer invertible in $\mathcal{O}_\mathrm{B}$ and let X denote a scheme of finite type over B. Then one begins with the *Kummer sequence*

$$1 \to \mu_m(1) \to \mathbb{G}_m \overset{m}{\to} \mathbb{G}_m \to 1, \tag{1.1}$$

which holds on the (small) étale site X_{et} of X, whenever m is invertible in $\mathcal{O}_\mathrm{B}$. (See [Gr, section 3] or [Mi, p. 66].) Taking étale cohomology, we obtain the corresponding long-exact sequence:

$$\to \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m}) \overset{m}{\to} \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m}) \to \cdots, \tag{1.2}$$

which holds on the étale site when m is invertible in $\mathcal{O}_\mathrm{B}$.

Definition 1.1. *The cohomological Brauer group* $\mathrm{Br}(\mathrm{X})$ *is the torsion subgroup of the cohomology group* $\mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m})$. *In other words,* $\mathrm{Br}(\mathrm{X}) = \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m})_{\mathrm{tors}}$. [2]

Next assume X is smooth over the base scheme B. Then, by Hilbert's Theorem 90, we obtain the isomorphisms:

$$\mathrm{Pic}(\mathrm{X}) \cong \mathrm{CH}^1(\mathrm{X}) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m}) \cong \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}), \tag{1.3}$$

where $\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z})$ denotes motivic cohomology (in degree 2 and weight 1) whose definition for smooth schemes of finite type over B is worked out in [Geis], and we recall this in the Appendix. Then one also obtains the short-exact sequence:

$$0 \to \mathrm{Pic}(\mathrm{X})/\mathrm{m} \cong \mathrm{NS}(\mathrm{X})/\mathrm{m} \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \to {}_m\mathrm{Br}(\mathrm{X}) \to 0, \tag{1.4}$$

[1]We want to point out that our statements only apply to classes in the Brauer group of the toric varieties and *not* to all classes in the Brauer group of the corresponding function field.

[2]If X is a regular integral Noetherian scheme, one may observe that $\mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m})_{\mathrm{tors}} = \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_\mathrm{m})$: see, for example, [CTS, Lemma 3.5.3]

where the map $\mathrm{Pic(X)/m} = \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mu_m(1))$ is the cycle map, and therefore, ${}_m\mathrm{Br(X)}$ identifies with the cokernel of the cycle map. Thus it follows that for smooth schemes X over B, ${}_m\mathrm{Br(X)}$ is trivial if and only if the above cycle map is surjective: our approach to the Brauer group adopted in this paper is to consider the above cycle map from motivic cohomology to etale cohomology, and involves a combination of motivic and étale cohomology techniques. Moreover, in view of this, *we will always restrict to smooth schemes of finite type over the given base scheme* B. However, apart from the restriction to smooth schemes, our approach making use of both motivic and étale cohomology techniques over Dedekind domains offers considerable advantages in various computations: these will become clear in later sections of the paper.

Acknowledgements. We are happy to acknowledge [SGS, 2.8 Theorem, 2.10 Theorem] as the inspiration behind our Theorem 2.1 and Corollary 2.3(iv). We are also happy to acknowledge [DF], [GS] and [GPS] as excellent references for us, while working on this project. We would also like to thank Michel Brion for various helpful email exchanges.

2. Brauer groups of split tori

The goal of this section is to prove Theorem 2.1 and to obtain a suitable interpretation of the summands on the right-hand-side in terms of cyclic algebras. Throughout the discussion, m will denote a fixed positive integer invertible in $\mathcal{O}_{\mathrm{B}}$.

Theorem 2.1. *Let* $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^r$ *denote a split torus defined over the base* B *as in 1.1. Then,*

$$ {}_m\mathrm{Br}(\mathbb{G}^r) \cong {}_m\mathrm{Br(B)} \oplus (\oplus^r \mathrm{H}^1_{\mathrm{et}}(\mathrm{B}, \mu_m(0))) \oplus (\oplus^{\binom{r}{2}} \mathrm{H}^0_{\mathrm{et}}(\mathrm{B}, \mu_m(-1))), \tag{2.1}$$

with the understanding that $\binom{r}{2} = 0$ *for* $r = 1$*, where* ${}_m\mathrm{A}$ *denotes the* m*-torsion part of the abelian group* A*. Here* $\mu_m(0)$ *is identified with the constant sheaf* $\mathbb{Z}/m$*.*

Proposition 2.2. *Let* X *denote a smooth scheme of finite type over* B*.*

- *Then the localization sequence*

$$\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \mathbb{A}^1, \mathbb{Z}/\mathrm{m}) \xrightarrow{j^*} \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}, \mathbb{Z}/\mathrm{m}) \longrightarrow \mathrm{H}^{*+1,\bullet}_{\mathrm{M},\mathrm{X}\times\{0\}}(\mathrm{X} \times \mathbb{A}^1, \mathbb{Z}/\mathrm{m}) \cong \mathrm{H}^{*-1,\bullet-1}_{\mathrm{M}}(\mathrm{X} \times \{0\}, \mathbb{Z}/\mathrm{m})$$

  (where $j : \mathrm{X} \times \mathbb{G}_{\mathrm{m}} \to \mathrm{X} \times \mathbb{A}^1$ *denotes the obvious open immersion) breaks up into short exact sequences, thereby providing the isomorphism:*

$$\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}, \mathbb{Z}/\mathrm{m}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \oplus \mathrm{H}^{*-1,\bullet-1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}).$$

- *A corresponding result holds in étale cohomology with coefficients in* $\mu_m(i)$*.*

Proof. Let $p : \mathrm{X} \times \mathbb{A}^1 \to \mathrm{X} \times \{1\}$ denote the obvious projection and let $i : \mathrm{X} \cong \mathrm{X} \times \{1\} \to \mathrm{X} \times \mathbb{G}_{\mathrm{m}}$ denote the corresponding closed immersion.

Then it is clear that the induced map

$$p^* : \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \{1\}, \mathbb{Z}/\mathrm{m}) \to \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \mathbb{A}^1, \mathbb{Z}/\mathrm{m})$$

is an isomorphism and the composition of the maps

$$i^* \circ j^* \circ p^* : \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \{1\}, \mathbb{Z}/\mathrm{m}) \to \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \{1\}, \mathbb{Z}/\mathrm{m})$$

is the identity, thereby proving that the map $j^* : \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \mathbb{A}^1, \mathbb{Z}/\mathrm{m}) \to \mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}, \mathbb{Z}/\mathrm{m})$ is a split injection. This proves the required assertion. $\square$

Making use of the last proposition, the following result computes the Brauer groups of schemes of the form $\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}$, $q \geq 1$, over a base scheme B as in (1.1). This plays a key role in our computations.

Corollary 2.3. *Let* X *denote a smooth scheme of finite type over* B. *Then the following hold:*

(i) *We obtain the isomorphism for any positive integer q:*
$$\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mathbb{Z}/\mathrm{m}) \cong \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \oplus (\oplus_{i=1}^{q} \mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m})).$$
Moreover, $\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \cong 0$ *so that the last isomorphism becomes:*
$$\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mathbb{Z}/\mathrm{m}) \cong \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}).$$

(ii) *The corresponding result for étale cohomology with respect to $\mu_m(i)$, where $\mu_m(0)$ is identified with the constant sheaf $\mathbb{Z}/m$, is:*
$$\mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mu_m(1)) \cong \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \oplus (\oplus_{i=1}^{q} \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}, \mu_m(0))) \oplus (\oplus_{i=1}^{\binom{q}{2}} \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{X}, \mu_m(-1))),$$
with the understanding that for $q = 1$, $\binom{q}{2} = 0$.

(iii) *One also obtains the following isomorphisms for étale cohomology with respect to $\mu_m(i)$, where $\mu_m(0)$ is identified with the constant sheaf $\mathbb{Z}/m$:*
$$\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mu_m(1)) \cong \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \oplus (\oplus_{i=1}^{q} \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{X}, \mu_m(0))).$$

(iv) *Consequently* ${}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}) \cong {}_m\mathrm{Br}(\mathrm{X}) \oplus (\oplus^{q} \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}, \mu_m(0))) \oplus (\oplus^{\binom{q}{2}} \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{X}, \mu_m(-1)))$, *with the same understanding that $\binom{q}{2} = 0$ for $q = 1$.*

Proof. The isomorphism in the first statement in (i) may be deduced from the last proposition by ascending induction on q. Here one may want to observe that the motivic complexes $\mathbb{Z}(j)$ are defined only for $j \geq 0$, i.e., $\mathbb{Z}(j) = 0$, for $j < 0$. Next we consider the remaining statements in (i).

Observe that the term $\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m})$
$$\begin{aligned}\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) &\cong \mathrm{H}^{3,1}_{\mathrm{M},\mathrm{X}\times\{0\}}(\mathrm{X} \times \mathbb{A}^1, \mathbb{Z}/\mathrm{m}) \\ &\cong \mathrm{H}^{3}_{\mathrm{Zar}}(\mathrm{X} \times \mathbb{A}^1, \mathrm{i}_*(\mathbb{Z}/\mathrm{m}(0)[-2]))\end{aligned}$$
by (7.3). This is because the codimension of $\mathrm{X} \times \{0\}$ in $\mathrm{X} \times \mathbb{A}^1$ is 1. Now $\mathbb{Z}/m(0)$, which is the motivic complex of weight 0 identifies with the constant sheaf $\mathbb{Z}/m$ (given by the ring of integers modulo m). The shift $[-2]$ shifts this sheaf to degree 2, so that $\mathbb{Z}/m(0)[-2]$ is the complex of Zariski sheaves concentrated in degree 2, where it is $\mathbb{Z}/m$. Since constant sheaves are flabby on the Zariski site of any irreducible scheme, the Zariski cohomology of $\mathrm{X} \times \{0\}$ with respect to this complex is trivial in degree 3 or higher. This observation completes the proof that $\mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \cong 0$ and completes the proof of the remaining statements in (i).

The statements in (ii) and (iii) on etale cohomology may be proven using ascending induction on q. One may also want to observe that $\mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{Y}, \mu_m(\mathrm{j})) = 0$, for all $i < 0$.

The last statement now follows from (i) and (ii), as well as the identification of the Brauer group ${}_m\mathrm{Br}(\mathrm{Y})$ for a smooth scheme Y with the cokernel of the cycle map $\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{Y}, \mathbb{Z}/\mathrm{m}) \to \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{Y}, \mu_m(1))$ as observed in the discussion following (1.4). $\square$

Remark 2.4. In [SGS, 2.8 Theorem, 2.10 Theorem], the authors prove a variant of the statement in Corollary 2.3(iv), under the assumption the base scheme B is a field. Moreover, their techniques involve the use of Milnor K-theory, and therefore seem to be restricted to the case where the base scheme B is a field and also a bit more involved than our arguments above, which involve largely Proposition 2.2, induction on the number of factors in a split torus and (1.4).

2.1. **External product pairing and the computation of the Brauer group of a split torus.** We proceed to provide an interpretation of the summands on the right in Corollary 2.3(iv) in terms of cyclic algebras: see [P, 1.5.7] or [GS, 2.5]. The following discussion is based on the one in [P, 1.5.7.4]. We will assume that X and Z are smooth schemes over B; in addition to assuming the integer m is invertible in $\mathcal{O}_{\mathcal{B}}$, we will further assume $\mathcal{O}_{\mathrm{B}}$ has a primitive m-th root of unity. We first consider the following external product pairing, where $\mu_m(0)$ is identified with the constant sheaf $\mathbb{Z}/m$:

$$\mathbf{x} : \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0)) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{Z} \times \mathrm{X}, \mu_m(1)). \tag{2.2}$$

One can also interpret the above pairing in terms of the following well-known construction of cyclic algebras. Observe that the boundary map

$$\delta : \mathit{cokernel}(\Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}}) \overset{m}{\to} \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(1))$$

(obtained from the Kummer sequence) is always injective. (In case $\mathrm{Pic}(\mathrm{Z}) \cong 0$, for example, if Z is the spectrum of a local ring or a Noetherian unique factorization domain, then the above map is an isomorphism, but we do not need to assume this.)

Let $a \in \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})$ and let $\mathrm{Y} \to \mathrm{X}$ denote a $\mathbb{Z}/m$-torsor corresponding to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))$. Let σ denote the generator of $\mathit{Aut}_{\mathrm{X}}(\mathrm{Y})) \cong \mathbb{Z}/\mathrm{m}$. Associated to Y and the class a (identified with $a \otimes 1 \in \mathcal{O}_{\mathrm{Z}} \otimes \mathcal{O}_{\mathrm{Y}} \cong \mathcal{O}_{\mathrm{Z} \times \mathrm{Y}}$), one defines the *cyclic algebra* $\mathcal{O}_{\mathrm{Z}\times\mathrm{Y}}[x]_\sigma/(x^m - a)$, where $x.y' = \sigma(y').x$, for all $y' \in \mathcal{O}_{\mathrm{Y}\times\mathrm{Z}}$. This defines a class in ${}_m\mathrm{Br}(\mathrm{Z} \times \mathrm{X})$ and identifies with the class defined as the image of $\delta(a) \in \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(1))$ and Y under the external product pairing in (2.2).

Next we take $\mathrm{Z} = \mathbb{G}_{\mathrm{m}}$, the multiplicative group scheme defined over B. Now $\mathcal{O}_{\mathbb{G}_m} = \mathcal{O}_{\mathrm{B}}[t, t^{-1}]$. Let $\mathrm{Y} \to \mathrm{X}$ denote a $\mathbb{Z}/m$-torsor corresponding to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))$ as in the last paragraph. Then, one may verify that the mapping $\mathrm{Y} \mapsto \mathcal{O}_{\mathrm{Y}\times\mathbb{G}_{\mathrm{m}}}[x]_\sigma/(x^m - t)$, is an injection $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \to {}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}})$, with inverse defined by the residue map associated to the divisor obtained by setting $t = 1$ in $\mathbb{G}_m$: see [CTS, p. 32]. (To be able to invoke [CTS, p. 32], one needs to first pull back classes in $\mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_m(1))$ and in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))$ to classes in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}), \mu_m(1))$ and $\mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}), \mu_m(0))$. Observe that the composite map $p_2^* : \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0)) \to \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}, \mu_m(0)) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathbb{G}_{\mathrm{m}} \times \mathrm{X}), \mu_m(0))$ is an injection.)

Next one may take $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$ to obtain the external product pairing:

$$\mathbf{x} : \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_m(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_m(0)) \to \mathrm{H}^2_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}, \mu_m(1)). \tag{2.3}$$

We proceed to interpret this pairing also in terms of cyclic algebras, under the assumption the base scheme B has the property that m is invertible in $\mathcal{O}_{\mathrm{B}}$ and that it has a primitive m-th root of unity ζ. Therefore, the sheaf μ_m identifies the with the constant sheaf $\mathbb{Z}/m$. Given a unit $b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$, let $\mathrm{Y} \to \mathbb{G}_{\mathrm{m}}$ denote the $\mathbb{Z}/m$-torsor given by $\mathrm{Spec}\,(\mathcal{O}_{\mathbb{G}_m}[x]/(x^m - b)) \to \mathbb{G}_m$: we equip this torsor with the automorphism σ given by sending $x \mapsto x\zeta$. Therefore, given two units $a, b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$, one may define a cyclic algebra $(a, b)_\zeta$, by applying the construction in the last paragraph with $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$, and the torsor $\mathrm{Y} \to \mathrm{X}$ given by the torsor $\mathrm{Spec}\,(\mathcal{O}_{\mathrm{X}}[x]/(x^m - b)) \to \mathrm{X} = \mathbb{G}_{\mathrm{m}}$.

At this point if X is any smooth scheme of finite type over B, pre-composing the external product pairing in (2.3) with the cup-product with $\mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\mathrm{m})$ defines classes in $\mathrm{H}^2_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}}; \mu_m(1))$, and hence classes in ${}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}})$. In terms of cyclic algebras this corresponds to letting $a = t_1$ and $b = t_2$ in the discussion in the last paragraph, and where $\mathcal{O}_{\mathbb{G}^2_m} = \mathcal{O}_{\mathrm{B}}[t_1, t_2, t_1^{-1}, t_2^{-1}]$. This defines the summand $(\oplus^2 \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mu_m(0))) \oplus \mathrm{H}^0_{\mathrm{et}}(\mathrm{X}, \mathbb{Z}/\mathrm{m})$ in ${}_m\mathrm{Br}(\mathrm{X} \times \mathbb{G}^2_{\mathrm{m}})$ appearing on the right-hand-side of Corollary 2.3(iv) with $q = 2$.

Now we take $\mathrm{Z} = \mathbb{G}^{\times r}_{\mathrm{m}}$, where $\mathbb{G}^{\times r}_m = \mathrm{Spec}\,\mathrm{K}[t_1, t_1^{-1}, \cdots, t_r, t_r^{-1}]$. In this case, the hypothesis $\mathrm{Pic(Z)} = 0$ is clearly satisfied. We may take $t_i \in \Gamma(\mathrm{Z}, \mathbb{G}_{\mathrm{m}}) = \mathrm{Hom}(\mathrm{Z}, \mathbb{G}_{\mathrm{m}})$: observe that the latter identifies with the ring of characters of Z. We will also take $\mathrm{X} = \mathrm{Spec}\,\mathrm{K}$. If $b \in \Gamma(\mathrm{Spec}\,\mathrm{K}, \mathbb{G}_m) = \mathrm{K}^*$, b corresponds to a $\mathbb{Z}/m$-torsor $\mathrm{Y} = \mathrm{Spec}\,(\mathrm{K[x]/(x^m - b)})$ over $\mathrm{Spec}\,\mathrm{K}$. Thus b corresponds to a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{K}, \mu_m(0))$. Now we may form the cyclic algebra $(b, t)_\zeta$. These cyclic algebras as b varies over the units in the field K and t varies among the characters t_i, $i = 1, \cdots, r$ of the split torus $\mathrm{T} = \mathbb{G}^r_{\mathrm{m}}$ correspond to the sum $\oplus^r \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{K}, \mu_m(0))$ in the right hand side of (2.1).

Next we take $\mathrm{Z} = \mathbb{G}_{\mathrm{m}}$ and $\mathrm{X} = \mathbb{G}_{\mathrm{m}}$, so that we may take $a = t_1$, and $b = t_2$, where $\mathbb{G}^2_m = \mathrm{Spec}\,\mathrm{K}[t_1, t_2, t_1^{-1}, t_2^{-1}]$. We will view b as corresponding to the $\mathbb{Z}/m$-torsor $\mathrm{Spec}\,(\mathcal{O}_{\mathbb{G}_{\mathrm{m}}}[\mathrm{x}]/(\mathrm{x^m - b}))$ over $\mathbb{G}_m$. Now, corresponding to the pair of coordinates t_1, t_2, we form the cyclic algebra $(t_1, t_2)_\zeta$. One may now repeat this construction taking two factors of $\mathbb{G}_m$ corresponding to the coordinates t_i, t_j, $i < j$ and form the cyclic algebras $(t_i, t_j)_\zeta$. As we vary t_i, t_j over all ordered pairs of coordinates, we obtain $\binom{r}{2}$ such cyclic algebras. These will account for each of the $\binom{r}{2}$ summands $\mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{K}, \mu_m(-1)))$. Consequently, we obtain the following Corollary to Theorem 2.1:

Corollary 2.5.

$$
{}_m\mathrm{Br}(\mathbb{G}^r_{\mathrm{m}}) = {}_m\mathrm{Br}(\mathrm{B}) \bigoplus \overbrace{\left(\bigoplus_{i \le i < j \le r} \mathbb{Z}/m\mathbb{Z} \cdot (t_i, t_j)_\zeta \right)}^{\mathcal{A}} \bigoplus \overbrace{\left(\bigoplus_{i=1}^{r} \sum_{b_i \in k^*} \mathbb{Z}/m\mathbb{Z} \cdot (b_i, t_i)_\zeta \right)}^{\mathcal{B}} \tag{2.4}
$$

2.2. **Localization sequences and the residue map.** Next we consider localization sequences for motivic and étale cohomology theories, which provide a convenient technique for computing Brauer groups. We begin with the following variant of the Snake Lemma: see [Iver, Snake Lemma 1.6].

Lemma 2.6. *Consider the commutative diagram*

$$
\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "\alpha"] & B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[r, "h'"] \arrow[d, "\gamma"] & D' \arrow[r] \arrow[d, "\delta"] & E' \arrow[d, "\eta"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & D \arrow[r] & E
\end{tikzcd}
$$

in an abelian category with exact rows. Then the following hold:

(i) *If α and β are isomorphisms and δ is a monomorphism, then the map γ is also a monomorphism.*

(ii) *If α is an epimorphism and η is a monomorphism, then*

$$kernel(\beta) \to kernel(\gamma) \to kernel(\delta) \to cokernel(\beta) \to cokernel(\gamma) \to cokernel(\delta)$$

is exact. In particular, if α is an epimorphism, η is a monomorphism and both β and δ are isomorphisms, then so is γ.

Proposition 2.7. *We start by considering the following situation. Consider* X *a smooth scheme of finite type over a base scheme* B *as in* (1.1) *which is of pure dimension over* B*. Fix a closed subscheme* Z *in* X *that is of pure codimension in* X *with open complement* U*. Then one obtains a commutative diagram*

$$\begin{tikzcd}
\mathrm{H}^{1,1}_{\mathrm{M}}(\mathrm{U}, \mathbb{Z}/m) \arrow[r] \arrow[d, "a_1"'] & \mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/m) \arrow[r] \arrow[d, "a_2"'] & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/m) \arrow[r] \arrow[d, "a_3"'] & \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{U}, \mathbb{Z}/m) \arrow[r] \arrow[d, "a_4"] & {} \\
\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{U}, \mu_m(1)) \arrow[r] & \mathrm{H}^{2}_{\mathrm{et,Z}}(\mathrm{X}, \mu_m(1)) \arrow[r] & \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X}, \mu_m(1)) \arrow[r] & \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{U}, \mu_m(1)) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/m) \arrow[r] \arrow[d, "a_5"'] & \mathrm{H}^{3,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/m) \arrow[d, "a_6"'] \\
{} \arrow[r] & \mathrm{H}^{3}_{\mathrm{et,Z}}(\mathrm{X}, \mu_m(1)) \arrow[r] & \mathrm{H}^{3}_{\mathrm{et}}(\mathrm{X}, \mu_m(1))
\end{tikzcd}$$

so that the following hold:

(1) *The maps a_1 and a_2 are isomorphisms always (under our hypotheses as in* (1.1)*.*
(2) *The maps a_3 and a_4 are always injective.*
(3) *Under the assumption that* Z *is also regular of pure codimension* 1*, we obtain isomorphisms* $\mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/\ell^{\mathrm{n}})) \cong \mathrm{H}^{0,0}_{\mathrm{M}}(\mathrm{Z}, \mathbb{Z}/\ell^{\mathrm{n}}))$ *and* $\mathrm{H}^{2}_{\mathrm{et,Z}}(\mathrm{X}, \mu_m(1))) \cong \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{Z}, \mu_m(0))$*.*
(4) $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/m) \cong 0$.
(5) $\mathrm{H}^{3,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/m) = 0$.
(6) *If* Z *has pure codimension* > 1*, then* $\mathrm{H}^{3}_{\mathrm{et,Z}}(\mathrm{X}, \mu_m(1)) \cong 0$ *as may be seen from cohomological purity.*

Proof. One can also see a proof of this result in [CTS, Theorems 3.7.1 and 3.7.2]. The proof we provide below is a bit different, as it also involves certain results on motivic cohomology over Dedekind domains as in the Appendix. The fact that the maps a_3 and a_4 are always injective follows from the Kummer sequence considered in (1.4). Observe that the scheme X is assumed to be smooth. The isomorphisms in the third statement are essentially Thom-isomorphisms, which exist as the schemes X and Z are assumed to be smooth.

The isomorphism $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/m) \cong 0$ in statement (4) may be obtained as follows. The localization sequence in (7.2) shows that $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/m)$ identifies with the cohomology of the complex $i_*(\mathbb{Z}/m^{\mathrm{Z}}(1-c)[-2c]$ computed on the Zariski site of the base scheme X. Here c is the codimension of Z in X. If $c > 1$, clearly this complex is trivial and hence conclusion follows in this case. If $c = 1$, one observes the isomorphism: $\mathrm{H}^{3,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/m) \cong \mathrm{H}^{3}_{\mathrm{Zar}}(\mathrm{X}, \mathrm{i}_*(\mathbb{Z}/m^{\mathrm{Z}}(0)[-2]) \cong \mathrm{H}^{1}_{\mathrm{Zar}}(\mathrm{Z}, \mathbb{Z}/m^{\mathrm{Z}}(0)) \cong 0$, since $\mathbb{Z}/m^{\mathrm{Z}}(0)$ is the constant sheaf $\mathbb{Z}/m$ and Z may be assumed to be irreducible.

The vanishing of the cohomology in the statement (5) may be obtained as follows. First, when the base scheme B is a field, this follows from the identification of the motivic cohomology $\mathrm{H}^{3,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/m)$ with $\mathrm{CH}^{1,2-3}(\mathrm{X}, \mathbb{Z}/m)$ which is trivial for obvious reasons. In general, as is shown in [Geis, Corollary 3.3], one takes the motivic complex $\mathbb{Z}/m(1)$, viewed as a complex of sheaves on the (small) flat site of the given scheme X and takes its pushforward to the Zariski site of the base scheme B, and then computes the Zariski cohomology of the resulting complex on B. The complex $\mathbb{Z}/m(1)$ is concentrated in degrees 0 and 1: in fact it is the complex $\mathbb{G}_m \overset{m}{\to} \mathbb{G}_m$ concentrated in degrees 0 and 1. Since B has Zariski cohomological dimension at most 1, the resulting complex on the Zariski site of B has no cohomology in degree 3 or above, proving the vanishing of the cohomology in the last statement. Moreover the vanishing of the local cohomology in the last statement is clear.

Therefore, it suffices to prove the first statement, which we proceed to do presently. We will first consider the map a_2. In case the codimension of Z in X is higher than 1, both the source and target of the map a_2 are trivial. Therefore, it suffices to consider the case when the codimension of Z in X is 1. First we will consider the case when Z is also regular. Now the localization sequence as in (7.2) readily provides the identification:

$$\mathrm{H}^{2,1}_{\mathrm{M,Z}}(\mathrm{X}, \mathbb{Z}/m) \cong \mathrm{H}^{0,0}_{\mathrm{M}}(\mathrm{Z}, \mathbb{Z}/m), \text{ while} \tag{2.5}$$

Thom isomorphism (or purity) provides the isomorphism:

$$\mathrm{H}^2_{\mathrm{et,Z}}(\mathrm{X}, \mu_m) \cong \mathrm{H}^0_{\mathrm{et}}(\mathrm{Z}, \mu_m(0)). \tag{2.6}$$

Clearly the right-hand-sides of both (2.5) and (2.6) are isomorphic to the sum $\oplus\mathbb{Z}/m$ indexed by the connected components of Z and are therefore isomorphic, thereby proving that the map a_2 is an isomorphism when Z is also regular.

In general, making use of the assumption the base scheme B is excellent, we may let $\mathrm{Z_s}$ denote the singular locus of Z which will be of codimension 1 or higher in Z. Then, in the localization sequence

$$\cdots \to \mathrm{H}^2_{\mathrm{Z_s,et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^2_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^2_{\mathrm{Z-Z_s,et}}(\mathrm{X}-\mathrm{Z_s}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z_s,et}}(\mathrm{X}, \mu_m(1)) \to \cdots$$

the first and last terms are trivial since the codimension of $\mathrm{Z_s}$ in X is at least 2. One obtains a similar localization sequence in motivic cohomology, using which we reduce to the case where Z is regular.

Next we consider the map a_1. Then the Beilinson-Lichtenbaum conjecture (which follows as a consequence of the *motivic Bloch-Kato conjecture*) first shows that the map a_1 is an isomorphism, when the base scheme B is the spectrum of a field. Observe that, this is the statement that the natural map

$$\mathbb{Z}/m(i) \simeq \tau_{\le i} R\epsilon_* \epsilon^*(\mathbb{Z}/m(i)) \simeq \tau_{\le i} R\epsilon_*(\mu_m(i)), \tag{2.7}$$

is a quasi-isomorphism, where ϵ is the obvious map of sites from the big étale site of the scheme B to the corresponding big Nisnevich site. It follows from the discussion in [Geis, Theorem 1.2] that the quasi-isomorphism in (2.7) extends to the case where B is a Dedekind domain. □

Corollary 2.8. *Assume* X *is a smooth scheme of pure dimension over* B*, with* Z *a closed subscheme of pure codimension in* X *with open complement* U*.*

(i) *Then, we obtain the exact sequence:*

$$0 \to {}_m\mathrm{Br}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{U}) \longrightarrow \mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)),$$

and in case Z *has pure codimension* 1 *in* X *and is also regular, one has the identification* $\mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(0))$.

(ii) *In case* Z *is of pure codimension 1, but only generically smooth, we obtain the exact sequence:*

$$0 \to {}_m\mathrm{Br}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{U}) \longrightarrow \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}-\mathrm{Z_s}, \mu_m(0)),$$

where $\mathrm{Z_s}$ *denotes the singular locus of* Z.

(iii) *In case the* Z *has pure codimension* >1 *in* X*, we obtain:*

$$ {}_m\mathrm{Br(X)} \overset{\cong}{\to} {}_m\mathrm{Br(U)}. $$

Proof. First we consider the case when Z has pure codimension 1 in X. Then one invokes Proposition 2.7 and Lemma 2.6 with $\alpha = a_1$, $\beta = a_2$, $\gamma = a_3$, $\delta = a_4$ and $\eta = a_5$ to obtain the first exact sequence stated in the corollary. Next we consider the case when Z is also regular. Then one observes the Thom isomorphism (in view of the assumption that Z is also smooth):

$$ \mathrm{H}^3_{\mathrm{et,Z}}(\mathrm{X}, \mu_m(1) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(0)). $$

Observe that the resulting map

$$ \mathrm{Br(U)}_m \longrightarrow \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(0)) $$

identifies with the residue map discussed in section 2.3. (Hence we denote this map by *res*.)

Next we consider (ii). Recall that the base scheme B is assumed to be excellent. In this case one obtains the long exact sequence:

$$ \cdots \to \mathrm{H}^3_{\mathrm{Z_s,et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z-Z_s,et}}(\mathrm{X}-\mathrm{Z_s}, \mu_m(1)) \cong \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}-\mathrm{Z_s}, \mu_m(0) \to \cdots $$

Observe that the codimension of $\mathrm{Z_s}$ in X is at least 2. Therefore, and making use of the assumption that B is excellent, one may prove readily that $\mathrm{H}^3_{\mathrm{Z_s,et}}(\mathrm{X}, \mu_m(1)) \cong 0$, which will prove the last map in the long exact sequence above in an injection. (In fact, one may filter the scheme $\mathrm{Z_s}$ further by $\{\mathrm{Z_{s,k}} \subseteq \mathrm{Z_{s,k-1}} \subseteq \cdots \mathrm{Z_{s,0}} = \mathrm{Z_s}\}$ so that the codimension of $\mathrm{Z_{s,k}}$ in X is at least $\mathrm{k}+1$. Then one may use a localization sequence

$$ \cdots \to \mathrm{H}^3_{\mathrm{Z_{s,k},et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z_{s,k-1},et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z_{s,k-1}-Z_{s,k},et}}(\mathrm{X}-\mathrm{Z_{s,k}}, \mu_m(1)) \to \cdots $$

and use descending induction on k to prove the required vanishing.) This completes the proof of (ii).

In case the codimension of Z in X is larger than 1, then $\mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \cong 0$, so that the statement (i) proves the last conclusion. ∎

In order to proceed with the computation of the Brauer group of a toric variety, we also need localization sequences, which is discussed next.

Proposition 2.9. (i) *The localization sequence in Corollary 2.8(i) is functorial in* X, U *and* Z *in the following sense: if* $\mathrm{X'} \to \mathrm{X}$ *is a map of smooth schemes and* $\mathrm{U'} = \mathrm{U} \times_{\mathrm{X}} \mathrm{X'}$, $\mathrm{Z'} = \mathrm{Z} \times_{\mathrm{X}} \mathrm{X'}$, *then one obtains a commutative diagram of localization sequences:*

$$ \begin{tikzcd} 0 \arrow[r] & {}_m\mathrm{Br(X)} \arrow[r] \arrow[d] & {}_m\mathrm{Br(U)} \arrow[r] \arrow[d] & \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(0)) \arrow[r] \arrow[d] & {} \\ 0 \arrow[r] & {}_m\mathrm{Br(X')} \arrow[r] & {}_m\mathrm{Br(U')} \arrow[r] & \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z'}, \mu_m(0)) \arrow[r] & {} . \end{tikzcd} $$

(ii) *Assume that* X *is a smooth scheme of finite type over the given base scheme* B *provided with the action of a smooth affine group scheme* G *with finitely many orbits. Let* U *denote the open* G *stable subscheme which is the (disjoint) union of the open orbits and let* $\{\mathcal{O}_i | i = 1, \cdots, m\}$ *denote the codimension* 1*-orbits. Let* $\mathrm{Z} = \cup_{\mathrm{i}=1}^{\mathrm{m}} \bar{\mathcal{O}}_{\mathrm{i}}$. *Then there exists a localization sequence:*

$$ 0 \to {}_m\mathrm{Br(X)} \to {}_m\mathrm{Br(U)} \to \oplus_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{H}^1_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_m(0)). $$

Moreover, the last map may then be identified with the residue map.

Proof. The first statement follows readily, since Z is also assumed to be smooth. In fact the localization sequence as in (i) holds more generally for any regular scheme X with U an open subscheme and Z the complement of U in X, which is assumed to be also regular as shown in [CTS, p. 91, (3.17)].

To see the last statement, one starts with the localization sequence:

$$0 \to {}_m\mathrm{Br}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{U}) \to \mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1))$$

whose existence should be clear from Corollary 2.8. Therefore, it suffices to prove that there is an injection

$$\mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \to \oplus \mathrm{H}^1_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_m(0)).$$

For this, first observe that if X is of dimension 1, then Z is of dimension 0 and hence smooth. Therefore, the map above would be an isomorphism in this case. Therefore, we may assume that X has dimension at least 2, and let Y denote the union of all the orbits of codimension 2 or more. Then one obtains the long-exact sequence:

$$\cdots \to \mathrm{H}^3_{\mathrm{Y,et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z-Y,et}}(\mathrm{X}-\mathrm{Y}, \mu_m(1)) \to \cdots \tag{2.8}$$

Now X is smooth, and therefore X − Y is also smooth. Z − Y is the disjoint union of orbits of codimension equal to 1, so it is also smooth. Therefore, by purity,

$$\mathrm{H}^3_{\mathrm{Z-Y,et}}(\mathrm{X}-\mathrm{Y}, \mu_m(1)) \cong \oplus_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{H}^1_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_m(0)).$$

Since Y has codimension at least 2 in X, $\mathrm{H}^3_{\mathrm{Y,et}}(\mathrm{X}, \mu_m(1)) \cong 0$. One may prove this statement using descending induction on the codimension of Y, since when Y has codimension = the dimension of X, Y consists of only 0-dimensional orbits, in which case the above statement is clear by purity. In general, one may use the localization sequence as in (2.8) and descending induction on the codimension of Y in X. This shows that the induced map

$$\mathrm{H}^3_{\mathrm{Z,et}}(\mathrm{X}, \mu_m(1)) \to \mathrm{H}^3_{\mathrm{Z-Y,et}}(\mathrm{X}-\mathrm{Y}, \mu_m(1)) \cong \oplus_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{H}^1_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_m(0))$$

is injective. The identification of the map

$${}_m\mathrm{Br}(\mathrm{U}) \to \oplus_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{H}^1_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_m(0))$$

with residue map follows from the discussion below. This completes the proof of the Proposition. □

2.3. **More on the residue map.** Assume the situation of Proposition 2.9. Then the resulting map

$${}_m\mathrm{Br}(\mathrm{U}) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Z}, \mu_m(0)) \tag{2.9}$$

has an explicit description in terms of *the residue map*, discussed in detail in [GS, Chapter 6] and also [CTS, Chapter I, 1.4], which we will summarize here.

First one observes that if p denotes the ideal sheaf in $\mathcal{O}_{\mathrm{X}}$ defining the divisor Z, then the localization of $\mathcal{O}_{\mathrm{X}}$ at p is a sheaf of discrete valuation rings. If K(X) denotes the corresponding function field of X, and $k(\mathrm{Z})$ denotes the function field of Z, then one observes the following:

(i) the field $k(\mathrm{Z}) \cong \mathcal{O}_{\mathrm{X},p}/p\mathcal{O}_{\mathrm{X},p}$ and the latter identifies with the residue field of the corresponding valuation on $\mathcal{O}_{\mathrm{X}}$.

Throughout the remaining statements let m denote a fixed positive integer invertible in $\mathcal{O}_{\mathrm{X}}$.

(ii) As X is smooth, the natural map ${}_m\mathrm{Br}(\mathrm{X}) \to {}_m\mathrm{Br}(\mathrm{K}(\mathrm{X}))$ is an injection: see [CTS, Theorem 3.5.5].

(iii) As Z - (the generic point of Z) is closed and of dimension < the dimension of Z, the natural map $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Z},\mu_m) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\, k(\mathrm{Z}),\mu_m)$ is also an injection.

(iv) A key result due to Merkurjev-Suslin identifies $\mathrm{K}_2^{\mathrm{Milnor}}(\mathrm{K(X)})/m\mathrm{K}_2^{\mathrm{Milnor}}(\mathrm{K(X)})$ with $\mathrm{H}^2_{\mathrm{et}}(\mathrm{K(X)},\mu_m(2))$ and $\mathrm{K}_1^{\mathrm{Milnor}}(k(\mathrm{Z}))/m\mathrm{K}_1^{\mathrm{Milnor}}(k(\mathrm{Z}))$ with $\mathrm{H}^1_{\mathrm{et}}(\mathrm{k(Z)},\mu_m(1))$. (See [MS], [GS, Chapter 8] and [CTS, Chapter 1, 1.4].)

(v) Another key result due to Merkurjev-Suslin shows that if the field K contains a primitive m-th root of unity ζ, then any class in the Brauer group ${}_m\mathrm{Br(K)}$ is *Brauer equivalent* to finite product of cyclic algebras of the form $(a_1,b_1)_\zeta \otimes_{\mathrm{K}} \cdots \otimes_{\mathrm{K}} (a_m,b_m)$, $a_i,b_i \epsilon \mathrm{K}^*$, $i=1,\cdots,m$. (See [GS, Theorem 2.5.7].)

(vi) One also has the following commutative square (see [GS, Proposition 7.5.1]), which proves the compatibility of the *residue map* in étale cohomology with the *tame symbol map* on Milnor K-theory as in:

$$\begin{tikzcd}
\mathrm{K}_2^{\mathrm{Milnor}}(\mathrm{K(X)})/m\mathrm{K}_2^{\mathrm{Milnor}}(\mathrm{K(X)}) \arrow[r, "Tame.sym"] \arrow[d, "h^2"] & \mathrm{K}_1^{\mathrm{Milnor}}(k(\mathrm{Z}))/m\mathrm{K}_1^{\mathrm{Milnor}}(\mathrm{k(Z)}) \arrow[d, "h^1"] \\
\mathrm{H}^2_{\mathrm{et}}(\mathrm{K(X)},\mu_m(2)) \arrow[r, "res"] & \mathrm{H}^1_{\mathrm{et}}(k(\mathrm{Z}),\mu_m(1))
\end{tikzcd} \tag{2.10}$$

where the map denoted $Tame.sym$ is the *tame symbol* defined as follows:

$$\delta_v : \mathrm{K(X)}^* \times \mathrm{K(X)}^* \to k(\mathrm{Z})^*, \delta_v(\mathrm{a},\mathrm{b}) = (-1)^{v(\mathrm{a}).v(\mathrm{b})}\frac{\mathrm{a}^{v(\mathrm{b})}}{\mathrm{b}^{v(\mathrm{a})}}\mathrm{mod(p)} \tag{2.11}$$

where p denotes the ideal defining Z in X. Moreover, h^2 and h^1 are the *Galois symbol maps*: see [GS, Definition 4.6.4].

(vii) Finally we also make the observation that, assuming K(X) contains a primitive m-th root of unity ζ, multiplication by ζ defines an isomorphism $\mu_m(i-1) \to \mu_m(i)$. Therefore, under this assumption and with the choice of a primitive m-th root of unity of ζ, the residue map above may be also viewed as a map

$$\mathrm{H}^2_{\mathrm{et}}(\mathrm{K(X)},\mu_m(1)) \xrightarrow{res} \mathrm{H}^1_{\mathrm{et}}(k(\mathrm{Z}),\mu_m(0)) \tag{2.12}$$

Moreover, in this case, the residue map has the following concrete interpretation:

$$residue((a,b)_\zeta) = \text{ the Galois extension of } k(\mathrm{Z}) \text{ given by adjoining } (\frac{a^{v_{\mathrm{Z}}(b)}}{b^{v_{\mathrm{Z}}(a)}})^{1/m}. \tag{2.13}$$

3. Brauer groups of smooth split toric varieties and their fans

In this section we will compute the Brauer groups of all smooth split toric varieties. Let X denote a smooth split toric variety over the given field K and let m denote a fixed positive integer invertible in K. Assume $r > 0$ is a fixed integer for which K has a primitive m-th root of unity ζ. Let T denote the open dense split torus in X of rank r. Next observe in view of Theorem 2.1 and Proposition 2.9, there are two types of contributions to the Brauer group of X:

(i) coming from the part generated by the $\binom{r}{2}$ cyclic algebras of the form $(t_i,t_j)_\zeta$, where t_i and t_j are two coordinates of T, with $i<j$, and

(ii) coming from the part generated by the r cyclic algebras of the form $(b,t_i)_\zeta$, where $b \in \mathrm{K}^*$ and t_i is a coordinate of T.

We will denote the subgroup of ${}_m\mathrm{Br}(\mathrm{T})$ generated by $\{\Pi_{1\le i<j\le r}(t_i,t_j)_\zeta^{e_{i,j}}|m > e_{i,j}\ge 0\}$ by $\mathcal{A}$, and the subgroup generated by $\{\Pi_{1\le i\le r}(b_i,t_i)_\zeta^{e_i}|m > e_i\ge 0, b_i\in \mathrm{K}^*\}$ by $\mathcal{B}$. Observe that any Azumaya algebra generated by the cyclic algebras $(t_i,t_j)_\zeta$, $1\le i<j\le r$ will be of the form $\Pi_{i<j}(t_i,t_j)_\zeta^{e_{i,j}}$, for some choice of integers $0\le e_{i,j}<m$ and that any Azumaya algebra generated by the cyclic algebras $(b_i,t_i)_\zeta$, $i=1,\cdots,r$ will be of the form $\Pi_{i=1}^r(b_i,t_i)_\zeta^{e_i}$, for some choice of integers $0\le e_i<m$.

Definition 3.1. (i) *Let $\mathbf{M}$ ($\mathbf{N}$) denote the lattice of characters (co-characters or 1-parameter subgroups, respectively) associated to the split torus* T. *For each $i=1,\cdots,r$, $\mathbf{m}_i$ will denote the character of* T *defined by*

$$\mathbf{m}_i(t_1,\cdots,t_r)=t_i.$$

Moreover, in this setting we let $\chi^{\mathbf{m}_i}=t_i$.

(ii) *Let Δ denote the fan associated to* X *and let $\mathbf{N}'$ denote the subgroup generated by $\bigcup_{\sigma\in\Delta}\sigma\cap\mathbf{N}$. (Observe that the generators of the rays of the fan $\{\rho_k|k=1,\cdots,n\}$ in $\mathbf{N}$ generate $\mathbf{N}'$.) Then $\mathbf{N}'=\mathbb{Z}a_1\mathbf{n}_1\oplus\cdots\mathbb{Z}a_u\mathbf{n}_u$ where $\mathbf{n}_1,\cdots,\mathbf{n}_u,\mathbf{n}_{u+1},\cdots,\mathbf{n}_r$ is a basis for $\mathbf{N}$ dual to the basis $\{\mathbf{m}_i|i=1,\cdots,r\}$ for $\mathbf{M}$, and $a_i\ge 1$ are integers with $a_i|a_{i+1}$, for $i=1,\cdots,u-1$. (This follows from the structure theorem for the finitely generated abelian group $\mathbf{N}/\mathbf{N}'$ and also the assumption that the toric variety* X *is regular, so that the minimal generators for each of the cones in the fan admit an extension to a $\mathbb{Z}$-basis for $\mathbf{N}$.) We call the set $\{a_1,\cdots,a_u\}$ the set of invariant factors of the given fan Δ.*

(iii) *For an element $b\in\mathrm{K}^*$, we let $\bar b$ denote the image of b in $\mathrm{K}^*/(\mathrm{K}^*)^m$. Let $ord_m(b)$ denote the order of $\bar b$, which is the least positive integer so that $b^{ord_m(b)}\in(\mathrm{K}^*)^m$.*

Then our main result is the following theorem. (One may want to observe that in the following theorem, the exponents $e_{i,j}$ and e_i are all strictly less than m, since m is the order of all the cyclic algebras appearing there.)

Theorem 3.2. *For a positive integer m invertible in* K*, the following hold, where $\zeta\in\mathrm{K}$ is a primitive m-th root of unity:*

(i) *${}_m\mathrm{Br}(\mathrm{X})\cap\mathcal{A}=$ the subgroup generated by $\{\Lambda=\Pi_{i<j}(t_i,t_j)_\zeta^{e_{i,j}}|m > e_{i,j}\ge 0\}$ satisfying the following conditions: for each $s=1,\ldots,min\{u,r-1\}$, if $m_s=hcf\{m,e_{1,s},e_{2,s},\cdots,e_{s-1,s},e_{s,s+1},\cdots,e_{s,r}\}$, then $(\frac{m}{m_s})|a_s$. In view of the assumption the toric variety is smooth, all $a_s=1$, and hence the last condition translates to $m_s=m$ for all s.*

(ii) *${}_m\mathrm{Br}(\mathrm{X})\cap\mathcal{B}$ is the subgroup generated by $\{\Lambda=\Pi_{i=1}^r(b_i,t_i)_\zeta^{e_i}|\ m > e_i\ge 0\}$, as $b_i\in\mathrm{K}^*$ varies among the corresponding classes in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K},\mu_m(0))$ so that the following conditions are satisfied: for each $s=1,\cdots,u$, if $m_s=hcf\{m,e_s,ord_m(b_s)\}$, then $(\frac{ord_m(b_s)}{m_s})|a_s$. In view of the assumption the toric variety is smooth, all $a_s=1$, and hence hence the last condition translates to $ord_m(b_s)=m_s$, for all s.*

(iii) *Moreover, ${}_m\mathrm{Br}(\mathrm{X})\cong{}_m\mathrm{Br}(\mathrm{Spec\,K})\oplus({}_m\mathrm{Br}(\mathrm{X})\cap\mathcal{A})\oplus({}_m\mathrm{Br}(\mathrm{X})\cap\mathcal{B})$. This can be identified with ${}_m\mathrm{Br}(\mathrm{Spec\,K})\oplus\mathrm{Hom}_{(\mathrm{Ab.groups})}(\wedge^2(\mathbf{N}/\mathbf{N}'),\mathbb{Z}/m\mathbb{Z})\oplus\mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}/\mathbf{N}',\mathrm{K}^*/\mathrm{K}^{*m})$.*

(iv) *Therefore, if ${}_m\mathrm{Br}_1(\mathrm{X})$[3] $=\ker({}_m\mathrm{Br}(\mathrm{X})\to{}_m\mathrm{Br}(\bar{\mathrm{X}}))$, where $\bar{\mathrm{X}}$ denotes the pull-back of* X *to the algebraic closure of* K*, then ${}_m\mathrm{Br}_1(\mathrm{X})\cong{}_m\mathrm{Br}(\mathrm{Spec\,K})\oplus({}_m\mathrm{Br}(\mathrm{X})\cap\mathcal{B})\cong{}_m\mathrm{Br}(\mathrm{Spec\,K})\oplus\mathrm{Hom}(\mathbf{N}/\mathbf{N}',\mathrm{K}^*/\mathrm{K}^{*m})$.*

---

[3]This is the m-torsion part of the *algebraic* Brauer group of X.

Corollary 3.3. *In case there is a cone σ in the fan Δ with dimension$(\sigma) \geq r-1$, then ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{A}$ is trivial. In case there is a cone σ in the fan Δ with dimension$(\sigma) \geq r$, then both ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{B}$ and ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{A}$ are trivial, so that ${}_m\mathrm{Br}(\mathrm{X}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\,\mathrm{K})$.*

The rest of this section will be devoted to a proof of the above theorem, when X is a smooth toric variety defined over the field K, the integer m is invertible in K and ζ is a primitive m-th root of unity in K. As one can see, we follow roughly the arguments in [DF](where they only considered the case of complex toric varieties), supplemented by Theorem 2.1.

We start with the following variant of the localization sequence in Proposition 2.9(ii), where ρ_k, $k = 1, \cdots, n$ denote the rays in the fan Δ, and $\mathrm{orb}(\rho_i)$ denotes the corresponding codimension 1-orbit of T in X:

$$0 \rightarrow {}_m\mathrm{Br}(\mathrm{X}) \rightarrow {}_m\mathrm{Br}(\mathrm{T}) \xrightarrow{res} \oplus_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{H}^1_{\mathrm{et}}(\mathrm{orb}(\rho_{\mathrm{i}}), \mu_m(0)) \tag{3.1}$$

Let ρ_k denote a ray in the fan associated to the given toric variety X. Given a cyclic algebra of the form $(\alpha, \beta)_\zeta$, where α, β belong to $\Gamma(\mathrm{T}, \mathbb{G}_{\mathrm{m}}) = \mathbf{M}$, the class of the residue, $res((\alpha, \beta)_\zeta)$ in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{orb}(\rho_{\mathrm{k}}), \mu_m(0))$ corresponds to the cyclic Galois extension of the function field $K(\mathrm{orb}(\rho_k))$ obtained by adjoining the m-th root of $\alpha^{v_k(\beta)}/\beta^{v_k(\alpha)}$, where v_k denotes the valuation associated with the toric divisor ρ_k. In case, $\beta = b \in \mathrm{K}^*$, $v_k(\beta) = 0$, so that it suffices to determine v_k on the characters of T.

We next consider the following Lemma.

Lemma 3.4. *(See* [DF, Lemma 1.5]*) Let η_k denote the primitive vector in $\mathbf{N} \cap \rho_k$ and $< \ \ , >: \mathbf{M} \times \mathbf{N} \rightarrow \mathbb{Z}$ denotes the natural perfect pairing. Let $\eta_k^\perp = \{\mathbf{m} \in \mathbf{M} | < \mathbf{m}, \eta_k >= 0\}$. Then*

(i) $\mathrm{K}(\overline{\mathrm{orb}}(\rho_k))$ *is the quotient field of $k[\eta_k^\perp]$, and*
(ii) $v_k(t_i) =< \mathbf{m}_i, \eta_k >$, $t_i \in \mathbf{M}$, *where the relationship between $\mathbf{m}_i$ and t_i is as in Definition 3.1(i).*

Again following [DF], we define the ramification map along ρ_k:

$$ram_{\rho_k} : \mathcal{A} \rightarrow \mathbf{M}/\mathbf{mM} \tag{3.2}$$

by:

$$ram_{\rho_k}((t_i, t_j)_\zeta) = v_k(t_j)\mathbf{m}_i - v_k(t_i)\mathbf{m}_j + m\mathbf{M}. \tag{3.3}$$

Let $\{\mathbf{n}_1, \cdots, \mathbf{n}_r\}$ denote the $\mathbb{Z}$-basis for $\mathbf{N}$ and $\{\mathbf{m}_1, \cdots, \mathbf{m}_r\}$ the dual basis for $\mathbf{M}$ chosen as in Definition 3.1. We will identify t_i with $\mathbf{m}_i$, $i = 1, \cdots, r$. Then we define the map

$$\phi : \mathcal{A} \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbf{N}, \mathbf{M}/m\mathbf{M}) \tag{3.4}$$

sending the class of the Azumaya algebra $\Lambda = \Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}}$ (where $e_{i,j}$ are some non-negative integers) to the $\mathbb{Z}$-linear map $\mathrm{T}_\Lambda : \mathbf{N} \rightarrow \mathbf{M}/(m\mathbf{M})$ whose matrix representation with respect to the chosen bases $\{\mathbf{n}_1, \cdots, \mathbf{n}_r\}$ and $\{\mathbf{m}_1, \cdots, \mathbf{m}_r\}$ is given by the matrix

$$\mathrm{M}_\Lambda = \begin{pmatrix} 0 & e_{1,2} & e_{1,3} & \cdots & . & e_{1,r} \\ -e_{1,2} & 0 & e_{23} & \cdots & . & e_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -e_{1,r} & -e_{2,r} & . & \cdots & -e_{r-1,r} & 0 \end{pmatrix}. \tag{3.5}$$

In fact T_Λ is the $\mathbb{Z}$-linear map so that

$$\mathrm{T}_\Lambda(\mathbf{n}_k) = \Sigma_{\mathrm{i}<\mathrm{j}}(\mathrm{e_{ij}} < \mathbf{m}_{\mathrm{j}}, \mathbf{n}_k > \mathbf{m}_{\mathrm{i}} - \mathrm{e_{ij}} < \mathrm{m_i}, \mathbf{n}_k > \mathbf{m}_{\mathrm{j}}) + \mathrm{m}\mathbf{M}.$$

Now it is easy to see (see for example, [DF, Proof of Lemma 1.7]), that the map $\phi : \mathrm{A} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbf{N}, \mathbf{M}/\mathrm{m}\mathbf{M})$ is *injective*. Moreover, for each $s = 1, \cdots, u$,

$$\mathrm{T}_\Lambda(a_\mathrm{s}\mathbf{n}_\mathrm{s}) = a_\mathrm{s} * \begin{pmatrix} e_{1,s} \\ e_{2,s} \\ \vdots \\ e_{s-1,s} \\ 0 \\ -e_{s,s+1} \\ \vdots \\ -e_{s,r} \end{pmatrix} \tag{3.6}$$

where the vector on the right is the s-th column of the matrix in (3.5). Therefore, we obtain the following criterion.

Criterion for *being unramified along* $a_s\mathbf{n}_s$: $ram_{a_i\mathbf{n}_s} : \mathcal{A} \to \mathbf{M}/m\mathbf{M}$, $s = 1, \cdots, u$:

$\mathrm{T}_\Lambda(a_\mathrm{s}\mathbf{n}_\mathrm{s})$ is in $m\mathbf{M}$, if and only if the following condition is satisfied: for each $i = 1, \cdots, u$,

$$\text{if } m_s = hcf\{m, e_{1,s}, e_{2,s}, \cdots, e_{s-1,s}, e_{s,s+1}, \cdots, e_{s,r}\}, \text{ then } (\frac{m}{m_s}) \text{ must divide } a_s. \tag{3.7}$$

We will next define the ramification map along ρ_k, $k = 1, \cdots, n$:

$$ram_{\rho_k} : \mathcal{B} \to \mathrm{K}^*/(\mathrm{K}^*)^\mathrm{m} \tag{3.8}$$

by:

$$ram_{\rho_k}((b_j, t_j)_\zeta^{e_j}) = b_j^{e_j v_k(t_j)} * (\mathrm{K}^*)^\mathrm{m}, j = 1, \cdots, r, \tag{3.9}$$

where $*$ denotes multiplication. In view of the above definitions, we will next define a map

$$\psi : \mathcal{B} \to \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathrm{K}^*/(\mathrm{K}^*)^\mathrm{m}) \tag{3.10}$$

as follows. Let $b_i \in \mathrm{K}^*$ viewed as a class in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\, k, \mu_m(0))$. Let $\Lambda = \Pi_{i=1}^r (b_i, t_i)_\zeta^{e_i}$ denote an Azumaya algebra in B. Then we let $\psi(\Lambda) : \mathbf{N} \to \mathrm{K}^*/(\mathrm{K}^*)^\mathrm{m}$ denote the map of abelian groups f_Λ defined by

$$f_\Lambda(a_s\mathbf{n}_s) = b_s^{a_s * e_s} = \Pi_{j=1}^r b_j^{e_j * a_s <\mathbf{m}_j, \mathbf{n}_s>}. \tag{3.11}$$

Lemma 3.5. *The $\mathbb{Z}$-linear map ψ is injective.*

Proof. One may first check that $f_\Lambda : \mathbf{N} \to \mathrm{K}^*/(\mathrm{K}^*)^\mathrm{m}$ is a group homomorphism. Therefore, it is enough to show that if $b_s^{e_s}$ belongs to $(\mathrm{K}^*)^\mathrm{m}$, then the class of the Azumaya algebra $((b_s, t_s)_\zeta)^{e_s}$ in B is trivial. Therefore, suppose $b_s^{e_s} = c^m$ for some $c \in \mathrm{K}^*$. Now one observes from [Miln, Theorem 15.1, and also p. 146] that as a class in B,

$$((b_s, t_s)_\zeta)^{e_s} = (b_s^{e_s}, t_s)_\zeta = (c^m, t_s)_\zeta = ((c, t_s)_\zeta)^m.$$

Since the order of the cyclic algebra $(c, t_s)_\zeta$ is m, it follows that the class $((c, t_s)_\zeta)^m$ is trivial. □

Criterion for being unramified along $ram_{a_s\mathbf{n}_s} : \mathcal{B} \to \mathrm{K}^*/\mathrm{K}^{*\mathrm{m}}$. For an element $b \in \mathrm{K}^*$, we let $\bar{b}$ denote the image of b in $\mathrm{K}^*/(\mathrm{K}^*)^\mathrm{m}$. Let $ord_m(b)$ denote the order of $\bar{b}$ as in Definition 3.1(iv). Then we obtain the following criterion for $ram_{a_s\mathbf{n}_s}(\Pi_{=1}^r (b_i, t_i)_\zeta^{e_i}) = 0$, $s = 1, \cdots, u$:

$$\begin{gathered} \text{if } m_s = hcf\{m, e_s, ord_m(b_s)\}, \text{ then } \frac{ord_m(b_s)}{m_s} \text{ must divide } a_s, \\ \text{where } ord_m(b_s) \text{ is defined as in Definition 3.1(iv).} \end{gathered} \tag{3.12}$$

Proof of Theorem 3.2. We begin with the commutative diagram (see [DF, Proof of Theorem 1.8]):

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r, "\phi"] & \mathrm{Hom}_{\mathbb{Z}}(\mathbf{N}, \mathbf{M}/m\mathbf{M}) \\ 0 \arrow[r] & {}_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{A} \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathbb{Z}}(\mathbf{N}/\mathbf{N}', \mathbf{M}/m\mathbf{M}) \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[u] \end{tikzcd} \tag{3.13}$$

Since $\mathbf{N}/\mathbf{N}' \cong \oplus_{\mathrm{i=1}}^{\mathrm{u}}(\mathbb{Z}/\mathbb{Z}a_i)\mathbf{n}_i \oplus \oplus_{\mathrm{i=u+1}}^{\mathrm{r}}\mathbb{Z}\mathbf{n}_i$, the conclusion in (i) follows.

Next we consider the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B} \arrow[r] & \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}}) \\ 0 \arrow[r] & {}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B} \arrow[r] \arrow[u] & \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}/\mathbf{N}', \mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}}) \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[u] \end{tikzcd} \tag{3.14}$$

Since $\mathbf{N}/\mathbf{N}' \cong \oplus_{\mathrm{i=1}}^{\mathrm{u}}(\mathbb{Z}/\mathbb{Z}a_i)\mathbf{n}_i \oplus \oplus_{\mathrm{i=u+1}}^{\mathrm{r}}\mathbb{Z}\mathbf{n}_i$, the conclusion in (ii) follows.

One may observe from the discussion of the ramification in (3.2) that the image of the residue map res on $\mathcal{A}$ is contained $\mathbf{M}/\mathbf{mM}$ and therefore is a transcendental over the field K. Similarly one may observe from the discussion in (3.8) that the image of the residue map res on $\mathcal{B}$ is in the field K. Therefore, no cancellation is possible between these. Therefore, one may observe that the product $(\Pi_{i<j}(t_i,t_j)_\zeta^{e_{i,j}})(\Pi_i(b,t_i)_\zeta^{e_i})$ belongs to ${}_m\mathrm{Br}(\mathrm{X})$ if and only if both factors $\Pi_{i<j}(t_i,t_j)_\zeta^{e_{i,j}}$ and $\Pi_i(b,t_i)_\zeta^{e_i}$ belong to ${}_m\mathrm{Br}(\mathrm{X})$. This proves the first statement in (iii).

One may prove the isomorphism

$$ {}_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{A} \cong \mathrm{Hom}(\wedge^2(\mathbf{N}/\mathbf{N}'), \mathbb{Z}/m\mathbb{Z}) \tag{3.15}$$

in the second statement in (iii) as follows.

Let $\mathrm{A} = \bigoplus_{1\leq \mathrm{i}<\mathrm{j}\leq \mathrm{r}} \mathbb{Z}/m\mathbb{Z}(\mathrm{t_i},\mathrm{t_j})_\zeta$. Define a map

$$\Phi : \mathrm{Hom}_{(Ab.groups)}(\wedge^2\mathbf{N}, \mathbb{Z}/m\mathbb{Z}) \to \mathrm{A} \tag{3.16}$$

as follows: Given a homomorphism $\varphi : \wedge^2\mathbf{N} \to \mathbb{Z}/m\mathbb{Z}$, let $e_{ij} \in \mathbb{Z}/m\mathbb{Z}$ be the image of $n_i \wedge n_j$ $(1 \leq i < j \leq r)$ under φ. The map Φ then maps φ to $\prod_{1\leq i<j\leq r}(t_i,t_j)_\zeta^{e_{ij}}$. Note that this is a homomorphism of groups: If $\varphi, \psi \in \mathrm{Hom}(\wedge^2(N), \mathbb{Z}/m\mathbb{Z})$ with $e_{ij} = \varphi(n_i \wedge n_j)$ and $f_{ij} = \psi(n_i \wedge n_j)$, then

$$\Phi(\varphi+\psi) = \prod_{1\leq i<j\leq r}(t_i,t_j)_\zeta^{e_{ij}+f_{ij}} = \prod_{1\leq i<j\leq r}(t_i,t_j)_\zeta^{e_{ij}} \cdot \prod_{1\leq i<j\leq r}(t_i,t_j)_\zeta^{f_{ij}} = \Phi(\varphi)\cdot\Phi(\psi). \tag{3.17}$$

Conversely, given an element $\prod_{1\leq i<j\leq r}(t_i,t_j)_\zeta^{e_{ij}} \in A$, we can define the inverse of Φ by mapping A to the homomorphism $\varphi : n_i \wedge n_j \mapsto e_{ij}$. Thus, we have an isomorphism of groups

$$\mathrm{Hom}_{(Ab.groups)}(\wedge^2(\mathbf{N}), \mathbb{Z}/m\mathbb{Z}) \cong \mathrm{A}. \tag{3.18}$$

We will now explain the relation between the group $\mathrm{Hom}_{(Ab.groups)}(\wedge^2(\mathbf{N}), \mathbb{Z}/m\mathbb{Z})$ and the injective homomorphism $\phi : A \to \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$ given in (3.4). Recall each $\Lambda = \prod_{1\le i<j\le r}(t_i, t_j)_\zeta^{e_{ij}}$ is associated with the matrix M_Λ as in (3.5). The injective homomorphism ϕ is then defined by sending Λ to the homomorphism $\phi(\Lambda) \in \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$ which maps $\mathbf{n}_i$ to

$$\phi(\Lambda)(\mathbf{n}_i) = \sum_{k<i} e_{ki}\mathbf{m}_k - \sum_{k>i} e_{ik}\mathbf{m}_k + m\mathbf{M}. \tag{3.19}$$

In other words, $\phi(\Lambda)$ maps each $\mathbf{n}_i$ to the ith column of the matrix M_Λ viewed as an element in $\mathbf{M}/\mathbf{mM}$. To relate this construction with $\mathrm{Hom}_{(Ab.groups)}(\wedge^2\mathbf{N}, \mathbb{Z}/m\mathbb{Z})$, we identify the group $\mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$ using the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M}) &\cong \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}) \\ &\cong \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathrm{Hom}_{(\mathbf{Ab.groups})}(\mathbf{N}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}) \\ &\cong \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathrm{Hom}_{(\mathbf{Ab.groups})}(\mathbf{N}, \mathbb{Z}/m\mathbb{Z})) \\ &\cong \mathrm{Hom}_{(Ab.groups)}(\mathbf{N} \otimes \mathbf{N}, \mathbb{Z}/m\mathbb{Z})\end{aligned} \tag{3.20}$$

where the last isomorphism is the tensor-hom adjunction and the penultimate isomorphism uses the freeness of the lattice $\mathbf{N}$. Let $\overline{\phi(\Lambda)}$ denote the image of $\phi(\Lambda)$ in $\mathrm{Hom}_{(Ab.groups)}(\mathbf{N}\otimes\mathbf{N}, \mathbb{Z}/m\mathbb{Z})$ under this identification. Then, $\overline{\phi(\Lambda)}$ can be described by the following rule:

$$\overline{\phi(\Lambda)}(\mathbf{n}_i \otimes \mathbf{n}_j) = \big(\phi(\Lambda)(\mathbf{n}_i)\big)(\mathbf{n}_j) = \begin{cases} -e_{ji} & j < i \\ 0 & j = i \\ e_{ij} & j > i. \end{cases} \tag{3.21}$$

Suppose $1 \le i < j \le r$ and consider the images of $\mathbf{n}_i \otimes \mathbf{n}_j$ and $\mathbf{n}_j \otimes \mathbf{n}_i$ of $\overline{\phi(\Lambda)}$; we have

$$\overline{\phi(\Lambda)}(\mathbf{n}_i \otimes \mathbf{n}_j) = e_{ij} \quad \text{because } i < j \tag{3.22}$$

and

$$\overline{\phi(\Lambda)}(\mathbf{n}_j \otimes \mathbf{n}_i) = -e_{ij} \quad \text{because } i < j. \tag{3.23}$$

On the other hand, if $i = j$, then we have $\overline{\phi(\Lambda)}(\mathbf{n}_i \otimes \mathbf{n}_j) = 0$. In other words, $\overline{\phi(\Lambda)}$ induces a unique homomorphism $\wedge^2\mathbf{N} \to \mathbb{Z}/m\mathbb{Z}$. This homomorphism is the image of $\prod_{1\le i<j\le r}(t_i, t_j)_\zeta^{e_{ij}}$ under the inverse of Φ defined in (3.16). In summary, we have shown that

Lemma 3.6. *The image of the homomorphism $\phi : \mathrm{A} \to \mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$ is $\mathrm{Hom}_{(Ab.groups)}(\wedge^2\mathbf{N}, \mathbb{Z}/m\mathbb{Z})$ when $\mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$ is identified with $\mathrm{Hom}_{(Ab.groups)}(\mathbf{N}\otimes\mathbf{N}, \mathbb{Z}/m\mathbb{Z})$.*

In (3.7), we already showed that Λ is unramified along $a_s\mathbf{n}_s$ if and only if $\mathrm{M}_\Lambda \cdot \mathrm{a_s}\mathbf{n}_\mathrm{s} = 0$. As $\mathrm{M}_\Lambda \cdot \mathrm{a_s}\mathbf{n}_\mathrm{s}$ is identified with $\overline{\phi(\Lambda)}(a_s\mathbf{n}_s \otimes -)$ using the above identification, Λ is unramified along all rays ρ_k in the fan if and only if

$$\overline{\phi(\Lambda)}(a_s\mathbf{n}_s \otimes \mathbf{n}_j) = 0 \text{ for all } j = 1, 2, \dots, r. \tag{3.24}$$

It follows that the class represented by Λ lies in the kernel of the residue map r if and only if the condition (3.24) holds for all $s = 1, \dots, u$, i.e., for each ray $\mathbf{n}_s$.

At this point, we recall that $\mathbf{N}'$ is the sublattice generated by the rays $\mathbf{n}_s$ and thus the condition (3.24) is equivalent to saying that

$$\overline{\phi(\Lambda)}(a_i\mathbf{n}_i \otimes \mathbf{n}_j) = 0 \text{ for all } i = 1, \cdots u, j = 1, \cdots r. \tag{3.25}$$

Finally we recall that there is an exact sequence

$$\mathbf{N'} \otimes \mathbf{N} \to \wedge^{\mathbf{2}}\mathbf{N} \to \wedge^{\mathbf{2}}(\mathbf{N}/\mathbf{N'}) \to \mathbf{0} \tag{3.26}$$

where the first arrow maps $\mathbf{n'} \otimes \mathbf{n}$ to $\mathbf{n'} \wedge \mathbf{n}$. Therefore, $\overline{\phi(\Lambda)}$ corresponds to an element in the kernel of the residue map if and only if it corresponds to an element of $\mathrm{Hom}_{(Ab.groups)}(\wedge^2(\mathbf{N}/\mathbf{N'}), \mathbb{Z}/m\mathbb{Z})$. This completes the proof of the isomorphism in (3.15).

Next we proceed to prove the isomorphism

$$_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{B} \cong \mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}/\mathbf{N'}, \mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}}) \tag{3.27}$$

in the second statement of (iii). First observe there is an injective map

$$_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{B} \to \mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}/\mathbf{N'}, \mathbf{K}^*/(\mathbf{K}^*)^{\mathbf{m}})$$

which takes an element $\Lambda = \prod_{j=1}^{r}(b_j, t_j)_\zeta^{e_i}$ to the map

$$f_\Lambda(n_i) = \prod_{j=1}^{r} b_j^{e_j\langle m_j, n_i\rangle} \cdot (\mathrm{K}^*)^{\mathrm{m}} = b_{\mathrm{i}}^{e_i} \cdot (\mathrm{K}^*)^{\mathrm{m}}. \tag{3.28}$$

This map can be shown to be also surjective as follows. A homomorphism $\varphi : \mathbf{N}/\mathbf{N'} \to \mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}}$ corresponds to a choice of r elements $b_i \in \mathrm{K}^*$ such that $b_i^{a_i} \in (\mathrm{K}^*)^{\mathrm{m}}$. Given such collection $\{b_i | i = 1, \cdots, r\}$ of elements, define a class

$$\Lambda_\varphi = \prod_{j=1}^{r}(b_j, t_j)_\zeta \in {}_m\mathrm{Br}(\mathrm{T_K}) \cap \mathcal{B}. \tag{3.29}$$

Then, under the map $\psi : \mathcal{B} \to \mathrm{Hom}_{(Ab.groups)}(\mathbf{N}, \mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}})$, Λ_φ is mapped to the homomorphism

$$f_{\Lambda_\varphi} : \mathbf{N} \to \mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}}, \quad \mathbf{n}_{\mathrm{i}} \mapsto \prod_{\mathrm{j}=1}^{r} b_j^{\langle \mathbf{m}_{\mathrm{j}}, \mathbf{n}_{\mathrm{i}}\rangle} \cdot (\mathrm{K}^*)^{\mathrm{m}} = b_i \cdot (\mathrm{K}^*)^{\mathrm{m}} \tag{3.30}$$

and thus

$$f_{\Lambda_\varphi}(a_i\mathbf{n}_i) = b_i^{a_i} \cdot (\mathrm{K}^*)^{\mathrm{m}} = (\mathrm{K}^*)^{\mathrm{m}} \quad \text{for all } \mathrm{i} = 1, 2, \ldots, \mathrm{r}. \tag{3.31}$$

Thus, f_{Λ_φ} is an element of $_m\mathrm{Br}(\mathrm{X}) \cap \mathcal{B}$ and the image of Λ_φ under f is φ. This completes the proof of theorem 3.2. □

Proof of Corollary 3.3. In view of the fact that the toric variety is regular, the invariant factor $a_i = 1$ for all i. Therefore, the conclusions are clear in view of Theorem 3.2.

4. Extension of the above results to toric schemes defined over discrete valuation rings and Dedekind domains

In this section we will extend the above results to toric schemes defined over more general bases. There will be two basic contexts we consider here:

(i) where the entire toric structure is defined over a Dedekind domain R satisfying the simplifying assumption in (4.1), *or*

(ii) where the base is the spectrum of a discrete valuation ring R defined as in [KKMSD, Chapter 4, section 3], and also where the toric scheme satisfies the simplifying assumption in (4.6).

Toric schemes over the spectra of any commutative ring R are defined in [Ro] as obtained by gluing affine toric schemes which are defined as the spectra associated to commutative rings defined using monoids. Moreover the following hypothesis is required to hold:
(4.1)
the toric scheme X is smooth, contains as an open dense subscheme the split torus $\mathrm{T} \cong \mathbb{G}_{\mathrm{m}}^{\times \mathrm{r}}$ (defined over Spec R), and that all the orbits of T on X are schemes that are smooth and faithfully flat over Spec R.

The first class of toric schemes we consider are such toric schemes defined over the spectra of Dedekind domains, which are excellent. We will assume that R denotes a Dedekind domain which is an *excellent* ring, the positive integer m is a unit in R, and R contains a primitive m-th root of unity which will be denoted ζ. In view of the above assumptions, the fan for the toric scheme X over Spec R is defined just as in Definition 3.1, and it corresponds to the fan of the toric scheme which is the fiber over the generic point of R = K.

Next we recall from the discussion at the end of section 1 that the results of Theorem 2.1, the pairings discussed following it, as well as Proposition 2.9, all extend to the case where the base field is replaced by a Dedekind domain R. As a result,

$$ {}_m\mathrm{Br}(\mathbb{G}_{\mathrm{m}}^{\times r}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\,\mathrm{R}) \oplus (\oplus^r \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(0))) \oplus (\oplus^{\binom{r}{2}} \mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(-1))) \tag{4.2} $$

Moreover, the $\binom{r}{2}$ summands $\mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(-1)))$ correspond to the cyclic algebras of the form $(t_i, t_j)_\zeta$, for $1 \le i < j \le r$ as t_i, t_j vary among the characters of T, while the r-summands $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(0))$ correspond to cyclic algebras of the form $(b_i, t_i)_\zeta$, where $b_i \in \mathrm{R}^*$ (which denotes the units in R) and t_i is a character of T.

We will denote the subgroup of ${}_m\mathrm{Br}(\mathrm{T})$ generated by $\{\Pi_{1 \le i<j \le r}(t_i, t_j)_\zeta^{e_{i,j}} | m > e_{i,j} \ge 0\}$ as t_i, t_j vary among the characters of the split torus T by A, and the subgroup generated by $\{\Pi_{1 \le i \le r}(b_i, t_i)_\zeta^{e_i} | m > e_i \ge 0$, and $b_i \in \mathrm{R}^*\}$ by B. Observe that any Azumaya algebra generated by the cyclic algebras $(t_i, t_j)_\zeta$, $1 \le i < j \le r$ will be of the form $\Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}}$, for some choice of integers $0 \le e_{i,j} < m$ and that any Azumaya algebra generated by the cyclic algebras $(b_i, t_i)_\zeta$, $i = 1, \cdots, r$ will be of the form $\Pi_{i=1}^r (b_i, t_i)_\zeta^{e_i}$, for some choice of integers $0 \le e_i < m$. Then we obtain the following theorem.

Theorem 4.1. *Let* R *denote a Dedekind domain, which we assume is an excellent ring. We will also assume that the positive integer m is a unit in* R *and that* R *contains a primitive m-th root of unity ζ. Then, under the assumption* (4.1) *the following hold:*

(i) ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A} =$ *the subgroup generated by* $\{\Lambda = \Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}} | m > e_{i,j} \ge 0\}$ *satisfying the following conditions: for each* $s = 1, \ldots, min\{u, r-1\}$, *if* $m_s = hcf\{m, e_{1,s}, e_{2,s}, \cdots, e_{s-1,s}, e_{s,s+1}, \cdots, e_{s,r}\}$, *then* $(\frac{m}{m_s})|a_s$. *In view of the assumption the toric scheme* X *is smooth, all* $a_s = 1$, *and hence hence the last condition translates to* $m_s = m$ *for all* s.

(ii) ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B}$ *is the subgroup generated by* $\{\Lambda = \Pi_{i=1}^r (b_i, t_i)_\zeta^{e_i} |\ m > e_i \ge 0\}$, *as* $b_i \in \mathrm{R}^*$ *varies among the corresponding classes in* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R}, \mu_m(0))$ *so that the following conditions are satisfied: for each* $s = 1, \cdots, u$, *if* $m_s = hcf\{m, e_s, ord_m(b_s)\}$, *then* $(\frac{ord_m(b_s)}{m_s})|a_s$. *In view of the assumption the toric scheme* X *is smooth, all* $a_s = 1$, *and hence hence the last condition translates to* $ord_m(b_s) = m_s$, *for all* s.

(iii) *Moreover,* ${}_m\mathrm{Br}(\mathrm{X}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\,\mathrm{R}) \oplus ({}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}) \oplus ({}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B})$.

Proof. Observe that the localization sequence in Proposition 2.9 holds also over the spectra of Dedekind domains. Therefore, in view of (4.2), one may observe that when X is a toric scheme defined over the Dedekind domain R, the Brauer group ${}_m\mathrm{Br}(\mathrm{X})$ is determined by the same exact sequence as in (3.1). Moreover, observe that ζ remains a primitive m-th root of unity in the fraction field K. But then we obtain the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_m\mathrm{Br}(\mathrm{X}) & \longrightarrow & {}_m\mathrm{Br}(\mathrm{T}) & \xrightarrow{res} & \oplus_{i=1}^{n}\mathrm{H}^1_{\mathrm{et}}(\mathrm{orb}(\rho_{\mathrm{i}}), \mu_m(0)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}_m\mathrm{Br}(\mathrm{X_K}) & \longrightarrow & {}_m\mathrm{Br}(\mathrm{T_K}) & \xrightarrow{res} & \oplus_{i=1}^{n}\mathrm{H}^1_{\mathrm{et}}(\mathrm{orb}(\rho_{\mathrm{i}})_{\mathrm{K}}, \mu_m(0)), \end{array} \tag{4.3}$$

where the first two vertical maps are clearly injective, and the last sum in the top row (the bottom row) varies over the codimension 1 orbits of T in X (of $\mathrm{T_K}$ in $\mathrm{X_K}$, respectively). The assumptions in (4.1) ensure that the each codimension 1-orbit in X corresponds $1-1$ to a codimension 1 orbit in $\mathrm{X_K}$. Therefore, the last vertical map is also injective. Thus to determine the kernel of the residue map in the top row, we reduce to determining the kernel of the residue map in the second row: but the toric scheme now is the toric variety $\mathrm{X_K}$ defined over the field of fractions K, with $\mathrm{T_K}$ denoting the open orbit and $\mathrm{orb}(\rho_i)_{\mathrm{K}}$ denoting the corresponding codimension 1-orbits. Therefore, under the assumption (4.1), we obtain the above results by invoking Theorem 3.2. □

Corollary 4.2. *In case there is a cone σ in the fan Δ with dimension$(\sigma) \geq r-1$, then ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}$ is trivial. In case there is a cone σ in the fan Δ with dimension$(\sigma) \geq r$, then both ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B}$ and ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}$ are trivial, so that ${}_m\mathrm{Br}(\mathrm{X}) \cong {}_m\mathrm{Br}(\mathrm{Spec}\,\mathrm{R})$.*

Proof. This follows by a reasoning as in the proof of the above theorem, and by invoking Corollary 3.3 with the field K there replaced by the field K, which is $\mathrm{R}_{(0)}$. □

Next we consider the second class of toric schemes, which are defined over the spectra of discrete valuation rings R which are also assumed to be excellent and defined as in [KKMSD, Chapter 4, section 3]: see also [GPS], [Qu] for similar discussions. We recall the following from [KKMSD, Chapter 4, section 3]. Let T denote a split torus of rank r over R, and let $\mathbf{M}$ ($\mathbf{N}$) denote the lattice of characters (co-characters, respectively) of T. Let $\widetilde{\mathbf{M}} = \mathbf{M} \times \mathbb{Z}$ and $\widetilde{\mathbf{N}} = \mathbf{N} \times \mathbb{Z}$. One adopts the definition of *split toric schemes* over R as in [KKMSD, Chapter 4, section 3]. These are irreducible normal schemes X over R on which a split torus T acts so that over the generic point of Spec R, X is in fact a toric variety defined over the field of fractions K of R. It is shown that there is a bijection between such affine toric schemes over Spec R and rational polyhedral cones $\sigma \subseteq \mathbf{N}(\mathrm{T}) \otimes \mathbb{R} \times \mathbb{R}_{\geq 0}$, where $\mathbb{R}_+ = \{z \in \mathbb{R} | z \geq 0\}$. Under this correspondence, it is shown there that the affine toric scheme X_σ corresponding to the cone σ is regular if and only if $\sigma \cap (\mathbf{N}(\mathrm{T}) \times \mathbb{Z})$ can be generated by a subset of a basis of $\mathbf{N}(\mathrm{T}) \times \mathbb{Z}$.

Moreover any such toric scheme over R is obtained by gluing affine toric schemes and such toric schemes are classified by fans in $\mathbf{N}(\mathrm{T}) \otimes \mathbb{R} \times \mathbb{R}_{\geq 0}$. Over the generic point of Spec R, such toric schemes correspond to fans in $\mathbf{N}(\mathrm{T}) \otimes \mathbb{R} \cong \mathbf{N}(\mathrm{T}) \otimes \mathbb{R} \times \{0\}$.

We will let

$$p : \widetilde{\mathbf{N}} \rightarrow \mathbb{Z} \text{ and } \pi : \widetilde{\mathbf{N}} \rightarrow \mathbf{N} \tag{4.4}$$

denote the obvious projections. Then we start with the following observations:

(i) Since the toric scheme X is assumed to be regular, and if the special fiber is reduced, it follows that $p(\rho) = 1$ for each ray ρ not contained in $\mathbf{N}_{\mathbb{R}} \times 0$: see [Qu, Corollary 2.1.21].
(ii) the irreducible components of the special fiber correspond to the rays of the fan not contained in $\mathbf{N}_{\mathbb{R}} \times 0$: see [Qu, [2.1.20].

Observe that

$$\mathrm{X} - \mathrm{T_K} = \Big(\bigcup_{\rho \in \Delta(1) \cap (\mathbf{N}_{\mathbb{R}} \times 0)} \mathrm{V}(\rho) \Big) \bigcup \mathrm{X}_k$$

where X_k is the special fiber of X over the closed point of $\mathrm{Spec}\,\mathrm{R}$. Note that X_k has pure codimension one in X because $\mathrm{X}_k = \mathrm{X} \times_{\mathrm{Spec}\,\mathrm{R}} \mathrm{Spec}\,(\mathrm{R}/(\pi))$.

Let $\tilde{\Delta}$ denote the fan in $\widetilde{\mathbf{N}}_{\mathbb{R}}$ associated to the given toric scheme. Since we assume that the special fiber is reduced, it follows that the irreducible components of X_k correspond to the rays of the fan $\tilde{\Delta}$ not contained in $\mathbf{N}_{\mathbb{R}} \times 0$. We will let $\{\rho_1, \ldots, \rho_\nu\}$ be the set of rays of $\tilde{\Delta}$ contained in $\mathbf{N}_{\mathbb{R}} \times 0$ and let $\{\tau_1, \ldots, \tau_m\}$ be the set of rays of $\tilde{\Delta}$ not contained in $\mathbf{N}_{\mathbb{R}} \times 0$. We define $\mathbf{N}'$ to be the sublattice generated by $\bigcup_{i=1}^{\nu} (\rho_i \cap \mathbf{N})$ as before. We choose a basis $\{\mathbf{n}_1, \ldots, \mathbf{n}_u, \mathbf{n}_{u+1}, \cdots, \mathbf{n}_r\}$ of $\mathbf{N}$ as before, i.e., in such a way that

$$\mathbf{N}' = \bigoplus_{i=1}^{u} \mathbb{Z} a_i \mathbf{n}_i \tag{4.5}$$

with positive integers $a_1 | a_2 | \cdots | a_u$. Let $\{\mathbf{m}_i | i = 1, \cdots, r\}$ be the dual basis of $\mathbf{M}$. Hence, $\widetilde{\mathbf{M}} = \bigoplus_{i=1}^{r} \mathbb{Z} \cdot \mathbf{m}_i \bigoplus \mathbb{Z}\mathbf{e}$, where $\mathbf{e} = (0, 1)$.

At this point we will make the following *simplifying assumption*[4]:

(4.6) the toric scheme X is regular, the special fiber of X over the residue field k is reduced,

and that the subgroup generated by $\pi(\tau_j), j = 1, \ldots, m,$ is contained in $\mathbf{N}'$.

Throughout the following discussion, we will assume that the prime integer ℓ is a unit in R and R contains a primitive ℓ^n-th root of unity which will be denoted ζ. We will also assume that the above primitive ℓ^n-th root of unity in R is in fact the lift of a primitive ℓ^n-th root of unity in the residue field k. Mainly for a change, rather than considering the m-torsion as in the last section, we will consider the ℓ^n-torsion part of the Brauer groups, in the remaining discussion in this section.

In view of the localization sequence as in Proposition 2.9 (see also [CTS, Theorem 3.7.2]) one has the exact sequence

$$0 \to {}_{\ell^n}\mathrm{Br}(\mathrm{X}) \to {}_{\ell^n}\mathrm{Br}(\mathrm{T_K}) \overset{res}{\to} \overbrace{\Big(\bigoplus_{i=1}^{\nu} \mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathrm{V}(\rho_i)), \mu_{\ell^n}(0)) \Big)}^{\mathrm{C}} \bigoplus \overbrace{\Big(\bigoplus_{j=1}^{m} \mathrm{H}^1_{\mathrm{et}}(\mathrm{K}(\mathrm{V}(\tau_j)), \mu_{\ell^n}(0)) \Big)}^{\mathrm{D}} \tag{4.7}$$

Thus, as in the case of the toric varieties over a field, the Brauer group of the toric scheme X is identified with the kernel of the map res, which is the residue map. As before, we have the complete description of

[4]Observe that the last condition in (4.6) is satisfied if the fan Δ defining the toric variety $\mathrm{X_K}$ has the same dimension as the rank of $\mathrm{T_K} = \mathrm{r}$. One should view the above condition as making sure the toric structure of the special fiber is not too different from the toric structure of the generic fiber.

${}_{\ell^n}\mathrm{Br}(\mathrm{T_K})$ as follows:

$$(4.8)\qquad {}_{\ell^n}\mathrm{Br}(\mathrm{T_K}) = {}_{\ell^n}\mathrm{Br}(\mathrm{Spec}\,\mathrm{K}) \bigoplus \overbrace{\Big(\bigoplus_{i\leq i<j\leq r} \mathbb{Z}/\ell^n\mathbb{Z}\cdot (t_i,t_j)_\zeta\Big)}^{\mathrm{A}} \bigoplus \overbrace{\Big(\bigoplus_{i=1}^{r}\sum_{b_i\in K^*} \mathbb{Z}/\ell^n\mathbb{Z}\cdot (b_i,t_i)_\zeta\Big)}^{\mathrm{B}}$$

where we use the same convention on the cyclic algebras $(b_i, t_i)_\zeta$ and $(t_i, t_j)_\zeta$ as before.

One may observe from the discussion of the ramification in (3.2) that the image of the residue map *res* on A is contained $\mathbf{M}/\ell^{\mathbf{n}}\mathbf{M}$ and therefore is a transcendental over the field K. Similarly one may observe from the discussion in (3.8) that the image of the residue map *res* on B is in the field K. Therefore, no cancellation is possible between these. Therefore, in view of the decompositions in (4.7) and (4.8), and the above observation, determining the kernel of the residue map *res* on ${}_{\ell^n}\mathrm{Br}(\mathrm{T_K})$ amounts to determining the kernel of the following restrictions of the residue map:

(i) $res_{\mathrm{A,C}} : \mathrm{A} \to \mathrm{C}$,
(ii) $res_{\mathrm{A,D}} : \mathrm{A} \to \mathrm{D}$,
(iii) $res_{\mathrm{B,C}} : \mathrm{B} \to \mathrm{C}$,
(iv) $res_{\mathrm{B,D}} : \mathrm{B} \to \mathrm{D}$ as well as
(v) $res \circ \pi^* : {}_{\ell^n}\mathrm{Br}(\mathrm{Spec}\,\mathrm{K}) \xrightarrow{\pi^*} {}_{\ell^n}\mathrm{Br}(\mathrm{T_K}) \xrightarrow{\mathrm{res}} \mathrm{C}\oplus\mathrm{D}$.

Proposition 4.3. *Let $\rho \in \tilde{\Delta}(1)$ be a ray and let η be the primitive vector of ρ. Let $\eta^\perp = \{\mathbf{m} \in \widetilde{\mathbf{M}} | <\mathbf{m}, \eta >= 0\}$.*

(i) *The function field $\mathrm{K}(\mathrm{V}(\rho))$ is the quotient field of $k[\eta^\perp]$ if $\rho \nsubseteq \mathbf{N}_\mathbb{R}\times 0$ and the quotient field of $K[\eta^\perp \cap \mathbf{M}]$ if $\rho \subseteq \mathbf{N}_\mathbb{R}\times 0$.*
(ii) *The valuation of $\chi^{\mathbf{m}}\pi^r \in K(\mathrm{X})$ along $V(\rho)$ is*

$$v_\rho(\chi^{\mathbf{m}}\pi^r) =< \widetilde{\mathbf{m}}, \eta > \qquad \textit{where } \widetilde{\mathbf{m}} = (\mathbf{m}, r).$$

Proof. Let $\mu \in \widetilde{\mathbf{M}}$ be such that $<\mu, \eta >= 1$. Then we have a split exact sequence

$$0 \to \mathbb{Z}\eta \to \widetilde{\mathbf{N}} \to \widetilde{\mathbf{N}}/\mathbb{Z}\eta \to 0$$

and its dual

$$0 \to \eta^\perp \to \widetilde{\mathbf{M}} \xrightarrow{m\mapsto <m,\eta>} \mathbb{Z} \to 0,$$

so that $\widetilde{\mathbf{M}} = \eta^\perp \oplus \mathbb{Z}\mu$.

Recall $\mathbf{e} = (0, 1) \in \widetilde{\mathbf{N}}$. If $\rho \subseteq \mathbf{N}_\mathbb{R}\times 0$, then the orbit closure $\mathrm{V}(\rho)$ embeds into $\mathrm{U}_\rho = \mathrm{Spec}\,\mathrm{R}([\eta^\perp \oplus \mathbf{N}\mu]/(\chi^{\mathrm{e}} - \pi))$ via the ring homomorphism

$$\mathrm{R}[\eta^\perp \oplus \mathbf{N}\mu]/(\chi^{\mathbf{e}} - \pi) \to \mathrm{R}[\eta^\perp]/(\chi^{\mathbf{e}} - \pi).$$

Hence, the prime ideal corresponding to $V(\rho)$ is (χ^μ).

If $\rho \nsubseteq \mathbf{N}_\mathbb{R}\times 0$, then the orbit closure $\mathrm{V}(\rho)$ embeds into U_ρ via the ring homomorphism

$$\mathrm{R}[\eta^\perp \oplus \mathbf{N}\mu]/(\chi^{\mathbf{e}} - \pi) \to k[\eta^\perp].$$

Note that since $\rho \nsubseteq \mathbf{N}_\mathbb{R}\times 0$, $e \notin \eta^\perp$. Thus, this homomorphism is well-defined and its kernel is again given by (χ^μ).

If $\rho \subseteq \mathbf{N}_\mathbb{R}\times 0$, then $-\mathbf{e} \in \eta^\perp$ and so π is invertible in $\mathrm{R}[\eta^\perp]/(\chi^{\mathbf{e}} - \pi)$ which shows that $\mathrm{R}[\eta^\perp]/(\chi^{\mathbf{e}} - \pi) = K[\eta^\perp \cap \mathbf{M}]$.

If $\rho \not\subseteq \mathbf{N}_{\mathbb{R}} \times 0$, then $\mathrm{V}(\rho)$ is a toric variety over k whose coordinate ring is given by $k[\eta^{\perp}]$. Hence, the function field of $\mathrm{V}(\rho)$ is the quotient field of $k[\eta^{\perp}]$. This proves (i).

For (ii), let $\chi^{\mathbf{m}}\pi^r \in \mathrm{K(X)}$ and write $\widetilde{\mathbf{m}} = (\mathbf{m}, r)$. Since $\widetilde{\mathbf{M}} = \eta^{\perp} \oplus \mathbb{Z}\mu$, we can find $x \in \eta^{\perp}$ and $n \in \mathbb{Z}$ such that $\widetilde{\mathbf{m}} = x + n\mu$. Since the prime ideal corresponding to $\mathrm{V}(\rho)$ is (χ^{μ}) in either case, the valuation of $\chi^{\mathbf{m}}\pi^r = \chi^{\widetilde{\mathbf{m}}} = \chi^x(\chi^{\mu})^n \in \mathrm{R}[\eta^{\perp} \oplus \mathbf{N}\mu]/(\chi^{\mathbf{e}} - \pi)$ is given by n. This integer can also be obtained as

$$< \widetilde{\mathbf{m}}, \eta >=< x + n\mu, \eta >=< x, \eta > +n < \mu, \eta >= n.$$

This shows that $v_{\rho}(\chi^{\mathbf{m}}\pi^r) =< \widetilde{\mathbf{m}}, \eta >$. □

The above results enable us to obtain an explicit description of the map *res* as follows. For $\rho \in \Delta(1)$, let η denote the primitive vector of the ray ρ so that $\rho_i = \mathbb{R}_{\geq 0}\eta_i$ and $\tau_j = \mathbb{R}_{\geq 0}\eta_j$ for $i = 1, \dots, \nu$ and $j = 1, \dots, m$. Recall that ρ_i, $i = 1, \cdots, \nu$ denote the rays in the fan $\tilde{\Delta}$ that are contained in $\mathbf{N}_{\mathbb{R}} \times 0$ and that τ_j, $j = 1, \cdots, m$ denote the rays in $\tilde{\Delta}$ that are *not* contained in $\mathbf{N}_{\mathbb{R}} \times 0$. For $\mathbf{m}_i \in \mathbf{M}$, we will also denote the image of $\mathbf{m}_i$ in $\widetilde{\mathbf{M}}$ by the same symbol $\mathbf{m}_i$. Then, observing that $t_i = \chi^{\mathbf{m}_i}$, $i = 1, \cdots, r$, we obtain for $1 \leq i < j \leq r$:

$$res((t_i, t_j)_{\zeta}) = \big((\chi^{<\mathbf{m}_j, \rho_u>\mathbf{m}_i - <\mathbf{m}_i, \rho_u>\mathbf{m}_j})_u, (\chi^{<\mathbf{m}_j, \eta_l>\mathbf{m}_i - <\mathbf{m}_i, \eta_l>\mathbf{m}_j})_l)_{u=1,\dots,n, l=1\dots,m}. \tag{4.9}$$

Definition 4.4. *We say that the algebra* $\Lambda = \Pi_{1 \leq i<j \leq r}^r (t_i, t_j)_{\zeta}^{e_{i,j}}$ *is unramified along* $\rho_i, i = 1, \cdots, n$ $(\tau_j, j = 1, \cdots, m)$*, if the* ρ_i*-th* (τ_j*-th*) *coordinate of* $res(\Lambda)$ *is zero. One defines the algebra* $\Lambda = \Pi_{i=1}^r (b_i, t_i)^{e_i}$ *with* $b_i \in \mathrm{K}^*$ *to be unramified along* $\rho_i, i = 1, \cdots, \nu$ $(\tau_j, j = 1, \cdots, m)$ *similarly.*

Computation of kernel$(res|_A)$. From the formula (4.9), we deduce that a class $\Lambda = \prod_{1 \leq i<j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}} \in {}_{\ell^n}\mathrm{Br}(\mathrm{T_K})$ lies in ${}_m\mathrm{Br(X)}$ if and only if $< \mathbf{m}_j, \rho_u > \mathbf{m}_i - < \mathbf{m}_i, \rho_u > \mathbf{m}_j \in \ell^n \mathbf{M}$. Note that this property does not depend on the base field. Moreover, since $\mathbf{m}_i$ viewed as an element of $\widetilde{\mathbf{M}}$ has the last coordinate 0 and $\rho_u \in \mathbf{N}_{\mathbb{R}} \times 0$, the pairing $< \mathbf{m}_i, \rho_u >$ is equal to the pairing $< \mathbf{m}_i, \rho_u) >$ of two elements $\mathbf{m}_i \in \mathbf{M}$ and $\rho_u \in \mathbf{N}$. Hence, we deduce the following:

The kernel of $res_{\mathrm{A,C}} : \mathrm{A} \to \mathrm{C}$ is given by those cyclic algebras $\Lambda = \prod_{1 \leq i<j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}}$ satisfying exactly the condition in Theorem 4.1(i).

Next recall from the simplifying assumption (4.6) that $\pi(\tau_j) \in \mathbf{N}'$ for all $j = 1, \cdots, m$. Therefore, any cyclic algebra Λ in the kernel of $res_{\mathrm{A,C}} : \mathrm{A} \to \mathrm{C}$ is unramified along $\pi(\eta_j)$, for any $j = 1, \cdots, m$ showing that the kernel of $res_{\mathrm{A,C}} : \mathrm{A} \to \mathrm{C}$ is contained in the kernel of $res_{\mathrm{A,D}} : \mathrm{A} \to \mathrm{D}$. To see this, fix a ray $\tau \in \tilde{\Delta}(1)$ that is not contained in $\mathbf{N}_{\mathbb{R}} \times 0$ and let η denote the primitive vector along τ in $\tilde{\mathbf{N}}$. Since $\pi(\eta)$ lies in $\mathbf{N}'$ (by our assumption in (4.6), there exist some integers $\alpha_1, \dots, \alpha_u$ such that $\pi(\eta) = \sum_{j=1}^u \alpha_j a_j \mathbf{n}_j$. On the other hand, because $m_i \in \mathbf{M}$, we have

$$< \mathbf{m}_i, \eta >=< \mathbf{m}_i, \pi(\eta) >= \sum_{j=1}^{u} < \mathbf{m}_i, \alpha_j a_j \mathbf{n}_j >= \alpha_i a_i \quad \text{for each } i = 1, \dots, r. \tag{4.10}$$

Now, $\Lambda = \prod_{1 \leq i<j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}}$ is unramified along τ if and only if $\ell^n | \alpha_i a_i e_{i,j}$, for all $1 \leq i < j \leq r$. But, using the assumption that Λ is unramified along each ρ_i, we see from Theorem 4.1(i) that $\ell^n | a_i e_{i,j}$, for all $1 \leq i < j \leq n$. Therefore, $\ell^n | \alpha_i a_i e_{i,j}$, for all $1 \leq i < j \leq n$, proving the required conclusion. These observations therefore prove the following result.

Proposition 4.5. *The subgroup* $\mathrm{A}\cap{}_{\ell^n}\mathrm{Br}(\mathrm{X})$ *is isomorphic to a subgroup of* ${}_{\ell^n}\mathrm{Br}(\mathrm{X_K})$*, where* $\mathrm{X_K}$ *is the toric variety associated to the fan* $\Delta\cap(\mathbf{N}_{\mathbb{R}}\times 0)$.

Computation of kernel$(res|_B)$. Next we proceed to determine the kernel of the residue map $res_{\mathrm{B,C}}:\mathrm{B}\to\mathrm{C}$. Therefore, let $b_i\in\mathrm{K}^*$ and $i=1,\ldots,r$. Let the cyclic algebra $(b_i,t_i)_\zeta$ represent a class in $\mathrm{B}\subseteq{}_{\ell^n}\mathrm{Br}(\mathrm{T_K})$. For $\rho\in\tilde{\Delta}(1)$ contained in $\mathbf{N}_{\mathbb{R}}\times 0$, and $b_i\in\mathrm{K}^*$, $v_\rho(b_i)=0$. Therefore, the image of $(b_i,t_i)_\zeta^{e_i}$ under the residue map ${}_{\ell^n}\mathrm{Br}(\mathrm{T_K})\to\mathrm{H}^1(\mathrm{K}(\mathrm{V}(\rho)),\mu_{\ell^n}(0))$ is the element

$$b_i^{e_iv_\rho(t_i)}/\chi^{e_iv_\rho(b_i)\mathbf{m}_i}=b_i^{e_iv_\rho(t_i)}=b_i^{e_i<\mathbf{m}_i,\rho>}\in\mathrm{K}^*$$

Choosing $\rho=a_j\mathbf{n}_j$, $j=1,\cdots,n$, we see that the algebra $\Lambda=\Pi_{i=1}^r(b_i,t_i)_\zeta^{e_i}$ belongs to the kernel of the residue map $res_{\mathrm{B,C}}:\mathrm{B}\to\mathrm{C}$ precisely when the condition in Theorem 4.1(ii) holds.

Next let $b_i=u_i\pi^{n_i}$ where $u_i\in\mathrm{R}^*$ and $n_i\in\mathbb{Z}$. Then for any ray $\rho\in\tilde{\Delta}$ which is contained in $\mathbf{N}_{\mathbb{R}}\times 0$,

$$b_i^{e_iv_\rho(t_i)}=(u_i\pi^{n_i})^{e_iv_\rho(t_i)}=(u_i)^{e_iv_\rho(t_i)}(\pi)^{e_in_iv_\rho(t_i)}=u_i^{e_i<\mathbf{m}_i,\rho>}.\chi^{\mathbf{e}e_in_i<\mathbf{m}_i,\rho>}\tag{4.11}$$

since $\pi=\chi^{\mathbf{e}}$. Therefore, the class of the algebra $\Pi_{i=1}^r(b_i,t_i)_\zeta^{e_i}$ in the subgroup B identifies with the class of the algebra $\Pi_{i=1}^r(u_i,t_i)_\zeta^{e_i}$ in B provided

$$\Sigma_{i=1}^re_in_i\text{ is divisible by }\ell^n.\tag{4.12}$$

Next we proceed to determine the kernel of the residue map $res_{\mathrm{B,D}}:\mathrm{B}\to\mathrm{D}$. Let $b_i\in\mathrm{K}^*$. Let $\tau\in\tilde{\Delta}(1)$ be not contained in $\mathbf{N}_{\mathbb{R}}\times 0$, with η denoting the primitive vector along τ.

Then the image of the cyclic algebra $(b_i,t_i)_\zeta^{e_i}$ under the residue map $\mathrm{B}\to\mathrm{H}^1(\mathrm{K}(\mathrm{V}(\tau)),\mu_{\ell^n}(0))\subseteq\mathrm{D}$ is the image of the element

$$x=b_i^{e_iv_\tau(t_i)}/\chi^{e_iv_\tau(b_i)\mathbf{m}_i}\in\mathcal{O}^*_{X,V(\tau)}$$

in $\mathrm{K}(\mathrm{V}(\tau))/(\mathrm{K}(\mathrm{V}(\tau)))^{*\ell^{\mathrm{n}}}$ and $\mathrm{K}(\mathrm{V}(\tau))$ is the quotient field of $k[\eta^\perp]$. (Recall $t_i=\chi^{\mathbf{m}_i}$, $i=1,\cdots,r$.)

Write $b_i=u_i\pi^{n_i}$ where $u_i\in\mathrm{R}^*$ and $n_i\in\mathbb{Z}$. Noting that $\chi^{\mathbf{e}}=\pi$, we can rewrite x as

$$u_i^{e_i<\mathbf{m}_i,\eta>}\cdot\left(\frac{\chi^{\mathbf{e}e_i<\mathbf{m}_i,\eta>}}{\chi^{e_i\mathbf{m}_i}}\right)^{n_i}=u_i^{e_i<\mathbf{m}_i,\eta>}\chi^{n_ie_i(<\mathbf{m}_i,\eta>\mathbf{e}-\mathbf{m}_i)}.\tag{4.13}$$

From (4.13), we see that for a class $\Lambda=\prod_{i=1}^r(b_i,t_i)_\zeta^{e_i}$ to lie in the kernel of the residue map $res_{\mathrm{B,D}}:\mathrm{B}\to\mathrm{D}$ it is necessary and sufficient that the following condition holds:

$$\prod_{i=1}^r\bar{u}_i^{e_i<\mathbf{m}_i,\eta_j>}\cdot\chi^{\sum_{i=1}^re_in_i(<\mathbf{m}_i,\eta_j>\mathbf{e}-\mathbf{m}_i)}\in\mathrm{K}(\mathrm{V}(\tau_{\mathrm{j}}))^{*\ell^{\mathrm{n}}}\tag{4.14}$$

for all $j=1,\ldots,m$ where $b_i=u_i\pi^{n_i},u_i\in\mathrm{R}^*$ with $\bar{u}_i$ the image of u_i in k^*, and $n_i\in\mathbb{Z}$. Since $\mathrm{K}(\mathrm{V}(\tau_{\mathrm{j}}))$ is the quotient field of $k[\eta_j^\perp]$, (4.14) holds if and only if

$$\sum_{i=1}^re_in_i(<\mathbf{m}_i,\eta_j>\mathbf{e}-\mathbf{m}_i)=\sum_{i=1}^re_in_i<\mathbf{m}_i,\eta_j>\mathbf{e}-\sum_{i=1}^re_in_i\mathbf{m}_i\in\ell^n\widetilde{\mathbf{M}}\text{ and,}\tag{4.15}$$
$$\prod_{i=1}^r\bar{u}_i^{e_i<\mathbf{m}_i,\eta_j>}\in(k^*)^{\ell^n}.$$

Since $\mathbf{m}_1,\ldots,\mathbf{m}_r,\mathbf{e}$ form a basis for $\widetilde{\mathbf{M}}$, the first condition in (4.15) in is equivalent to requiring that

$$\ell^n|e_in_i\text{ for all }i=1,\ldots,r.\tag{4.16}$$

Moreover, under this assumption, we may assume that $b_i=u_i\in\mathrm{R}^*$ for all $i=1,\ldots,r$.

Lemma 4.6. *Let $u_i \in \mathrm{R}^*$ for $i = 1, \cdots, r$. Then the condition that the algebra $\Lambda = \prod_{i=1}^{r}(u_i, t_i)_{\zeta}^{e_i}$ is unramified along each of the rays ρ_i, $i = 1, \cdots, \nu$ in $\tilde{\Delta}(1)$ contained in $\mathbf{N}_{\mathbb{R}} \times 0$ implies that Λ is also unramified along each of the rays $\tau_j, j = 1, \cdots, m$ in $\tilde{\Delta}(1)$ that are not contained in $\mathbf{N}_{\mathbb{R}} \times 0$.*

Proof. The required argument is similar to the one used in the discussion centered around (4.10), but we provide it here for the readers' convenience. To see this, fix a ray $\tau \in \tilde{\Delta}(1)$ that is not contained in $\mathbf{N}_{\mathbb{R}} \times 0$ and let η denote the primitive vector along τ in $\tilde{\mathbf{N}}$. Since $\pi(\eta)$ lies in $\mathbf{N}'$ (by our assumption in (4.6), there exist some integers $\alpha_1, \ldots, \alpha_u$ such that $\pi(\eta) = \sum_{j=1}^{u} \alpha_j a_j \mathbf{n}_j$. On the other hand, because $m_i \in \mathbf{M}$, we have

$$(4.17) \qquad < \mathbf{m}_i, \eta >=< \mathbf{m}_i, \pi(\eta) >= \sum_{j=1}^{u} < \mathbf{m}_i, \alpha_j a_j \mathbf{n}_j >= \alpha_i a_i \quad \text{for each } i = 1, \ldots, r.$$

Now, Λ is unramified along τ if and only if $\prod_{i=1}^{r} \overline{u}_i^{e_i <\mathbf{m}_i, \eta>} \in k^{*\ell^n}$. But, using (4.17), we can write

$$\prod_{i=1}^{r} \overline{u}_i^{e_i <\mathbf{m}_i, \eta>} = \prod_{i=1}^{r} \overline{u}_i^{\alpha_i a_i e_i} = \prod_{i=1}^{r} \left(\overline{u}_i^{a_i e_i}\right)^{\alpha_i}.$$

Now, if $u_i^{a_i e_i} \in \mathrm{K}^{*\ell^{\mathrm{n}}}$, then $\overline{u}_i^{a_i e_i} \in k^{*\ell^n}$ and, thus, $\prod_{i=1}^{r} \overline{u}_i^{e_i <\mathbf{m}_i, \eta>} \in k^{*\ell^n}$ as well. □

Let B' denote the subgroup of B generated by the algebras of the form $\Lambda = \Pi_{i=1}^{r}(u_i, t_i)_{\zeta}^{e_i}$, as u_i varies among the units in R. The above discussion shows that for such an algebra $\Lambda \in \mathrm{B}'$ to lie in the kernel of $res_{\mathrm{B,D}} : \mathrm{B} \to \mathrm{D}$, it is sufficient that it be unramified along the rays of $\tilde{\Delta}$ that are contained in $\mathbf{N}_{\mathbb{R}} \times 0$.

Proposition 4.7. *The subgroup $\mathrm{B} \cap {}_{\ell^{\mathrm{n}}}\mathrm{Br(X)}$ is generated by the classes $\Lambda = \prod_{i=1}^{r}(b_i, t_i)_{\zeta}^{e_i}$ satisfying the following conditions:*

(i) *$\ell^n | e_i n_i$, for each $i = 1, \cdots, r$, where $b_i = u_i \pi^{n_i}$, $u_i \in \mathrm{R}^*$, and*
(ii) *$b_i \in \mathrm{K}^*$, $i = 1, \ldots, r$, and $0 \le e_i < \ell^n$, so that for each $s = 1, \cdots, u$, if $k_s \le n$ is the largest integer for which $\ell^{k_s} | e_s$, then $\ell^{max\{0, ord_{\ell^n}(u_s) - k_s\}} | a_s$, and where $ord_{\ell^n}(u)$ denotes the order of the image of u in $\mathrm{K}^*/(\mathrm{K}^*)^{\ell^{\mathrm{n}}}$.*

Proof. Clearly $\mathrm{B} \cap {}_{\ell^{\mathrm{n}}}\mathrm{Br(X)} = \mathrm{kernel}(\mathrm{res_{B,C}} : \mathrm{B} \to \mathrm{C}) \cap \mathrm{kernel}(\mathrm{res_{B.D}} : \mathrm{B} \to \mathrm{D})$. Next observe that the condition in (4.16) (which is the condition (i)) implies the condition in (4.12). Since the condition (4.16) is necessary for Λ to be in the kernel of the residue map $res_{\mathrm{B,D}} : \mathrm{B} \to \mathrm{D}$, the discussion centered around (4.11) and (4.12) shows one may reduce to considering algebras of the form $\Lambda = \prod_{i=1}^{r}(u_i, t_i)_{\zeta}^{e_i}$, with $u_i \in \mathrm{R}^*$.

Moreover, we just showed that an algebra of the form $\Lambda = \Pi_{i=1}^{r}(u_i, t_i)_{\zeta}^{e_i}$ is unramified along the rays τ_j, $j = 1, \cdots, m$, if it is unramified along the rays ρ_i, $i = 1, \cdots, \nu$: clearly the latter condition is equivalent to $\Lambda = \Pi_{i=1}^{r}(u_i, t_i)_{\zeta}^{e_i}$ being in the kernel of the residue map $res_{\mathrm{B,C}} : \mathrm{B} \to \mathrm{C}$. Finally an algebra of the form $\Lambda = \Pi_{i=1}^{r}(u_i, t_i)_{\zeta}^{e_i}$ is in the kernel of the residue map $res_{\mathrm{B,C}} : \mathrm{B} \to \mathrm{C}$, if and only if it satisfies the condition that for each $0 \le e_i < \ell^n$, so that for each $s = 1, \cdots, u$, if $k_s < n$ is the largest integer for which $\ell^{k_s} | e_s$, then $\ell^{max\{0, ord_{\ell^n}(u_s) - k_s\}} | a_s$, and where $ord_{\ell^n}(u)$ denotes the order of the image of u in $\mathrm{K}^*/(\mathrm{K}^*)^{\ell^{\mathrm{n}}}$. Observe that for an element $c \in \mathrm{K}^*$, $ord_{\ell^n}(c)$ denotes the order of its in image in $\mathrm{K}^*/(\mathrm{K}^*)^{\ell^{\mathrm{n}}}$. This completes the proof of the proposition. □

Computation of kernel$(res|_{{}_{\ell^n}\mathrm{Br(Spec\,K)}})$. Finally, we need to determine the kernel of the map res restricted to ${}_{\ell^{\mathrm{n}}}\mathrm{Br(Spec\,K)} \subseteq {}_{\ell^{\mathrm{n}}}\mathrm{Br(T_K)}$. Again we break this into two steps: (i) *kernel*$(res : {}_{\ell^n}\mathrm{Br(Spec\,K)} \to \mathrm{C})$ and (ii) *kernel*$(res : {}_{\ell^n}\mathrm{Br(Spec\,K)} \to \mathrm{D})$.

Since the rays ρ_i, $i = 1, \cdots, \nu$ are all contained in $\mathbf{N}_{\mathbb{R}} \times \mathbf{0}$, these rays correspond to the rays in the fan of the toric variety $_{\mathrm{K}}$. Therefore, it is clear that the residue map $res : {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,K}) \rightarrow \mathrm{C}$ is trivial proving ${}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,K}) = \mathrm{kernel}(\mathrm{res} : {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,K}) \rightarrow \mathrm{C})$.

Next we determine the kernel of the residue map $res : \mathrm{Br}(\mathrm{Spec\,K}) \rightarrow \mathrm{D}$. For that, we begin with the commutative diagram:

$$\begin{CD} 0 @>>> {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,R}) @>>> {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,K}) @>{res}>> \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,k, \mu_{\ell^n}(0)) \\ @. @VVV @VVV @VVV \\ 0 @>>> {}_{\ell^n}\mathrm{Br}(\mathrm{X}) @>>> {}_{\ell^n}\mathrm{Br}(\mathrm{T_K}) @>{res}>> \mathrm{C} \oplus \mathrm{D} \end{CD} \tag{4.18}$$

The left-most vertical map is induced by the structure map $\mathrm{X} \rightarrow \mathrm{Spec\,R}$ and the next vertical map is induced by the structure map $\pi : \mathrm{T_K} \rightarrow \mathrm{Spec\,K}$. The exactness of the top row is shown in [CTS, Remark 3.6.5]. That the diagram commutes follows from Proposition 2.9(iii). One may now observe from [Qu, Proposition 2.1.3] that the cones of the fan not contained in $\mathbf{N}_{\mathbb{R}} \times 0$ correspond to the orbits of the torus T_k: since T_k is abelian, these correspond to (possibly) smaller dimensional tori over k. Hence it follows that the special fiber X_k has a k-rational point, so that the last vertical map in (4.18) is injective. Observe that this map, maps into the summand D. Therefore, the kernel of the composition of the the middle vertical map followed by the residue map $res : {}_{\ell^n}\mathrm{Br}(\mathrm{T_K}) \rightarrow \mathrm{D}$ equals the kernel of the top map $res : {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,K}) \rightarrow \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,k, \mu_{\ell^n}(0))$: clearly this is ${}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,R})$ by the exactness of the top row. These show that ${}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cap {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,K}) = {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,R})$.

Therefore, summarizing the above discussion, we obtain the following theorem:

Theorem 4.8. *Assume that* R *is a discrete valuation ring with field of fractions* K *and residue field k. Assume ℓ is a prime invertible in* R *and that* R *contains a primitive ℓ^n-th root of unity ζ. Let* X *denote a regular toric scheme over* Spec R *in the sense of* [KKMSD, Chapter 4, section 3] *and satisfying the assumption* (4.6)*. Let $a_1|a_2|\cdots|a_u$ denote the invariant factors of the given fan as in* (4.5)*. Then*

$${}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cong {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,R}) \oplus ({}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}) \oplus ({}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cap \mathrm{B})$$

where $({}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cap \mathrm{A})$ is the subgroup generated by $\{\Lambda = \Pi_{i<j}(t_i, t_j)_{\zeta}^{e_{i,j}} | \ell^n > e_{i,j} \geq 0\}$ satisfying the following conditions: for each $i = 1, \ldots, min\{u, r-1\}$, if k_i is the largest integer $\leq n$ so that $\ell^{k_i}|e_{1,i}, \ell^{k_i}|e_{2,i}, \cdots, \ell^{k_i}|e_{i-1,i}, \ell^{k_i}|e_{i,i+1}, \cdots, \ell^{k_i}|e_{i,r}$, then $\ell^{n-k_i}|a_i$, and

$\mathrm{Br}(\mathrm{X})_{\mathrm{m}} \cap \mathrm{B}$ *is the subgroup generated by $\{\Lambda = \Pi_{i=1}^{r}(b_i, t_i)_{\zeta}^{e_i} |\ \ell^n > e_i \geq 0\}$, satisfying the following conditions:*

(i) *$\ell^n | e_i n_i$, for each $i = 1, \cdots, r$, where $b_i = u_i \pi^{n_i}$, $u_i \in \mathrm{R}^*$, and*

(ii) *$b_i \in \mathrm{K}^*$, $i = 1, \ldots, r$, and $0 \leq e_i < \ell^n$, so that for each $s = 1, \cdots, u$, if $k_s \leq n$ is the largest integer for which $\ell^{k_s}|e_s$, then $\ell^{max\{0, ord_{\ell^n}(b_s) - k_s\}}|a_s$, and where $ord_{\ell^n}(b)$ denotes the order of the image of b in $\mathrm{K}^*/(\mathrm{K}^*)^{\ell^n}$.*

Corollary 4.9. *Assume the hypotheses of Theorem 4.1. In case there is a cone σ in the fan Δ with dimension$(\sigma) \geq r - 1$, then ${}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}$ is trivial.*

In case there is a cone σ in the fan Δ with dimension$(\sigma) \geq r$, then both ${}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cap \mathrm{B}$ and ${}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}$ are trivial, so that ${}_{\ell^n}\mathrm{Br}(\mathrm{X}) \cong {}_{\ell^n}\mathrm{Br}(\mathrm{Spec\,R})$.

Proof. The proof of the first statement is identical to the proof of the first statement in Corollary 4.2 or Corollary 3.3. In view of the condition (i) in Theorem 4.8 that $\ell^n | e_i n_i$, for all $i = 1, \cdots, r$, the arguments as in (4.11) and (4.12) show we reduce to considering cyclic algebras of the form $\Pi_{i=1}^r(u_i,t_i)_\zeta^{e_i}$, where each $u_i \in \mathrm{R}^*$. Since each $a_s = 1$, for all $s = 1, \cdots, u = r$, it follows from condition (ii) in Theorem 4.8 that $ord_{\ell^n}(u_s) \le k_s$, for each $s = 1, \cdots, r$. Since $\ell^{k_s}|e_s$, for each $s = 1, \cdots, r$, it follows that $\ell^{ord_m(u_s)}|e_s$, for each $s = 1, \cdots, r$. Therefore, it follows readily that each $(u_s,t_s)_\zeta^{e_s}$ is trivial. This proves the second statement. □

5. The period-index problem for toric varieties

This section is devoted to a discussion of the period-index problem for smooth toric varieties, beginning with tori. Our interest in this problem started with the paper of Hotchkiss (see [Hotch]) where he disproved the period-index conjecture for complex analytic tori. See [dJ04], [Lie08] and [AW] for basic background on this problem.

The starting point of the discussion in this section is the calculation of the Brauer group of a split torus as in Corollary 2.4. We will assume throughout the following discussion that the base field K has a primitive m-th root of unity, for a fixed positive integer m. (Clearly such an assumption is satisfied when the field is algebraically closed, but in the second half of this section, we will also consider base fields that are not necessarily algebraically closed.) We begin with the following preliminary results.

Proposition 5.1. *Let* T *denote a torus of dimension n over the algebraically closed field K. Let ζ denote a primitive m-root of unity in* K*, where m is prime to the characteristic of* K*. Let $\beta_i = \Pi_{j=i+1}^n(t_i,t_j)_\zeta^{a_{i,j}}$. Then the smallest degree of the extension field over* K *in which β_i splits is m/u_i, where $u_i = hcf\{a_{i,i+1}, \cdots, a_{i,n}, m\}$.*

Proof. Let $b_{i,j}$ denote positive integers chosen so that $a_{i,j} = u_i b_{i,j}$, for $j = i+1, \cdots, n$. Therefore,

$$a_{i,j}(m/u_i) = (u_i b_{i,j})(m/u_i) = b_{i,j}m,$$

so that

$$\beta_i^{m/u_i} = \Pi_{j=i+1}^n(t_i,t_j)_\zeta^{b_{i,j}m}.$$

This proves that the cyclic algebra β_i splits in a field extension of degree m/u_i.

Suppose β_i splits in a field extension of degree q. Then $m|a_{i,j}q$, $j = i+1, \cdots, n$. Let $m = m'u_i$, and $a_{ij} = u_i b'_{i,j} v'_i$ where v'_i is chosen as in the following lemma. Therefore, the condition $m|a_{i,j}q$, $j = i+1, \cdots, n$ implies $m'|b_{i,j}v'_i q$, $j = i+1, \cdots, n$. Now the following lemma shows that $hcf\{m', b'_{i,i+1}v'_i, \cdots, b'_{i,n}v'_i\} = 1$. Therefore, it follows that $m'|q$. This proves that $m' = m/u_i$ is the smallest degree of the extension field in which β_i splits. □

Lemma 5.2. *Let u_i be as in Proposition 5.1, $v_i = hcf\{a_{i,i+1}, \cdots, a_{i,n}\} = u_i v'_i$, $m' = \frac{m}{u_i}$ and $b'_{i,j} = \frac{a_{i,j}}{v_i}$, $j = i+1, \cdots, n$. Then the following hold:*

(i) $hcf\{b'_{i,i+1}, \cdots, b'_{i,n}, m'\} = 1$, *and,*
(ii) $hcf\{m', v'_i\} = 1$.

Proof. Let $w = hcf\{b'_{i,i+1}, \cdots, b'_{i,n}, m'\}$. Then $b'_{i,j} = c_{i,j}w$ and

$$a_{i,j} = b'_{i,j}.v_i = c_{i,j}wv_i, v_i = u_i v'_i, j = i+1, \cdots, n.$$

Therefore, $wu_i|a_{i,j}$, $j = i+1, \cdots, n$. Since $w|m'$, it follows $wu_i|m'u_i = m$. Thus $wu_i|u_i$, where u_i in fact is the $hcf\{a_{i,i+1}, \cdots, a_{i,n}, m\}$. It follows that $w = 1$, proving the first statement.

Let $w' = hcf\{m', v_i'\}$. Then $m' = w'm''$ and $v_i' = w'v_i''$. The last equality shows $w'|v_i'$ and hence $w'|v_i$. Recall $a_{i,j} = b'_{i,j}v_i = b'_{i,j}v_i'u_i$. Since $w'|v_i'$, it follows that $u_iw'|a_{i,j}$, for all $j = i+1, \cdots, n$. Since $m = m'u_i = w'm''u_i$, it follows that $u_iw'|m$ as well. Therefore, $u_iw'|u_i$, and hence $w' = 1$, thereby proving the second statement. □

The following is our main result on the period-index problem for tori over algebraically closed fields.

Theorem 5.3. *Let* K *denote an algebraically closed field and let* T *denote an* n*-dimensional torus over* K*.*

(i) *Assume* $char(\mathrm{K}) = 0$*. For each positive integer* m*, the* m*-torsion part of the Brauer group of* T*, namely* ${}_m\mathrm{Br}(\mathrm{T})$ *is the free* $\mathbb{Z}/m\mathbb{Z}$*-module with basis given by the* $\binom{n}{2}$ *cyclic algebras* $(t_i, t_j)_\zeta$*,* $1 \leq i < j \leq n$*, where* ζ *is an* m*-th root of unity. Therefore, if* $\alpha = \Pi_{i<j}(t_i, t_j)_\zeta^{a_{i,j}}$*,* $a_{i,j} \geq 0$*, and with* $u = hcf\{a_{i,j}, m | i, j\}$*, the order of* α *in* ${}_m\mathrm{Br}(\mathrm{T})$ *is* $m' = \frac{m}{u}$*.*

(ii) $Index(\alpha)$ *may be calculated as follows. For* $i = 1, \cdots, n-1,$*, let* $u_i = hcf\{a_{i,i+1}, \cdots a_{i,n}, m\}$*. Then the index of* $\Pi_{j=i+1}^n (t_i, t_j)_\zeta^{a_{ij}}$ *is given by* $\frac{m}{u_i}$*. Therefore, the index of* $\alpha = \Pi_{i=1}^{n-1}(\frac{m}{u_i}) = \frac{m^{n-1}}{u_1 \cdots u_{n-1}}$*. Consequently* $Index(\alpha) | Period(\alpha)^{n-1}$*.*

(iii) *Assume* $char(\mathrm{K}) = \mathrm{p} > 0$*. For each positive integer* m *relatively prime to* p*, the same conclusions as in (i) hold.*

Proof. We first prove that the period of α is $m' = m/u$. To see this first observe that $a_{i,j} = c_{i,j}u$ for some sequence of integers $c_{i,j}$. Therefore, $a_{i,j}m' = c_{i,j}um' = c_{i,j}u(m/u) = c_{ij}m$. Therefore, $\alpha^m = *$. Next we proceed to show that if p is an integer so that $\alpha^p = *$, then $m'|p$, which will complete the proof that the period of α is m'. Therefore, let p denote an integer so that $\alpha^p = *$. Then $m|a_{i,j}p$, for all i, j. Writing $a_{i,j} = c_{i,j}u$, this means

$$m = m'u | a_{i,j}p = c_{ij}up, \text{ for all } i < j.$$

Therefore, $m'|c_{i,j}p$, for all $i < j$. Once we show that $1 = hcf\{m', c_{i,j} | i < j\}$, it will follow that $m'|p$ proving the period of α is m'.

Therefore, let $hcf\{m', c_{i,j} | i < j\} = w$. Then $m' = wm''$, for some m'' and $c_{i,j} = d_{ij}w$. Therefore,

$$m = m'u = wm''u, \text{ and } c_{i,j} = d_{i,j}w$$

for some integers $d_{i,j}$. Therefore, $a_{i,j} = c_{i,j}u = d_{i,j}wu$. Thus $wu|a_{i,j}$ for all $i < j$ and $wu|m$. Therefore, $wu|u$, since $u = hcf\{m, a_{i,j} | i < j\}$. This readily implies $w = 1$. These complete the proof of (i) in view of Corollary 2.4.

We list the $\binom{n}{2}$-factors $(t_i, t_j)_\zeta^{a_{i,j}}$, $i < j$, in matrix form

$$\begin{pmatrix} (t_1, t_2)_\zeta^{a_{1,2}} & (t_1, t_3)_\zeta^{a_{2,3}} & \cdots & (t_1, t_n)_\zeta^{a_{1,n}} \\ 1 & (t_2, t_3)_\zeta^{a_{2,3}} & \cdots & (t_2, t_n)_\zeta^{a_{2,n}} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & & & (t_{n-1}, t_n)_\zeta^{a_{n-1,n}} \end{pmatrix}$$

Now we invoke Proposition 5.1 to the product of the cyclic algebras in each row. We observe then that product of the $(n-1)$-factors in the first row is split by adding an m/u_1-th root of t_1, the product of the $(n-2)$ factors in the second row is split by adding an m/u_2-th root of t_2, and so on, with just the one factor

$(t_{n-1}, t_n)_\zeta^{a_{n-1,n}}$ in the $n-1$-th row which is split by adding an m/u_{n-1}-th root of t_{n-1}, so that all these terms together account for the $\binom{n}{2}$-factors $(t_i, t_j)_\zeta^{a_{i,j}}$ in α. Therefore,

$$Index(\alpha) = \Pi_{i=1}^{n-1}(\frac{m}{u_i}) = \frac{m^{n-1}}{u_1 \cdots u_{n-1}}.$$

Since $u|u_i$, for all $i = 1, \cdots, n-1$, it follows that $u^{n-1}|\Pi_{i=1}^{n-1}u_i$ and therefore

$$Index(\alpha) = \frac{m^{n-1}}{u_1 \cdots u_{n-1}} | \frac{m^{n-1}}{u^{n-1}} = Period(\alpha)^{n-1},$$

thereby completing the proof of (ii). The proof of (iii) is similar and is therefore skipped. □

Corollary 5.4. *Let* X *denote a smooth toric variety of dimension* n *over an algebraically closed field* K*. Then for each integer* m *relatively prime to* $char(\mathrm{K})$*, and any class* $\alpha \in {}_m\mathrm{Br}(\mathrm{X})$*,*

$$Index(\alpha)|Period(\alpha)^{n-1}.$$

Proof. This is clear from Theorem 5.3 in view of corollary 2.4. □

Remark 5.5. One may observe that the recent result of Hotchkiss (see [Hotch]) shows that the period-index conjecture is false for complex analytic tori, by showing the existence of certain classes in the Brauer group of a complex analytic torus of rank r at least 3, for which the index does not divide period^{r-1}. Therefore, the results of Theorem 5.3 show that such counter-examples do not show up for algebraic tori.

Next we proceed to extend our results to toric varieties defined over fields of cohomological dimension 1, which includes fields that are quasi-algebraically closed and finite fields. Let K denote a field of cohomological dimension 1 and let T denote an n-dimensional split torus over K. We will assume K contains a primitive m-th root of unity. Corollary 2.4 shows that the m-torsion part of the Brauer group $\mathrm{Br(T)_m}$ has two main components: (i) the part coming from cyclic algebras of the form $(t_i, t_j)_\zeta^{a_{i,j}}$ as t_i, t_j run over characters of the given split torus T and (ii) the part coming from cyclic algebras of the form $(b, t_i)_\zeta^{a_i}$, as $b \in \mathrm{K}^*$ and t_i denotes a character of the torus T.

Theorem 5.6. (i) *Assume* $char(\mathrm{K}) = 0$*. Let* S *denote a finite subset of* K^* *so that the map*

$$coker(\Gamma(\mathrm{Spec\,K}, \mathbb{G}_\mathrm{m}) \xrightarrow{m} \Gamma(\mathrm{Spec\,K}, \mathbb{G}_\mathrm{m})) \xrightarrow{\delta} \mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K}, \mu_m)$$

sends the set S *injectively into the nontrivial elements in* $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K}, \mu_m)$*. If* $\alpha = \Pi_{b \epsilon \mathrm{S}}\Pi_{i=1}^n (b, t_i)_\zeta^{a_i}$*,* $a_i \geq 0$*, and with* $u = hcf\{a_i, m | i = 1, \cdots, n\}$*, the order of* α *in* ${}_m\mathrm{Br}(\mathrm{T})$ *is* $m' = \frac{m}{u}$*.*

(ii) *Let* S *denote a finite subset of* K^* *and let* α *denote a class as in (i). Assume that* $|\mathrm{S}| \geq \mathrm{n}$*. For* $i = 1, \cdots, n-1,$*, let* $u_i = hcf\{a_i, m\}$*. Then the index of* $\alpha_i = \Pi_{b \epsilon \mathrm{S}}(b, t_i)_\zeta^{a_i}$ *is given by* $\frac{m}{u_i}$*. Let* $\alpha = \Pi_{i=1}^n \alpha_i = \Pi_{i=1}^n \Pi_{b \epsilon \mathrm{S}}(b, t_i)_\zeta^{a_i}$*. Then* $Index(\alpha) = \Pi_{i=1}^n(\frac{m}{u_i}) = \frac{m^n}{u_1 \cdots u_n}$*. Consequently* $Index(\alpha)|Period(\alpha)^n$*.*

(iii) *Assume* $char(\mathrm{K}) = \mathrm{p} > 0$*, let* m *be prime to* $char(\mathrm{K})$ *and that* K *contains a primitive* m*-th root of unity. Then the same conclusions as in (i) and (ii) hold for classes in* ${}_m\mathrm{Br}(\mathrm{T})$*.*

Proof. We first observe that the period of the cyclic algebra $(b, t_i)_\zeta$ is m: this may be proven exactly as in [Mag, Proof of Lemma 5]. We will next prove that the period of α is m/u. This will be a variant of the proof of the first statement in Theorem 5.3. To see this first observe that $a_i = c_i u$ for some sequence of integers c_i. Therefore, $a_i m' = c_i u m' = c_i u(m/u) = c_i m$. Therefore, $\alpha^m = *$. Next we proceed to show that if p is an integer so that $\alpha^p = *$, then $m'|p$, which will complete the proof that the period of α is m'. Therefore, let p denote an integer so that $\alpha^p = *$. Then $m|a_i p$, for all i, since the period of the cyclic algebra $(b, t_i)_\zeta$ is m. Writing $a_i = c_i u$, this means

$$m = m'u|a_i p = c_i up, \text{ for all } i = 1, \cdots, n.$$

Therefore, $m'|c_i p$, for all $i = 1, \cdots, n$. Once we show that $1 = hcf\{m', c_i|i = 1, \cdots, n\}$, it will follow that $m'|p$ proving the period of α is m'.

Therefore, let $hcf\{m', c_i|i = 1, \cdots, n\} = w$. Then $m' = wm''$, for some m'' and $c_i = d_i w$. Therefore,

$$m = m'u = wm''u, \text{ and } c_i = d_i w$$

for some integers d_i. Therefore, $a_i = c_i u = d_i wu$. Thus $wu|a_i$ for all $i = 1, \cdots, n$ and $wu|m$. Therefore, $wu|u$, since $u = hcf\{m, a_i|i = 1, \cdots, n\}$. This readily implies $w = 1$. These complete the proof of (i) in view of Corollary 4.2.

It should be clear from the above discussion that the index of $\alpha_i = m/u_i$, for each $i = 1, \cdots, n$. Therefore, the index of α is $\Pi_{i=1}^n \frac{m}{u_i} = \frac{m^n}{u_1 \cdots u_n}$. Since $u|u_i$, for each $i = 1, \cdots, n$, it follows that $u^n|u_1 \cdots u_n$, proving that $Index(\alpha)|Perod(\alpha)^n = (\frac{m}{u})^n = \frac{m^n}{u^n}$. We skip the proof of the last statement which should be clear. □

Corollary 5.7. *Assume the base field* K *is the function field of a curve over an algebraically closed field. Let* X *denote a smooth split toric variety over* K *of dimension n. Then for each integer m relatively prime to $char$*(K)*, and any class $\alpha \in {}_m\mathrm{Br}(\mathrm{X})$,*

$$Index(\alpha)|Period(\alpha)^n.$$

Proof. First one reduces to the case of a split torus over K. Let C denote the given affine curve over the algebraically closed field k. Let $\mathrm{U = C - \{x_i|i = 1, \cdots, n\}}$ and $\mathrm{X = \{x_i|i = 1, \cdots, n\}}$. Then the localization sequence

$$0 \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{C}, \mu_m) \to \mathrm{H}^1_{\mathrm{et}}(\mathrm{U}, \mu_m) \to \mathrm{H}^2_{\mathrm{et,X}}(\mathrm{C}, \mu_m) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{C}, \mu_m) \cong 0,$$

and the isomorphism $\mathrm{H}^2_{\mathrm{et,X}}(\mathrm{C}, \mu_m) \cong \oplus_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{H}^0_{\mathrm{et}}(\mathrm{x_i}, \mu_m) \cong \oplus_{\mathrm{i=1}}^{\mathrm{n}} \mathbb{Z}/\mathrm{m}$ show that on taking $n \to \infty$, $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K(C)}, \mu_m)$ is of infinite rank as a free $\mathbb{Z}/m$-module. Now the statement follows from corollary 2.4, Theorem 5.6 and the observation that any algebraically closed field of characteristic p contains primitive m-th roots of unity for all m prime to p. □

Corollary 5.8. *Assume the base field is the finite field $\mathbb{F}_q$ of characteristic p and let* X *denote a split smooth toric variety over $\mathbb{F}_q$ of dimension n. Let $m|q-1$. If $\alpha \in {}_m\mathrm{Br}(\mathrm{X})$,*

$$Index(\alpha)|Period(\alpha)^{n-1}.$$

Proof. Here the key observation is that $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{F}_\mathrm{q}, \mu_m) = \mathbb{Z}/\mathrm{m}$. Therefore, the period and index of the class $\alpha = \Pi_{i=1}^n (b, t_i)_\zeta$ is m, where $b \epsilon \mathbb{F}_q^*$ corresponds to the generator of $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{F}_\mathrm{q}, \mu_m)$. The conclusion now follows from Theorems 5.3 and 5.6 once one observes that $\mathbb{F}_q$ contains a primitive m-th root of unity. □

6. Examples

We will assume throughout this section that $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K}, \mu_m(0))$ is *not* trivial.

Example 6.1. The simplest example of a smooth split toric variety X for which ${}_m\mathrm{Br}(\mathrm{X})$ does not reduce to ${}_m\mathrm{Br}(\mathrm{Spec\,K})$ is $\mathrm{X} = \mathbb{P}^1 \times \mathbb{G}_{\mathrm{m}}$. Though this example may be too simple when K is algebraically closed, we begin with this, since the details seem worth considering in the context of Theorem 3.2. In this case the fan is 1-dimensional and it is the same as the fan for $\mathbb{P}^1$, namely the the x-axis in the plane, with the two 1-dimensional cones given by the positive x-axis and the negative x-axis. The open torus $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\times 2}$.

Since the 2 cones are of dimension 1, the first part of the proof of Theorem 3.2 shows that the terms $\mathrm{H}^0_{\mathrm{et}}(\mathrm{Spec\,K}, \mu_m(-1))$ do not contribute to ${}_m\mathrm{Br}(\mathrm{X})$.

There are however, three distinct orbits. The two 1-dimensional orbits $1 \times \mathbb{G}_m$ and $\infty \times \mathbb{G}_m$, where 1 and ∞ are the two 0-dimensional orbits for the 1-dimensional torus $\mathbb{G}_m$ in $\mathbb{P}^1$. In addition, one has the open orbit $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\times 2}$. Let t_1, t_2 denote the coordinates on T so that t_1 defines the 1-parameter subgroup corresponding to the open torus in $\mathbb{P}^1$. Both the 1-dimensional cones correspond to the orbit which is given by putting $t_2 = 1$. As a result though there are 2 summands $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K}, \mu_m(0))$ in ${}_m\mathrm{Br}(\mathrm{X})$, only the summand corresponding to the cyclic algebras $(b, t_1)_\zeta$ cancel off on taking the residue map. As a result, since $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K}, \mu_m(0))$ is assumed to be not trivial, ${}_m\mathrm{Br}(\mathrm{X})$ will be spanned by the cyclic algebras of the form $(b, t_2)_\zeta$, as b varies over the units in the based field K, in addition to ${}_m\mathrm{Br}(\mathrm{Spec\,K})$ which will occur as a split summand.

Example 6.2. One can generate higher dimensional examples by taking $\mathrm{X} = \mathbb{P}^{r_1} \times \mathbb{G}_{\mathrm{m}}^{r_2}$, for $r_1, r_2 \geq 1$, or by taking $\mathrm{X} = \mathrm{Y} \times \mathbb{G}_{\mathrm{m}}^{r_2}$, where Y is any smooth split toric variety of dimension r_1 over Spec K and $r_2 \geq 1$. Assume further that there is a cone of dimension r_1 in the fan for Y. In this case the fan for the toric variety X is of dimension r_1, while the lattices **M** and **N** associated to the toric variety X are both of rank $r_1 + r_2$. Therefore, the invariant factors $a_1, a_2, \cdots, a_{r_1}$ will all be 1. It follows that:

(i) ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A} =$ the subgroup generated by $\{\Lambda = \Pi_{i<j}(t_i, t_j)_\zeta^{e_{i,j}} \mid |i = r_1 + 1, \cdots r_1 + r_2\}$, and
(ii) ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B} =$ is generated by $\{\Lambda(b) = \Pi_{i=1}^{r}(b_i, t_i)_\zeta^{e_i} \mid |i = r_1 + 1, \cdots, r_1 + r_2\}$ as $b_i \in k^*$ varies among the corresponding classes in $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec\,K}, \mu_m(0))$.
(iii) Therefore, ${}_m\mathrm{Br}(\mathrm{X}) = {}_m\mathrm{Br}(\mathrm{Spec\,K}) \oplus ({}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}) \oplus ({}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{B})$. In particular, ${}_m\mathrm{Br}(\mathrm{X}) \neq {}_m\mathrm{Br}(\mathrm{Spec\,K})$.

Remark 6.3. *One may observe that in Example 6.1, $r_1 = 1$, and $r_2 = 1$, so that there are no cyclic algebras of the form $(t_i, t_j)_\zeta$, with $i = 2$, $2 < j \leq 2$. Therefore, ${}_m\mathrm{Br}(\mathrm{X}) \cap \mathrm{A}$ is trivial in this case.*

Example 6.4. As rather simple examples of toric schemes over a Dedekind domain considered in Theorem 4.1, one may consider the affine space $\mathbb{A}^n_{\mathrm{R}}$, the projective space $\mathbb{P}^n_{\mathrm{R}}$ and $\mathbb{A}^{n_1} \times \mathbb{G}_m^{n_2}$, all defined over the Dedekind domain R. Clearly these satisfy the simplifying assumptions in (4.1).

Example 6.5. Here one may take the ring of p-adic integers for a fixed prime p. This is a complete DVR with residue field $\mathbb{Z}/p\mathbb{Z}$. As another example, one may take the formal power series ring $\mathrm{R} = \mathbb{Z}/\mathrm{p}\mathbb{Z}[[\mathrm{x}]]$ for a fixed prime p. This is also a DVR and has the special property that the residue field $k = \mathbb{Z}/p\mathbb{Z}$ is a subring of R. Therefore, the discussion in [Qu, Remark 2.1.10] applies in this case. This means that if $\tilde{\Delta}$ is a fan in $\tilde{\mathbf{N}}$, the map of fans $\tilde{\Delta} \to \mathbb{Q}_{\geq 0}$ induces a map

$$\mathrm{X}(\tilde{\Delta})_k \to \mathbb{A}^1_k$$

of toric varieties over the field $k = \mathbb{Z}/p\mathbb{Z}$, defined by the homomorphism of k-algebras $k[x] \to \mathrm{R}, \mathrm{x} \mapsto \pi$. In fact in this case if X_{R} denotes the corresponding toric scheme over R,

$$\mathrm{X}_{\mathrm{R}} \cong \mathrm{X}(\tilde{\Delta})_k \times_{\mathbb{A}^1_k} (\mathrm{Spec\,R}).$$

7. Appendix: Motivic cohomology over regular Noetherian base schemes

First one may observe that the higher cycle complex may be defined over any base scheme B: if X is a scheme of finite type over B, and $c \geq 0$ is a fixed integer, one defines $Z^c(\mathrm{X}, .)$ to be the chain complex defined in degree n, by

$$\{\mathrm{Z} = \text{ a pure codim c cycle on } \mathrm{X} \times_{\mathrm{B}} \Delta_{\mathrm{B}}[\mathrm{n}] \,|\, \mathrm{Z} \text{ intersects the faces of } \mathrm{X} \times \Delta_{\mathrm{B}}[\mathrm{n}] \text{ properly}\}. \tag{7.1}$$

Definition 7.1. *We let $\mathbb{Z}(c)$ denote the co-chain complex $Z^c(\mathrm{X}, .)[-2\mathrm{c}]$ in cohomological degree m, that is, $\mathbb{Z}(c) = Z^c(\mathrm{X}, 2\mathrm{c} - \mathrm{m})$. (Observe that $\mathbb{Z}(c)$ is contravariantly functorial for flat maps.) If X denotes a smooth scheme of finite type over B, we will let $\mathbb{Z}^{\mathrm{X}}(c)$ denote the restriction of the complex $\mathbb{Z}(c)$ to the small Zariski or Nisnevich site of X. We call the complex $\mathbb{Z}(c)$ the motivic complex of weight c. If $c < 0$ is an integer, we define $\mathbb{Z}(c)$ to be $\{0\}$.*

Next we will assume that $\mathrm{B} = \mathrm{Spec}\,\mathrm{R}$, where R is a Dedekind domain.

Proposition 7.2. *Assume in addition that X denotes a smooth scheme of finite type over B. Then $\mathbb{Z}^{\mathrm{X}}(1)[1]$ identifies with $\mathbb{G}_m^{\mathrm{X}}$, which denotes the restriction of the sheaf $\mathbb{G}_m$ to the small Nisnevich site of X.*

Proof. This is discussed in [Bl, section 6], where the discussion does not assume the base scheme is a field. □

Definition 7.3. *(Motivic cohomology) Let X denote a scheme of finite type over B. We let $\mathrm{H}^{\mathrm{i,j}}_{\mathrm{M}}(\mathrm{X}) = \mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathbb{Z}(\mathrm{j}))$, where $\mathrm{H}_{\mathrm{Zar}}$ denotes the hypercohomology on the Zariski site.*

It is observed in [Geis, Corollary 3.3], that one obtains the identification $\mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathbb{Z}(\mathrm{j})) \cong \mathrm{H}_{\mathrm{Zar}}(\mathrm{B}, \pi_*(\mathbb{Z}(\mathrm{j})))$, where $\pi : \mathrm{X} \to \mathrm{B}$ denotes the structure map.

Let X denote a scheme of finite type over B and let Z denote a closed subscheme of X of pure codimension c with open complement U. Let $i : \mathrm{Z} \to \mathrm{X}$ and $j : \mathrm{U} \to \mathrm{X}$ denote the corresponding immersions. Then it is shown in [Geis, Corollary 3.3] that one obtains the distinguished triangle

$$0 \to i_*\mathbb{Z}^{\mathrm{Z}}(n-c)[-2c] \to \mathbb{Z}^{\mathrm{X}}(n) \to j_*\mathbb{Z}^{\mathrm{U}}(n) \tag{7.2}$$

in the derived category of Zariski sheaves on X. In particular, this provides the identification of the terms in the long-exact sequence forming the top row in the diagram:

$$\begin{tikzcd}
\mathrm{H}^{\mathrm{i}}_{\mathrm{Z}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^{\mathrm{i}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^{\mathrm{i}}(\mathrm{U}, \mathbb{Z}(\mathrm{n})) \arrow[d, "\cong"] \\
\mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathrm{i}_*\mathbb{Z}^{\mathrm{Z}}(\mathrm{n}-\mathrm{c})[-2\mathrm{c}]) \arrow[r] & \mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathbb{Z}^{\mathrm{X}}(\mathrm{n})) \arrow[r] & \mathrm{H}^{\mathrm{i}}_{\mathrm{Zar}}(\mathrm{X}, \mathrm{j}_*\mathbb{Z}^{\mathrm{X}}(\mathrm{n}))
\end{tikzcd} \tag{7.3}$$

Proposition 7.4. *Assume the above situation. If $n < c = codim_{\mathrm{X}}(\mathrm{Z})$, where $codim_{\mathrm{X}}(\mathrm{Z})$ denotes the codimension of Z in X, then $\mathrm{H}^{\mathrm{i}}_{\mathrm{Z}}(\mathrm{X}, \mathbb{Z}(\mathrm{n})) = 0$. A corresponding result also holds when $\mathbb{Z}(n)$ is replaced by $\mathbb{Z}/m(n)$, when m is invertible in R.*

Proof. The first statement is clear from the identification of the first terms in the commutative diagram (7.3), since when $n < c$, the complex $\mathbb{Z}^{\mathrm{Z}}(n-c)$ is trivial. This proves the first statement.

To obtain the second statement, one first tensors the localization sequence in (7.2) with $\mathbb{Z}/m$ (i.e., the ring of integers modulo m) to obtain the corresponding localization sequence involving the motivic complexes

$\mathbb{Z}/m(n)$. This then provides the commutative diagram corresponding to the one in (7.3) where the integral motivic complex $\mathbb{Z}(n)$ is replaced by $\mathbb{Z}/m(n)$. $\square$

8. COI

The authors have no conflict of interest to declare that are relevant to this article.

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