Equivariant Intersection Cohomology

Recall the definition of the intersection cohomology complex

Throughout the talk we will restrict to algebraic varieties defined over a field, which we may assume (for the sake of simplicity) is $\mathbb{C}$; much of our work will also hold in positive characteristic.

X : a possibly singular variety provided with a filtration:

$$U_1 \overset{j_1}{\longrightarrow} U_2....U_n \overset{j_n}{\longrightarrow} U_{n+1} = X,\ dim_k X = n$$

so that each j_i is an open immersion and each $U_{i+1} - U_i$ is smooth and of codimension $= i$ in X.

$\mathcal{L}$: a local system on U_1

p:(non-negative even integers) $\to$ (integers) is a non-decreasing function so that

$p(2k+2) - p(2k) = 0$, 1 or 2, $p(2) = -n+1$.

Now $IC_p(\mathcal{L})$ is characterized in the derived category by:

i). $IC_p(\mathcal{L})_{|U_0} \simeq \mathcal{L}[n]$

ii). $\mathcal{H}^i(IC_p(\mathcal{L}))_{|U_{k+1}-U_k} \cong 0$ if $i \geq p(2k)$

iii). $\mathcal{H}^i_{U_{k+1}-U_k}(IC_p(\mathcal{L})) \cong 0$ if $i \leq p(2k)$

iv). $\mathcal{H}^i(IC_p(\mathcal{L})) = 0$ if $i > n$ or if $i < -n$

Now Deligne's construction:

$$IC_p(\mathcal{L}) = \sigma_{<p(2n)} Rj_{n*} \ldots \sigma_{<p(2)} Rj_{1*}(\mathcal{L}[n])$$

$$IH^*_p(X;\, \mathcal{L}) = \mathbb{H}^*(X;\, IC_p(\mathcal{L}))$$

Incorporate a group action

Let G act on X. (Now $\mu, pr_2 : G \times X \to X$, $\sigma : X \to G \times X$.) Now we obtain the simplicial space $EG\underset{G}{\times}X$ given by: $(EG\underset{G}{\times}X)_n = G^n \times X$,

$$d_i(g_0, ..., g_{i-1}, g_i, g_{i+1}, ..., g_{n-1}, x) = (g_1, ..., g_{n-1}, x) \text{ if } i = 0$$

$$= (g_0, ..., g_{i-1}.g_i, g_{i+1}, ..., g_{n-1}, x) \text{ if } 0 < i < n$$

$$= (g_0, ..., g_{n-2}, g_{n-1}.x) \text{ if } i = n$$

$s_i(g_0, ..., g_{i-1}, g_i, ..., g_{n-2}, x) = (g_0, ..., g_{i-1}, e, g_i, ..., g_{n-2}, x)$, $0 \le i \le n-1$

$Top(EG\underset{G}{\times}X)$ = the Grothendieck topology with

objects: U_n in $Top((EG\underset{G}{\times}X)_n)$

morphisms: given U_n in $Top((EG\underset{G}{\times}X)_n)$ and U_m in $Top((EG\underset{G}{\times}X)_m)$ a map $U_n \to U_m$ is a map lying over some structure map $(EG\underset{G}{\times}X)_n \to (EG\underset{G}{\times}X)_m$.

A *sheaf* F on $Top(EG\underset{G}{\times}X)$ is a collection of sheaves $\{F_n|n\}$ so that each F_n is a sheaf on $Top((EG\underset{G}{\times}X)_n)$ and provided with maps $\phi_\alpha : \alpha^*(F_m) \to F_n$ for each structure map α. (These are required to satisfy a compatibility condition.)

Such a sheaf is G-*equivariant* if each of the maps ϕ_α is an isomorphism. The category of G-equivariant sheaves is an abelian sub-category closed under extensions: therefore one defines $D_b^G(X;\, \mathbb{Q})$ (= the equivariant derived category) to be the full sub-category of $D_b(EG\underset{G}{\times}X;\, \mathbb{Q})$ consisting of complexes whose cohomology sheaves are G-equivariant.

Note: if $f : X \to Y$ is a G-equivariant map, the induced map $EG\underset{G}{\times}X \to EG\underset{G}{\times}Y$ is denoted f^G. This is given by $(f^G)_n = f \times id^n$. Now one may define the derived functors Rf^G_*: observe that $Rf^G_* = \{R(f^G)_{n*}|n\}$.

Equivariant intersection cohomolgy:

$IC^G_p(\mathcal{L}) = \sigma_{<p(2n)} Rj^G_{n*} ... \sigma_{<p(2)} Rj^G_{1*}(\mathcal{L}[n])$, $\mathcal{L}$ - a G-equivariant local system on $EG \underset{G}{\times} U_1$

$IH^*_{G,p}(X;\, \mathcal{L}) = \mathbb{H}^*(EG \underset{G}{\times} X;\, IC^G_p(\mathcal{L}))$

Main results on equivariant intersecton cohomology

Proposition 1. $IH^*_{G,p}(X;\, \mathcal{L})$ is a module over $H^*(BG;\, \mathbb{Q})$

Theorem 1. There exists a Leray-spectral sequence:

$$E^{s,t}_2 = H^s(BG;\, R^t\pi_*(IC^G_p(\mathcal{L}))) \Rightarrow IH^{s+t}_{G,p}(X;\, \mathcal{L})$$

where $\pi : EG \underset{G}{\times} X \to BG$ is the obvious map. Moreover if $\bar{x} \epsilon (BG)_n$,

$$(R^t \pi_* IC_p^G(\mathcal{L})) \cong IH_p^t(X; \mathcal{L})$$

Theorem 2. The above spectral sequence with $\mathcal{L} = \underline{\mathbb{Q}}$ degenerates in the following cases and provides the isomorphism:

$$IH^*_{G,p}(X; \mathbb{Q}) \cong H^*(BG; \mathbb{Q}) \otimes IH^*_p(X; \mathbb{Q})$$

(a) G acts trivially on X and p is arbitrary

(b) G is connected, $p = m$ and X is projective

Next we consider torus actions. Let $G = T$, $i : X^T \to X$. $S = H^*(BT; \mathbb{Q}) - 0$.

Theorem 3. $S^{-1} IH^*_{T,p}(X\colon\ \mathbb{Q}) \cong S^{-1} H^*(BT; \mathbb{Q}) \otimes \mathbb{H}^*(X^T; Ri^! IC_p^T(\mathbb{Q}))$

Next we consider the action of a complex reductive group G.

Theorem 4. Assume that G acts on X with finite stabilizers and so that a geometric quotient X/G exits as a scheme. Then

$$IH^*_{G,p}(X;\mathbb{Q}) \cong IH^*_p(X/G;\mathbb{Q})$$

Next we consider equivariant derived categories in more detail. Let $D^G_{b,c}(X;\ \mathbb{Q})$ denote the full subcategory of $D^G_b(X;\mathbb{Q})$ with constructible cohomology sheaves.

Theorem 5. For each fixed perversity p, there exists a non-standard t-structure on $D^G_{b,c}(X;\mathbb{Q})$ whose *heart* will be called the category of G-*equivariant perverse sheaves* on X and denoted $C^G_p(X)$.

The simple objects in the above abelian category $C_p^G(X)$ are given by the complexes $IC_p^G(\mathcal{L}_C)[d_C]$, where C is a G-stable locally closed smooth sub-variety of X of dimension d_C and $\mathcal{L}$ is an irreducible G-equivariant local system on $EG\underset{G}{\times}C$.

Theorem 6. The category $C_m^G(X)$ is equivalent to the subcategory of $C_m(X)$ consisting of perverse sheaves F on X provided with the following data:

there exists an isomorphism $\phi : \mu^*(F) \to pr_2^*(F)$ (of perverse sheaves on $G \times X$) so that $\sigma^*(\phi) = id$ and there exists a cocycle condition between the pull-backs $d_0^*(\phi)$, $d_1^*(\phi)$ and $d_2^*(\phi)$ as perverse sheaves on $G \times G \times X$. Morphisms are maps of perverse sheaves on X that preserve the above structure.

Connections with the theory of $\mathcal{D}$-modules

Assume X is a *smooth* algebraic variety and X_{an} = the associated analytic space. An algebraic (left) $\mathcal{D}_X$-module M is strongly G-equivariant if there exists an isomorphism $\phi : \mu^*(M) \to pr_2^*(M)$ (as $\mathcal{D}_{G\times X} \cong \mathcal{D}_G \boxtimes \mathcal{D}_X$-modules) so that $\sigma^*(\phi) = id$ and there exists a co-cycle condition between the pull-backs $d_0^*(\phi)$, $d_1^*(\phi)$ and $d_2^*(\phi)$ as $\mathcal{D}_{G\times G\times X}$-modules.

Now the De-Rham functor DR induces an equivalence of categories:

$$Mod_{r.h}^{s-G}(\mathcal{D}_X) \xrightarrow{\simeq} C_m^G(X_{an})$$

where the left hand side = the category of strongly G-equivariant regular holonomic $\mathcal{D}_X$-modules. Under this correspondence the equivariant intersection cohomology complexes associated to G-stable locally closed smooth sub-varieties C of X and G-equivariant irreducible local systems on $EG \underset{G}{\times} C$ correspond to simple objects in the category on the left.

Sample of some of the applications

1. *Vanishing of odd dimensional intersection cohomology*

Theorem 7. Assume X is a projective variety provided with the action of an algebraic torus T. Assume further that X is provided with a T-stable stratification $\{S\}$ so that

a) each S is connected

b) each $\mathcal{H}^i(IC_m(\mathbb{Q}))$ is locally constant on each stratum S

c) each stratum S has at least one fixed point for the T-action and X^T is discrete

Now:

(i) $IH_m^i(X;\ \mathbb{Q}) = 0$ for all *odd* i if and only if $\mathcal{H}^i(IC_m(\mathbb{Q})) = 0$ for all *odd* i

(ii) Moreover $IH_m^i(X;\ \mathbb{Q}) = 0$ for all odd i, if there exists a T-equivariant resolution of singularities $\tilde{X} \to X$ so that $(\tilde{X})^T$ is also discrete.

Corollary. The conclusions of the above theorem hold for Schubert varieties and the varieties $\bar{O}(w)$ (= the closure of the orbit $O(w)$ for the diagonal action of a reductive group G on the flag-manifold $G/B \times G/B$

2. *Applications to geometric invariant theory.*

Let X be reductive acting on the projective variety X. X^{ss} (X^s)= the set of semi-stable (stable, respectively) points with respect to a G-linearized ample line bundle.

Theorem (Kirwan)

(i) There exists a G-stable stratification of X indexed by a partially ordered set $\mathcal{B}$ where the open stratum is X^{ss}. The closure of a stratum S_β is contained in the union of $\{S_\gamma | \gamma \geq \beta\}$. One can write $\mathcal{B} = \{\beta_0, \beta_1, ..., \beta_s\}$ so that, for $0 \leq j \leq s$, $U_j = S_{\beta_0} \cup ... \cup S_{\beta_j}$ is open in X and S_{β_j} is closed in U_j.

(ii) If X is also smooth, one obtains:

$0 \to H^n_G(U_j, U_{j-1}; \mathbb{Q}) \to H^n_G(U_j; \mathbb{Q}) \to H^n(U_{j-1}; \mathbb{Q}) \to 0$

(iii) In general:

$0 \to IH^n_{G,m}(U_j, U_{j-1}; \mathbb{Q}) \to IH^n_{G,m}(U_j; \mathbb{Q}) \to IH^n_{G,m}(U_{j-1}; \mathbb{Q}) \to 0$

(iv) Moreover $H^*_G(X;\mathbb{Q})(in(ii)) \cong H^*(BG;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ and $IH^*_{G,m}(X;\mathbb{Q})(in(iii)) \cong H^*(BG;\mathbb{Q}) \otimes IH^*_m(X;\ \mathbb{Q})$. If $X^{ss} = X^s$ also, (see Theorem 4), $IH^*_{G,m}(X^{ss};\mathbb{Q}) \cong IH^*_m(X//G;\mathbb{Q})$.

Therefore one may compute the Poincaré series for $IH^*(X//G;\mathbb{Q})$ using the Poincaré-series in equivariant intersection cohomology of the various pairs (U_j, U_{j-1}). (The Poincaré series in equivariant intersection cohomology of a pair (U_j, U_{j-1}) with a G-action is:

$$IP^G_t(U_j, U_{j-1}) = \Sigma_i dim IH^i_{G,m}(U_j, U_{j-1};\mathbb{Q})t^i)$$

Let $T =$ a maximal torus in G. Now the Hilbert-Mumford criterion for semi-stability implies:

$$X^{ss} = \bigcap_{g\ \epsilon\ G} gX^{ss}_T$$

$X^{ss}_T =$ the semi-stable points for the T-action.

Theorem(B-J)

$$IH^*_{G,m}(X^{ss};\mathbb{Q}) \cong (IH^*_{T,m}(X^{ss}_T;\mathbb{Q}))^a$$

where a denotes the anti-invariant part = the part corresponding to the sign-representation of W. To see the W-action on $IH^*_{T,m}(X^{ss}_T;\mathbb{Q})$ observe:

$$H^*(BT;\mathbb{Q}) \cong H^*(G/T;\mathbb{Q}) \otimes H^*(BG;\mathbb{Q}).$$

Now $IH^*_{T,m}(X^{ss}_T;\mathbb{Q})$ is a module over $H^*(BT;\mathbb{Q})$.

In particular if $X^{ss} = X^s$, then:

$$IH^*_m(X//G;\mathbb{Q}) \cong (IH^*_{T,m}(X^{ss}_T;\mathbb{Q}))^a$$

Home-page: http://www.math.ohio-state.edu/~ joshua