On Murre's conjectures for blow-ups: a constructive approach

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Abstract

We show, by an explicit construction of the relevant Chow-Kunneth projectors, that if X is the quotient of a smooth projective variety by a finite group action and Y is obtained from X by blowing up a finite set of points, then (under appropriate hypotheses) each of Murre's conjectures holds for Y if and only if it holds for X. The novelty of our approach is that, finite dimensionality of the corresponding motives is never used or needed and that our construction explicitly provides the Chow-Kunneth projectors for X in terms of the Chow-Kunneth projectors for Y and vice-versa. This is applied to two classes of examples: one where the group action is trivial and the other to Kummer manifolds over algebraically closed fields of characteristic different from 2, which are obtained by blowing up the corresponding Kummer varieties along the 2-torsion points. In particular, our results imply that the Kummer manifolds satisfy part of Murre's vanishing conjecture (B) in all dimensions and the full vanishing conjecture in dimensions 3 and 4.

1 Introduction

Among the many deep conjectures about motives are *Murre's conjectures*, first enunciated in [MU] (see Section 1.1 for a full statement). These conjectures are equivalent to the Bloch-Beilinson conjectures and have only been established for certain classes of varieties – among them curves, surfaces, the product of a curve and a surface, abelian varieties of dimension at most 4 [MU], and a handful of other cases (see for example, [GM], [KIM], [VI]).

The work in this article was motivated by a desire to study Murre's conjectures for blow-ups, particularly in the context of our prior work (see [AJ]) where we provided an explicit construction of Chow-Kunneth projectors for quotients of Abelian varieties by finite groups. In particular, our approach is constructive, providing explicitly a construction of the Chow-Kunneth projectors for the blow-up in

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terms of the Chow-Kunneth projectors for the original variety and never requiring or invoking finite dimensionality of the motives of the corresponding varieties. A discussion on Murre's conjectures appears in section 2 of the paper.

Our main result is the following theorem:

Theorem 1.1. *Suppose X' is a smooth variety of dimension $d > 0$ over a field k and G a finite group acting on X'. Let $X = X'/G$ denote the quotient variety and $q_X : X' \to X$ the quotient map. Let $T \subseteq X$ be a finite set of k-rational points and $f : Y \to X$ the blow-up of X along T. Suppose*

- *The morphism f is induced by a morphism of smooth varieties, i.e. $Y = Y'/G$ for some smooth variety Y' and there is some morphism $f' : Y' \to X'$ such that $q_X \circ f' = f \circ q_Y$, where $q_Y : Y' \to Y$ is the obvious map.*
- *Each component of the exceptional divisor of the blow-up $f : Y \to X$ is isomorphic to $\mathbb{P}^{d-1}$.*

Then Y has a Chow-Künneth decomposition if and only if X has a Chow-Künneth decomposition, where the Chow-Kunneth decomposition for Y and X are related by (2.2.2) *and Corollary 2.10. Furthermore, the Chow-Künneth decomposition for Y satisfies Poincaré duality (respectively, Murre's conjecture **B**, **B'**, **C**, **D**) if and only if that on X does.*

A notable application of this theorem is to *Kummer manifolds.* Given an abelian variety A over an algebraically closed field of characteristic different from 2, its associated Kummer variety K_A can be described as the quotient of A by the involution $a \mapsto -a$. If A has dimension $d > 0$, K_A has 2^{2d} singular points, which are the images of the 2-torsion points under the quotient map $A \to K_A$. Blowing these points up yields a smooth variety K'_A which we call the associated *Kummer manifold.*

Corollary 1.2. *Let A be an abelian variety of dimension d over an algebraically closed field of characteristic different from 2 and let K'_A be its associated Kummer manifold. Then K'_A has a Chow-Künneth decomposition satisfying Poincaré duality. The projector ρ_i acts as zero on $CH^j_{\mathbb{Q}}(K'_A)$ for $i < j$ and also for $i > j + d$, i.e. Murre's conjecture **B'** holds for the action of the projectors $\{\rho_i | i\}$. In particular, when $d \leq 4$, ρ_i acts trivially on $CH^j_{\mathbb{Q}}(X)$ for $i < j$ and also for $i > 2j$ so that Murre's vanishing conjecture **B** holds for K'_A.*

Here is short outline of the paper. In the rest of this section, we set up the basic framework by reviewing basic results in this area and establishing the terminology used in the rest of the paper. Section 2 forms the technical heart of the paper, where an explicit Chow-Kunneth decomposition for blow-ups of (what we call) pseudo-smooth varieties at a finite number of rational points is established.

In section 3, we show that, in the context of section 2.2, each of Murre's other conjectures holds for the blown-up variety if and only if it holds for the original variety. In section 4, we consider several applications. The first application is to so-called Kummer manifolds. The second application is to show that Murre's conjectures hold for the variety obtained by blowing up a finite number of k-rational points on a smooth variety if and only if they hold for the original variety.

We also want to point out that the notion of finiteness for motives can be used to predict the existence of a Chow-Künneth decomposition for Kummer manifolds of dimension 4 or less, satisfying Murre's conjectures, **B** and **C**. However explicitly constructing such projectors using finite dimensionality is tedious and of exponential complexity in the dimension of the abelian variety as we showed in [AJ, section 2]. Therefore, the explicit construction of the Chow-Künneth projectors is often quite useful: see, for example, [KMP]. In addition, it does not seem likely that finite dimensionality provides a means to prove Murre's conjectures **B'** in arbitrary dimensions as we have.

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1.1 Basic framework and terminology: Chow motives for pseudo-smooth varieties

Let k be a field. For convenience, we refer to the quotient of a smooth variety by the action of a finite group (scheme) as a *pseudo-smooth variety.* It is shown in [F, Example 16.1.13] that the basic machinery of intersection theory and the usual formalism for correspondences extends naturally from smooth varieties to pseudo-smooth varieties, provided one uses rational coefficients. Thus, we may define the category $\mathcal{M}_k(\mathbb{Q})$ of (rational) *Chow motives of pseudo-smooth projective varieties* in the same way as for smooth projective varieties (see [F, Chapter 16], [MA], or [SCH]). Throughout this article, we use the notation $CH^i(X)$ for the Chow groups of (an algebraic scheme) X and write $CH^i_{\mathbb{Q}}(X) = CH^i(X) \otimes \mathbb{Q}$. It is worth noting if a finite group G acts on a smooth variety X, the machinery of equivariant intersection theory allows us to identify the equivariant Chow groups $CH^*_G(X)_{\mathbb{Q}}$ with $CH^*_{\mathbb{Q}}(X/G)$. Thus, the extension of the usual formalism of correspondences to pseudo-smooth varieties can also be derived from the analogous theory in the equivariant context.

Since we will make use of many projection maps in the sequel, we reserve the symbol p for these, with the superscript indicating the domain and the subscript the range. For example, if k is a field and X, Y, Z are pseudo-smooth varieties over k, the map

$$p_{13}^{XYZ} : X \times Y \times Z \to X \times Z \tag{1.1.1}$$

is the map $(x, y, z) \mapsto (x, z)$. This will be denoted p_{13} if the choice of X, Y and Z are clear. A subscript of $\emptyset$ indicates the structure morphism; for example, $p_{\emptyset}^{XY}$ is the structure morphism $X \times Y \to \mathrm{Spec}\ k$. Given cycles $\alpha \in CH^i(X)$ and $\beta \in CH^j(Y)$, we refer to their exterior product $\alpha \times \beta = p_1^{XY*}\alpha \cdot p_2^{XY*}\beta$ as a *product cycle* on $X \times Y$ *of type* (i, j); by abuse of terminology, we sometimes also refer to linear combinations of such elements as product cycles. (These are referred to as *degenerate correspondences* in [F].) When $\gamma \in CH^*(X \times Y)$ and $\delta \in CH^*(Y \times Z)$, we define their composition $\delta \bullet \gamma \in CH^*(X \times Z)$ by the formula

$$\delta \bullet \gamma = p_{13}^{XYZ}{}_*(p_{12}^{XYZ*}\gamma \cdot p_{23}^{XYZ*}\delta).$$

If $s : X \times Y \to Y \times X$ is the exchange of factors, we define the *transpose* of $\alpha \in CH^*(X \times Y)$ by $\alpha^t := s^*(\alpha)$. We write Δ_X for the diagonal in $X \times X$.

We say that a variety X of dimension d has a *Chow-Künneth decomposition* if the diagonal class $[\Delta_X] \in CH^d_{\mathbb{Q}}(X \times X)$ has a decomposition into mutually orthogonal idempotents, each of which maps onto the appropriate Künneth component under the cycle map. More precisely, there exist $\pi_i \in CH^d_{\mathbb{Q}}(X \times X)$, $0 \leq i \leq 2d$, such that:

(i) $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$;

(ii) $\pi_i \bullet \pi_i = \pi_i$ for all i, and $\pi_i \bullet \pi_j = 0$ for $i \neq j$;

(iii) If H^* is a Weil cohomology theory, then for each i, the image of π_i under the cycle map $cl_X : CH^d_{\mathbb{Q}}(X \times X) \to H^{2d}(X \times X; \mathbb{Q})$ is the $(2d - i, i)$ Künneth component of the diagonal class.

We say that Chow-Künneth decomposition as above satisfies *Poincaré duality* if $\pi_{2d-i} = {\pi_i}^t$ for $0 \leq i \leq 2d$. A variety X as above has a *strong Künneth decomposition* if there exist elements $\lambda_{i,j}, \mu_{i,j} \in CH^i_{\mathbb{Q}}(X)$, $0 \leq i \leq d$, such that $[\Delta_X] = \sum_{i=0}^{d} \sum_{j} \lambda_{i,j} \times \mu_{d-i,j}$. It is easy to check (cf. [AJ, Prop. 3.4]) that a strong Künneth decomposition is a Chow-Künneth decomposition.

Murre's Conjectures

Let X be a pseudo-smooth projective variety. Then

A. X has a Chow-Künneth decomposition.

B. If $i < j$ or $i > 2j$, then π_i acts as 0 on $CH^j_{\mathbb{Q}}(X)$.

C. If we define $F^0 CH^j_{\mathbb{Q}}(X) = CH^j_{\mathbb{Q}}(X)$ and $F^k CH^j_{\mathbb{Q}}(X) = \mathrm{Ker}\ \pi_{2j+1-k_*}|_{F^{k-1}CH^j_{\mathbb{Q}}(X)}$ for $k > 0$, then the resulting filtration is independent of the particular choice of projectors π_i.

D. For any filtration as defined in **C**, $F^1 CH^j_{\mathbb{Q}}(X)$ is the subgroup of cycles in $CH^j_{\mathbb{Q}}(X)$ homologically equivalent to zero.

We also consider the following property which is weaker than conjecture **B**:

B'. π_i acts as zero on $CH^j_{\mathbb{Q}}(X)$ for $i < j$ or for $i > j + d$, where $d = dim_k(X)$.

It has long been known [MA] that when X is a smooth projective variety, $T \subseteq X$ is a nonsingular subvariety of pure codimension $r + 1 > 1$, and $f : Y \to X$ is the blow-up of X along T, there are morphisms giving a split exact sequence (cf. [SCH, 2.7]):

$$0 \to h(T) \otimes \mathbb{L}^{r+1} \to h(X) \oplus (h(T) \otimes \mathbb{L}) \to h(Y) \to 0$$

where $\oplus$ and $\otimes$ are (respectively) the coproduct and tensor product structures on the category of motives of smooth projective varieties, and $\mathbb{L}$ is the Lefschetz motive. Since the motive of Y is so closely related to that of X, it is natural to ask how the validity of (any of) Murre's conjectures on one variety is related to the same for the other variety.

Our main result (see Theorem 1.1) shows that if T is finite and X is pseudo-smooth, then, under appropriate hypotheses, this is indeed the case. The proof is somewhat technical, although it is not difficult to sketch the key ideas. Letting $d = \dim X = \dim Y$, observe that the $\mathbb{Q}$-vector space $CH^d_{\mathbb{Q}}(Y \times Y)$ is a (noncommutative) ring under composition of correspondences. We show first that there is a direct sum decomposition $CH^d_{\mathbb{Q}}(Y \times Y) \cong A \oplus B$, where $A = (f \times f)^* CH^d_{\mathbb{Q}}(X \times X)$ and B consists of cycles supported on $E \times Y \cup Y \times E$, and moreover that A and B are "orthogonal", in the sense that $\alpha \bullet \beta = \beta \bullet \alpha = 0$ when $\alpha \in A$, $\beta \in B$. Using this decomposition, we can show, starting from a Chow-Künneth decomposition for X, how to construct one on Y and vice versa. Each of the Chow-Künneth projectors in Y will have one component in A and another in B: the former is easy to construct, but the latter takes some work and relies on the hypothesis that the exceptional divisor has a strong Künneth decomposition. The next step is to prove that for $i = 0, \ldots, d$, there is an analogous decomposition $CH^i_{\mathbb{Q}}(Y) \cong A_i \oplus B_i$ such that correspondences in A act trivially on B_i and correspondences in B act trivially on A_i. The result on the other parts of Murre's conjectures then follows with little difficulty. We note that Vial [VI] has also studied the motive of a blow-up (when X is smooth), but from a different perspective. For further discussion of motives, we refer the reader to [MA] and [SCH].

2 An explicit Chow-Künneth decompositions for blow-ups

2.1 Intersection theory on pseudo-smooth varieties

As described in [F, Example 16.1.13], one can define pullback maps, pushforward maps, and intersection products for pseudo-smooth varieties, and many basic results (including in particular the projection formula) carry over from the smooth case into this setting. It is somewhat less clear that certain other properties which we will need – in particular, the exchange of pushforwards and pullbacks in Cartesian squares – extend to the pseudo-smooth situation; so in the interest of completeness of exposition we provide a complete proof.

In the following, when a group G acts on a quasi-projective variety V' (over some field), we denote by $g_{V'} : V' \to V'$ the map describing the action of $g \in G$ on V. Furthermore, we let V denote the quotient V'/G and $q_V : V' \to V$ the quotient map. If G is a finite group acting on smooth projective varieties V' and W' and $V = V'/G$, $W = W'/G$, we say that a morphism $f : V \to W$ is *induced* by a morphism $f' : V' \to W'$ if $q_W \circ f' = f \circ q_V$. We will need a technical result; for smooth varieties (i.e. when G is trivial), this is a well-known property of the Gysin morphism:

Lemma 2.1. *Let G be a finite group acting on smooth projective varieties V' and W' of dimension d over some field; let $V = V'/G$, $W = W'/G$, and let $f : V \to W$ be a morphism induced by some $f' : V' \to W'$. Then*

$$p_{13}^{VVV}{}_*(f \times 1 \times f)^* = (f \times f)^* p_{13}^{WVW}{}_*.$$

Proof. We wish to show that the diagram below commutes:

$$
\begin{tikzcd}
CH^*_{\mathbb{Q}}(V \times V \times V) \arrow[r, "p_{13}^{VVV}{}_*"] & CH^{*-d}_{\mathbb{Q}}(V \times V) \\
CH^*_{\mathbb{Q}}(W \times V \times W) \arrow[u, "(f\times 1\times f)^*"] \arrow[r, "p_{13}^{WVW}{}_*"] & CH^{*-d}_{\mathbb{Q}}(W \times W) \arrow[u, "(f\times f)^*"']
\end{tikzcd}
$$

The G-actions on V' and W' naturally induce actions of $G \times G$ on $V' \times V'$ and $W' \times W'$ and actions of $G \times G \times G$ on $V' \times V' \times V'$ and $W' \times V' \times W'$. Moreover, by [F, 8.3], there is an isomorphism $q^*_{V\times V}$: $CH^*_{\mathbb{Q}}(V \times V) \to CH^*_{\mathbb{Q}}(V' \times V')^{G\times G}$ with inverse $\dfrac{1}{|G|} q_{V\times V\,*}$, and similarly for the other product varieties mentioned above. We also note that our hypotheses imply that the various pullback and pushforward maps are compatible with the above isomorphism: for example, there is a commutative diagram:

$$
\begin{tikzcd}
CH^*_{\mathbb{Q}}(V \times V) \arrow[r, "q^*_{V\times V}", "\cong"'] & CH^*_{\mathbb{Q}}(V' \times V')^{G\times G} \\
CH^*_{\mathbb{Q}}(W \times W) \arrow[u, "(f\times f)^*"] \arrow[r, "q^*_{W\times W}", "\cong"'] & CH^*_{\mathbb{Q}}(W' \times W')^{G\times G} \arrow[u, "(f'\times f')^*"']
\end{tikzcd}
$$

Applying this reasoning multiple times, we deduce that there is a cube

$$
\begin{tikzcd}
CH^*_{\mathbb{Q}}(V \times V \times V) \arrow[rr, "p_{13}^{VVV}{}_*"] \arrow[dr, "\cong"] & & CH^{*-d}_{\mathbb{Q}}(V \times V) \arrow[dr, "\cong"] & \\
& CH^*_{\mathbb{Q}}(V' \times V' \times V')^{G\times G\times G} \arrow[rr, "p_{13}^{VVV}{}_*"'] & & CH^{*-d}_{\mathbb{Q}}(V' \times V')^{G\times G} \\
CH^*_{\mathbb{Q}}(W \times V \times W) \arrow[uu, "(f\times 1\times f)^*"] \arrow[rr, "p_{13}^{WVW}{}_*"] \arrow[dr, "\cong"] & & CH^{*-d}_{\mathbb{Q}}(W \times W) \arrow[uu, "(f\times f)^*"] \arrow[dr, "\cong"] & \\
& CH^*_{\mathbb{Q}}(W' \times V' \times W')^{G\times G\times G} \arrow[uu, "(f'\times 1\times f')^*"] \arrow[rr, "p_{13}^{WVW}{}_*"] & & CH^{*-d}_{\mathbb{Q}}(W' \times W')^{G\times G} \arrow[uu, "(f'\times f')^*"']
\end{tikzcd}
$$

in which the diagonal maps are the various isomorphisms induced via pulling back by the corresponding quotient morphisms. The two side faces and the top and bottom commute by the reasoning described above. Note that all of the varieties appearing on the front face are smooth. Moreover, since any morphism between smooth projective varieties is a local complete intersection morphism in the sense of [F, p. 439], commutativity of the front face follows from [F, Prop. 6.6(c)]. Finally, the diagonal maps are all isomorphisms; so an elementary diagram chase allows us to deduce that the rear face commutes. $\square$

We proceed by studying how composition of correspondences behaves under pullback.

Proposition 2.2. *Suppose G is a finite group acting on smooth projective varieties V' and W', both of dimension d over a field k, and let $V = V'/G$, $W = W'/G$. Suppose $f : V \to W$ is an induced morphism of degree m. For $\alpha, \beta \in CH^d(W \times W)$,*

$$(f \times f)^*(\alpha) \bullet (f \times f)^*(\beta) = m(f \times f)^*(\alpha \bullet \beta).$$

Proof.

$$\begin{aligned}
(f \times f)^*\alpha \bullet (f \times f)^*\beta &= {p_{13}^{VVV}}_*({p_{12}^{VVV}}^*(f \times f)^*\beta \cdot {p_{23}^{VVV}}^*(f \times f)^*\alpha) \\
&= {p_{13}^{VVV}}_*((f \times f \times f)^*{p_{12}^{WWW}}^*\beta \cdot (f \times f \times f)^*{p_{23}^{WWW}}^*\alpha) \\
&= {p_{13}^{VVV}}_*(f \times f \times f)^*({p_{12}^{WWW}}^*\beta \cdot {p_{23}^{WWW}}^*\alpha) \\
&= {p_{13}^{VVV}}_*(f \times 1 \times f)^*(1 \times f \times 1)^*({p_{12}^{WWW}}^*\beta \cdot {p_{23}^{WWW}}^*\alpha) \\
&= (f \times f)^*{p_{13}^{WVW}}_*(1 \times f \times 1)^*({p_{12}^{WWW}}^*\beta \cdot {p_{23}^{WWW}}^*\alpha) \\
&= (f \times f)^*{p_{13}^{WWW}}_*(1 \times f \times 1)_*(1 \times f \times 1)^*({p_{12}^{WWW}}^*\beta \cdot {p_{23}^{WWW}}^*\alpha) \\
&= m(f \times f)^*{p_{13}^{WWW}}_*({p_{12}^{WWW}}^*\beta \cdot {p_{23}^{WWW}}^*\alpha) \\
&= m(f \times f)^*(\alpha \bullet \beta).
\end{aligned} \tag{2.1.1}$$

where the fifth equality is obtained by invoking Lemma 2.1.

□

Corollary 2.3. *With hypotheses as in Proposition 2.2, suppose furthermore that f is a birational morphism. If $\alpha \in CH^d(W \times W)$ is an idempotent (for the operation $\bullet$), then so is $(f \times f)^*\alpha$. If $\alpha, \beta \in CH^d(W \times W)$ and $\alpha \bullet \beta = 0$, then $(f \times f)^*(\alpha) \bullet (f \times f)^*(\beta) = 0$.*

The following fact about product cycles is surely well known; we include a proof in the interest of completeness of exposition.

Lemma 2.4. *Let V be a pseudo-smooth irreducible projective variety of dimension d over some field k. Suppose $\alpha \in CH^i(V)$, $\beta \in CH^{d-i}(V)$, $\gamma \in CH^j(V)$, $\delta \in CH^{d-j}(V)$. Then*

$$(\alpha \times \beta) \bullet (\gamma \times \delta) = (\gamma \times \beta) \cdot {p_\emptyset^{VV}}^* {p_\emptyset^V}_*(\delta \cdot \alpha).$$

If $i \neq j$, then $(\alpha \times \beta) \bullet (\gamma \times \delta) = 0$.

If $i = j$, then $(\alpha \times \beta) \bullet (\gamma \times \delta) = m(\gamma \times \beta)$ for some $m \in \mathbb{Z}$.

Proof.

$$\begin{aligned}
(\alpha \times \beta) \bullet (\gamma \times \delta) &= {p_{13}^{VVV}}_*({p_{12}^{VVV}}^*({p_1^{VV}}^*\gamma \cdot {p_2^{VV}}^*\delta) \cdot {p_{23}^{VVV}}^*({p_1^{VV}}^*\alpha \cdot {p_2^{VV}}^*\beta)) \\
&= {p_{13}^{VVV}}_*({p_{13}^{VVV}}^*{p_1^{VV}}^*\gamma \cdot {p_2^{VVV}}^*\delta \cdot {p_2^{VVV}}^*\alpha \cdot {p_{13}^{VVV}}^*{p_2^{VV}}^*\beta) \\
&= {p_{13}^{VVV}}_*({p_{13}^{VVV}}^*({p_1^{VV}}^*\gamma \cdot {p_2^{VV}}^*\beta) \cdot {p_2^{VVV}}^*\delta \cdot {p_2^{VVV}}^*\alpha) \\
&= {p_1^{VV}}^*\gamma \cdot {p_2^{VV}}^*\beta \cdot {p_{13}^{VVV}}_*{p_2^{VVV}}^*(\delta \cdot \alpha) \\
&= {p_1^{VV}}^*\gamma \cdot {p_2^{VV}}^*\beta \cdot {p_\emptyset^{VV}}^*{p_\emptyset^V}_*(\delta \cdot \alpha) \\
&= (\gamma \times \beta) \cdot {p_\emptyset^{VV}}^*{p_\emptyset^V}_*(\delta \cdot \alpha).
\end{aligned}$$

If $i \neq j$, $p^V_{\emptyset\,*}(\delta \cdot \alpha) \in CH^{i-j}(\text{Spec } k) = 0$. If $i = j$, $p^{V}_{\emptyset}{}^{V*} p^V_{\emptyset\,*}(\delta \cdot \alpha) \in CH^0(V \times V) \cong \mathbb{Z}$.

□

2.2 The main construction

The objective of this section is to describe, more or less explicitly, how (under the appropriate hypotheses) a Chow-Künneth decomposition for a pseudo-smooth projective variety can be used to construct one on the blow-up of this variety along a point, and conversely.

We begin by setting up the framework for our construction and proving some auxiliary results. Let X' be a smooth projective variety of dimension $d > 0$ over a field k, G a finite group acting on X' and $X = X'/G$. Let a be some k-rational point of X, $T = \{a\}$, and $f : Y \to X$ the blow-up of X along T.

Suppose further the following hypotheses are satisfied:

(i) The morphism $f : Y \to X$ is induced by some morphism of smooth varieties.

(ii) The exceptional divisor Z of the blow-up is isomorphic to $\mathbb{P}^{d-1}$.

Implicit in (i) is the assumption that Y is the quotient of a smooth variety by some action of G; in particular, Y is itself pseudo-smooth.

Now $Y \times Y$ is the blow-up of $X \times X$ along the closed subvariety $S = S_1 \cup S_2$, where $S_1 = T \times X$ and $S_2 = X \times T$; the exceptional divisor of this blow-up is $E = E_1 \cup E_2$, where $E_1 = Z \times Y$ and $E_2 = Y \times Z$. Thus we have commutative diagrams:

$$\begin{tikzcd} E \arrow[r, "\tilde{j}"] \arrow[d, "\tilde{g}"] & Y \times Y \arrow[d, "f \times f"] \\ S \arrow[r, "\tilde{i}"] & X \times X \end{tikzcd}$$

and

$$\begin{tikzcd} E_1 \arrow[r, hook, "j \times 1"] \arrow[d, "g \times 1"] & Y \times Y \arrow[d, "f \times f"] \\ S_1 \arrow[r, hook, "i \times 1"] & X \times X \end{tikzcd} \qquad \begin{tikzcd} E_2 \arrow[r, hook, "1 \times j"] \arrow[d, "1 \times g"] & Y \times Y \arrow[d, "f \times f"] \\ S_2 \arrow[r, hook, "1 \times i"] & X \times X \end{tikzcd}$$

Note that even when X is smooth we cannot use the blow-up exact sequence to relate the Chow groups of $Y \times Y$ to those of $X \times X$, because S is not regularly imbedded in $X \times X$. Instead, we use localization; letting $U_X = X \times X - S$ and $U_Y = Y \times Y - E$, there is a commutative diagram with exact rows:

$$\begin{tikzcd}
CH^{d-1}_{\mathbb{Q}}(E) \arrow[r, "\tilde{j}_*"] \arrow[d, "\tilde{g}_*"] & CH^d_{\mathbb{Q}}(Y\times Y) \arrow[r] \arrow[d, "(f\times f)_*"] & CH^d_{\mathbb{Q}}(U_Y) \arrow[r] \arrow[d, "\cong"] & 0 \\
CH^0_{\mathbb{Q}}(S) \arrow[r, "\tilde{i}_*"] & CH^d_{\mathbb{Q}}(X\times X) \arrow[r] & CH^d_{\mathbb{Q}}(U_X) \arrow[r] & 0
\end{tikzcd} \tag{2.2.1}$$

Let $i_1 : S_1 \hookrightarrow S$, $i_2 : S_2 \hookrightarrow S$, $i'_1 : S_1 \cap S_2 \hookrightarrow S_1$, and $i'_2 : S_1 \cap S_2 \hookrightarrow S_2$ denote the various inclusion maps. Then, noting that $S_1 \cap S_2 = T \times T$ has dimension 0, we have an exact sequence (see [F, Example 1.3.1 (c)]:

$$CH^{-d}_{\mathbb{Q}}(S_1 \cap S_2) \overset{(i'_{1*}, i'_{2*})}{\longrightarrow} CH^0_{\mathbb{Q}}(S_1) \oplus CH^0_{\mathbb{Q}}(S_2) \overset{i_{1*}-i_{2*}}{\longrightarrow} CH^0_{\mathbb{Q}}(S) \to 0,$$

Now $d > 0$, making the the first term of the sequence 0; so, there is an isomorphism $t : CH^0_{\mathbb{Q}}(S_1) \oplus CH^0_{\mathbb{Q}}(S_2) \overset{\cong}{\to} CH^0_{\mathbb{Q}}(S)$. Now $CH^0_{\mathbb{Q}}(S_1) \cong \mathbb{Q}$ is generated by $[S_1]$ and $CH^0_{\mathbb{Q}}(S_2) \cong \mathbb{Q}$ is generated by $[S_2]$. Hence, for $v, w \in \mathbb{Q}$ we have $(\tilde{i}_* \circ t)(v[S_1], w[S_2]) = v[S_1] + w[S_2]$, where on the right we interpret the terms $[S_i]$ as cycles on X.

Since a is a k-rational point, the inclusion map $i : \{a\} = T \hookrightarrow X$ may be identified with a section of the structure morphism $X \to \mathrm{Spec}\ k$; thus, both maps $(i \times 1)_*$ and $(1 \times i)_*$ are (split) injective.

Lemma 2.5. *The map $\tilde{i}_*$ in (2.2.1) is injective.*

Proof. It suffices to show that $u = \tilde{i}_* \circ t$ is injective. To this end, suppose $u(v[S_1], -w[S_2]) = 0$. Then $v[S_1] = w[S_2]$ on X, and so $v[S_1] \bullet [S_1] = w[S_1] \bullet [S_2]$. Note that $[S_1] = T \times [X]$ is a product cycle (on $X \times X$) of type $(0, d)$ and $[S_2] = [X] \times T$ is a product cycle of type $(d, 0)$. Direct calculation using Lemma 2.4 shows that $[S_1] \bullet [S_1] = [S_1]$ and $[S_1] \bullet [S_2] = 0$. Hence $v[S_1] = 0$; that is, $v(i \times 1)_*[S_1] = 0$. Because $(i \times 1)_*$ is injective, it follows that $v = 0$. Hence $w[S_2] = w(1 \times i)_*[S_2] = 0$, and by injectivity of $(1 \times i)_*$ we have $w = 0$ also. □

Define subgroups $A = (f \times f)^* CH^d_{\mathbb{Q}}(X \times X)$ and $B = \tilde{j}_*(\mathrm{Ker}\ \tilde{g}_*)$ of $CH^d_{\mathbb{Q}}(Y \times Y)$. The next result gives an "orthogonality principle" which is central to our construction.

Proposition 2.6. *If $\alpha \in A$ and $\beta \in B$, then $\alpha \bullet \beta = \beta \bullet \alpha = 0$.*

Proof. For $\alpha \in A$ and $\beta \in B$, write $\alpha = (f \times f)^*\delta$ and $\beta = \tilde{j}_*\varepsilon$ where $\delta \in CH^d_{\mathbb{Q}}(X \times X)$ and $\varepsilon \in CH^{d-1}_{\mathbb{Q}}(E)$. As before, we may write $\varepsilon = h_{1*}(\varepsilon_1) + h_{2*}(\varepsilon_2)$, where $\varepsilon_i \in CH^{d-1}_{\mathbb{Q}}(E_i)$, $i = 1, 2$. Thus $\beta = (j \times 1)_*\varepsilon_1 + (1 \times j)_*\varepsilon_2$. Now it is easy to check that $CH^0(S) \cong CH^0(S_1) \oplus CH^0(S_2) \cong \mathbb{Z} \oplus \mathbb{Z}$ and that $\tilde{g}(\varepsilon) = ((g \times 1)_*\varepsilon_1, (1 \times g)_*\varepsilon_2)$. Since $\tilde{g}(\varepsilon) = 0$, it follows that $(g \times 1)_*\varepsilon_1 = 0$ and $(1 \times g)_*\varepsilon_2 = 0$.

Hence

$$
\begin{aligned}
\alpha \bullet (1\times j)_*\varepsilon_2 &= p_{13}^{YYY}{}_*(p_{12}^{YYY*}(1\times j)_*\varepsilon_2 \cdot p_{23}^{YYY*}(f\times f)^*\delta) \\
&= p_{13}^{YYY}{}_*((1\times j\times 1)_* p_{12}^{YZY*}\varepsilon_2 \cdot (f\times f\times f)^* p_{23}^{XXX*}\delta) \\
&= p_{13}^{YYY}{}_*(1\times j\times 1)_*(p_{12}^{YZY*}\varepsilon_2 \cdot (1\times j\times 1)^*(f\times f\times f)^* p_{23}^{XXX*}\delta).
\end{aligned}
$$

To obtain the last equality above, we have used the projection formula. Now the last term equals:

$$
\begin{aligned}
&p_{13}^{YZY}{}_*(p_{12}^{YZY*}\varepsilon_2 \cdot (1\times (f\circ j)\times 1)^*(f\times 1\times f)^* p_{23}^{XXX*}\delta) \\
&= p_{13}^{YZY}{}_*(p_{12}^{YZY*}\varepsilon_2 \cdot (1\times (i\circ g)\times 1)^*(f\times 1\times f)^* p_{23}^{XXX*}\delta) \\
&= p_{13}^{YXY}{}_*(1\times (i\circ g)\times 1)_*(p_{12}^{YZY*}\varepsilon_2 \cdot (1\times (i\circ g)\times 1)^*(f\times 1\times f)^* p_{23}^{XXX*}\delta) \\
&= p_{13}^{YXY}{}_*((1\times (i\circ g)\times 1)_* p_{12}^{YZY*}\varepsilon_2 \cdot (f\times 1\times f)^* p_{23}^{XXX*}\delta) \\
&= p_{13}^{YXY}{}_*(p_{12}^{YXY*}(1\times i)_*(1\times g)_*\varepsilon_2 \cdot (f\times 1\times f)^* p_{23}^{XXX*}\delta) \\
&= 0.
\end{aligned}
$$

A similar computation shows $(1\times j)_*\varepsilon_2 \bullet \alpha = 0$.

Now $\alpha^t \in A$ also, and since $(g\times 1)_*\varepsilon_1 = 0$, we have $(1\times g)_*\varepsilon_1^t = 0$. Thus, $\alpha \bullet (j\times 1)_*\varepsilon_1 = ((\alpha \bullet (j\times 1)_*\varepsilon_1)^t)^t = ((1\times j)_*\varepsilon_1^t \bullet \alpha^t)^t = 0$ and similarly $(j\times 1)_*\varepsilon_1 \bullet \alpha = 0$. ☐

We will need a basic result about noncommutative rings.

Lemma 2.7. *Let R be a noncommutative ring with 1. Suppose I and J are subgroups of the additive group of R such that $I+J=R$, and that I and J are mutually orthogonal, i.e. for all $i \in I$ and $j \in J$, $ij = ji = 0$. Then R is the internal direct sum of I and J, I and J are two-sided ideals of R, and there is a ring isomorphism $R \cong I \times J$.*

Proof. The hypothesis $I+J=R$ allows us to write $1 = 1_I + 1_J$ for some $1_I \in I$ and $1_J \in J$. Now if $x \in I\cap J$, then $x = x\cdot 1 = x1_I + x1_J$. By the orthogonality hypothesis, $x1_I$ and $x1_J$ are both 0; hence $x=0$ and so R is the internal direct sum of (the abelian groups) I and J.

Next, observe that if $i \in I$, then $i1_I = i(1-1_J) = i\cdot 1 - i\cdot 1_J = i\cdot 1 = i$ (by orthogonality); so 1_I is a right identity for multiplication on I. Similar arguments show that 1_I is a left identity and that 1_J is a two-sided identity for multiplication on J. Next, we claim that I is closed under multiplication: indeed, if $i_1, i_2 \in I$ and $i_1 i_2 = i' + j'$ where $i' \in I$, $j' \in J$, then $i' + j' = i_1 i_2 = i_1(i_2 1_I) = (i_1 i_2)1_I = (i'+j')1_I = i'1_I + j'1_I = i'1_I = i'$. Thus, $j' = 0$ and so $i_1 i_2 \in I$. A similar argument shows that J is also closed under multiplication, and hence that I and J are rings.

Next consider $i \in I$ and $r \in R$. Writing $r = i' + j'$ with $i' \in I$, $j' \in J$, we see that $ri = (i'+j')i = i'i + j'i = i'i \in A$. A similar argument shows $ir \in A$ and hence that A is a two-sided ideal; analogous reasoning shows that B is also a two-sided ideal. Finally, the internal direct sum decomposition yields a group isomorphism $f : I \times J \to R$ defined by $(i,j) \mapsto i + j$; the orthogonality hypothesis and the construction of 1_I and 1_J easily imply that f is in fact a ring isomorphism. ☐

Corollary 2.8. *There is an isomorphism of rings $CH^d_{\mathbb{Q}}(Y \times Y) \cong A \times B$ given explicitly by $\gamma \mapsto ((f \times f)^*(f \times f)_*\gamma, \gamma - (f \times f)^*(f \times f)_*\gamma)$. Furthermore, A and B are two-sided ideals of $CH^d_{\mathbb{Q}}(Y \times Y)$.*

Proof. By Lemma 2.7, it suffices to show that for $\gamma \in CH^d_{\mathbb{Q}}(Y \times Y)$, $(f \times f)^*(f \times f)_*\gamma \in A$ and $\gamma - (f \times f)^*(f \times f)_*\gamma \in B$. The former is obvious from the definition of A. By a chase on the diagram (2.2.1), it follows that $(f \times f)^*(f \times f)_*(\gamma) - \gamma = \tilde{j}_*(\varepsilon)$ for some $\varepsilon \in CH^{d-1}_{\mathbb{Q}}(E)$. Moreover, $(f \times f)_*\tilde{j}_*(\varepsilon) = 0$; hence $\tilde{i}_*\tilde{g}_*(\varepsilon) = 0$. Since $\tilde{i}_*$ is injective by Lemma 2.5, it follows that $\varepsilon \in Ker(\tilde{g}_*)$. Thus, $\tilde{j}_*(\varepsilon) \in B$, as desired.

We are now in a position to state and prove our main result.

Theorem 2.9. *Assume the hypotheses 2.2. If X has a Chow-Künneth decomposition, then Y also has a Chow-Künneth decomposition. Moreover, if the assumed Chow-Künneth decomposition for X satisfies Poincaré duality, then so does the Chow-Künneth decomposition for Y.*

Proof. Suppose $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$ is a Chow-Künneth decomposition for X; we will show how to construct a Chow-Künneth decomposition for Y and define $\sigma = [\Delta_Y] - (f \times f)^*[\Delta_X]$. By Corollary 2.3, $\{(f \times f)^*\pi_i\}_{i=0}^{2d}$ is a set of orthogonal idempotents in $CH^d_{\mathbb{Q}}(Y \times Y)$ and hence σ is an idempotent element of $CH^d_{\mathbb{Q}}(Y \times Y)$. Furthermore, by Corollary 2.8,

$$\sigma = [\Delta_Y] - (f \times f)^*[\Delta_X] = [\Delta_Y] - (f \times f)^*(f \times f)_*[\Delta_Y] \in B.$$

Letting $h_1 : E_1 \hookrightarrow E$ and $h_2 : E_2 \hookrightarrow E$ denote the inclusion maps, there is a surjection (cf. [F, Example 1.3.1(c)]):

$$CH^{d-1}_{\mathbb{Q}}(E_1) \oplus CH^{d-1}_{\mathbb{Q}}(E_2) \xrightarrow{h_{1*} - h_{2*}} CH^{d-1}_{\mathbb{Q}}(E)$$

so we may write $\tau = h_{1*}\tau_1 + h_{2*}\tau_2$, with $\tau_i \in CH^{d-1}_{\mathbb{Q}}(E_i)$, $i = 1, 2$. Since $\tilde{j} \bullet h_1 = j \times 1$ and $\tilde{j} \bullet h_2 = 1 \times j$, we have $\sigma = (j \times 1)_*\tau_1 + (1 \times j)_*\tau_2$. (This is the only part of the construction which is not explicit, and depends on the choices made for τ_1 and τ_2.)

Now let $\ell \in CH^1_{\mathbb{Q}}(\mathbb{P}^{d-1})$ be the class of a generic hyperplane. Recall [MA, p.455] that $\mathbb{P}^{d-1}$ has a strong Künneth decomposition given by $[\Delta_{\mathbb{P}^{d-1}}] = \displaystyle\sum_{i=0}^{d-1} \ell^i \times \ell^{d-i-1}$.

Keeping in mind our hypothesis $Z = \mathbb{P}^{d-1}$ and the definitions $E_1 = Z \times Y$ and $E_2 = Y \times Z$, we define, for $i = 1, \ldots, d$,

$$\eta_i' = \tau_1 \bullet (\ell^{i-1} \times \ell^{d-i}) \in CH^{d-1}_{\mathbb{Q}}(E_1)$$

and for $i = 0, \ldots, d-1$,

$$\theta_i' = (\ell^{d-i-1} \times \ell^i) \bullet \tau_2 \in CH^{d-1}_{\mathbb{Q}}(E_2).$$

These definitions are made so as to ensure that $\tau_1 = \sum_{i=1}^{d} \eta'_i$ and $\tau_2 = \sum_{i=0}^{d-1} \theta'_i$. By [F, Example 16.1.2], we have

$$\eta'_i = \ell^{i-1} \times {p_2^{ZY}}_*({p_1^{ZY}}^* \ell^{d-i} \cdot \tau_1) \text{ and}$$

$$\theta'_i = {p_1^{YZ}}_*(\tau_2 \cdot {p_2^{YZ}}^* \ell^{i}) \times \ell^{d-i-1}.$$

Now define $\eta_i = (j \times 1)_* \eta'_i$ and $\theta_i = (1 \times j)_* \theta'_i$. By an easy argument (cf. [F, Proposition 1.10]) we have:

$$\eta_i = j_* \ell^{i-1} \times {p_2^{ZY}}_*({p_1^{ZY}}^* \ell^{d-i} \cdot \tau_1) \text{ and}$$
$$\theta_i = {p_1^{YZ}}_*(\tau_2 \cdot {p_2^{YZ}}^* \ell^{i}) \times j_* \ell^{d-i-1}.$$

Note that η_d is a product cycle on $Y \times Y$ of type $(d, 0)$, θ_0 is a product cycle on $Y \times Y$ of type $(0, d)$, and for i, $1 \leq i \leq d-1$, both η_i and θ_i are product cycles on $Y \times Y$ of type $(i, d-i)$.

Define $\gamma_0 = \theta_0$, $\gamma_d = \eta_d$, and for i, $1 \leq i \leq d-1$, $\gamma_i = \eta_i + \theta_i$. One easily checks that $\sum_{j=0}^{d} \gamma_j = \sigma$. By Lemma 2.4, $\gamma_i \bullet \gamma_j = 0$ when $i \neq j$ and hence

$$\sigma \bullet \gamma_i = \gamma_i \bullet \gamma_i = \gamma_i \bullet \sigma$$

for $i = 0, \ldots, d$. By Lemma 2.4, we also have: $\gamma_i \bullet \gamma_i = m_i a \gamma_i$ for some $m_i \in \mathbb{Z}$, $i = 0, \ldots, d$.

Thus,

$$\gamma_i \bullet \gamma_i = \sigma \bullet \gamma_i = (\sigma \bullet \sigma) \bullet \gamma_i = \sigma \bullet (\sigma \bullet \gamma_i) = \sigma \bullet (\gamma_i \bullet \gamma_i) = m_i(\sigma \bullet \gamma_i) = m_i(\gamma_i \bullet \gamma_i).$$

This forces each m_i to be 0 or 1.

Now define $J = \{j : 0 \leq j \leq d,\ m_j = 0\}$. Because

$$\sum_{j=0}^{d} m_j \gamma_j = \sigma \bullet \sigma = \sigma = \sum_{j=0}^{d} \gamma_j$$

we must have $\sum_{j \in J} \gamma_j = 0$. By replacing all γ_j, $j \in J$, with 0, we can ensure that the formulae

$$\sigma = \sum_{j=0}^{d} \gamma_j, \quad \gamma_i \bullet \gamma_i = \gamma_i, \quad \gamma_i \bullet \gamma_j = 0 \text{ when } i \neq j$$

still hold.

Now since B is an ideal of $CH^d_{\mathbb{Q}}(Y \times Y)$ by Corollary 2.8 and $\sigma \in B$, the previous relations imply $\gamma_i = \gamma_i \bullet \gamma_i = \sigma \bullet \gamma_i \in B$.

Thus Proposition 2.6 implies $\gamma_i \bullet (f \times f)^* \pi_j = (f \times f)^* \pi_j \bullet \gamma_i = 0$ for $0 \leq i \leq d$ and $0 \leq j \leq 2d$.

Finally, define, for j, $0 \leq j \leq 2d$,

$$\rho_j = \begin{cases} (f \times f)^* \pi_j + \gamma_{d-j/2} & \text{if } j \text{ is even} \\ (f \times f)^* \pi_j & \text{if } j \text{ is odd} \end{cases} \tag{2.2.2}$$

The computations above show that $[\Delta_Y] = \sum_{j=0}^{2d} \rho_j$ satisfies properties (i) and (ii) in the definition of Chow-Künneth decomposition.

It remains to show that for any Weil cohomology theory H^* and every j, $0 \leq j \leq 2d$, $cl_{Y \times Y}(\rho_j)$ is the $(2d-j, j)$ Künneth component of $[\Delta_Y] \in H^{2d}(Y \times Y; \mathbb{Q})$. Using the Künneth isomorphism to make the identification $H^{2d}(Y \times Y; \mathbb{Q}) \cong \bigoplus_{i=0}^{2d} H^{2d-i}(Y; \mathbb{Q}) \otimes_{\mathbb{Q}} H^i(Y; \mathbb{Q})$, it suffices to show that $cl_{Y \times Y}(\rho_j) \in H^{2d-j}(Y; \mathbb{Q}) \otimes_{\mathbb{Q}} H^j(Y; \mathbb{Q})$.

Now π_j is a projector in the original Chow-Künneth composition for X; so $cl_{X \times X}(\pi_j) \in H^{2d-j}(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H^j(X; \mathbb{Q})$. Hence, using properties of the cycle map from the definition of Weil cohomology (see for example, [KL, Section 3]), we have $cl_{Y \times Y}(f \times f)^* \pi_j = (f \times f)^* cl_{X \times X}(\pi_j) \in H^{2d-j}(Y; \mathbb{Q}) \otimes_{\mathbb{Q}} H^j(Y; \mathbb{Q})$. Moreover, $\gamma_{d-j/2}$ is a product cycle of type $(d-j/2, j/2)$; hence $\gamma_{d-j/2} = \sum_{m=0}^{r} \lambda_m \times \mu_m$, where $\lambda_m \in CH_{\mathbb{Q}}^{d-j/2}(Y)$ and $\mu_m \in CH_{\mathbb{Q}}^{j/2}(Y)$. Again using properties of the cycle map,

$$cl_{Y \times Y}(\gamma_{d-j/2}) = \sum_{m=0}^{r} cl_{Y \times Y}(\lambda_m \times \mu_m) = \sum_{m=0}^{r} cl_Y(\lambda_m) \otimes cl_Y(\mu_m) \in H^{2d-j}(Y; \mathbb{Q}) \otimes_{\mathbb{Q}} H^d(Y; \mathbb{Q}).$$

Thus, regardless of whether j is odd or even, $cl_{Y \times Y}(\rho_j) \in H^{2d-j}(Y; \mathbb{Q}) \otimes_{\mathbb{Q}} H^j(Y; \mathbb{Q})$.

Finally, suppose the Chow-Künneth decomposition for X satisfies Poincaré duality. Then ${\pi_i}^t = \pi_{2d-i}$ for $0 \leq i \leq 2d$; so certainly $(f \times f)^* {\pi_i}^t = (f \times f)^* \pi_{2d-i}$. It remains to show that $\gamma_j^t = \gamma_{d-j}$ for $0 \leq j \leq d$. Tracing through the construction of the γ_j; we see that $\sigma = [\Delta_Y] - \sum_{i=0}^{2d} (f \times f)^* \pi_i$ is self-transpose, i.e. $\sigma^t = \sigma$. We claim that the elements $\tau_i \in CH_{\mathbb{Q}}^{d-1}(E_i)$, $i = 1, 2$ can be selected such that $\tau_2 = \tau_1^t$. To see this, let τ_i be (as in the construction) any elements such that $\sigma = (j \times 1)_* \tau_1 + (1 \times j)_* \tau_2$. Then $\sigma = \sigma^t = (1 \times j)_* \tau_1^t + (j \times 1)_* \tau_2^t$ and hence

$$\sigma = (j \times 1)_* \frac{1}{2}(\tau_1 + \tau_2^t) + (1 \times j)_* \frac{1}{2}(\tau_1^t + \tau_2) = (j \times 1)_* \frac{1}{2}(\tau_1 + \tau_2^t) + (1 \times j)_* \frac{1}{2}(\tau_1 + \tau_2^t)^t.$$

This shows that we may replace (τ_1, τ_2) by $(\frac{1}{2}(\tau_1 + \tau_2^t), \frac{1}{2}(\tau_1^t + \tau_2))$. Now direct computation shows that $\eta_0 = \theta_d^t$ and $\eta_j = \theta_j^t$ for $1 \leq j \leq d-1$. Hence $\gamma_j^t = \gamma_{d-j}$ for $0 \leq j \leq d$ and the Chow-Künneth decomposition for Y satisfies Poincaré duality. □

Corollary 2.10. *With hypotheses as in Theorem 2.9, suppose $[\Delta_Y] = \sum_{j=0}^{2d} \nu_j$ is a Chow-Künneth decomposition for Y. Then $[\Delta_X] = \sum_{j=0}^{2d} (f \times f)_* \nu_j$ is a Chow-Künneth decomposition for X and $\nu_j = (f \times f)^* (f \times f)_* \nu_j + b_j$ for some $b_j \in B$. Moreover, if the Chow-Künneth decomposition for Y satisfies Poincaré duality, then so does the Chow-Künneth decomposition for X.*

Proof. The formula for $[\Delta_X]$ follows by applying $(f\times f)_*$ to both sides of the expression for $[\Delta_Y]$ and noting that $f\times f$ has degree 1. It remains to check that $\{(f\times f)_*\nu_j\}_{j=0}^{2d}$, is a set of mutually orthogonal idempotents in $CH^d_{\mathbb{Q}}(X\times X)$.

By Corollary 2.8, $CH^d_{\mathbb{Q}}(Y\times Y)$ we may write each ν_j uniquely as $\nu_j=(f\times f)^*(f\times f)_*\nu_j+b_j$, where $b_j\in B$. Now using Proposition 2.6, the relation $\nu_i\bullet\nu_j=0$ for $i\neq j$ implies that

$$(f\times f)^*(f\times f)_*\nu_i\bullet(f\times f)^*(f\times f)_*\nu_j=0 \text{ and } b_i\bullet b_j=0$$

also hold. By Proposition 2.2,

$$(f\times f)^*((f\times f)_*\nu_i\bullet(f\times f)_*\nu_j)=0.$$

Now injectivity of the map $(f\times f)^*$ implies that $(f\times f)_*\nu_i\bullet(f\times f)_*\nu_j=0$ when $i\neq j$. Finally,

$$(f\times f)_*\nu_i\bullet(f\times f)_*\nu_i=[\Delta_X]\bullet(f\times f)_*\nu_i=(f\times f)_*\nu_i.$$

The remaining assertions are clear from the construction. □

3 Murre's Conjectures

We retain the notation and assumptions of the previous section. The goal of this section is to prove that each of Murre's Conjectures holds for X if and only if it holds for Y. The case of Murre's Conjecture **A** (existence of a Chow-Künneth decomposition) was completed in the previous section. In the interest of making the proofs easier to follow, we use Greek letters for elements of $CH^*_{\mathbb{Q}}(Y\times Y)$ or $CH^*_{\mathbb{Q}}(X\times X)$ and Roman letters for elements of $CH^*_{\mathbb{Q}}(Y)$ or $CH^*_{\mathbb{Q}}(X)$.

We will need some results analogous to – but less complicated than – those of the previous section.

Lemma 3.1. *If $\alpha\in CH^*_{\mathbb{Q}}(X\times X)$ and $b\in CH^*_{\mathbb{Q}}(X)$, then*

$$f^*(\alpha\bullet b)=(f\times f)^*(\alpha)\bullet f^*b.$$

Proof.

$$\begin{aligned}(f\times f)^*(\alpha)\bullet f^*(\beta)&={p_2^{YY}}_*({p_1^{YY}}^*f^*\beta\cdot(f\times f)^*\alpha)\\&={p_2^{YY}}_*(f\times f)^*({p_1^{XX}}^*\beta\cdot\alpha)={p_2^{YY}}_*(1\times f)^*(f\times 1)^*({p_1^{XX}}^*\beta\cdot\alpha)\end{aligned}\tag{3.0.3}$$

By means of an argument similar to that employed in Lemma 2.1, one sees that ${p_2^{YY}}_*(1\times f)^*=f^*{p_2^{YX}}_*$. Applying this fact, the above expression equals:

$$\begin{aligned}&f^*{p_2^{YX}}_*(f\times 1)^*({p_1^{XX}}^*\beta\cdot\alpha)=f^*{p_2^{XX}}_*(f\times 1)_*(f\times 1)^*({p_1^{XX}}^*\beta\cdot\alpha)\\&=f^*{p_2^{XX}}_*(f\times 1)_*(f\times 1)^*({p_1^{XX}}^*\beta\cdot\alpha)=f^*(\alpha\bullet\beta)\end{aligned}\tag{3.0.4}$$

□

For i, $0 \leq i \leq d$, define subgroups $A_i = f^* CH^i_{\mathbb{Q}}(X)$ and $B_i = j_*(\mathrm{Ker}\ g_* : CH^{i-1}_{\mathbb{Q}}(Z) \to CH^{i-d}_{\mathbb{Q}}(T))$ of $CH^i_{\mathbb{Q}}(Y)$. Observe that $B_i = CH^{i-1}_{\mathbb{Q}}(Z) \cong \mathbb{Q}$ if $i < d$ and $B_i = 0$ if $i = d$. Then

Proposition 3.2. *$CH^i_{\mathbb{Q}}(Y)$ is the internal direct sum of A_i and B_i. Moreover, if $\alpha \in A$, $\beta \in B$, $a \in A_i$ and $b \in B_i$, then $\alpha \bullet b = \beta \bullet a = 0$.*

Proof. Let $V_X = X - T$ and $V_Y = Y - Z$. Then $V_X \cong V_Y$ and we have a commutative diagram:

$$\begin{tikzcd}
CH^{d-1}_{\mathbb{Q}}(Z) \arrow[r, "j_*"] \arrow[d, "g_*"] & CH^d_{\mathbb{Q}}(Y) \arrow[r] \arrow[d, "f_*"] & CH^d_{\mathbb{Q}}(V_Y) \arrow[r] \arrow[d, "\cong"] & 0 \\
CH^0_{\mathbb{Q}}(T) \arrow[r, "i_*"] & CH^d_{\mathbb{Q}}(X) \arrow[r] & CH^d_{\mathbb{Q}}(V_X) \arrow[r] & 0
\end{tikzcd}$$

The property $CH^i_{\mathbb{Q}}(Y) = A_i + B_i$ follows from a straightforward diagram chase. Now write $\alpha = (f \times f)^* u$, $a = f^* v$, and $b = j_* y$. Then

$$\begin{aligned}
\alpha \bullet b &= (f \times f)^* u \bullet j_* y = p_2^{YY}{}_*(p_1^{YY*} j_* y \cdot (f \times f)^* u) \\
&= p_2^{YY}{}_*((j \times 1)_* p_1^{ZY*} y \cdot (f \times 1)^* (1 \times f)^* u \\
&= p_2^{YY}{}_*(j \times 1)_*(p_1^{ZY*} y \cdot (j \times 1)^* (f \times 1)^* (1 \times f)^* u)
\end{aligned} \tag{3.0.5}$$

Because $g : Z \to T = \mathrm{Spec}\ k$ is simply the structure morphism, we have $p_2 \circ (j \times 1) = g \times 1$; thus the above expression may be rewritten:

$$\begin{aligned}
&(g \times 1)_*(p_1^{ZY*} y \cdot (g \times 1)^* (i \times 1)^* (1 \times f)^* u \\
&= (g \times 1)_* p_1^{ZY*} y \cdot (i \times 1)^* (1 \times f)^* u) \\
&= p_\emptyset^{Y*} g_* y \cdot (i \times 1)^* (1 \times f)^* u = 0.
\end{aligned} \tag{3.0.6}$$

Using Lemma 3.1 and Proposition 2.6, we have:

$$\beta \bullet a = \beta \bullet f^*([\Delta_X] \bullet v) = \beta \bullet ((f \times f)^* [\Delta_X] \bullet f^* v) = (\beta \bullet (f \times f)^* [\Delta_X]) \bullet f^* v = 0.$$

□

From this point onward, we make the identifications

$$CH^i_{\mathbb{Q}}(Y) \cong A_i \oplus B_i \text{ and } CH^d_{\mathbb{Q}}(Y \times Y) \cong A \oplus B$$

Corollary 3.3. *Suppose $((f \times f)^* \alpha, \beta) \in CH^d_{\mathbb{Q}}(Y \times Y)$ and $(f^* x, y) \in CH^i_{\mathbb{Q}}(Y)$. Then*

$$((f \times f)^* \alpha, \beta) \bullet (f^* x, y) = (f^*(\alpha \bullet x), \beta \bullet y)$$

Proposition 3.4. *Let $\{\rho_i | i = 0, \cdots, 2d\}$ denote the Chow-Kunneth projectors constructed as in* (2.2.2) *for Y starting with the Chow-Kunneth projectors $\{\pi_i | i = 0, \cdots, 2d\}$ for X. Then the projectors $\{\rho_i | i = 0, \cdots, 2d\}$ satisfy Murre's conjecture **B** (**B'**) if and only if the projectors $\{\pi_i | i = 0, \cdots, 2d\}$ satisfy Murre's conjecture **B** (**B'**, respectively).*

Proof. First suppose X has a Chow-Künneth decomposition $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$ satisfying Murre's Conjecture **B**, i.e. $\pi_\ell \bullet CH^j_{\mathbb{Q}}(X) = 0$ when $\ell < j$ or $\ell > 2j$, and let $[\Delta_Y] = \sum_{i=0}^{2d}((f \times f)^*\pi_i, \beta_i)$ be the Chow-Künneth decomposition for Y as given by the construction in the proof of Theorem 2.9, where $\beta_i = \gamma_{d-i/2}$ if i is even or 0 if i is odd. Now fix j, $0 \le j \le d$, and consider $(f^*x, y) \in CH^j_{\mathbb{Q}}(Y)$. Note that $((f \times f)^*\pi_\ell, \beta_\ell) \bullet (f^*x, y) = (f^*(\pi_\ell \bullet x), \beta_\ell \bullet y)$. If $\ell < j$ or $\ell > 2j$, then $\pi_\ell \bullet x = 0$. When ℓ is odd, clearly $\beta_\ell \bullet y = 0$; so assume ℓ is even. Then β_ℓ is a product cycle of type $(d - \ell/2, \ell/2)$; so it suffices to show that for any $u \in {CH_{\mathbb{Q}}}^{d-\ell/2}(Y)$ and $v \in {CH_{\mathbb{Q}}}^{\ell/2}(Y)$, $(u \times v) \bullet y = 0$ when $\ell < j$ or $\ell > 2j$. Then

$$(u \times v) \bullet y = {p_2^{YY}}_*({p_1^{YY}}^* y \cdot {p_1^{YY}}^* u \cdot {p_2^{YY}}^* v) = {p_2^{YY}}_* {p_1^{YY}}^*(y \cdot u) \cdot v = {p_\emptyset^Y}^* {p_\emptyset^Y}_*(y \cdot u) \cdot v.$$

Note that $y \cdot u \in CH^{j+d-\ell/2}_{\mathbb{Q}}(Y)$. If $\ell < j$, then $j + d - \ell/2 > d$; so $y \cdot u = 0$. If $l > 2j$, then $j - \ell/2 < 0$ so that ${p_\emptyset^Y}_*(y \cdot u) \in CH^{j-\ell/2}_{\mathbb{Q}}(\operatorname{Spec} k) = 0$. Thus, this Chow-Künneth decomposition for Y satisfies Murre's Conjecture **B**.

Next suppose X has a Chow-Künneth decomposition $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$ satisfying Murre's Conjecture **B'**, i.e. $\pi_\ell \bullet CH^j_{\mathbb{Q}}(X) = 0$ when $\ell < j$ or $\ell > j + d$. The only difference in the argument is for $\ell > j + d$. But $\ell > j + d$ implies $j - \ell/2 < j - (j + d)/2 = j/2 - d/2$. Since $j \le d$, this means $j - \ell/2 < 0$, and hence ${p_\emptyset^Y}_*(y \cdot u) \in CH^{j-\ell/2}_{\mathbb{Q}}(\operatorname{Spec} k) = 0$ once more. Thus, this Chow-Künneth decomposition for Y satisfies Murre's Conjecture **B'**.

Conversely, suppose $[\Delta_Y] = \sum_{i=0}^{2d}((f \times f)^*\pi, \beta_i)$ is a Chow-Künneth decomposition for Y satisfying Murre's Conjecture **B** (Conjecture **B'**). This means that if $(f^*x, y) \in CH^\ell_{\mathbb{Q}}(Y)$, then $(f \times f)^*\pi_j \bullet f^*x = 0$ when $\ell < j$ or $\ell > 2j$ ($\ell > j + d$, respectively). By Lemma 3.1 we have $f^*(\pi_j \bullet x) = 0$, and since f^* is injective, $\pi_j \bullet x = 0$. Corollary 2.10 then guarantees that $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$ is a Chow-Künneth decomposition for X satisfying Murre's Conjecture **B** (Conjecture **B'**, respectively).

□

Proposition 3.5. *Murre's Conjecture **C** holds for X if and only if it holds for Y. Similarly, Conjecture **D** holds for X if and only if it holds for Y.*

Proof. Assume first that X has a Chow-Künneth decomposition satisfying Murre's Conjecture **C**. Now let $[\Delta_Y] = \sum_{\ell=0}^{2d}(f \times f)^*\pi_\ell + \beta_\ell$ be a Chow-Künneth decomposition for Y coming from Theorem 2.9. By Proposition 3.2 and Corollary 3.3, we have $((f \times f)^*\pi_\ell + \beta_\ell) \bullet CH^i_{\mathbb{Q}}(Y) = (f \times f)^*\pi_\ell \bullet A_i + \beta_\ell \bullet B_i = f^*(\pi_\ell \bullet A_i) + \beta_\ell \bullet B_i$. In particular, this implies that the filtration induced by this Chow-Künneth decomposition (as defined in the Introduction) is described by

$$F^m CH^i_{\mathbb{Q}}(Y) = f^* F^m CH^i_{\mathbb{Q}}(X) + B_{i,m},$$

where $B_i = B_{i,0} \supseteq B_{i,1} \supseteq \dots$ is a descending chain of subgroups. By Murre's Conjecture **C** for X, the term $F^m CH^i_{\mathbb{Q}}(X)$ is independent of the original choice of Chow-Künneth decomposition. Also, by [MU, Lemma 1.4.4], $F^1 CH^i_{\mathbb{Q}}(Y)$ is contained in the subgroup of $CH^i_{\mathbb{Q}}(Y)$ consisting of cycles homologically equivalent to zero. If $i = d$, then $B_i = 0$; so $B_{i,j} = 0$ for all j. If $i < d$, then $B_i = CH^{i-1}_{\mathbb{Q}}(Z) \cong \mathbb{Q}$ is a one-dimensional $\mathbb{Q}$-vector space, and its only cycle homologically equivalent to zero is 0 itself. Hence

$B_{i,j} = 0$ for all $j > 1$, showing that the filtration $F^m CH^i_{\mathbb{Q}}(Y)$ is independent of the original choice of Chow-Künneth decomposition on Y. This proof also shows that if Conjecture **D** holds for X, i.e. $F^1 CH^i_{\mathbb{Q}}(X) = CH^i_{\mathbb{Q}}(X)_{hom}$, then likewise $F^1 CH^i_{\mathbb{Q}}(Y) = CH^i_{\mathbb{Q}}(Y)_{hom}$.

Conversely, suppose Y has a Chow-Künneth decomposition satisfying Murre's Conjecture **C**. If $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$ is a Chow-Künneth decomposition on X, use Theorem 2.9 to construct a Chow-Künneth decomposition $[\Delta_Y] = \sum_{i=0}^{2d} (f \times f)^* \pi_i + \beta_i$ on Y. By assumption, the filtration defined by this Chow-Künneth decomposition, i.e. $F^m CH^i_{\mathbb{Q}}(Y) = f^* F^m CH^i_{\mathbb{Q}}(X) + B_{i,m}$ is independent of the original choice of Chow-Künneth decomposition on X; hence $f_* F^m CH^i_{\mathbb{Q}}(Y) = F^m CH^i_{\mathbb{Q}}(X)$ is also independent on this choice, and so Conjecture **C** holds for X. The assertion concerning Conjecture **D** follows similarly.

□

4 Examples

We conclude by applying our results to two classes of examples.

4.1 *Resolutions of Kummer varieties.*

Let k be an algebraically closed field of characteristic $\neq 2$. Following the notation established earlier, we denote by K_A the Kummer variety associated to the abelian variety A (defined over k) and K'_A the resolution of K_A obtained by blowing up its (finite) singular locus. We recall the main result of [AJ]:

Theorem 4.1. *[AJ, Theorem 1.1, Lemma 2.10, Corollary 2.11] Let A be an abelian variety of dimension d over a field and G a finite group acting on A. Then there exists a Chow-Künneth decomposition $[\Delta_{A/G}] = \sum_{i=0}^{2d} \rho_i$ satisfying Poincaré duality such that ρ_i acts as 0 on $CH^j_{\mathbb{Q}}(A/G)$ if $i < j$ or $i > j + d$. In particular, if $d \leq 4$, then ρ_i acts as zero on $CH^j_{\mathbb{Q}}(A/G)$ if $i < j$ or $i > 2j$.*

Now the Kummer variety K_A is obtained from an abelian variety A by taking the quotient by the group G generated by the involution $\iota : a \mapsto -a$. The resolution K'_A is obtained from K_A by blowing up the image of the 2-torsion points under the quotient map $\pi : A \to K_A$. As observed in [DL, p. 4], the Kummer variety of A is embedded in $\mathbb{P}^{2^d - 1}$ (using a symmetric theta divisor), so that the image of any 2-torsion point is a singular point étale locally isomorphic to the affine cone over the second Veronese variety of $\mathbb{P}^{d-1}$. (To see this, observe that the negation involution of the abelian variety A acts locally by $(z_1, ..., z_g) \to (-z_1, \ldots, -z_g)$ because it acts so on the tangent space. The ring of invariants is generated by polynomials $z_i z_j$.) It follows that the exceptional divisor of the blow-up of the Kummer variety at a 2-torsion is isomorphic to $\mathbb{P}^{d-1}$.

Let $a \in A$ denote a 2-torsion point and $\tilde{K}_A$ the blow-up of K_A along $\{\pi(a)\}$. Let $\tilde{A}$ denote the blow-up of the abelian variety A along $\{a\}$. Then the universal property of the blow-up, along with the observation that G acts trivially on K_A, shows that one obtains an induced map $f : \tilde{A}/G \to \tilde{K}_A$. Since the exceptional divisors of both blow-ups are isomorphic to $\mathbb{P}^{d-1}$, the above map is quasi-finite. Moreover the Kummer variety is known to be a normal variety: in [SAS] it is shown to be projectively normal. Thus, f is a quasi-finite proper birational map with $\tilde{K}_A$ normal; so, Zariski's main theorem implies that f is an isomorphism. (cf. [AM, Corollary 3.9].) This proves that $\tilde{K}_A$ is also a pseudo-smooth variety satisfying the hypotheses of 2.2. Moreover, a similar argument shows that the intermediate schemes obtained by blowing up each of the 2-torsion points on K_A iteratively also satisfy the same hypotheses. Therefore, Theorem 2.9 may be applied inductively; when combined with Proposition 3.4, we obtain:

Corollary 4.2. *Let A be an abelian variety over an algebraically closed field of characteristic different from 2 and K'_A its associated Kummer variety. Then K'_A has a Chow-Künneth decomposition satisfying Poincaré duality and Murre's conjectureB'. When K'_A has dimension at most 4, Murre's vanishing conjecture **B** holds for K'_A.*

4.2 *Blow-ups of smooth varieties along a finite locus of k-rational points*

If X is a smooth projective variety over an arbitrary field k (i.e. G is the trivial group) and a is any k-rational point on X, the blow-up Y of X along $\{a\}$ is smooth, with exceptional divisor isomorphic to $\mathbb{P}^{\dim X-1}$. Thus, the hypotheses of Theorem 2.9 are fulfilled, and validity of each of Murre's conjectures on X is equivalent to its validity on Y. Since Y is smooth, this procedure may be iterated to extend this result to the blow-up of a smooth variety along any finite locus of k-rational points.

For example, one may blow up an abelian variety along any finite set of k-rational points. The resulting variety will then satisfy all those conjectures of Murre which hold for the abelian variety.

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