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The present paper is a continuation of earlier work by Gunnar Carlsson and the first author on a motivic variant of the classical Becker–Gottlieb transfer and an additivity theorem for such a transfer by the present authors. Here, we establish a motivic variant of the classical Segal–Becker theorem relating the classifying space of a 1-dimensional torus with the spectrum defining algebraic K-theory.

1. Introduction

A classical result from [Segal 1973] shows that the classifying space of the infinite unitary group, namely BU, is a split summand of $\lim_{m\to\infty}\Omega^m_{S^1}((S^1)^m\wedge\mathbb{CP}^\infty)$. A year later, [Becker 1974] contained a similar result for the infinite orthogonal group in place of the infinite unitary group U and BO(2) in the place of $\mathbb{CP}^\infty$.

The purpose of this paper is to consider similar problems in the motivic world and for algebraic K-theory, making use of a theory of the motivic Becker–Gottlieb transfer worked out in [Carlsson and Joshua 2020] and the additivity theorem for such a transfer worked out in [Joshua and Pelaez 2020]. We adopt the terminology and conventions from [Carlsson and Joshua 2020] as well as other terminology that has now become standard. As such, the base scheme is a perfect field k and we restrict to the category of smooth schemes of finite type over k. This category is denoted $\mathrm{Sm}(k)$ and is provided with the *Nisnevich topology*. $\mathrm{PSh}_*(k)$ is the category of pointed simplicial presheaves on this site. This category is made motivic by inverting the affine line $\mathbb{A}^1$ as in [Morel and Voevodsky 1999]; the pointed simplicial presheaves in this category are referred to as motivic spaces. We let $\boldsymbol{T}=\mathbb{P}^1$, pointed at ∞. We denote $\boldsymbol{T}^{\wedge n}$ throughout by $\boldsymbol{T}^n$. Then a motivic spectrum E denotes a sequence $\{E_n \mid n\geq 0\}$ of motivic spaces provided with structure maps $\boldsymbol{T}\wedge E_n\to E_{n+1}$. The category of motivic spectra is denoted $\mathbf{Spt}(k)$, or just $\mathbf{Spt}$ if the choice of k is clear. Then a motivic spectrum E is called an $\Omega_{\boldsymbol{T}}$-spectrum if it is levelwise fibrant and the adjoint to the structure maps given by $\{E_n\to\Omega_{\boldsymbol{T}}(E_{n+1})\mid n\}$ are all motivic weak-equivalences.

Then the first observation (see [Voevodsky 1998, §6.2]) is that algebraic K-theory is represented by the motivic spectrum with $\mathbb{Z}\times\mathrm{BGL}_\infty$ as the motivic space in

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each degree, with the structure map given by the Bott periodicity: $\mathbb{Z} \times \mathrm{BGL}_\infty \simeq \Omega_{\boldsymbol{T}}(\mathbb{Z} \times \mathrm{BGL}_\infty)$. We denote this motivic spectrum by $\boldsymbol{K}$. Therefore,

$$K^0(X) \simeq [\Sigma^\infty_{\boldsymbol{T}}(X_+), \boldsymbol{K}] \cong [X, \mathbb{Z} \times \mathrm{BGL}_\infty],$$

where the first and second [,] denote the hom in the stable motivic homotopy category and the corresponding unstable pointed motivic homotopy category, respectively.

We observe in Proposition 2.1 that there is an $\Omega_{\boldsymbol{T}}$-motivic spectrum whose 0-th term is given by the motivic space BGL_∞. Assuming this, the first main result of this paper is the following theorem, which we call the *motivic Segal–Becker theorem* in view of the fact that such a result was proven for topological complex K-theory, making use of complex unitary groups, in [Segal 1973] and for real K-theory, making use of orthogonal groups, in [Becker 1974]. (In fact, Becker's proof, making use of the transfer, also applies to topological complex K-theory.) For a motivic space P, we let $Q(P) = \lim_{n\to\infty} \Omega^n_{\boldsymbol{T}} \boldsymbol{T}^{\wedge n}(P)$. Of key importance for us is the map

$$\lambda : Q(\mathrm{B}\mathbb{G}_m) \to Q(\varinjlim_n \mathrm{BGL}_n) = Q(\mathrm{BGL}_\infty) \xrightarrow{q} \mathrm{BGL}_\infty, \tag{1.1}$$

where the map q is the obvious one induced by the fact that BGL_∞ is the 0-th space of an $\Omega_{\boldsymbol{T}}$-spectrum (see Proposition 2.1). The map $Q(\mathrm{B}\mathbb{G}_m) \to Q(\varinjlim_n \mathrm{BGL}_n) = Q(\mathrm{BGL}_\infty)$ is induced by the inclusion, $\mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{GL}_\infty$, where the first map is the diagonal imbedding.

Theorem 1.2 (the motivic Segal–Becker theorem for algebraic K-theory). (i) *Assume that the base scheme is a field k of characteristic* 0. *Then the map in* (1.1) *induces a surjection for every pointed motivic space X that is a compact object in the unstable pointed motivic homotopy category*:

$$[X, Q(\mathrm{B}\mathbb{G}_m)] \to [X, \mathrm{BGL}_\infty].$$

(*Recall that a motivic space X is a compact object in the unstable pointed motivic homotopy category if* $\mathrm{Map}(X, \ )$ *commutes with all small colimits in the second argument*, *where* $\mathrm{Map}(\ ,\ )$ *denotes the simplicial mapping space.*)

(ii) *Assume that the base scheme is a perfect field k of positive characteristic $p > 0$.*[1] *Then*, *after inverting p*, *the map in* (1.1) *induces a surjection for every pointed motivic space X that is a compact object in the corresponding unstable pointed motivic homotopy category*:

$$[X, Q(\mathrm{B}\mathbb{G}_m)] \to [X, \mathrm{BGL}_\infty].$$

[1] The assumption that k be perfect may be dropped in view of recent results such as in [Elmanto and Khan 2020] and [Bachmann and Hoyois 2021, Theorem 10.12].

Remarks 1.3. (1) Localizing at the prime p in the unstable pointed motivic homotopy category, as used in statement (ii) and elsewhere in this paper, is discussed in detail in [Asok et al. 2020]. One may also observe that, though [,] as used in statement (ii) denotes Hom in the unstable pointed motivic homotopy theory, since the target space is an infinite $\boldsymbol{T}$-loop space, the above Hom identifies readily with a Hom in the motivic stable homotopy category, after making use of the adjunction between taking $\Omega_{\boldsymbol{T}}$-loops and $\boldsymbol{T}$-suspension.

(2) One should view the above results as a rather weak-form of the Segal–Becker theorem, in the sense that we are able to prove only the surjectivity and (not split surjectivity) of the above maps, and also only for objects X that are compact objects in the corresponding unstable pointed motivic homotopy category. We hope to consider questions on split surjectivity in a sequel to this paper, as it seems to involve considerable additional work and certain techniques used in establishing such splittings classically do not seem to extend readily to the motivic framework.

(3) It is possible there is an analogue of the above theorem for Hermitian K-theory [Hornbostel 2005; Schlichting 2017] which is represented by the classifying space of the infinite orthogonal groups. In fact, much of the proof for the case of algebraic K-theory seems to carry over to the Hermitian case, the main difficulty being to prove an analogue of Theorem 3.20. We hope to return to this question elsewhere.

Our approach to all of the above is via a theory of motivic transfers. Such a theory of motivic and étale variants of the classical Becker–Gottlieb transfer were developed in [Carlsson and Joshua 2020] and the additivity of transfers was established in a general framework in [Joshua and Pelaez 2020], though special cases such as Snaith splitting for the suspension spectrum of BGL_n appears in [Kleen 2018]. Theorem 1.2 is proven by making intrinsic use of this transfer, just as was done by Becker and Gottlieb, making use of the classical Becker–Gottlieb transfer. See [Becker 1974] and also [Becker and Gottlieb 1975; 1976].

In fact, we summarize the main ideas of the proof of Theorem 1.2 (*as well as an overview of the paper*) *as follows*: The splitting provided by the motivic Becker–Gottlieb transfer as in Proposition 2.2 enables us to prove Proposition 2.8. This shows the map

$$\begin{aligned}\bar{q} = q \circ Q(p) : Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T)) &= Q(\varinjlim_n \mathrm{BN}_{\mathrm{GL}_n}(T_n)) \\ &\to Q(\varinjlim_n \mathrm{BGL}_n) = Q(\mathrm{BGL}_\infty) \xrightarrow{q} \mathrm{BGL}_\infty\end{aligned}$$

induces a surjection

$$[X, Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T))] \xrightarrow{\bar{q}_*} [X, \mathrm{BGL}_\infty]$$

for every compact object X in the unstable pointed motivic homotopy category. (Here $\mathrm{N}_{\mathrm{GL}_n}(T_n)$ denotes the normalizer of the maximal torus of diagonal matrices in GL_n.) Then we show in Propositions 2.13 and 2.27 that the map $\bar{q}_*$ above factors through λ_*, where λ is the map in (1.1), thereby proving the theorem.

It may be worth pointing out this involves *a second, somewhat different use of the transfer, this time as defined in* (3.5), *and with a key property proven in Corollary 3.24.* These occupy most of Section 2 of the paper. While Proposition 2.8 is rather straightforward given the properties of the motivic Becker–Gottlieb transfer, Proposition 2.27 is a bit involved: here one needs to know the relationship between maps defined by the transfer as in (3.5) and Gysin maps, for at least finite étale maps in orientable motivic cohomology theories. This is discussed in Section 3 of the paper.

1A. ***Basic assumptions and terminology.*** We assume throughout that the base scheme is a perfect field. (The assumption that k be perfect can be dropped, if one prefers, in view of recent results such as in [Elmanto and Khan 2020] and [Bachmann and Hoyois 2021, Theorem 10.12].) Then $\mathbf{Spt} = \mathbf{Spt}(k)$ denotes the category of motivic spectra on the big Nisnevich site of k, with $\mathcal{SH} = \mathcal{SH}(k)$ denoting the corresponding motivic stable homotopy category. If k is of characteristic 0, no further assumptions are needed.

However, if $\mathrm{char}(k) = p > 0$, then we only consider $\mathcal{SH}[p^{-1}]$, which is the motivic stable homotopy category on k, with the prime p inverted. (The main reason for this restriction is that a theory of Spanier–Whitehead duality holds only after inverting p in this case.) Given a motivic spectrum E and a motivic space X, the generalized motivic cohomology represented by E is given by the bigraded theory

$$h^{p,q}(X, E) = [\Sigma_T^{\infty} X, (S^1)^{p-q} \wedge \mathbb{G}_m^{\wedge q} E] \tag{1.4}$$

with [,] denoting the Hom in the motivic stable homotopy category.

In both the above cases, we do not require the existence of a symmetric monoidal structure on the category of spectra itself; that is, it is sufficient to assume the smash product of spectra is homotopy associative and homotopy commutative.

1B. ***Geometric classifying spaces.*** We begin by recalling briefly the construction of the *geometric classifying space of a linear algebraic group*; see for example, [Totaro 1999, §1; Morel and Voevodsky 1999, §4]. Let G denote a linear algebraic group over $S = \operatorname{Spec} k$, that is, a closed subgroup-scheme in GL_n over S for some n. For a (closed) imbedding $i : G \to \mathrm{GL}_n$ as a closed subgroup-scheme, *the geometric classifying space $B_{gm}(G; i)$ of G with respect to i* is defined as follows. For $m \geq 1$, let $\mathrm{EG}^{gm,m} = U_m(G) = U(\mathbb{A}^{nm})$ be the open subscheme of $\mathbb{A}^{nm}$ where the diagonal action of G determined by i is free. By choosing m large enough, one can always ensure that $U(\mathbb{A}^{nm})$ is nonempty and the quotient $U(\mathbb{A}^{nm})/G$ is a quasiprojective scheme. We further choose such a family $\{U(\mathbb{A}^{nm}) \mid m\}$ so that it satisfies the hypotheses in [Morel and Voevodsky 1999, Definition 2.1, Section 4.2] defining an *admissible gadget*.

Let $\mathrm{BG}^{gm,m} = V_m(G) = U_m(G)/G$ denote the quotient S-scheme (a quasiprojective variety) for the action of G on $U_m(G)$ induced by this (diagonal) action of G on $\mathbb{A}^{nm}$; the projection $U_m(G) \to V_m(G)$ defines $V_m(G)$ as the quotient of $U_m(G)$ by the free action of G and $V_m(G)$ is thus smooth. We have closed imbeddings $U_m(G) \to U_{m+1}(G)$ and $V_m(G) \to V_{m+1}(G)$ corresponding to the imbeddings $\mathrm{Id} \times \{0\} : \mathbb{A}^{nm} \to \mathbb{A}^{nm} \times \mathbb{A}^n$. We set $\mathrm{EG}^{gm} = \{U_m(G) \mid m\} = \{\mathrm{EG}^{gm,m} \mid m\}$ and $\mathrm{BG}^{gm} = \{V_m(G) \mid m\}$, which are ind-objects in the category of schemes. (If one prefers, one may view each $\mathrm{EG}^{gm,m}$ (resp. $\mathrm{BG}^{gm,m}$) as a sheaf on the big Nisnevich (étale) site of smooth schemes over k and then view EG^{gm} (resp. BG^{gm}) as the corresponding colimit taken in the category of sheaves on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ or $(\mathrm{Sm}/k)_{\text{ét}}$.)

Definition 1.5. We denote EG^{gm} by EG and BG^{gm} by BG throughout the paper.

Given a scheme X of finite type over S with a G-action, we let $U_m(G) \times_G X$ denote the *balanced product*, where (u, x) and (ug^{-1}, gx) are identified for all $(u, x) \in U_m \times X$ and $g \in G$. Since the G-action on $U_m(G)$ is free, $U_m(G) \times_G X$ exists as a geometric quotient which is also a quasiprojective scheme in this setting, in case X is assumed to be quasiprojective; see [Mumford et al. 1994, Proposition 7.1]. (In case X is an algebraic space of finite type over S, the above quotient also exists, but as an algebraic space of finite type over S.)

Next we recall a particularly nice way to construct geometric classifying spaces for closed subgroups of GL_n making use of the Stiefel varieties.

Definition 1.6 (Stiefel varieties and Grassmannians). Let n denote a fixed positive integer and let $i \geq 0$ denote an integer. We let $\mathrm{St}_{n+i,n}$ denote the set of all $(n+i) \times n$-matrices of rank n, or equivalently the set of all injective linear transformations $\mathbb{A}^n \to \mathbb{A}^{n+i}$. We view this as an open subscheme of the affine space $\mathbb{A}^{(n+i)\times n}$. The group GL_n acts on $\mathrm{St}_{n+i,n}$ through its action on $\mathbb{A}^n$: we view this as a right action on the set of all $(n+i) \times n$-matrices. This is a free action and the quotient is the Grassmann variety of n-planes in $\mathbb{A}^{n+i}$, and denoted $\mathrm{Grass}_{n+i,n}$.

As observed in [Morel and Voevodsky 1999, p. 138], for each fixed positive integer n, the family $\{\mathrm{St}_{n+i,n} \mid i \geq 0\}$ satisfies the conditions in [Morel and Voevodsky 1999, Definition 2.1, p. 133], so that it defines what is there called an *admissible gadget*. Thus $\{\mathrm{St}_{n+i,n}/H \mid i \geq 0\}$ forms finite dimensional approximations to the classifying space for any closed subgroup H of GL_n. Therefore, we make the following definitions.

Definition 1.7. (i) $\mathrm{BH}^{gm,i} = \mathrm{St}_{n+i,n}/H$ and $\mathrm{BH} = \lim_{i\to\infty} \mathrm{BH}^{gm,i}$.

(ii) $\mathrm{BGL}_\infty = \lim_{i\to\infty} \lim_{n\to\infty} \mathrm{St}_{n+i,n}/\mathrm{GL}_n = \lim_{n\to\infty} \mathrm{St}_{2n,n}/\mathrm{GL}_n$.

For any linear algebraic group G, we let BG denote the geometric classifying space defined above (as in Definition 1.5 or equivalently in Definition 1.7) as an ind-scheme. (We may view this as a motivic space.)

2. The motivic Segal–Becker theorem: proof of Theorem 1.2

We begin with the following observation due to Voevodsky.

Proposition 2.1. *There exists a motivic $\Omega_{\boldsymbol{T}}$-spectrum $\widetilde{\boldsymbol{K}}$, whose* 0*-th space is given by* BGL_∞.

Proof. The required spectrum is just $f_1(\boldsymbol{K})$, where $\boldsymbol{K}$ denotes the $\Omega_{\boldsymbol{T}}$-spectrum representing algebraic K-theory; see [Voevodsky 2002a, Theorem 2.2]. That this is the case follows from [Voevodsky 2002b, Lemma 2.2] (which holds unconditionally over any field by [Levine 2008, Theorem 7.5.1]) and [Voevodsky 2002b, Theorem 4.1, Lemma 4.6 and its proof]. □

2A. ***Changing base points.*** Recall motivic spaces are assumed to be pointed simplicial presheaves. However, it is often necessary for us to consider a motivic space Y viewed as an unpointed simplicial presheaf and then provide it with an extra base point $+$. A typical example we run into in this paper is when Y is the geometric classifying space of a linear algebraic group (denoted BG, recalling Definition 1.5) or a finite degree approximation of it (denoted $\mathrm{BG}^{gm,m}$), both of which are *pointed*. However, while considering a motivic Becker–Gottlieb transfer involving BG (resp. $\mathrm{BG}^{gm,m}$), one needs to consider $\Sigma^\infty_{\boldsymbol{T}}\mathrm{BG}_+$ (resp. $\Sigma^\infty_{\boldsymbol{T}}\mathrm{BG}^{gm,m}_+$), which is the $\boldsymbol{T}$-suspension spectrum of BG (resp. $\mathrm{BG}^{gm,m}$) provided with an extra base point $+$.

Observe that there is a natural map $r : \mathrm{BG}_+ \to \mathrm{BG}$ sending $+$ to the base point of BG. Let $a : \Sigma^\infty_{\boldsymbol{T}}\mathrm{BG} \to \Sigma^\infty_{\boldsymbol{T}}\mathrm{BG}_+$ denote a map such that

$$\Sigma^\infty_{\boldsymbol{T}} r \circ a = \mathrm{id}_{\Sigma^\infty_{\boldsymbol{T}}\mathrm{BG}}\,.$$

(Since the definition of such a map a is straightforward, we skip the details.)

Proposition 2.2. *Let $h^{*,\bullet}$ denote a generalized motivic cohomology theory defined with respect to a motivic spectrum* (*with p inverted*, *if* $\mathrm{char}(k) = p > 0$.) *Let G denote a linear algebraic group*, *which is also special in the sense of Grothendieck* (see [Chevalley et al. 1958]). *Then*, *with* $\mathrm{N}(T)$ *denoting the normalizer of a split maximal torus in G*, *one obtains the commutative square*

$$\begin{array}{ccc} h^{*,\bullet}(\Sigma^\infty_{\boldsymbol{T}}\mathrm{BN}(T)) & \xrightarrow{r^*} & h^{*,\bullet}(\Sigma^\infty_{\boldsymbol{T}}\mathrm{BN}(T)_+) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ h^{*,\bullet}(\Sigma^\infty_{\boldsymbol{T}}\mathrm{BG}) & \xrightarrow{r^*} & h^{*,\bullet}(\Sigma^\infty_{\boldsymbol{T}}\mathrm{BG}_+) \end{array} \tag{2.3}$$

where the map $p : \mathrm{BN}(T) \to \mathrm{BG}$ *is the map induced by the inclusion* $\mathrm{N}(T) \to G$*. The right vertical map and the horizontal maps are all split monomorphisms. Therefore, the left vertical map is also a monomorphism.*

Proof. That the right vertical map is a split monomorphism is a consequence of the motivic Becker–Gottlieb transfer as proved in [Carlsson and Joshua 2020, §9.2], as well as [Joshua and Pelaez 2020, Theorem 1.6] and [Ananyevskiy 2021, Theorem 5.1]. Moreover, all of these depend on the key identification of the Grothendieck–Witt group with the motivic π_0 of the motivic sphere spectrum from [Morel 2004; 2012]. The restriction that the characteristic of the base field k be different from 2 is removed in [Bachmann and Hoyois 2021, Theorem 10.12].

Let the motivic Euler characteristic of $G/\mathrm{N}(T)$ be denoted $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ henceforth. Now one may recall from [Joshua and Pelaez 2020, Theorem 1.6] that we showed $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ is 1 in the Grothendieck–Witt group $\mathrm{GW}(\mathrm{Spec}\, k)$ (or $\mathrm{GW}(\mathrm{Spec}\, k)[p^{-1}]$ if $\mathrm{char}(k) = p > 0$), provided k has a square root of -1. Hence this conclusion holds whenever the base field k is algebraically or quadratically closed. In positive characteristics p, one may see that this already shows that $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ is a *unit* in the group $\mathrm{GW}(\mathrm{Spec}\, k)[p^{-1}]$, without the assumption on the existence of a square root of -1 in k. For this, one may first observe the commutative diagram

$$\begin{tikzcd} \mathrm{GW}(\mathrm{Spec}\, \bar{k})[p^{-1}] \arrow[r, "\mathrm{rk}", "\cong"'] & \mathbb{Z}[p^{-1}] \\ \mathrm{GW}(\mathrm{Spec}\, k)[p^{-1}] \arrow[r, "\mathrm{rk}"] \arrow[u] & \mathbb{Z}[p^{-1}] \arrow[u, "\mathrm{id}"'] \end{tikzcd} \tag{2.4}$$

where $\bar{k}$ is an algebraic closure of k. Here the left vertical map is induced by the change of base fields from k to $\bar{k}$, and rk denotes the *rank* map. Since the motivic Euler characteristic of $G/\mathrm{N}(T)$ over $\mathrm{Spec}\, k$ maps to the motivic Euler characteristic of the corresponding $G/\mathrm{N}(T)$ over $\mathrm{Spec}\, \bar{k}$, it follows that the rank of $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ over $\mathrm{Spec}\, k$ is in fact 1. By [Ananyevskiy 2021, Lemma 2.9(2)], this shows that the $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ over $\mathrm{Spec}\, k$ is in fact a unit in $\mathrm{GW}(\mathrm{Spec}\, k)[p^{-1}]$, that is, when k has positive characteristic. (For the convenience of the reader, we summarize a few key facts discussed in [Ananyevskiy 2021, proof of Lemma 2.9(2)]. It is observed there that when the base field k is *not* formally real, then

$$I(k) = \mathrm{kernel}(\mathrm{GW}(k) \xrightarrow{\mathrm{rk}} \mathbb{Z})$$

is the nilradical of $\mathrm{GW}(k)$ [Baeza 1978, Theorem V.8.9, Lemma V.7.7 and Theorem V. 7.8]. Therefore, if $\mathrm{char}(k) = p > 0$, and the rank of $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ is 1 in $\mathbb{Z}[p^{-1}]$, then $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ is $1 + q$ for some nilpotent element q in $I(k)[p^{-1}]$ and the conclusion follows.)

In characteristic 0, the commutative diagram

$$\begin{tikzcd} \mathrm{GW}(\operatorname{Spec}\bar{k}) \arrow[r, "\mathrm{rk}", "\cong"'] & \mathbb{Z} \\ \mathrm{GW}(\operatorname{Spec}k) \arrow[u] \arrow[r, "\mathrm{rk}"] & \mathbb{Z} \arrow[u, "\mathrm{id}"'] \end{tikzcd} \tag{2.5}$$

shows that once again the rank of $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ is 1. Therefore, to show that the class $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ is a unit in $\mathrm{GW}(\operatorname{Spec}k)$, it suffices to show its signature is 1; this is proven in [Ananyevskiy 2021, Theorem 5.1(1)]. (Again, for the convenience of the reader, we summarize some details from the proof of [Ananyevskiy 2021, Theorem 5.1(1)]. When the field k is not formally real, the discussion in the last paragraph applies, so that by [Ananyevskiy 2021, Lemma 2.12] one reduces to considering only the case when k is a real closed field. In this case, one lets $\mathbb{R}^{\mathrm{alg}}$ denote the real closure of $\mathbb{Q}$ in $\mathbb{R}$. Then, one knows that the given real closed field k contains a copy of $\mathbb{R}^{\mathrm{alg}}$ and that there exists a reductive group scheme $\widetilde{G}$ over $\operatorname{Spec}\mathbb{R}^{\mathrm{alg}}$ such that $G = \widetilde{G}\times_{\operatorname{Spec}\mathbb{R}^{\mathrm{alg}}} \operatorname{Spec}k$. Let $G_{\mathbb{R}} = \widetilde{G}\times_{\operatorname{Spec}\mathbb{R}^{\mathrm{alg}}} \operatorname{Spec}\mathbb{R}$. Then one also observes that the Grothendieck–Witt groups of the three fields k, $\mathbb{R}^{\mathrm{alg}}$ and $\mathbb{R}$ are isomorphic, and the motivic Euler characteristics

$$\chi^{\mathbb{A}^1}(G/\mathrm{N}(T)), \qquad \chi^{\mathbb{A}^1}(\widetilde{G}/\widetilde{\mathrm{N}(T)}), \qquad \chi^{\mathbb{A}^1}(G_{\operatorname{Spec}\mathbb{R}}/\mathrm{N}(T)_{\operatorname{Spec}\mathbb{R}})$$

over the above three fields identify under the above isomorphisms, so that one may assume the base field k is $\mathbb{R}$. Then it is shown in [Ananyevskiy 2021, proof of Theorem 5.1(1)] that, in this case, knowing the rank and signature of the motivic Euler characteristic $\chi^{\mathbb{A}^1}(G/\mathrm{N}(T))$ are 1 suffices to prove it is a unit in the Grothendieck–Witt group.)

These complete the proof that the right vertical map in (2.3) is a split monomorphism.

The horizontal maps in (2.3) are split by the map a^*. Since the diagram commutes, it follows that the left vertical map in (2.3) is also a monomorphism. This proves the proposition. □

Remark 2.6. There is an extension of the above theorem for linear algebraic groups that are not special, as discussed in [Joshua and Pelaez 2020, Theorem 1.5] and [Carlsson and Joshua 2020, Theorem 1.5(1)]. But then we require the field be infinite to prevent certain situations like those mentioned in [Morel and Voevodsky 1999, §4.2, Example 2.10] from occurring. However, for the applications in this paper, we only need to consider the linear algebraic groups $\{\mathrm{GL}_n \mid n\}$, which are all special.

2B. For the rest of the discussion, we restrict to the family of groups $\{\mathrm{GL}_n \mid n\}$. Then the main result of this section is the proof of Theorem 1.2. We break the proof into several propositions. Given two motivic spaces $\mathcal{X}, \mathcal{Y}$, we let $[\mathcal{X}, \mathcal{Y}]$ denote the Hom in the unstable pointed motivic homotopy category, with p inverted, if $p > 0$ is the characteristic of the base field k. For a motivic space P, we let $Q(P) = \lim_{n\to\infty} \Omega^n_{\boldsymbol{T}}(\boldsymbol{T}^{\wedge n}(P))$.

Let

$$p : \mathrm{BN}_{\mathrm{GL}_\infty}(T) = \varinjlim_n \mathrm{BN}_{\mathrm{GL}_n}(T_n) \to \varinjlim_n \mathrm{BGL}_n = \mathrm{BGL}_\infty \tag{2.7}$$

denote the map induced by the inclusion of $\mathrm{N}_{\mathrm{GL}_n}(T_n)$ in GL_n.

Proposition 2.8. (i) *Assume the base field k is of characteristic* 0. *Then the map*

$$\begin{aligned}\bar{q} = q \circ Q(p) : Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T)) &= Q(\varinjlim_n \mathrm{BN}_{\mathrm{GL}_n}(T_n)) \\ &\to Q(\varinjlim_n \mathrm{BGL}_n) = Q(\mathrm{BGL}_\infty) \xrightarrow{q} \mathrm{BGL}_\infty\end{aligned}$$

induces a surjection for every pointed motivic space X which is a compact object in the unstable pointed motivic homotopy category:

$$[X, \varinjlim_n Q(\mathrm{BN}_{\mathrm{GL}_n}(T_n))] \to [X, \mathrm{BGL}_\infty].$$

(ii) *Assume the base field k is perfect and of positive characteristic p. Then the same conclusion holds after inverting the prime p.*

Proof. We follow [Becker 1974, §4] in this proof. The proof of the second statement follows along the same lines as the proof of the first statement. Therefore, we discuss a proof of only the first statement. Clearly the map $\bar{q}$ provides a map of the corresponding spectra:

$$\Sigma^\infty_{\boldsymbol{T}}(\mathrm{BN}_{\mathrm{GL}_\infty}(T)) = \Sigma^\infty_{\boldsymbol{T}}(\lim_{n\to\infty} \mathrm{BN}_{\mathrm{GL}_n}(T_n)) \to \widetilde{\boldsymbol{K}}, \tag{2.9}$$

where $\widetilde{\boldsymbol{K}}$ is the motivic $\Omega_{\boldsymbol{T}}$-spectrum whose 0-th space is BGL_∞. Let ϕ denote the above map in (2.9).

Let h denote the motivic cohomology theory defined by the mapping cone of the above map ϕ. Then, for every motivic space X, we obtain a long exact sequence:

$$\cdots \to [X, Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T))] \xrightarrow{\bar{q}_*} [X, \mathrm{BGL}_\infty] \xrightarrow{c} h^{0,0}(X) \to \cdots. \tag{2.10}$$

For each $n \geq 0$, let

$$u_n \in [\mathrm{BGL}_n, \mathrm{BGL}_\infty] \tag{2.11}$$

be the class of the map induced by the imbedding $\mathrm{GL}_n \to \mathrm{GL}_\infty$. Then it suffices to show that each such u_n is in the image of the induced map $\bar{q}_*$, which is equivalent to showing that $c(u_n) = 0$. To see this let $v_n : \mathrm{BGL}_n \to Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T))$ be such that $\bar{q}_*([v_n]) = [\bar{q} \circ v_n] = u_n$. Here, if α is a map, $[\alpha]$ denotes the stable homotopy class of α. Since X is assumed to be compact, $[X, \mathrm{BGL}_\infty] = \lim_{n\to\infty}[X, \mathrm{BGL}_n]$.

Therefore, giving an $\alpha \in [X, \mathrm{BGL}_\infty]$ is equivalent to giving an $\alpha_n : X \to \mathrm{BGL}_n$ (for some n), so that $\alpha = u_n \circ \alpha_n$. Now let $\beta_n = v_n \circ \alpha_n$. Then

$$\bar{q}_*([\beta_n]) = [\bar{q} \circ v_n \circ \alpha_n] = [u_n \circ \alpha_n] = [\alpha].$$

Now we observe the commutative diagram

$$\begin{tikzcd}
{[\mathrm{BGL}_n, Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T))]} \arrow[r, "\bar{q}_*"] \arrow[d, "p_n^*"] & \widetilde{K}^{0,0}(\mathrm{BGL}_n) \arrow[r, "c"] \arrow[d, "p_n^*"] & h^{0,0}(\mathrm{BGL}_n) \arrow[d, "p_n^*"] \\
{[\mathrm{BN}_{\mathrm{GL}_n}(T_n), Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T))]} \arrow[r, "\bar{q}_*"] & \widetilde{K}^{0,0}(\mathrm{BN}_{\mathrm{GL}_n}(T_n)) \arrow[r, "c"] & h^{0,0}(\mathrm{BN}_{\mathrm{GL}_n}(T_n))
\end{tikzcd} \tag{2.12}$$

where $p_n : \mathrm{BN}_{\mathrm{GL}_n}(T_n) \to \mathrm{BGL}_n$ is the obvious map induced by the imbedding $\mathrm{N}_{\mathrm{GL}_n}(T_n) \to \mathrm{GL}_n$. Recall that p_n^* is a monomorphism, by Proposition 2.2. Therefore, now it suffices to prove $p_n^*(c(u_n)) = 0$, for each n. But the commutativity of the above diagram, shows that this is equivalent to showing that $c(p_n^*(u_n)) = 0$. At this point, we observe the commutative diagram

$$\begin{tikzcd}
\mathrm{BN}_{\mathrm{GL}_n}(T_n) \arrow[r, "p_n"] \arrow[d, "i_n"] & \mathrm{BGL}_n \arrow[d, "u_n"] & \\
\mathrm{BN}_{\mathrm{GL}_\infty}(T) \arrow[r, "p"] \arrow[d, "j"] & \mathrm{BGL}_\infty \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T)) \arrow[r, "Q(p)"] & Q(\mathrm{BGL}_\infty) \arrow[r, "q"] & \mathrm{BGL}_\infty
\end{tikzcd}$$

where i_n and j are the obvious maps. The map $\mathrm{BGL}_\infty \to Q(\mathrm{BGL}_\infty)$ is provided by taking the colimit of the maps $\mathrm{BGL}_\infty \to \Omega_{\boldsymbol{T}}^n \Sigma_{\boldsymbol{T}}^n(\mathrm{BGL}_\infty)$, while the map $Q(\mathrm{BGL}_\infty) \to \mathrm{BGL}_\infty$ is provided by the fact BGL_∞ is an infinite $\boldsymbol{T}$-loop space. The fact that the composite map $\mathrm{BGL}_\infty \to Q(\mathrm{BGL}_\infty) \to \mathrm{BGL}_\infty$ is homotopic to the identity map follows from the adjunction between $\wedge \boldsymbol{T}$ and $\Omega_{\boldsymbol{T}}$, along with the $\Omega_{\boldsymbol{T}}$-infinite loop space structure on BGL_∞. In view of the commutative diagram above, $p_n^*(u_n) = u_n \circ p_n = q \circ Q(p) \circ j \circ i_n = \bar{q}_*(j \circ i_n)$. Since the two rows in the diagram (2.12) are exact, it follows that indeed $c(\bar{q}_*(j \circ i_n)) = 0$, thereby completing the proof that $c(p_n^*(u_n)) = 0$. $\square$

Let $\lambda : Q(\mathrm{B}\mathbb{G}_m) \to Q(\mathrm{BGL}_\infty) \to \mathrm{BGL}_\infty$ denote the map considered in (1.1).

Proposition 2.13. (i) *Assume the base field k is of characteristic* 0. *Let X be any pointed motivic space. Then there is a map* $\zeta : Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T)) \to Q(\mathrm{B}\mathbb{G}_m) = Q(\mathbb{P}^\infty)$, *so that the triangle*

$$\begin{tikzcd} {[X,\, Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T))]} \arrow[r, "\zeta_*"] \arrow[dr, "\bar{q}_*"'] & {[X,\, Q(\mathrm{B}\mathbb{G}_m)]} \arrow[d, "\lambda_*"] \\ & {[X, \mathrm{BGL}_\infty]} \end{tikzcd} \tag{2.14}$$

commutes.

(ii) *Assume the base field k is perfect and of positive characteristic p. Then the same conclusion holds after inverting the prime p.*

Proof of Theorem 1.2. Before we prove the above proposition, we proceed to show how to complete the proof of Theorem 1.2, given the above proposition. We simply observe that, for a compact object X, the composition of the maps in

$$[X,\, Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T))] \xrightarrow{\zeta_*} [X,\, Q(\mathrm{B}\mathbb{G}_m))] \xrightarrow{\lambda_*} [X, \mathrm{BGL}_\infty]$$

is a surjection, thereby proving that the map λ_* is also a surjection, which completes the proof of the theorem. $\square$

2C. Therefore, it suffices to prove Proposition 2.13, which we now proceed to do. Moreover, we only consider the characteristic 0 case explicitly, as the positive characteristic case follows exactly along the same lines, once the characteristic is inverted. Let $\mathrm{St}_{2n,n}$ denote the *Stiefel variety* of n-frames (that is, n-linearly independent vectors) $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ in $\mathbb{A}^{2n}$. The group GL_n acts on this variety, by acting on such frames by sending $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ to $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) * g$, $g \in \mathrm{GL}_n$. (We view this as a right action because the Stiefel variety $\mathrm{St}_{2n,n}$ identifies with the variety of all $2n \times n$-matrices of rank n, with each vector $\boldsymbol{v}_i$ in an n-frame $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ written as the i-th column.) This is a free action and the quotient is the *Grassmannian* $\mathrm{Grass}_{2n,n}$. The Stiefel variety $\mathrm{St}_{2n,n}$ is an open subvariety of the affine space $\mathbb{A}^{2n^2}$ and the complement has codimension $n+1$ in $\mathbb{A}^{2n^2}$.

Next we consider the ind-scheme

$$\mathbb{A}^2 \to \mathbb{A}^4 \to \mathbb{A}^6 \to \cdots \to \mathbb{A}^{2n} \xrightarrow{i_n} \mathbb{A}^{2n+2} \cdots, \tag{2.15}$$

where the closed immersion $\mathbb{A}^{2n} \to \mathbb{A}^{2n+2}$ sends $(x_1, \dots, x_{2n})$ to $(x_1, \dots, x_{2n}, 0, 0)$. Let $\boldsymbol{e}_i$, $i = 1, \dots, 2n, 2n+1, 2n+2$, denote the standard basis vectors in $\mathbb{A}^{2n+2}$. Then we obtain a closed immersion

$$i_n : \mathrm{St}_{2n,n} \to \mathrm{St}_{2n+2,n+1}, \tag{2.16}$$

by sending an n-frame $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ in $\mathbb{A}^{2n}$ to the $n+1$-frame

$$(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \bar{\boldsymbol{e}}_{n+1} = \boldsymbol{e}_{2n+1} + \boldsymbol{e}_{2n+2}).$$

This induces a closed immersion of the Grassmannians, $\mathrm{Grass}_{2n,n} \to \mathrm{Grass}_{2n+2,n+1}$. Therefore, the discussion in Section 1B shows that the above Stiefel varieties may be used as finite dimensional approximations to EGL_n and the Grassmannian, $\mathrm{Grass}_{2n,n}$, could be used as a finite dimensional approximation to BGL_n.

Next we make the identifications

$$\begin{aligned} \mathrm{BGL}_n^{gm,n} &= \mathrm{St}_{2n,n} \,/\, \mathrm{GL}_n, \\ \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} &= \mathrm{St}_{2n,n} \,/\mathrm{N}_{\mathrm{GL}_n}(T_n), \\ \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n} &= \mathrm{St}_{2n,n} \,/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})), \end{aligned} \tag{2.17}$$

where we imbed GL_{n-1} in GL_n as the last $(n-1)\times(n-1)$-block, then imbed $\mathrm{N}_{\mathrm{GL}_n}(T_n)$ in GL_n, and $\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}(T_{n-1})}$ in $\mathbb{G}_m \times \mathrm{GL}_{n-1}$. Now $\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})$ is a subgroup of index n in $\mathrm{N}_{\mathrm{GL}_n}(T_n)$, so that the projection

$$r_n : \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n} \to \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}$$

is a finite étale cover of degree n. The fact that the terms appearing on the right-hand sides in (2.17) are indeed approximations to the classifying spaces of the corresponding linear algebraic groups follows from the discussion in Section 1B.

Next consider the map $\mathrm{St}_{2n,n} \to \mathrm{St}_{2n,1}$ sending an n-frame $(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$ to $\boldsymbol{v}_1$. Clearly this factors through the quotient of $\mathrm{St}_{2n,n}\,/1 \times \mathrm{GL}_{n-1}$, where GL_{n-1} acts only on the last $n-1$-vectors in the n-frame $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n)$. Therefore, we obtain the map

$$\phi_n : \mathrm{St}_{2n,n}\,/(1 \times \mathrm{GL}_{n-1}) \to \mathrm{St}_{2n,1}\,. \tag{2.18}$$

Moreover, the above map ϕ_n is compatible with the obvious action of $\mathbb{G}_m$ on $\mathrm{St}_{2n,n}\,/\,\mathrm{GL}_{n-1}$, where it acts on the vector $\boldsymbol{v}_1$ in an n-frame $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n)$, and it acts on the 1-frame $\boldsymbol{v}$ in $\mathrm{St}_{2n,1}$. Taking quotients, this defines the map

$$\bar{\phi}_n : \mathrm{St}_{2n,n}\,/(\mathbb{G}_m \times \mathrm{GL}_{n-1}) \to \mathrm{St}_{2n,1}\,/\mathbb{G}_m. \tag{2.19}$$

One may then observe the commutative square

$$\begin{tikzcd} \mathrm{St}_{2n,n}\,/(\mathbb{G}_m \times \mathrm{GL}_{n-1}) \arrow[r, "\bar{\phi}_n"] \arrow[d] & \mathrm{St}_{2n,1}\,/\mathbb{G}_m \arrow[d] \\ \mathrm{St}_{2n+2,n+1}\,/(\mathbb{G}_m \times \mathrm{GL}_n) \arrow[r, "\bar{\phi}_{n+1}"] & \mathrm{St}_{2n+2,1}\,/\mathbb{G}_m \end{tikzcd} \tag{2.20}$$

where the left vertical map is the closed immersion defined by $i_n : \mathrm{St}_{2n,n} \to \mathrm{St}_{2n+2,n+1}$ and the right vertical map is induced by the closed immersion $\mathrm{St}_{2n,1} \to \mathrm{St}_{2n+2,1}$.

One may also observe that clearly $\mathrm{St}_{2n,1}\,/\mathbb{G}_m$ is an approximation to the classifying space of $\mathbb{G}_m$, so that we let

$$\mathrm{B}\mathbb{G}_m^{gm,n} = \mathrm{St}_{2n,1}\,/\mathbb{G}_m. \tag{2.21}$$

We also let

$$\begin{aligned} u_n : \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n} &= \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \\ &\to \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{GL}_{n-1}), \\ \bar{u}_n = \bar{\phi}_n \circ u_n : \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n} &= \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \\ &\to \mathrm{St}_{2n,1}/\mathbb{G}_m = \mathrm{B}\mathbb{G}_m^{gm,n}. \end{aligned} \tag{2.22}$$

Then we also obtain the commutative diagram

$$\begin{tikzcd} \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \arrow[r, "u_n"] \arrow[d] & \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{GL}_{n-1}) \arrow[r, "\bar{\phi}_n"] \arrow[d] & \mathrm{St}_{2n,1}/\mathbb{G}_m \arrow[d] \\ \mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n)) \arrow[r, "u_{n+1}"] & \mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{GL}_n) \arrow[r, "\bar{\phi}_{n+1}"] & \mathrm{St}_{2n+2,1}/\mathbb{G}_m \end{tikzcd} \tag{2.23}$$

As pointed out in the introduction, apart from the transfer for passage from $\mathrm{BGL}_n^{gm,n}$ to $\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}$, for the proof of Theorem 1.2, one also needs to invoke a transfer map for the finite étale map

$$\begin{aligned} r_n : \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n} &= \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \\ &\to \mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n) = \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}. \end{aligned}$$

We also need to know that such a transfer map has reasonable properties, like compatibility with base change, and agreement with Gysin maps defined on orientable generalized motivic cohomology theories. The purpose of the last short section of the paper is to setup such a transfer and establish these basic properties for it: see (3.5) for the definition of such a transfer. Let

$$\tau_n : \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{BN}_{\mathrm{GL}_n}(T_n)_+^{gm,n} \to \Sigma_{\boldsymbol{T}}^{\infty}(\mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n})_+ \tag{2.24}$$

denote the corresponding transfer defined as in (3.5), and let

$$\begin{aligned} \zeta_n : \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{BN}_{\mathrm{GL}_n}(T_n)_+^{gm,n} &\xrightarrow{\tau_n} \Sigma_{\boldsymbol{T}}^{\infty}(\mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n})_+ \\ &\xrightarrow{\pi_n} \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\mathbb{G}_m^{gm,n} \xrightarrow{j_n} \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\mathbb{G}_m \end{aligned} \tag{2.25}$$

denote the composition, where the map π_n is the composition of the map $\Sigma_{\boldsymbol{T}}^{\infty}\bar{u}_{n+}$ followed by the map that sends the base point $+$ to the base point of $\mathrm{B}\mathbb{G}_m^{gm,n}$ as in Section 2A. The last map, denoted j_n, is the obvious one sending a finite dimensional approximation of $\mathrm{B}\mathbb{G}_m$ to the direct limit of such approximations. Let

$$\bar{q}_n : Q(\mathrm{BN}_{\mathrm{GL}_n}(T_n)_+^{gm,n}) \to \mathrm{BGL}_\infty \tag{2.26}$$

denote the composition

$$Q(\mathrm{BN}_{\mathrm{GL}_n}(T_n)_+^{gm,n}) \to Q(\mathrm{BN}_{\mathrm{GL}_\infty}(T)) \xrightarrow{Q(p)} Q(\mathrm{BGL}_\infty) \xrightarrow{q} \mathrm{BGL}_\infty.$$

Then a key result is the next proposition, which we show also proves Proposition 2.13.

Proposition 2.27. *Assume the above situation. Then the following diagrams commute:*

$$
\begin{tikzcd}
\Sigma^{\infty}_{\boldsymbol{T}}\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}_{+} \arrow[d] \arrow[dr, "\zeta_n"] & \\
\Sigma^{\infty}_{\boldsymbol{T}}\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})^{gm,n+1}_{+} \arrow[r, "\zeta_{n+1}"'] & \Sigma^{\infty}_{\boldsymbol{T}}\mathrm{B}\mathbb{G}_m
\end{tikzcd}
\tag{2.28}
$$

$$
\begin{tikzcd}
{[X,\, Q(\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}_{+})]} \arrow[r, "\zeta_{n*}"] & {[X,\, Q(\mathrm{B}\mathbb{G}_m)]} \arrow[dd, bend left=60, "\lambda_*"] \\
{[\Sigma^{\infty}_{\boldsymbol{T}}X,\, \Sigma^{\infty}_{\boldsymbol{T}}(\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}_{+})]} \arrow[u, "\simeq"] \arrow[r, "\zeta_{n*}"] \arrow[dr, "\bar{q}_{n*}"'] & {[\Sigma^{\infty}_{\boldsymbol{T}}X,\, \Sigma^{\infty}_{\boldsymbol{T}}\mathrm{B}\mathbb{G}_m]} \arrow[u, "\simeq"'] \arrow[d, "\lambda'_*"] \\
& \widetilde{K}^{0,0}(X) = [X,\, \mathrm{BGL}_{\infty}]
\end{tikzcd}
\tag{2.29}
$$

where $\lambda' : \Sigma^{\infty}_{\boldsymbol{T}}\mathrm{B}\mathbb{G}_m \to f_1(\boldsymbol{K}) = \widetilde{\boldsymbol{K}}$ *is the map of spectra corresponding to the infinite loop-space map* $\lambda : Q(\mathrm{B}\mathbb{G}_m) \to \mathrm{BGL}_{\infty}$. *The left vertical map in* (2.28) *is the obvious map induced by the closed immersion* i_n *in* (2.16). *Moreover,* $[\ ,\ ]$ *in the middle row of* (2.29) *denotes* Hom *in the motivic stable homotopy category, while* $[\ ,\ ]$ *in the top row and the bottom row denotes* Hom *in the unstable pointed motivic homotopy category.*

Proof. We first prove the commutativity of the triangle in (2.28). For this, one begins with the cartesian square (which also defines P_n and the map r'_n):

$$
\begin{tikzcd}
P_n \arrow[r, "\tilde{i}"] \arrow[d, "r'_n"] & \begin{array}{c}\mathrm{BN}_{\mathrm{GL}_{n+1}}\widetilde{(T_{n+1})}^{gm,n+1} \\ = \mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n))\end{array} \arrow[d, "r_{n+1}"] \\
\begin{array}{c}\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} \\ = \mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n)\end{array} \arrow[r, "i"] & \begin{array}{c}\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})^{gm,n+1} \\ = \mathrm{St}_{2n+2,n+1}/\mathrm{N}_{\mathrm{GL}_{n+1}}(T_{n+1})\end{array}
\end{tikzcd}
\tag{2.30}
$$

Observe that the right vertical map, and therefore also the left vertical map, is a finite étale map of degree $n+1$. By Proposition 3.6 (which shows the naturality of the transfer with respect to base change), we observe that the square below homotopy commutes:

$$
\begin{array}{ccc}
\Sigma_{T}^{\infty} P_{n,+} & \xrightarrow{\Sigma_{T}^{\infty}\tilde{i}_{+}} & \Sigma_{T}^{\infty}\mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_{n+1}}(T_{n+1})}_{+}^{gm,n+1} \\
\uparrow{\scriptstyle \tau_n'} & & \uparrow{\scriptstyle \tau_{n+1}} \\
\Sigma_{T}^{\infty}\mathrm{BN}_{\mathrm{GL}_n}(T_n)_{+}^{gm,n} & \xrightarrow{\Sigma_{T}^{\infty} i_{+}} & \Sigma_{T}^{\infty}\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})_{+}{}^{gm,n+1}
\end{array}
\tag{2.31}
$$

Then a straightforward calculation, as discussed below, shows that

$$P_n = \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n} \sqcup \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}. \tag{2.32}$$

The main observation here is that under the identifications in (2.17), the map

$$\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} \to \mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})^{gm,n+1}$$

lifts to $\mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_{n+1}}(T_{n+1})}^{gm,n+1}$, which provides the required splitting. In fact, this splitting may be described in more detail as follows. Observe first that the imbedding $i : \mathrm{St}_{2n,n} \to \mathrm{St}_{2n+2,n+1}$ is defined by sending an n-frame $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ in $\mathrm{St}_{2n,n}$ to the $n+1$-frame $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n, \bar{\boldsymbol{e}}_{n+1})$ in $\mathrm{St}_{2n+2,n+1}$, where $\bar{\boldsymbol{e}}_{n+1}$ is the nonzero vector chosen as in (2.16) that lies in the ambient affine space $\mathbb{A}^{2n+2}$ and is outside of $\mathbb{A}^n \subseteq \mathbb{A}^{n+2}$. The induced map $\mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n) \to \mathrm{St}_{2n+2,n+1}/\mathrm{N}_{\mathrm{GL}_{n+1}}$ is the map $i : \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} \to \mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})^{gm,n+1}$ appearing in (2.30).

One may see that from i one obtains an induced map

$$\mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \to \mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n)), \tag{2.33}$$

since the action of $\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})$ on $\mathrm{St}_{2n,n}$ and the action of $\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n)$ on $\mathrm{St}_{2n+2,n+1}$ are compatible. (For this one identifies $\mathrm{St}_{2n,n}$ as imbedded in $\mathrm{St}_{2n+2,n+1}$ using the imbedding i_n considered in (2.16), and identifies the group $\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})$ with the subgroup $\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1}) \times 1$ of $\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n)$.) Moreover, one may see that this map is a closed immersion and that one obtains a commutative triangle

$$
\begin{tikzcd}
\mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \arrow[rr] \arrow[dr] & & P_n \arrow[dl, "r_n'"] \\
& \mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n) &
\end{tikzcd}
\tag{2.34}
$$

The left inclined map is a finite étale map of degree n, while the right inclined map is a finite étale map of degree $n+1$. Since the top horizontal map is also étale (see [Milne 1980, Chapter I, Corollary 3.6]) and a closed immersion, it is the open (and closed) imbedding of a connected component in P_n. Let the complement in P_n of $\mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1}))$ be denoted C_n. Then the induced map $C_n \to \mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n)$ is a bijective and finite étale map, so is an isomorphism, showing that $\mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n) = \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}$ is a split summand of

$$\mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n)) = \widetilde{\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})}^{gm,n+1}.$$

(One may also obtain an explicit description of the above splitting as given by sending the n-frame $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ in $\mathrm{St}_{2n,n}$ to the $n+1$-frame $(\bar{\boldsymbol{e}}_{n+1}, \boldsymbol{v}_1, \dots, \mathbf{v}_n)$ in $\mathrm{St}_{2n+2,n+1}$. $\mathbb{G}_m$ acts on $\bar{\boldsymbol{e}}_{n+1}$ by multiplication by scalars. Let s denote this imbedding of $\mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n)$ in $\mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n))$.)

Moreover, observe from the above description of the splitting (2.32) that under the composite map

$$\begin{aligned} \bar{u}_{n+1} \circ \tilde{i} : P_n \xrightarrow{\tilde{i}} \widetilde{\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})}^{gm,n+1} &= \mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_n}(T_n)) \\ &\xrightarrow{u_{n+1}} \mathrm{St}_{2n+2,n+1}/(\mathbb{G}_m \times \mathrm{GL}_n) \xrightarrow{\bar{\phi}_n} \mathrm{St}_{2n+2,1}/\mathbb{G}_m, \end{aligned} \tag{2.35}$$

the copy of $\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} = \mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n)$ in P_n (under the above splitting of P_n) is sent to the base point. (Observe that the n-frames $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ coming from $\mathrm{St}_{2n,n}$, under the above imbedding s get sent to the last n-frames.) Since the diagram

$$\begin{tikzcd}
\mathrm{B}\mathbb{G}_m^{gm,n} \arrow[r] & \mathrm{B}\mathbb{G}_m^{gm,n+1} \\
\widetilde{\mathrm{BN}_{\mathrm{GL}_n}(T_n)}^{gm,n} \arrow[u, "\bar{u}_n"] \arrow[r] & \widetilde{\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})}^{gm,n+1} \arrow[u, "\bar{u}_{n+1}"']
\end{tikzcd}$$

(which is the same as the diagram (2.23)) also commutes, combining these diagrams and composing with the inclusions into $\mathrm{B}\mathbb{G}_m$ proves the commutativity of the triangle (2.28). In fact, the commutativity of the triangle (2.28) is equivalent to the statement that the composition of maps along the left column followed by the top inclined map is the same (up to homotopy) as the composition of the bottom map followed by the maps in the right column in the following big diagram:

$$\begin{tikzcd}
 & & \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\mathbb{G}_m \\
\Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\mathbb{G}_m^{gm,n} \arrow[urr, "j_n"] \arrow[rr] & & \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\mathbb{G}_m^{gm,n+1} \arrow[u, "j_{n+1}"'] \\
\Sigma_{\boldsymbol{T}}^{\infty}\widetilde{\mathrm{BN}_{\mathrm{GL}_n}(T_n)}_+^{gm,n} \arrow[u, "\pi_n"] \arrow[r] & \Sigma_{\boldsymbol{T}}^{\infty} P_{n,+} \arrow[r, "\Sigma_{\boldsymbol{T}}^{\infty}\tilde{i}_+"] & \Sigma_{\boldsymbol{T}}^{\infty}\widetilde{\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})}_+^{gm,n+1} \arrow[u, "\pi_{n+1}"'] \\
\Sigma_{\boldsymbol{T}}^{\infty}\mathrm{BN}_{\mathrm{GL}_n}(T_n)_+^{gm,n} \arrow[u, "\tau_n"] \arrow[ur, "\tau_n'"] \arrow[rr] & & \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{BN}_{\mathrm{GL}_{n+1}}(T_{n+1})_+^{gm,n+1} \arrow[u, "\tau_{n+1}"']
\end{tikzcd}$$

The commutativity of the part of the above diagram on and above the second row has already been proven. The bottom square on the right commutes by (2.31). Proposition 3.6(i) proves that $\tau'_n = \tau_n + \mathrm{id}$, where id denotes the identity map of $\Sigma^{\infty}_{\boldsymbol{T}}\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}_{+}$. Finally, the observation in (2.35) completes the proof.

Next we consider the commutativity of the diagram (2.29). Since the top square there evidently commutes, it suffices to consider the commutativity of the bottom triangle there. *The key observation is that, in order to prove the commutativity of the bottom triangle in* (2.29), *it suffices to take* $X = \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}_{+}$ *and show that the triangle commutes for the class* $u \in [X, Q(\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}_{+})]$ *denoting the class corresponding to the identity map* $\Sigma^{\infty}_{\boldsymbol{T}}X_+ \to \Sigma^{\infty}_{\boldsymbol{T}}X_+$.

Let u denote the class considered in the last line. Then $\bar{q}_{n*}(u) = (q \circ Q(p))_*(u)$ denotes the class of the vector bundle of rank n associated to the principal $\mathrm{N}_{\mathrm{GL}_n}(T_n)$-bundle over $\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}$. One may see this readily as follows: First, the natural map $p : \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} \to \mathrm{BGL}_\infty$ corresponds to the rank n vector bundle over $\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}$ associated to the principal $\mathrm{N}_{\mathrm{GL}_n}(T_n)$-bundle over $\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}$. Then the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} \arrow[r, "p"] \arrow[d, "u"] & \mathrm{BGL}_\infty \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
Q(\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}) \arrow[r, "Q(p)"] & Q(\mathrm{BGL}_\infty) \arrow[r, "q"] & \mathrm{BGL}_\infty
\end{tikzcd}$$

completes the proof. *We denote by* α *this vector bundle over*

$$\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n} = \mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n). \tag{2.36}$$

Let β *denote the line bundle associated to the principal* $\mathbb{G}_m$*-bundle*

$$\mathrm{St}_{2n,n}/(1 \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \\ \to \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) = \mathrm{BN}_{\widetilde{\mathrm{GL}_n}(T_n)}{}^{gm,n}. \tag{2.37}$$

Then $\lambda'_*(\zeta_{n*}(u))$ is the image of β under the transfer map

$$\tau_n^* : \widetilde{K}^{0,0}\big(\mathrm{BN}_{\widetilde{\mathrm{GL}_n}(T_n)}{}^{gm,n}\big) \to \widetilde{K}^{0,0}(\mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}),$$

where ζ_n is the map in (2.25). This results from the following observations:

(i) The map $\lambda' \circ j_n : \Sigma^{\infty}_{\boldsymbol{T}}\mathrm{B}\mathbb{G}^{gm,n} \to \Sigma^{\infty}_{\boldsymbol{T}}\mathrm{B}\mathbb{G}_m^{\mathrm{gm}} \to f_1(\boldsymbol{K}) = \widetilde{\boldsymbol{K}}$ corresponds to a map $\tilde{\lambda}'_n : \mathrm{St}_{2n,1}/\mathbb{G}_m = \mathrm{B}\mathbb{G}_m^{gm,n} \to \mathrm{BGL}_\infty$.

(ii) The map $\tilde{\lambda}'_n : \mathrm{St}_{2n,1}/\mathbb{G}_m = \mathrm{B}\mathbb{G}_m^{gm,n} \to \mathrm{BGL}_\infty$ corresponds to the line bundle on $\mathrm{St}_{2n,1}/\mathbb{G}_m$ corresponding to the $\mathbb{G}_m$-bundle $\mathrm{St}_{2n,1} \to St_{2n,1}/\mathbb{G}_m$.

(iii) The above line bundle on $\mathrm{St}_{2n,1}/\mathbb{G}_m$ pulls back under the map $\bar{\phi}_n$ (see (2.19)) to the line bundle on $\mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{GL}_{n-1})$ corresponding to the $\mathbb{G}_m$-bundle $\mathrm{St}_{2n,n}/(1 \times \mathrm{GL}_{n-1}) \to \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{GL}_{n-1})$.

(iv) The above line bundle on $\mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{GL}_{n-1})$ pulls back to β by the map

$$u_n : \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}^{gm,n} = \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})) \to \mathrm{St}_{2n,n}/(\mathbb{G}_m \times \mathrm{GL}_{n-1})$$

(see (2.22)).

(v) Recall that $\pi_n = \Sigma_{\boldsymbol{T}}^{\infty}\bar{u}_{n+}$. The above observations now show that the composite map

$$\lambda' \circ j_n \circ \pi_n = \lambda' \circ j_n \circ \Sigma_{\boldsymbol{T}}^{\infty}\bar{u}_{n+} = \lambda' \circ j_n \circ \Sigma_{\boldsymbol{T}}^{\infty}u_{n+} \circ \Sigma_{\boldsymbol{T}}^{\infty}\bar{\phi}_{n+} : \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}_{+}^{gm,n} \xrightarrow{j_n \circ \pi_n} \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\mathbb{G}_m^{gm} \xrightarrow{\lambda'} f_1\boldsymbol{K} = \widetilde{\boldsymbol{K}}$$

corresponds to the bundle β.

(vi) Now

$$\begin{aligned}\lambda' \circ \zeta_n &= \lambda' \circ j_n \circ \pi_{n+} \circ \tau_n \\ &= \lambda' \circ j_n \circ \Sigma_{\boldsymbol{T}}^{\infty}\bar{u}_{n+} \circ \tau_n = \lambda' \circ j_n \circ \Sigma_{\boldsymbol{T}}^{\infty}u_{n+} \circ \Sigma_{\boldsymbol{T}}^{\infty}\bar{\phi}_{n+} \circ \tau_n\end{aligned}$$

(see (2.25)).

Therefore, it follows that $\lambda'_*(\zeta_{n*}(u))$ corresponds to the composite map

$$\Sigma_{\boldsymbol{T}}^{\infty}\mathrm{BN}_{\mathrm{GL}_n}(T_n)_{+}^{gm,n} \xrightarrow{\mathrm{id}} \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{BN}_{\mathrm{GL}_n}(T_n)_{+}^{gm,n} \xrightarrow{\tau_n} \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(T_n)}_{+}^{gm,n} \xrightarrow{j_n \circ \pi_n} \Sigma_{\boldsymbol{T}}^{\infty}\mathrm{B}\mathbb{G}_m^{gm} \xrightarrow{\lambda'} f_1\boldsymbol{K} = \widetilde{\boldsymbol{K}}. \quad (2.38)$$

Therefore, at this point, in order to prove the commutativity of the bottom triangle in (2.29), it suffices to prove that

$$\tau_n^*(\beta) = \alpha, \tag{2.39}$$

where τ_n^* denotes the transfer map induced by the transfer τ_n on the Grothendieck groups. This is a straightforward computation making use of the direct images of coherent sheaves under finite étale maps as discussed in the following paragraphs, as well as Corollary 3.24. (See [Becker 1974, p. 142] for very similar arguments in the topological case.)

Denoting the total space of the vector bundle α by $E(\alpha)$, observe that $E(\alpha) = \mathrm{St}_{2n,n} \times_{\mathrm{N}_{\mathrm{GL}_n}(T_n)} W$, where W corresponds to the n-dimensional representation of GL_n forming the fibers of the vector bundle α. We let W' denote the representation W, but viewed as a representation of $\mathrm{N}_{\mathrm{GL}_n}(T_n)$. Let T_{n-1} denote the $(n-1)$-dimensional split torus forming the last $(n-1)$-factors in the split maximal torus T_n. Observe that on further restricting to the action of the subgroup $H = \mathbb{G}_m \times \mathrm{N}_{\mathrm{GL}_{n-1}}(T_{n-1})$, W' is the representation of $\mathrm{N}_{\mathrm{GL}_n}(T_n)$ that is induced from a 1-dimensional representation V of the subgroup H, that is, if $\{\sigma_i H \mid i = 1, \ldots, n\}$ is the complete set of left cosets of H in $\mathrm{N}_{\mathrm{GL}_n}(T_n)$, then $W' \cong \bigoplus_{i=1}^{n} V_i$, with each $V_i = V$ and where $\mathrm{N}_{\mathrm{GL}_n}(T_n)$ acts on W' as follows. For $g \in \mathrm{N}_{\mathrm{GL}_n}(T_n)$, if $g.\sigma_i = \sigma_k h$,

$h \in H$, then $g.v_i = h_k$, with $v_i = v_k \in V$. (This may be seen by observing that the normalizer of the maximal torus $\mathrm{N}_{\mathrm{GL}_n}(T_n)$ is the semidirect product of the symmetric group Σ_n and the maximal torus T_n.)

Next observe that

$$p : \mathrm{BN}_{\widetilde{\mathrm{GL}_n}(T_n)}^{gm,n} = \mathrm{St}_{2n,n}/H \to \mathrm{St}_{2n,n}/\mathrm{N}_{\mathrm{GL}_n}(T_n) = \mathrm{BN}_{\mathrm{GL}_n}(T_n)^{gm,n}$$

is a finite étale map of degree n. Then β identifies with the line bundle, with structure group $\mathbb{G}_m$, defined by $\mathrm{St}_{2n,n} \times_H V$ on

$$\mathrm{BN}_{\widetilde{\mathrm{GL}_n}(T_n)}^{gm,n} = \mathrm{St}_{2n,n}/H.$$

Clearly $p_*(\beta) = \alpha$. Therefore, it suffices to show that the transfer τ_n^* equals p_* in this case. That is, it suffices to prove that the transfer τ_n^* on Grothendieck groups identifies with the push-forward in this case, which is proven in more generality in the next section of this paper (see Corollary 3.24). This, therefore, completes the proof of the proposition. □

Proof of Proposition 2.13. One observes from Proposition 2.27 that

$$\bar{q}_{n*} = \lambda_* \circ \zeta_{n*},$$

and that the direct system of maps $\{\zeta_{n*} \mid n\}$ are compatible. Therefore, it follows that the maps $\{\bar{q}_{n*} \mid n\}$ are also compatible, and taking the direct limit, we obtain $\bar{q}_* = \lim_{n\to\infty} \bar{q}_{n*} = \lambda_* \circ \lim_{n\to\infty} \zeta_{n*} = \lambda_* \circ \zeta_*$, which proves Proposition 2.13. □

3. The motivic transfer and the motivic Gysin maps associated to projective smooth morphisms

3A. One may observe from the discussion in (2.24) and (2.39) that we need to define a transfer for all finite étale maps between smooth schemes with reasonable properties, such as compatibility with base change, and then show that such a transfer induces the push-forward map at the level of algebraic K-theory. The definition of such a transfer map for finite étale maps is relatively straightforward. In fact there are existing constructions in the literature, as the referee has pointed out, which either provide such transfers directly or can be used to provide such transfers with a little bit of effort. We discuss some of these below in Remarks 3.8.

However, proving that such transfers coincide with the push-forward map on algebraic K-theory seems a bit involved: in the approach we take, one needs to first show that these transfers coincide with Gysin maps for all orientable generalized motivic cohomology theories, and then observe that such Gysin maps on algebraic K-theory agree with push-forward maps. A careful examination of the proof of the first statement shows that it takes more or less the same effort to define a transfer map for all *projective smooth* maps and show that it agrees with a Gysin map up to multiplication by a certain Euler class, which trivializes when the maps are finite étale. Therefore, we observe that as a consequence we are able to *derive the precise relationship between the transfer and Gysin maps associated to all projective smooth maps between smooth quasiprojective schemes on all orientable generalized motivic cohomology theories*. We believe this result is of independent interest, though not used in the rest of the paper in this generality.

Therefore, the general context in which we work in this section is the following. Let

$$p : E \to B \tag{3.1}$$

denote a *projective smooth map* of quasiprojective smooth schemes over the base field.

3B. ***The definition of a transfer for projective smooth maps.*** In order to motivate this construction, we quickly review the corresponding *Thom–Pontrjagin construction* in the context of classical algebraic topology. Here $p : E \to B$ denotes a smooth fiber bundle between compact manifolds E and B. Then one may obtain a closed imbedding of E in $B \times \mathbb{R}^N$ for N sufficiently large. We denote this imbedding by i. Therefore, one obtains the Thom–Pontrjagin collapse map

$$\mathrm{TP} : B_+ \wedge S^N \to \mathrm{Th}(\nu), \tag{3.2}$$

where ν denotes the normal bundle associated to the closed imbedding i. (One may recall that this is the starting point of the classical Atiyah duality [Atiyah 1961; Spanier and Whitehead 1955; Dold and Puppe 1983] as well as its étale variant as in [Joshua 1986; 1987] in the context of étale homotopy theory as in [Artin and Mazur 1969].)

We proceed to define a corresponding construction in the motivic context, making use of *the Voevodsky collapse* in the place of the Thom–Pontrjagin collapse. In the situation in (3.1), as the schemes E and B are assumed to be quasiprojective, one obtains a closed immersion $i : E \to B \times \mathbb{P}^N$ for a large enough N. Therefore, the discussion in [Voevodsky 2003, Proposition 2.7, Lemma 2.10 and Theorem 2.11] (see also [Carlsson and Joshua 2020, §10.4]) provides the Voevodsky collapse map

$$V : B_+ \wedge \boldsymbol{T}^n \to \mathrm{Th}(\nu) \tag{3.3}$$

for a suitably large n, and where ν denotes the vector bundle on E, which we call the *virtual normal bundle*; see [Carlsson and Joshua 2020, §10.8]. (See also [Hoyois 2017, §5.3] for a discussion on the collapse, which in this framework is originally due to Voevodsky.)

Let $\tau = \tau_{E/B}$ denote the relative tangent bundle associated to $p : E \to B$. Assume the relative dimension of p is d. Then it follows from [Voevodsky 2003, Proposition 2.7 through Theorem 2.11] (see also [Carlsson and Joshua 2020, §10.4, Definition 10.8]) that $\nu \oplus \tau$ is a trivial bundle on pull-back to $\widetilde{E}$, where $\widetilde{E}$ is a (functorial) affine replacement of E provided by the technique of [Jouanolou 1973].

Definition 3.4. Therefore, we may define the Becker–Gottlieb transfer in the situation of (3.1) as follows:

$$\mathrm{tr} : B_+ \wedge \boldsymbol{T}^n \xrightarrow{V} \mathrm{Th}(\nu) \xrightarrow{i_\nu} \mathrm{Th}(\nu \oplus \tau) \simeq E_+ \wedge \boldsymbol{T}^n, \tag{3.5}$$

where i_ν is the map induced by the obvious inclusion $\nu \to \nu \oplus \tau$. (See, for example, the proof of [Becker and Gottlieb 1975, Theorem 4.3].)

Proposition 3.6 (some basic properties of the transfer). (i) *Assume that in* (3.1), $E = E_0 \sqcup E_1$. *Denoting the corresponding transfers* $\mathrm{tr}_i : B_+ \wedge \boldsymbol{T}^n \to E_{i,+} \wedge \boldsymbol{T}^n$ *and* $\mathrm{tr} : B_+ \wedge \boldsymbol{T}^n \to E_+ \wedge \boldsymbol{T}^n$, *we have* $\mathrm{tr}^* = \mathrm{tr}_0^* + \mathrm{tr}_1^*$ *in any generalized motivic cohomology theory.*

(ii) *In case* $E = B$ *in* (3.1) *and the map* p *is the identity map on* B, *then* $\mathrm{tr}^* = \mathrm{id}$ *on any orientable generalized motivic cohomology theory.*

(iii) *Assume that the square*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ B' & \longrightarrow & B \end{array} \tag{3.7}$$

is cartesian. Then we obtain the following homotopy commutative diagram of transfer maps:

$$\begin{array}{ccccc} B_+ \wedge \boldsymbol{T}^n & \xrightarrow{V} & \mathrm{Th}(\nu) & \xrightarrow{i_\nu} & \mathrm{Th}(\nu \oplus \tau) \simeq E_+ \wedge \boldsymbol{T}^n \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ B'_+ \wedge \boldsymbol{T}^n & \xrightarrow{V'} & \mathrm{Th}(\nu') & \xrightarrow{i'_\nu} & \mathrm{Th}(\nu' \oplus \tau') \simeq E'_+ \wedge \boldsymbol{T}^n \end{array}$$

where V' *and* i'_ν *are the maps corresponding to* V *and* i_ν *when* B *and* E *are replaced by* B' *and* E'.

Proof. The proofs of the first and last statements are straightforward from the construction of the transfer. The second statement follows Corollary 3.24 by taking the map p to be the identity. □

Remarks 3.8. Here we briefly discuss other possible constructions of the transfer associated to *finite étale maps* $p : E \to B$, where E and B are quasiprojective smooth schemes over the base field k. One may find one such construction in [Röndigs and Østvær 2008, §2.3], as pointed out by the referee, where it is verified that this transfer is compatible with base change. (Making use of the 6-functor formalism in motivic homotopy theory as in [Ayoub 2007], the second author has also sketched a construction of a transfer for finite étale maps. As this is not all that different from the one in [Röndigs and Østvær 2008, §2.3], we do not discuss this any further here.) Therefore, what one needs to show is that this transfer on algebraic K-theory agrees with the push-forward. As this is not discussed in [Röndigs and Østvær 2008], all one can say is that a proof of this fact will likely follow the same steps as outlined above and discussed below in detail, except that the relative tangent bundle to the map p is trivial in this case.

The referee has also pointed out that the discussions in [Elmanto et al. 2021] and [Hoyois et al. 2021], making use of framed correspondences, provide a transfer map for finite étale maps that identify with push-forwards (of vector bundles) on algebraic K-theory.

In the rest of this section, we do the following:

(i) Making use of the same Voevodsky collapse used in the construction of the transfer, we proceed to define a Gysin map associated to projective and smooth maps between smooth quasiprojective schemes in all *orientable* generalized motivic cohomology theories.

(ii) Then we show that this Gysin map agrees with the Gysin maps defined by more traditional means, typically by factoring the given map $p : E \to B$ as the composition of a closed immersion of E into a relative projective space $B \times \mathbb{P}^n$ followed by the projection $\pi : B \times \mathbb{P}^n \to B$.

(iii) At this point, standard comparison results (see [Panin 2009, §2.9.1] which invokes [Thomason and Trobaugh 1990, 3.16, 3.17 and 3.18]) show that the above Gysin maps identify with the push-forward maps on the algebraic K-theory of smooth quasiprojective schemes.

(iv) Finally, we show that on orientable generalized motivic cohomology theories, the map induced by the transfer and the Gysin maps constructed below differ only by multiplication by the Euler class of the relative tangent bundle to the map p. As a result, when p is a finite étale map, the relative tangent bundle to the map p trivializes and the map induced by the transfer agrees with the push-forward on algebraic K-theory.

3C. ***Gysin maps associated to projective smooth maps on orientable generalized motivic cohomology theories.*** We begin by quickly reviewing the corresponding situation in algebraic topology. For any generalized cohomology theory h^*, the Thom–Pontrjagin collapse in (3.2) induces the map

$$\mathrm{TP}^* : h^*(\mathrm{Th}(\nu)) \to h^*(B_+ \wedge S^N).$$

We further assume that h^* is an orientable cohomology theory in the sense that it has a Thom class $T(\nu) \in h^c(\mathrm{Th}(\nu))$ (where c is the codimension of E in $B \times S^N$), so that cup product with this class defines the Thom isomorphism $h^*(E) \to h^{*+c}(\mathrm{Th}(\nu))$.

In this case one also observes the suspension isomorphism $h^*(B_+ \wedge S^N) \cong h^{*-N}(B)$. Thus the composition

$$p_* : h^*(E) \xrightarrow{\cup T(\nu)} h^{*+c}(\mathrm{Th}(\nu)) \xrightarrow{\mathrm{TP}^*} h^{*+c}(B_+ \wedge S^N) \cong h^{*+c-N}(B) \tag{3.9}$$

defines a Gysin map. One may observe that if the relative dimension of E over B is d, then $\mathrm{c} = N - d$, so that $h^{*+c-N}(B) = h^{*-d}(B)$ as required of a Gysin map.

We proceed to define a corresponding Gysin map in the motivic context, *for orientable generalized motivic cohomology theories* in the sense of [Panin and Yagunov 2002, §2] (see also [Panin 2009]), making use of the Voevodsky collapse in place of the Thom–Pontrjagin collapse. In the situation in (3.1), as the schemes E and B are assumed to be quasiprojective, one obtains a closed immersion $i : E \to B \times \mathbb{P}^N$ for a large enough N. In this context, we recall the Voevodsky collapse

$$V : B_+ \wedge \boldsymbol{T}^n \to \mathrm{Th}(\nu) \tag{3.10}$$

as discussed above in (3.3). It should be clear that, with this collapse map replacing the Thom–Pontrjagin collapse, and generalized motivic cohomology theories that are orientable (and bigraded), one obtains a *Gysin map*

$$\begin{aligned} p_* : h^{*,\bullet}(E) &\xrightarrow{\cup T(\nu)} h^{*+2c,\bullet+c}(\mathrm{Th}(\nu)) \\ &\xrightarrow{V^*} h^{*+2c,\bullet+c}(B_+ \wedge \boldsymbol{T}^n) \cong h^{*+2c-2n,\bullet+c-n}(B) = h^{*-2d,\bullet-d}(B) \end{aligned} \tag{3.11}$$

if d is the relative dimension of E over B, $T(\nu)$ is the Thom-class of the bundle ν, and c is the rank of the vector bundle ν.

Next we proceed to show that the Gysin map defined above indeed agrees with Gysin maps that are defined by other more traditional means, such as in [Panin 2009] or [Panin and Yagunov 2002, §4 and §5]. (See also [Déglise 2008].) For this, we need to first recall the framework for defining the Voevodsky collapse. One may observe from [Voevodsky 2003, pp. 69–70] that one needs to consider the sequence of closed immersions

$$E \xrightarrow{i} B \times \mathbb{P}^d \xrightarrow{\mathrm{id} \times \Delta} B \times \mathbb{P}^d \times \mathbb{P}^d \xrightarrow{\mathrm{id} \times \mathrm{Segre}} B \times \mathbb{P}^{d^2+2d}, \tag{3.12}$$

where Segre denotes the Segre imbedding. *We let $m = d^2 + 2d$ henceforth.* Let ν denote the normal bundle to the above composite closed immersion and let c denote the codimension of this closed immersion. Then one obtains the following sequence of maps:

$$\begin{aligned} h^{*,\bullet}(E) &\xrightarrow{\cup T(\nu)} h^{*+2c,\bullet+c}(\mathrm{Th}(\nu)) \cong h^{*+2c,\bullet+c}(B \times \mathbb{P}^m/(B \times \mathbb{P}^m - E)) \\ &\xrightarrow{\mathrm{Gysin}_1'} h^{*+2c,\bullet+c}(B \times \mathbb{P}^m) \xrightarrow{\mathrm{Gysin}_2} h^{*+2c-2m,\bullet+c-m}(B). \end{aligned} \tag{3.13}$$

Here the map denoted Gysin_1' precomposed with the cup product with the Thom class $T(\nu)$ is the usual Gysin map associated to the composite closed immersion

$E \to B \times \mathbb{P}^m$ (see [Panin and Yagunov 2002, §4]) and the map denoted Gysin_2 is the usual Gysin map associated to the projection $B \times \mathbb{P}^m \to B$; see [Panin and Yagunov 2002, Definition 5.1].

3C1. *Deformation to the normal cone.* In order to relate the Gysin maps in (3.11) and (3.13), we first invoke the technique of deformation to the normal cone from [Panin 2009, §1.2.1, Theorem 1.2]. (See also [Panin and Smirnov 2000, §2.2.8].) Let $i : Y \to X$ denote a closed immersion of smooth schemes of finite type over k with normal bundle $\mathcal{N}$. Then there exists a smooth scheme $\widetilde{X}$ together with a smooth map $p : \widetilde{X} \to \mathbb{A}^1$ and a closed immersion $i : Y \times \mathbb{A}^1 \to \widetilde{X}$, so that the composition $p \circ i : Y \times \mathbb{A}^1 \to \mathbb{A}^1$ coincides with the projection $Y \times \mathbb{A}^1 \to \mathbb{A}^1$. Moreover, the following additional properties hold:

(1) The fiber of p over $1 \in \mathbb{A}^1$ is isomorphic to X and the base change of i by the imbedding $1 \in \mathbb{A}^1$ corresponds to the given imbedding $Y \to X$.

(2) The fiber of p over $0 \in \mathbb{A}^1$ is isomorphic to $\mathcal{N}$ and the base change of i by the imbedding $0 \in \mathbb{A}^1$ corresponds to the 0-section imbedding $Y \to \mathcal{N}$.

(3) If $Z \to Y$ is a closed immersion of a smooth subscheme of Y, then one obtains the diagram

$$(\mathcal{N}, \mathcal{N} - Z) \xrightarrow{i_0} (\widetilde{X}, \tilde{X} - Z \times \mathbb{A}^1) \xleftarrow{i_1} (X, X - Z),$$

and hence the following diagram for any orientable generalized motivic cohomology theory $h^{*,\bullet}$ with both the horizontal maps being isomorphisms:[2]

$$h^{*,\bullet}(\mathcal{N}, \mathcal{N} - Z) \xleftarrow{i_0^*} h^{*,\bullet}(\widetilde{X}, \widetilde{X} - Z \times \mathbb{A}^1) \xrightarrow{i_1^*} h^{*,\bullet}(X, X - Z). \tag{3.14}$$

(4) Moreover, in the above situation the normal bundle to the composite closed immersion $Z \to Y \to \mathcal{N}$ is isomorphic to the sum $\mathcal{N}_{Z,Y} \oplus \mathcal{N}_{|Z}$, where $\mathcal{N}_{Z,Y}$ denotes the normal bundle associated to the closed immersion $Z \to Y$.

Proposition 3.15. *Assume the above situation. Then the Gysin map defined in* (3.11) *agrees with the Gysin map defined in* (3.13).

Proof. We follow the constructions in [Voevodsky 2003, pp. 69–70]. Accordingly the second $\mathbb{P}^d$ in $\mathbb{P}^d \times \mathbb{P}^d$ in (3.12) is identified with the dual projective space and $\boldsymbol{H}$ denotes the incidence hyperplane in $\mathbb{P}^d \times \mathbb{P}^d$. Then it is observed there that $\widetilde{\mathbb{P}}^d = \mathbb{P}^d \times \mathbb{P}^d - \boldsymbol{H}$ considered as a scheme over $\mathbb{P}^d$ by the projection to the first factor (p_1) is an affine space bundle; in fact, this is an instance of what is known as Jouanolou's trick.

Let N denote the normal bundle to the Segre imbedding of $B \times \mathbb{P}^d \times \mathbb{P}^d$ in $B \times \mathbb{P}^{\mathrm{n}}$. If $j : B \times \widetilde{\mathbb{P}}^d \to B \times \mathbb{P}^d \times \mathbb{P}^d$ denotes the open immersion, we let $j^*(N)$ be the

[2]This may be viewed as a cohomology variant of the purity theorem; see [Morel and Voevodsky 1999, Theorem 2.23, p. 115].

pull-back of N to $B \times \widetilde{\mathbb{P}}^d$. Since $\widetilde{\mathbb{P}}^d \to \mathbb{P}^d$ is a torsor for a vector bundle, and all vector bundles on affine spaces are trivial by the affirmative solution to the Serre conjecture, one may see that $j^*(N) \cong p_1^*(\Delta^*(N))$, where $\Delta : B \times \mathbb{P}^d \to B \times \mathbb{P}^d \times \mathbb{P}^d$ is the diagonal imbedding.

Let $\mathcal{E}$ denote the pull-back $p_1^*(\tau_{B\times\mathbb{P}^d/B})$ (where $\tau_{B\times\mathbb{P}^d/B}$ denotes the relative tangent bundle to $B \times \mathbb{P}^d$ over B), and let ν_1 denote the normal bundle to the closed immersion $i : E \to B \times \mathbb{P}^d$. Let $p : \widetilde{E} \to E$ denote the map induced by $p_1 : B \times \widetilde{\mathbb{P}}^d \to B \times \mathbb{P}^d$ when $\widetilde{E}$ is defined by the cartesian square

$$\begin{tikzcd}
\widetilde{E} \arrow[r] \arrow[d, "p"] & B \times \widetilde{\mathbb{P}}^d \arrow[d, "p_1"] \\
E \arrow[r] & B \times \mathbb{P}^d
\end{tikzcd}$$

Let $\tilde{\nu}_1$ denote the normal bundle to the induced closed immersion $\widetilde{E} \to B \times \widetilde{\mathbb{P}}^d$. Then denoting the Thom class of the bundle $\mathcal{E}_1 = (\mathcal{E} \oplus j^*(N))_{|\widetilde{E}} \oplus \tilde{\nu}_1$ by $T(\mathcal{E}_1)$, we obtain the sequence of maps

$$h^{*,\bullet}(\widetilde{E}) \xrightarrow{\cup T(\mathcal{E}_1)} h^{*+2c,\bullet+c}(\mathrm{Th}(\mathcal{E}_1)) \to h^{*+2c,\bullet+c}(\mathrm{Th}(\mathcal{E} \oplus j^*(N))). \tag{3.16}$$

The last map is obtained from the observation that the normal bundle to the composite closed immersion $\widetilde{E} \to B \times \widetilde{\mathbb{P}}^d \xrightarrow{\text{0-section}} \mathcal{E} \oplus j^*(N)$ is $\mathcal{E}_1$ (see 3C1(4) above). At this point we make use of the identification $j^*(N) = p_1^*(\Delta^*(N))$, so that

$$\mathcal{E}_1 = p^*(i^*(\tau_{B\times\mathbb{P}^d/B} \oplus \Delta^*(N)) \oplus \nu_1) \quad \text{and} \quad \mathcal{E} \oplus N = p_1^*(\tau_{B\times\mathbb{P}^d/B} \oplus \Delta^*(N)),$$

which then readily provides the commutativity of the following diagram with $\mathcal{E}_0 = i^*(\tau_{B\times\mathbb{P}^d/B} \oplus \Delta^*(N)) \oplus \nu_1$:

$$\begin{tikzcd}
h^{*,\bullet}(\widetilde{E}) \arrow[dr, "\cup T(\mathcal{E}_1)"] \arrow[dd, "\cong"] & & \\
& h^{*+2c,\bullet+c}(\mathrm{Th}(\mathcal{E}_1)) \arrow[dr] \arrow[dd, "\cong"] & \\
h^{*,\bullet}(E) \arrow[dr, "\cup T(\mathcal{E}_0)"] & & h^{*+2c,\bullet+c}(\mathrm{Th}(\mathcal{E} \oplus j^*(N))) \arrow[dd, "\cong"] \\
& h^{*+2c,\bullet+c}(B \times \mathbb{P}^m/(B \times \mathbb{P}^m - E)) \arrow[dr] & \\
& & h^{*+2c,\bullet+c}(B \times \mathbb{P}^m/(B \times \mathbb{P}^m - (B \times \Delta\mathbb{P}^d)))
\end{tikzcd} \tag{3.17}$$

The left-most vertical map above is an isomorphism as $p : \widetilde{E} \to E$ is an affine space bundle. To see that the next vertical map is an isomorphism, one needs to observe that

$$\mathrm{Th}(\mathcal{E}_1) = \mathrm{Th}\big(p^*(i^*(\tau_{B\times\mathbb{P}^d/B} \oplus \Delta^*(N)) \oplus \nu_1)\big) \simeq \mathrm{Th}(i^*(\tau_{B\times\mathbb{P}^d/B} \oplus \Delta^*(N)) \oplus \nu_1).$$

At this point we make use of the isomorphisms in (3.14) and 3C1(4) to obtain the isomorphism

$$h^{*,\bullet}\big(\mathrm{Th}(i^*(\tau_{B\times\mathbb{P}^d/B}\oplus\Delta^*(N))\oplus\nu_1)\big)\cong h^{*,\bullet}((B\times\mathbb{P}^m)/(B\times\mathbb{P}^m-E)).$$

To see the last vertical map in (3.17) is an isomorphism, one first observes that

$$\mathrm{Th}(\mathcal{E}\oplus j^*(N))=\mathrm{Th}(p_1^*(\tau_{B\times\mathbb{P}^d/B}\oplus\Delta^*(N)))\simeq\mathrm{Th}(\tau_{B\times\mathbb{P}^d/B}\oplus\Delta^*(N)),$$

and then adopts a similar argument to obtain the isomorphism

$$h^{*,\bullet}(\mathrm{Th}(\tau_{B\times\mathbb{P}^d/B}\oplus\Delta^*(N))\cong h^{*,\bullet}\big((B\times\mathbb{P}^m)/((B\times\mathbb{P}^m)-(B\times\Delta\mathbb{P}^d))\big).$$

We also obtain the following commutative diagram:

$$\begin{tikzcd}
h^{*+2c,\bullet+c}\big(B\times\mathbb{P}^m/((B\times\mathbb{P}^m-B\times(\mathbb{P}^d\times\mathbb{P}^d))\cup B\times\boldsymbol{H}_\infty)\big) \arrow[dr] \arrow[dd] & & \\
& h^{*+2c,\bullet+c}(B\times\mathbb{P}^m/(B\times\boldsymbol{H}_\infty)) \arrow[dr, "\cong"] \arrow[dd] & \\
h^{*+2c,\bullet+c}(B\times\mathbb{P}^m/(B\times\mathbb{P}^m-(B\times\mathbb{P}^d\times\mathbb{P}^d))) \arrow[dr] & & h^{*+2c,\bullet+c}(B_+\wedge\boldsymbol{T}^m) \arrow[dd, "\cong"] \\
& h^{*+2c,\bullet+c}(B\times\mathbb{P}^m) \arrow[dr, "\mathrm{Gysin}"] & \\
& & h^{*+2c-2m,\bullet+c-m}(B)
\end{tikzcd}\tag{3.18}$$

where $\boldsymbol{H}_\infty$ is a hyperplane in $B\times\mathbb{P}^m$ which pulls back to the incidence hyperplane in $B\times\mathbb{P}^d\times\mathbb{P}^d$ under the Segre imbedding. Next we recall the following identification (see [Voevodsky 2003, proof of Lemma 2.10]):

$$\mathrm{Th}(\mathcal{E}\oplus j^*(N))\simeq B\times\mathbb{P}^m/\big((B\times\mathbb{P}^m-B\times(\mathbb{P}^d\times\mathbb{P}^d))\cup B\times\boldsymbol{H}_\infty\big).$$

This shows that in this case, one obtains a composite *collapse map*

$$V:B_+\wedge\boldsymbol{T}^m\to\mathrm{Th}(\mathcal{E}\oplus j^*(N))\to\mathrm{Th}(\mathcal{E}_1),\tag{3.19}$$

and hence that one may compose the maps forming the top row of the diagram (3.17) followed by the maps forming the top row of the diagram (3.18). In view of the fact that $p:\widetilde{E}\to E$ is an affine replacement, $\mathrm{Th}(\mathcal{E}_1)\simeq\mathrm{Th}(\mathcal{E}_0)$, so that the collapse map in (3.19) defines a collapse

$$V:B_+\wedge\boldsymbol{T}^m\to\mathrm{Th}(\mathcal{E}_0),$$

which differs from the collapse map in (3.3) only by the addition of a trivial bundle, and hence a $\boldsymbol{T}$-suspension of some finite degree on both the source and the target; see [Voevodsky 2003, proof of Proposition 2.7 and Theorem 2.11]. Therefore, one may now observe that the composition of the maps in the top rows of the two diagrams followed by the suspension isomorphism forming the right-most vertical map in the second diagram identifies with the map p_* in (3.11).

Observe that there is a natural map

$$h^{*+2c,\bullet+c}\big(B \times \mathbb{P}^m/(B \times \mathbb{P}^m - (B \times \Delta\mathbb{P}^d))\big) \\ \to h^{*+2c,\bullet+c}\big(B \times \mathbb{P}^m/(B \times \mathbb{P}^m - (B \times \mathbb{P}^d \times \mathbb{P}^d))\big).$$

Therefore one may compose the maps forming the bottom rows of the two diagrams (3.17) and (3.18). The composition of the maps forming the bottom rows of the two diagrams defines the Gysin map in (3.13). The commutativity of the two diagrams proves these two maps are the same. ☐

Theorem 3.20. *Let $h^{*,\bullet}$ denote a generalized motivic cohomology which is orientable in the above sense. Let* tr *denote the transfer as in* (3.5)*. Then if* $\mathrm{eu}(\tau)$ *denotes the Euler class of the bundle τ, we obtain the relation*

$$\mathrm{tr}^*(\alpha) = p_*(\alpha \cup \mathrm{eu}(\tau)), \quad \alpha \in h^{*,\bullet}(E), \tag{3.21}$$

where p_* denotes the Gysin map defined above in (3.11)*.*

Proof. As shown in [Becker and Gottlieb 1975, Theorem 4.3], and adopting the terminology as in (3.3) and (3.11), it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
h^{*,\bullet}(E) \arrow[rr, "\cup\,\mathrm{eu}(\tau)"] \arrow[d, "\cong"] & & h^{*+2d,\bullet+d}(E) \arrow[r, "p_*"] \arrow[d, "\cup T(\nu)"] & h^{*,\bullet}(B) \arrow[d, "\cong"] \\
h^{*+2n,\bullet+n}(E_+\wedge \boldsymbol{T}^n) \arrow[r, "\cong"] & h^{*+2n,\bullet+n}(\mathrm{Th}(\nu\oplus\tau)) \arrow[r, "i^*"] & h^{*+2n,\bullet+n}(\mathrm{Th}(\nu)) \arrow[r, "V^*"] & h^{*+2n,\bullet+n}(B_+\wedge \boldsymbol{T}^n)
\end{tikzcd} \tag{3.22}$$

Here d is the relative dimension of E over B and i denotes the map of Thom-spaces induced by the inclusion $\nu \to \nu \oplus \tau$. The definition of the Gysin map as in (3.11) readily proves the commutativity of the right square, so that it suffices to prove the commutativity of the left square. This results from the commutativity of the following diagram:

$$
\begin{tikzcd}
h^{*,\bullet}(E) \arrow[r, "\cup \mathrm{eu}(\tau)"] \arrow[d, "\cup T(\tau)"] & h^{*+2d,\bullet+d}(E) \arrow[r, "\cup T(\nu)"] \arrow[d, "\cong"] & h^{*+2n,\bullet+n}(\mathrm{Th}(\nu)) \arrow[d, "\cong"] \\
h^{*+2d,\bullet+d}(\mathrm{Th}(\tau)) \arrow[r] \arrow[d, "\cup T(\nu)"] & h^{*+2d,\bullet+d}(E(\tau)) \arrow[r, "\cup T(\pi_1^*(\nu))"] \arrow[d, "\cup T(\pi_1^*(\nu))"] & h^{*+2n,\bullet+n}(\mathrm{Th}(\pi_1^*(\nu))) \arrow[dl, "\mathrm{id}"] \\
h^{*+2n,\bullet+n}(\mathrm{Th}(\tau \oplus \nu)) \arrow[r] & h^{*+2n,\bullet+n}(\mathrm{Th}(\pi_1^*(\nu))) &
\end{tikzcd}
\tag{3.23}
$$

Here, if α denotes a vector bundle, then $\mathrm{Th}(\alpha)$ denotes the Thom space and $T(\alpha)$ the Thom class of α. Observe that the composition of the top row and the right vertical map in the left square of (3.22) equals the composition of the maps in the top row of (3.23). The composition of the map in the left column and the first bottom map in (3.22) clearly equals the composition of the two vertical maps in the left-most column of (3.23). Since $E(\tau)$ denotes the total space of the vector bundle τ, we obtain the isomorphism $h^{*+2d,\bullet+d}(E) \xrightarrow{\cong} h^{*+2d,\bullet+d}(E(\tau))$ and also the isomorphism $h^{*+2n,\bullet+n}(\mathrm{Th}(\nu)) \xrightarrow{\cong} h^{*+2n,\bullet+n}(\mathrm{Th}(\pi_1^*(\nu)))$, where $\pi_1 : E(\tau) \to E$ denotes the projection. Moreover, under the above isomorphisms, the map denoted i^* in (3.22) identifies with the bottom-most map in (3.23). These observations prove the commutativity of the diagram (3.23) and hence the commutativity of the diagram (3.22) as well. □

Corollary 3.24. *Let $p : E \to B$ denote a finite étale map between smooth quasiprojective schemes. If $h^{*,\bullet}$ is an orientable generalized motivic cohomology theory defined by a motivic spectrum, then one has the equality*

$$\mathrm{tr}^* = p_*,$$

where tr^* *denotes the map induced by the motivic Becker–Gottlieb transfer* tr (see (3.5)) *in the above cohomology theory and p_* denotes the Gysin map. Moreover, for algebraic K-theory, the Gysin map p_* agrees with the finite push-forward defined for coherent sheaves.*

Proof. The first statement is an immediate consequence of Theorem 3.20, once one observes that the Euler class $\mathrm{eu}(\tau)$ is trivial, which follows from the fact that p is finite étale and τ denotes the relative tangent bundle of the map p. The second statement on the Gysin map p_* for algebraic K-theory follows from [Panin 2009, §2.9.1], invoking [Thomason and Trobaugh 1990, 3.16, 3.17 and 3.18]. Observe that push-forward by finite étale maps sends vector bundles to vector bundles, and for smooth quasiprojective schemes over k, the K-theory of coherent sheaves identifies with the K-theory of vector bundles. □

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References

[Ananyevskiy 2021] A. Ananyevskiy, "On the $\mathbb{A}^1$-Euler characteristic of the variety of maximal tori in a reductive group", preprint, 2021. arXiv 2011.14613v2

[Artin and Mazur 1969] M. Artin and B. Mazur, *Étale homotopy*, Lecture Notes in Math. **100**, Springer, 1969. MR Zbl

[Asok et al. 2020] A. Asok, J. Fasel, and M. J. Hopkins, "Localization and nilpotent spaces in $\mathbb{A}^1$-homotopy theory", preprint, 2020. arXiv 1909.05195v2

[Atiyah 1961] M. F. Atiyah, "Thom complexes", *Proc. London Math. Soc.* (3) **11** (1961), 291–310. MR Zbl

[Ayoub 2007] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*, *I*, Astérisque **314**, Soc. Math. France, Paris, 2007. MR Zbl

[Bachmann and Hoyois 2021] T. Bachmann and M. Hoyois, *Norms in motivic homotopy theory*, Astérisque **425**, Soc. Math. France, Paris, 2021. MR Zbl

[Baeza 1978] R. Baeza, *Quadratic forms over semilocal rings*, Lecture Notes in Math. **655**, Springer, 1978. MR Zbl

[Becker 1974] J. C. Becker, "Characteristic classes and K-theory", pp. 132–143 in *Algebraic and geometrical methods in topology* (Binghamton, NY, 1973), edited by L. F. McAuley, Lecture Notes in Math. **428**, Springer, 1974. MR Zbl

[Becker and Gottlieb 1975] J. C. Becker and D. H. Gottlieb, "The transfer map and fiber bundles", *Topology* **14** (1975), 1–12. MR Zbl

[Becker and Gottlieb 1976] J. C. Becker and D. H. Gottlieb, "Transfer maps for fibrations and duality", *Compositio Math.* **33**:2 (1976), 107–133. MR Zbl

[Carlsson and Joshua 2020] G. Carlsson and R. Joshua, "Motivic and étale Spanier–Whitehead duality and the Becker–Gottlieb transfer", preprint, 2020. arXiv 2007.02247v2

[Chevalley et al. 1958] C. Chevalley, A. Grothendieck, and J.-P. Serre, *Séminaire C. Chevalley*, 2*e année*: *Anneaux de Chow et applications*, Secrétariat mathématique, Paris, 1958. MR

[Déglise 2008] F. Déglise, "Around the Gysin triangle, II", *Doc. Math.* **13** (2008), 613–675. MR Zbl

[Dold and Puppe 1983] A. Dold and D. Puppe, "Duality, trace and transfer", pp. 81–97 in *Topology*, vol. 154, 1983. In Russian; translated in *Proceedings of the international conference on geometric topology* (Warsaw,1978), PWN, Warsaw (1980), 81–102. MR Zbl

[Elmanto and Khan 2020] E. Elmanto and A. A. Khan, "Perfection in motivic homotopy theory", *Proc. Lond. Math. Soc.* (3) **120**:1 (2020), 28–38. MR Zbl

[Elmanto et al. 2021] E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, and M. Yakerson, "Motivic infinite loop spaces", *Camb. J. Math.* **9**:2 (2021), 431–549. MR Zbl

[Hornbostel 2005] J. Hornbostel, "$\mathbb{A}^1$-representability of Hermitian K-theory and Witt groups", *Topology* **44**:3 (2005), 661–687. MR Zbl

[Hoyois 2017] M. Hoyois, "The six operations in equivariant motivic homotopy theory", *Adv. Math.* **305** (2017), 197–279. MR Zbl

[Hoyois et al. 2021] M. Hoyois, J. Jelisiejew, D. Nardin, B. Totaro, and M. Yakerson, "The Hilbert scheme of infinite affine space and algebraic K-theory", preprint, 2021. arXiv 2002.11439v3

[Joshua 1986] R. Joshua, "Spanier–Whitehead duality in étale homotopy", *Trans. Amer. Math. Soc.* **296**:1 (1986), 151–166. MR Zbl

[Joshua 1987] R. Joshua, "Becker–Gottlieb transfer in étale homotopy", *Amer. J. Math.* **109**:3 (1987), 453–497. MR Zbl

[Joshua and Pelaez 2020] R. Joshua and P. Pelaez, "Additivity and double coset formulae for the motivic and étale Becker–Gottlieb transfer", preprint, 2020. arXiv 2007.02249v2

[Jouanolou 1973] J. P. Jouanolou, "Une suite exacte de Mayer–Vietoris en K-théorie algébrique", pp. 293–316 in *Algebraic K-theory*, *I*: *Higher K-theories* (Seattle, WA, 1972), edited by H. Bass, Lecture Notes in Math. **341**, Springer, 1973. MR Zbl

[Kleen 2018] V. Kleen, "A motivic Snaith decomposition", preprint, 2018. arXiv 1812.02352v1

[Levine 2008] M. Levine, "The homotopy coniveau tower", *J. Topol.* **1**:1 (2008), 217–267. MR Zbl

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Math. Series **33**, Princeton Univ. Press, 1980. MR Zbl

[Morel 2004] F. Morel, "On the motivic π_0 of the sphere spectrum", pp. 219–260 in *Axiomatic, enriched and motivic homotopy theory*, edited by J. P. C. Greenlees, NATO Sci. Ser. II Math. Phys. Chem. **131**, Kluwer Acad. Publ., Dordrecht, 2004. MR Zbl

[Morel 2012] F. Morel, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Math. **2052**, Springer, 2012. MR Zbl

[Morel and Voevodsky 1999] F. Morel and V. Voevodsky, "$\mathbb{A}^1$-homotopy theory of schemes", *Inst. Hautes Études Sci. Publ. Math.* **1999**:90 (1999), 45–143. MR Zbl

[Mumford et al. 1994] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergeb. der Math. **34**, Springer, 1994. MR Zbl

[Panin 2009] I. Panin, "Oriented cohomology theories of algebraic varieties, II", *Homology Homotopy Appl.* **11**:1 (2009), 349–405. MR Zbl

[Panin and Smirnov 2000] I. Panin and A. Smirnov, "Push-forwards in oriented cohomology theories of algebraic varieties", preprint, 2000, available at https://faculty.math.illinois.edu/K-theory/0459/preprint3.pdf.

[Panin and Yagunov 2002] I. Panin and S. Yagunov, "Rigidity for orientable functors", *J. Pure Appl. Algebra* **172**:1 (2002), 49–77. MR Zbl

[Röndigs and Østvær 2008] O. Röndigs and P. A. Østvær, "Rigidity in motivic homotopy theory", *Math. Ann.* **341**:3 (2008), 651–675. MR Zbl

[Schlichting 2017] M. Schlichting, "Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem", *J. Pure Appl. Algebra* **221**:7 (2017), 1729–1844. MR Zbl

[Segal 1973] G. Segal, "The stable homotopy of complex projective space", *Quart. J. Math. Oxford Ser.* (2) **24** (1973), 1–5. MR Zbl

[Spanier and Whitehead 1955] E. H. Spanier and J. H. C. Whitehead, "Duality in homotopy theory", *Mathematika* **2** (1955), 56–80. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, "Higher algebraic K-theory of schemes and of derived categories", pp. 247–435 in *The Grothendieck Festschrift*, vol. III, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Totaro 1999] B. Totaro, "The Chow ring of a classifying space", pp. 249–281 in *Algebraic K-theory* (Seattle, WA, 1997), edited by W. Raskind and C. Weibel, Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Voevodsky 1998] V. Voevodsky, "$\mathbb{A}^1$-homotopy theory", pp. 579–604 in *Proceedings of the International Congress of Mathematicians* (Berlin, 1998), vol. I, 1998. MR Zbl

[Voevodsky 2002a] V. Voevodsky, "Open problems in the motivic stable homotopy theory, I", pp. 3–34 in *Motives, polylogarithms and Hodge theory* (Irvine, CA, 1998), vol. I, edited by F. Bogomolov and L. Katzarkov, Int. Press Lect. Ser. **3**, International Press, Somerville, MA, 2002. MR Zbl

[Voevodsky 2002b] V. Voevodsky, "A possible new approach to the motivic spectral sequence for algebraic K-theory", pp. 371–379 in *Recent progress in homotopy theory* (Baltimore, MD, 2000), edited by D. M. Davis et al., Contemp. Math. **293**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

[Voevodsky 2003] V. Voevodsky, "Motivic cohomology with $\mathbb{Z}/2$-coefficients", *Publ. Math. Inst. Hautes Études Sci.* **2003**:98 (2003), 59–104. MR Zbl

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