HIGHER INTERSECTION THEORY ON ALGEBRAIC STACKS:I

Roy Joshua

Abstract. In this paper and the sequel we establish a theory of Chow groups and higher Chow groups on algebraic stacks locally of finite type over a field and establish their basic properties. This includes algebraic stacks in the sense of Deligne-Mumford as well as Artin. An intrinsic difference between our approach and earlier approaches is that the higher Chow groups of Bloch enter into our theory early on and depends heavily on his fundamental work. Our theory may be more appropriately called *the (Lichtenbaum) motivic homology and cohomology* of algebraic stacks. One of the main themes of these papers is that such a motivic homology does provide a reasonable intersection theory for algebraic stacks (of finite type over a field), with several key properties holding integrally and extending to stacks locally of finite type. While several important properties of our higher Chow groups, like covariance for projective representable maps (that factor as the composition of a closed immersion into the projective space associated to a locally free coherent sheaf and the obvious projection), an intersection pairing and contravariant functoriality for all smooth algebraic stacks are shown to hold integrally, our theory works best with rational coefficients.

The *main results of Part I* are the following. The higher Chow groups are defined in general with respect to an atlas, but are shown to be independent of the choice of the atlas for smooth stacks if one uses finite coefficients with torsion prime to the characteristics or in general for Deligne-Mumford stacks. (Using some results on motivic cohomology, we extend this integrally to all smooth algebraic stacks, in Part II.) *Using cohomological descent, we extend Bloch's fundamental localization sequence for quasi-projective schemes to long exact localization sequences of the higher Chow groups modulo torsion for all Artin stacks: this is one of the main results of the paper.* We show that these higher Chow groups modulo torsion are covariant for all proper representable maps between stacks of finite type while being contravariant for all representable flat maps and, in Part II, that they are independent of the choice of an atlas for all stacks of finite type over the given field k. The comparison with motivic cohomology as is worked out in Part II, enables us to provide an explicit comparison of our theory for quotient stacks associated to actions of linear algebraic groups on quasi-projective schemes with the corresponding Totaro-Edidin-Graham equivariant intersection theory. As an application of our theory we compute the higher Chow groups of Deligne-Mumford stacks and show that they are isomorphic modulo torsion to the higher Chow groups of their coarse moduli spaces. As a by-product of our theory we also produce localization sequences in (integral) higher Chow groups for all schemes locally of finite type over a field: these higher Chow groups are defined as the Zariski hypercohomology with respect to the cycle complex.

Table of Contents

Supported by the Max Planck Institut für Mathematik, the IHES and the ICTP

0. Introduction

In this paper and the sequel we establish a theory of higher Chow groups for all Artin stacks locally of finite type over a field along with an (integer valued) intersection pairing for all *smooth Artin stacks.* We show that this theory reduces, *modulo torsion*, to the usual theory of Chow groups for all Deligne-Mumford stacks that have a quasi-projective coarse-moduli space. Recall that the higher cycle complex of Bloch has reasonable properties (at least in a widely accepted sense) only on restriction to the category of quasi-projective schemes over k. We extend the definition of the higher Chow groups first to all schemes locally of finite type over a field k as the Zariski hypercohomology with respect to the higher cycle complex: already this recovers many of the nice properties of the higher Chow groups including a localization sequence for all schemes of finite type over a field (see Theorem 4) as well as an intersection pairing (as in [J-1] Theorem 2). Next we extend the theory to algebraic spaces and to algebraic stacks locally of finite type over a field k as hypercohomology on the étale site of a classifying simplicial algebraic space associated to the stack. The higher Chow groups we define are also contravariantly functorial for representable flat maps while being covariant for all proper representable maps modulo torsion.

We summarize the main results here. Let (alg.stacks) denote the category of algebraic stacks in the sense of Artin and locally of finite type over a given field k.

Theorem 1. (See (4.5.1), (4.5.2), (4.6.1), (4.6.3), (3.8.4) and also [J-1] Theorem 2.) Associated to each algebraic stack $\mathfrak{S}$ and presentation $x : X \to \mathfrak{S}$, there exist two bi-graded abelian groups, $CH^*(\mathfrak{S}, x, .)$ and $CH^*(\mathfrak{S}, x, .; \mathbb{Q})$ having the following properties:

(i) If $\mathfrak{S}$ is an Artin stack, $CH_*(\mathfrak{S}, x, .)$ depends, in general, on the choice of the atlas $x : X \to \mathfrak{S}$. It is independent of the choice of the atlas (and intrinsic to the stack) for all Deligne-Mumford stacks. It is also intrinsic to the stack for all smooth Artin stacks while $CH_*(\mathfrak{S}, .; \mathbb{Q})$ is intrinsic to the stack for all algebraic stacks of finite type over k. (We establish this only with finite coefficients in this paper; the general case is considered in [J-1] Theorem 2.)

(ii) $CH_*(\quad, .)$ and $CH_*(\quad, .; \mathbb{Q})$ are contravariant for all *representable flat maps* (with an appropriate shift). They are also contravariant for all maps between smooth stacks: again this is established only in [J-1]. $CH_*(\quad, .; \mathbb{Q})$ is covariant for all representable proper maps between stacks of finite type. It is also covariant for all finite maps between Deligne-Mumford stacks and $CH_*(\quad, .)$ is covariant for strongly projective morphisms (i.e. morphisms that factor as a closed immersion into *Proj* of a locally free coherent sheaf followed by the projection to the base) between stacks locally of finite type.

(iii) Both $CH_*(\quad, .)$ and $CH_*(\quad, .; \mathbb{Q})$ are homotopy invariant, i.e. $CH_*(\mathfrak{S}, x, .) \simeq CH_*(\mathfrak{S} \times \mathbb{A}^1, x \times \mathbb{A}^1, .)$ and similarly for $CH_*(\quad, .; \mathbb{Q})$.

(iv) If $\mathfrak{S}' \to \mathfrak{S}$ is a *closed immersion* of algebraic stacks with $\mathfrak{S}'' = \mathfrak{S} - \mathfrak{S}'$, $x : X \to \mathfrak{S}$ is a given atlas for $\mathfrak{S}$ with $x' = x \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ and $x'' = x \underset{\mathfrak{S}}{\times} \mathfrak{S}''$, then one obtains a long-exact sequence:

$$... \to CH_*(\mathfrak{S}', x', n; \mathbb{Q}) \to CH_*(\mathfrak{S}, x, n; \mathbb{Q}) \to CH_*(\mathfrak{S}'', x'', n; \mathbb{Q}) \to CH_*(\mathfrak{S}', x', n-1; \mathbb{Q}) \to ...$$

(v) There exists a natural map $CH_*(\mathfrak{S}, x; .) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to CH_*(\mathfrak{S}, x; \mathbb{Q})$ which is an isomorphism if $\mathfrak{S}$ is a Deligne-Mumford stack (in particular, an algebraic space or scheme) of finite type. □

In view of (i) of the above theorem, we may omit the presentation from the higher Chow groups for Deligne-Mumford stacks. Throughout the paper we will use the terminology, *a property holds for the functor $CH_*(\quad, .)$ modulo torsion*, to mean it holds for the functor $CH_*(\quad, .; \mathbb{Q})$. A *coarse-moduli space* for an algebraic stack will always mean one satisfying the hypotheses in (1.3.4)(vii).

Theorem 2 (See (4.1.1), (4.3.4), (4.6.5), (5.1.2) and (5.1.3).) (i) There is a natural augmentation from the *naive* Chow groups of a stack, $CH_*^{naive}(\mathfrak{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, to the groups $CH_*(\mathfrak{S}, 0; \mathbb{Q})$ in Theorem 1. (See (4.3.3) for the definition of the naive Chow groups.) This augmentation is an isomorphism in the following cases: all quasi-projective schemes over k and all separated Deligne-Mumford stacks of finite type over k that have a quasi-projective coarse moduli space.

(ii) If $CH_*(X, .)$ of a scheme X, of finite type over a field k, are defined as the (homotopy groups of the) Zariski hypercohomology with respect to the higher cycle complex of Bloch, there exists an obvious augmentation from $CH_*(X, .) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ to $CH_*(X, .; \mathbb{Q})$. (Here $CH_*(X, .; \mathbb{Q})$ is the functor in Theorem 1.) This is an isomorphism.

(iii) Let $\mathfrak{S}$ denote a separated Deligne-Mumford stack of finite type over k so that a coarse-moduli space $\mathfrak{M}_{\mathfrak{S}}$ exists satisfying the hypotheses of (1.3.4)(vii). If $\mathfrak{M}_{\mathfrak{S}}$ is a scheme of finite type over k, then $CH_*(\mathfrak{S}, .; \mathbb{Q}) \cong CH_*(\mathfrak{M}_{\mathfrak{S}}, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ where the right hand side is defined as in (ii). More generally when the coarse moduli space $\mathfrak{M}_{\mathfrak{S}}$ is only an algebraic space (see [KM]), one obtains the isomorphism $CH_*(\mathfrak{S}, .; \mathbb{Q}) \cong CH_*(\mathfrak{M}_{\mathfrak{S}}, .; \mathbb{Q})$ where the right-hand-side is defined as in (4.6.6). □

The following corollary is to serve as a sample of the applications.

Corollary 3. (i) Let G denote a smooth affine group scheme acting on a scheme locally of finite type over k and let $i : Y \rightarrow X$ denote the closed immersion of a G-stable closed subscheme with $j : U = X - Y \rightarrow X$ the open immersion of its complement. Then one obtains a long-exact-sequence:

$$... \rightarrow CH_*([Y/G], Y, n; \mathbb{Q}) \rightarrow CH_*([X/G], X, n; \mathbb{Q}) \rightarrow CH_*([U/G], U, n; \mathbb{Q}) \rightarrow CH_*([Y/G], Y, n-1; \mathbb{Q}) \rightarrow ....$$

where X (Y, U) denotes the obvious atlas for the stack $[X/G]$ ($[Y/G]$, $[U/G]$, respectively). (Observe in view of Theorem 1 (i), that if X is smooth or of finite type over k, the groups $CH_*([X/G], X, .; \mathbb{Q})$ are in fact independent of the atlas X.)

(ii) If G is a linear algebraic group and X is quasi-projective provided with a linearized G-action (i.e. X admits a G-equivariant locally closed immersion into a projective space onto which the G-action extends to a linear action, for example if G is connected and X is normal by [Su]), one obtains the isomorphism $CH_*([X/G], .; \mathbb{Q}) \cong CH^G_*(X, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ where the right hand side is the equivariant intersection theory considered in [EG] and [Tot]. In particular, $CH_*([X/G], n; \mathbb{Q}) \cong 0$ if $n < 0$.

(iii) Let G denote a finite group acting on a separated scheme X of finite over k so that $\mathfrak{M}_{[X/G]}$ is a coarse moduli space. Then $CH_*([X/G], .; \mathbb{Q}) \cong CH_*(\mathfrak{M}_{[X/G]}, .; \mathbb{Q})$. (For example $CH_*(BG, .; \mathbb{Q}) \cong CH_*(Spec \quad k, .; \mathbb{Q})$.) More generally, let G denote a smooth affine group scheme acting locally properly on a separated scheme X of finite type over k so that the stack $[X/G]$ is Deligne-Mumford (for example the stabilizers are all reduced and finite). Let $\mathfrak{M}_{[X/G]}$ denote a coarse moduli space as in (1.3.4)(vii). Then one obtains the isomorphisms: $CH_*(\mathfrak{M}_{[X/G]}, .; \mathbb{Q}) \cong CH_*([X/G], .; \mathbb{Q})$.

(iv) Let X denote a smooth projective curve over an algebraically closed field k and let $r > 0$ denote an integer. Let $\mathcal{SL}_X(r)$ denote the moduli stack of rank r vector bundles on X with trivial determinant and let $\mathcal{SL}_X(r)^{ss}$ denote the open sub-stack of semi-stable bundles. Let $z : Z \rightarrow \mathcal{SL}_X(r)$ denote an atlas for the first stack and let z'', z' denote the induced atlases for $\mathcal{SL}_X(r)^{ss}$, $\mathcal{SL}_X(r) - \mathcal{SL}_X(r)^{ss}$, respectively. Then one obtains a long-exact sequence:

$$... \rightarrow CH_*(\mathcal{SL}_X(r) - \mathcal{SL}_X(r)^{ss}, z', n; \mathbb{Q}) \rightarrow CH_*(\mathcal{SL}_X(r), z, n; \mathbb{Q}) \rightarrow CH_*(\mathcal{SL}_X(r)^{ss}, z'', n; \mathbb{Q}) \rightarrow ...$$

(v) Let X denote a smooth projective variety which is *convex* in the sense of [F-P] p.6. Let $\bar{M}_{g,n}(X, \beta)$ denote the stack of stable families of maps of n-pointed genus g-curves to X and let $\overline{\mathfrak{M}}_{g,n}(X, \beta)$ denote the corresponding coarse-moduli space. Here β denotes a class in $CH^1(X)$. Then one obtains an isomorphism: $CH_*(\overline{\mathfrak{M}}_{g,n}(X, \beta), .; \mathbb{Q}) \cong CH_*(\bar{M}_{g,n}(X, \beta), .; \mathbb{Q})$. □

The example in (iv) is important for several reasons. First, as shown in [BL], one cannot interpret these stacks (a priori) as quotient stacks unless one is willing to consider ind-schemes. Moreover these stacks are *only locally of finite type*. Secondly these stacks are shown to be important to various problems in mathematical physics. (iv) provides a technique for computing their (higher) Chow groups. The example in (ii) shows that our theory of higher Chow groups extends the theory of Totaro-Edidin-Graham (see [EG], [Tot]) from quotient stacks associated to actions of algebraic groups to all algebraic stacks, at least if one works modulo torsion.

As a by product of our theory we also obtain a straightforward extension of the higher Chow groups to all schemes locally of finite type over a field k. This is summarized in the following theorem. Let (schemes) denote the category of all schemes locally of finite type over a given field k.

Theorem 4. (See (5.2).)

There exist functors

$CH_*(\quad, .), \quad CH_*(\quad, .; \mathbb{Q})$:(schemes) $\rightarrow$ (bi-graded abelian groups)

having the following properties.

(i) $CH_*(\ \ ,.)$ is contravariant (with an appropriate shift) for flat maps and covariant for all proper maps between schemes of finite type type. (Between smooth schemes, it is also contravariant (with an appropriate shift) for all maps: see [J-1] Theorem 3.)

(ii) These groups are intrinsic to the scheme (and are defined as the Zariski hypercohomology with respect to the higher cycle complex). There exits a natural morphism $CH_*(X,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \rightarrow CH_*(X,.;\mathbb{Q})$ which is an isomorphism if X is a scheme of finite type over k.

(iii) If $Y \rightarrow X$ is a closed immersion of schemes with $U = X - Y$, one obtains a long-exact sequence

$... \rightarrow CH_*(Y,n) \rightarrow CH_*(X,n) \rightarrow CH_*(U,n) \rightarrow CH_*(Y,n-1) \rightarrow ...$

(iv) The functor $CH_*(\ \ ,.;\mathbb{Q})$ inherits all the above properties of $CH_*(\ \ ,.)$. In addition, $CH_*(X,n;\mathbb{Q}) = 0$ if $n < 0$ for all schemes of finite type over a field k. (A priori, these groups may be non-trivial integrally in negative degrees. This needs a strong form of the Riemann-Roch theorem as outlined in (4.4.5).)

(v) If X is a quasi-projective scheme over k, these higher Chow groups are isomorphic to the ones defined in [Bl-1]. (In particular, in this case $CH_*(X,n) = 0$ for all $n < 0$.) □

We would like to call attention to some of the advantages to our approach. First of all, as seen above, our Chow groups are defined along with the higher Chow groups in a uniform manner and they come with localization sequences (and hence Mayer-Vietoris sequences) at least modulo torsion. Therefore they are computable and also conceptually straightforward to define for all stacks locally of finite type over a field; being able to consider stacks that are *locally of finite type and not necessarily of finite type seems essential since there are several important stacks that are only locally of finite type. Moreover, it applies to all quotient stacks, not necessarily those that are defined by linear algebraic group actions.* (See the examples (ii) and (iii) in (1.4) below as well as (5.3).) Cohomological descent modulo torsion on the étale site shows they also agree with the usual Chow groups for schemes (and Deligne-Mumford stacks with a quasi-projective scheme as a coarse moduli space) modulo torsion. Moreover, the intersection pairing is rather easy to define - see Part II. The only disadvantage of our theory is that the integral version does not coincide with the usual Chow groups even for schemes, but rather is given by the étale hypercohomology with respect to an appropriate pre-sheafification of the higher cycle complex of Bloch. From a motivic point of view one may regard our definition as the (Lichtenbaum) motivic homology and cohomology of stacks. Our approach to the higher Chow groups for schemes is technically quite simple, but does provide a working definition of the higher Chow groups with reasonable properties for all schemes locally of finite type over a field (at least modulo torsion).

There are several inherent difficulties with stacks that seem to indicate our results may be more or less optimal. In fact, since the diagonal of an Artin stack is not a local imbedding, the technique of the Gysin map does not apply to define an intersection pairing in general for all smooth stacks. However, see [Kr] where such an intersection pairing is defined for a large family of smooth stacks. One key advantage of our theory is that it is indeed possible to define an intersection pairing for *all smooth Artin stacks.* An algebraic stack is defined by 'gluing' together schemes in the étale or smooth topology rather than the Zariski topology. This makes it necessary to work on the étale site; for cohomological descent on the étale site one needs to work modulo torsion, in general. For an Artin stack, an atlas can only be a smooth map $x : X \rightarrow \mathfrak{S}$ with X an algebraic space. Moreover there are, in general, maps that are not flat between objects on the smooth site of the stack $\mathfrak{S}$. Therefore, the cycle complex is not a presheaf on the smooth site and pre-sheafification of this complex seems to loose some of its nicer properties. The étale site of a Deligne-Mumford stack has only étale maps as morphisms and therefore the cycle complex is indeed a presheaf on this site. These make it necessary to include the choice of an atlas in the definition of the higher Chow groups for Artin stacks in general. (However the motivic complexes are contravariant for all maps; the comparison of the higher Chow groups with motivic cohomology (see [Voev-1] and [Fr-S]) enables one to replace the cycle complex often by the complex giving motivic cohomology. The precise details are worked out in the second part of this paper.)

While we discuss only few applications here, it is to be expected that our results will have a variety of applications, for example, to Riemann-Roch problems on stacks (see [Toe], [J-2]), the theory of Gromov-Witten invariants considered for example in [B-M] as well as to quantum cohomology (see [F-P]). These will be explored elsewhere.

Here is an *outline* of the paper. We review basic definitions and examples of algebraic stacks in section 1. This is followed by a discussion of the sites we consider in this paper associated to algebraic stacks. The third section begins with a discussion on hypercohomology and cohomological descent, which plays a key role in obtaining the localization sequences. *The main result of this section is Theorem (3.8.3) which provides us localization sequences as a consequence of cohomological descent. Though cohomological descent is by now a standard technique, its use in providing localization sequences is new.* We have provided a rather detailed discussion of cohomological descent, mainly to make the paper accessible to non-experts; the knowledgeable reader may skip some of this and go straight to Theorem (3.8.3). In the fourth section, we define and establish several fundamental properties for the higher Chow groups. A few applications are discussed in section 5 and section 6 contains a self-contained discussion on homotopy inverse limits: these are the substitutes for the total complex of a double complex when considering unbounded complexes.

Acknowledgments. As should be clear, we have made extensive use of many results due to various authors so that it is difficult to fully acknowledge them here. These papers could never have been written without several helpful discussions with Spencer Bloch, Michel Brion, Dan Edidin, Andrew Kresch and Bertrand Toen. In fact, the key idea of comparison with motivic cohomology as appears in Part II, we owe to Spencer Bloch while Bertrand Toen helped us with the comparison with the Totaro-Edidin-Graham theory as well as several other key ideas. Finally we would also like to thank the referee for several insightful suggestions for improvements.

1. Review of algebraic stacks: basic definitions and examples

In this section we quickly review the basics of algebraic stacks and set-up the basic framework for the rest of our work. We will refer to [L-MB1] for the basic theory of algebraic stacks and to [Kn] for the basic theory of algebraic spaces. We also recall some standard examples of stacks.

(1.0). Throughout the paper we will restrict to *schemes and algebraic spaces locally of finite type over a field k of arbitrary characteristic (and hence quasi-separated).* (See [Kn] p. 51.) (For the most part, it suffices to consider only objects of finite type.) Let $(schemes/k)$ denote this category of schemes. We will (usually) provide $(schemes/k)$ with the $fppf$ topology. Recall this is the topology generated by the following pre-topology. If X is a scheme over k, the coverings of X are finite families $\{X_i \xrightarrow{\phi_i} X|i\}$ so that each map ϕ_i is a map over k, is flat, locally of finite presentation and $X = \underset{i}{\cup}\phi_i(X_i)$. A sheaf of sets on this topology will be often referred to as a *space*.

(1.1). We will assume the basic terminology on *algebraic spaces* from ([Kn], chapter 2). The category of algebraic spaces locally of finite type over k will be denoted $(alg.spaces/k)$. Observe that if X is a scheme locally of finite type over k, the associated functor $(schemes/k)^{op} \rightarrow$ (sets) represented by X is an algebraic space. Thus the category of schemes admits an imbedding as a full subcategory of the category of algebraic spaces.

The class of maps between algebraic spaces that are of locally of finite type (étale, smooth) is stable under composition and base-change. Most local properties of algebraic spaces are given in terms of the corresponding properties of a representable étale covering. We will provide the category $(alg.spaces/k)$ with the fppf topology.

(1.2.1) Let C denote a category with finite limits and let S be a category fibered in groupoids over C as in ([D-M] (section 4)). Assume that for every $\varphi : U \rightarrow V$ in C, and for every $y \in Ob(S_V)$ a map $f : x \rightarrow y$ lifting φ has been chosen. Then x will be denoted $\varphi^*(y)$. Now $\varphi^* : S_V \rightarrow S_U$ is a functor and one is given a natural isomorphism $(\varphi \circ \psi)^* \cong \psi^* \circ \varphi^*$ (satisfying certain obvious compatibility conditions) if $\varphi : U \rightarrow V$ and $\psi : Z \rightarrow U$ are in C.

(1.2.2) Let C denote a category with finite limits and provided with a Grothendieck topology. A *stack in groupoids over* C is a category fibered in groupoids over C satisfying the conditions in ([D-M] Definition (4.1)). Observe that the stacks in groupoids over C (denoted $(stacks/C)$) forms a 2-category: the 1-morphisms are functors from one stack to another and the 2-morphisms are morphisms of such functors. Let C also denote the 2-category having the same objects and morphisms as C and where the 2-morphisms are all the identities; thus C may be identified as a sub 2-category of the the 2-category $(stacks/C)$. If K is an object of C, K provides a stack, namely the category whose sections over U ϵ C is the discrete category of morphisms in C from U to K. Such a stack is said to be *represented* by the object K.

(1.2.3). A 1-morphism $F : \mathfrak{S}_1 \rightarrow \mathfrak{S}_2$ of stacks over C is *representable* if for every X in C and any 1-morphism $x : X \rightarrow \mathfrak{S}_2$, the fiber-product $X \underset{\mathfrak{S}_2}{\times} \mathfrak{S}_1$ is a representable stack. (Recall that if A ϵ C,

$$X \underset{\mathfrak{S}_2}{\times} \mathfrak{S}_1(A) = \{(f : A \to X,\ u \,\epsilon\, Ob(\mathfrak{S}_1(A))|F(u) = f^*(x),\ x \text{ regarded as an object of } \mathfrak{S}_2(X)\}$$

regarded as a category in the trivial manner (i.e. all morphisms are the identities); the above condition says that the functor $f \to Ob(X \underset{\mathfrak{S}_2}{\times} \mathfrak{S}_1(A))$ is representable by some $g : Y \to X$.)

(1.2.4). Let P be a property of morphisms in C, stable under base-change and of a local nature on the target. *A representable map* $F : \mathfrak{S}_1 \to \mathfrak{S}_2$ of stacks over C has the property P if the map $F' : X \underset{\mathfrak{S}_1}{\times} \mathfrak{S}_2 \longrightarrow X$ induced by base change for any 1-morphism $x : X \longrightarrow \mathfrak{S}_2$, $X \,\epsilon\, C$, has the same property P.

(1.2.5). Finally observe the following: let $\mathfrak{S}$ be a stack over C. Then the diagonal map $\mathfrak{S} \to \mathfrak{S} \times \mathfrak{S}$ is representable if and only if for every X, $Y \,\epsilon\, C$, and 1-morphisms $x : X \to \mathfrak{S}$, $y : Y \to \mathfrak{S}$, the fiber-product $X \underset{\mathfrak{S}}{\times} Y$ is representable.

From now onwards, C will denote the category $(alg.spaces/k)$ provided with the big étale topology, with k a field.

(1.3.1). **Definition**. An *algebraic stack* $\mathfrak{S}$ is a stack in groupoids over the category (alg. spaces/k) so that

(a) $\Delta : \mathfrak{S} \to \mathfrak{S} \underset{Spec\ k}{\times} \mathfrak{S}$ is representable, separated and quasi-compact and

(b) there exists a representable *smooth and surjective map* $x : X \to \mathfrak{S}$ with X an algebraic space. (i.e. for every $Y \to \mathfrak{S}$, the fibered product $X \underset{\mathfrak{S}}{\times} Y$ is a representable stack represented by an algebraic space and the obvious induced map $X \underset{\mathfrak{S}}{\times} Y \to Y$ is smooth and surjective.) We will often refer to $x : X \to \mathfrak{S}$ as an *atlas* or a *smooth atlas*. (Most local properties of algebraic stacks are expressed in terms of the corresponding properties of an atlas.)

(1.3.2) An algebraic stack is *Deligne-Mumford* if the map $x : X \to \mathfrak{S}$ is *étale surjective*. This is equivalent to requiring that the diagonal $\mathfrak{S} \to \mathfrak{S} \underset{Spec\ k}{\times} \mathfrak{S}$ is unramified. A general stack as in (1.4) will often be referred to as an *Artin* stack.

Remarks. (1.3.3) Since all our stacks are locally of finite type over a field they are also locally of finite presentation. Moreover any map between stacks that are locally of finite type is also locally of finite presentation.

(1.3.4) **Definitions**. (i) A representable map $i : \mathfrak{S}' \to \mathfrak{S}$ of algebraic stacks is a *closed (open, locally closed) immersion* if for any map $x : X \to \mathfrak{S}$, with X a scheme, the induced map $i' : X' = X \underset{\mathfrak{S}'}{\times} \mathfrak{S} \to X$ is a closed (open, locally closed respectively) immersion of schemes.

(ii) A representable map $i : \mathfrak{S}' \to \mathfrak{S}$ of algebraic stacks is a *local imbedding* if for any map $x : X \to \mathfrak{S}$, with X a scheme, the induced map $i' : X' = X \underset{\mathfrak{S}'}{\times} \mathfrak{S} \to X$ is an unramified map of finite type.

(iii) Let $\mathfrak{S}$ denote an algebraic stack over k and let $\mathcal{E}$ denote a coherent sheaf on $\mathfrak{S}$. Let $P(\mathcal{E}) = Proj(\mathcal{E})$ denote the corresponding projective space over $\mathfrak{S}$. An algebraic stack $\mathfrak{S}'$ is quasi-projective over the stack $\mathfrak{S}$ if there exists a locally-closed immersion $i : \mathfrak{S}' \to P(\mathcal{E})$ for some coherent sheaf $\mathcal{E}$ over $\mathfrak{S}$.

(iv) A representable map $p : \mathfrak{S}' \to \mathfrak{S}$ of algebraic stacks is *proper*, if for any map $x : X \to \mathfrak{S}$ with X a scheme, the induced map $X' = X \underset{\mathfrak{S}}{\times} \mathfrak{S}' \to X$ is proper. It is *projective* if it factors as the composition of a closed immersion $i : \mathfrak{S}' \to P(\mathcal{E})$ and the obvious projection $\pi : P(\mathcal{E}) \to \mathfrak{S}$ for some coherent sheaf $\mathcal{E}$ over $\mathfrak{S}$. (We do not consider non-representable proper maps in this paper.)

(v) Let $x : X \to \mathfrak{S}$ denote an atlas for an Artin stack. One may define *the relative local dimension* of X over $\mathfrak{S}$, $dim_{\mathfrak{S},\zeta} X$ at a point ζ of X as follows. Let $y : Y \to \mathfrak{S}$ denote a smooth map with Y a scheme. Let $\bar{\zeta} = x(\zeta)$ and let ζ' be a point of Y so that $y(\zeta') = \bar{\zeta}$. Then we let $dim_{\mathfrak{S},\zeta} X = dim_{Y,\zeta'}(Y \underset{\mathfrak{S}}{\times} X) = dim_{(\zeta',\zeta)}(Y \underset{\mathfrak{S}}{\times} X) - dim_{\zeta'} Y$. Observe that this is constant on each connected component of X. (We skip the direct verification that this depends only on x.)

(vi) One may define the *local dimension* of the stack $\mathfrak{S}$ at a point $\bar{\zeta}$ of $\mathfrak{S}$ to be $dim_{\zeta}(X) - dim_{\mathfrak{S},\zeta} X$ where $x(\zeta) = \bar{\zeta}$. (Once again we skip the verification that this depends only on the stack $\mathfrak{S}$ and the point $\bar{\zeta}$.) The dimension of the stack $\mathfrak{S}$ is defined to be the supremum of all the local dimensions at all the points of $\mathfrak{S}$. (Observe that, if $\mathfrak{S}$ is a Deligne-Mumford stack with $x : X \rightarrow \mathfrak{S}$ an atlas, the dimension of $\mathfrak{S}$ = the dimension of X. Moreover, observe that the dimension of a stack could be negative; for example, if G denotes an algebraic group defined over a field k, the stack [Spec k/G] has dimension = - the dimension of G over k. Observe also that if $\mathfrak{S}$ has components of different dimensions, the dimension of $\mathfrak{S}$ is the supremum of the dimensions of the components.)

(vii) A *coarse moduli-space* for an algebraic stack $\mathfrak{S}$ will be a *proper map* $\pi : \mathfrak{S} \rightarrow \mathfrak{M}_{\mathfrak{S}}$ (with $\mathfrak{M}_{\mathfrak{S}}$ an algebraic space) which is a uniform categorical quotient and a uniform geometric quotient in the sense of [KM] 1.1 Theorem. (This may be different from the notion adopted in [Vis] or [Gi].) It is shown in [KM] that if the stack $\mathfrak{S}$ is Deligne-Mumford, of finite type over k and the obvious map $I_{\mathfrak{S}} \rightarrow \mathfrak{S}$ is finite, then a coarse moduli space exists with all of the above properties, except the map π may not be proper (i.e. finite). However, if $\mathfrak{S}$ is also separated over k, then the map π will also be proper (i.e. finite). To see this observe (see [Vis]) that one may find an étale covering $\mathfrak{M}' \rightarrow \mathfrak{M}_{\mathfrak{S}}$ so that the induced map $\pi' : \mathfrak{S} \underset{\mathfrak{M}_{\mathfrak{S}}}{\times} \mathfrak{M}' \rightarrow \mathfrak{M}'$ is finite. (In fact one may assume that the stack $\mathfrak{S} \underset{\mathfrak{M}_{\mathfrak{S}}}{\times} \mathfrak{M}'$ is the quotient stack associated a finite group action.) Therefore, in this case π itself is finite and a coarse moduli space in our sense exists.

(1.4) **Examples of algebraic stacks**. The following will be the main examples of Artin stacks that we consider in this paper.

(i) *Quotient stacks*: let G denote a smooth affine group scheme acting on an algebraic space X locally of finite type over k. Then the quotient stack $[X/G]$ is the stack whose sections over a scheme T are principal G-bundles on T provided with a G-equivariant map to X. Morphisms are maps of such principal G-bundles commuting with maps to X. This is denoted $[X/G]$ and the map $X \rightarrow [X/G]$ induced by the trivial principal G-bundle $G \times X$ is an atlas.

(ii) *Principal G-bundles.* Let G be as before and let X denote a fixed scheme of finite type over k. The stack of principal G-bundles on X, $\mathcal{M}_G$, has as sections over a scheme T, principal G-bundles on $X \times T$. A morphism from such an object E to another object given by a scheme T' and a principal G-bundle E' on $X \times T'$ is a map $f : T \rightarrow T'$ along with an isomorphism $E \cong (id_X \times f)^*(E')$.

(iii) *Vector bundle stacks of a fixed rank r.* Let X denote a given scheme of finite type over k and let $\mathcal{GL}_r^X$ denote the stack whose sections over a scheme T are vector bundles of rank r on $X \times T$. A morphism from such an object to another object given by a scheme T' and a rank r vector bundle E' on $X \times T'$ is a map $f : T \rightarrow T'$ along with an isomorphism $E \cong (id_X \times f)^*(E')$.

The last two are examples of stacks that are only locally of finite type, but not of finite type. The following are some of the interesting examples of Deligne-Mumford stacks.

(iv) quotient stacks as in (i) where the stabilizers are all finite and reduced.

(v) The stack of smooth curves of genus g. The groupoid M_g (over Spec $\mathbb{Z}$) is defined as follows. The objects of M_g are smooth and proper morphisms of schemes $C \rightarrow T$ whose geometric fibers are geometrically connected smooth curves of genus g. An arrow from $C \rightarrow T$ to $C' \rightarrow T'$ is a commutative diagram:

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

which induces an isomorphism $C \cong T \underset{T'}{\times} C'$. The functor $M_g \rightarrow$ (*schemes/Spec* $\mathbb{Z}$) sends $(C \rightarrow T) \rightarrow T$.

One may also consider the stack Z_g of smooth pointed curves of genus g. An object of Z_g is a morphism $C \rightarrow T$ as above together with a section $T \rightarrow C$. Maps between two such objects are cartesian squares respecting the sections. (See [DM].)

(vi) Let X be a smooth projective variety over an algebraically closed field, g, n two non-negative integers and $\beta \in A_1(X)$ = the class of a dimension 1 cycle in X. The stack of stable maps of genus g, $\bar{M}_{g,n}(X, \beta)$ (see [F-P] and [B-M]) is a Deligne-Mumford stack, not necessarily smooth.

2. Sites and topoi associated to algebraic stacks

(2.1) **Definitions**. (a) Let $\mathfrak{S}$ denote an algebraic stack as above with an atlas $x : X \to \mathfrak{S}$ as in (1.4.1). The *smooth site* of $\mathfrak{S}$ will be denoted $\mathfrak{S}_{\underline{smt}}$ and will be defined as follows. The objects of $\mathfrak{S}_{\underline{smt}}$ will be representable smooth maps $Y \xrightarrow{y} \mathfrak{S}$ with Y *an algebraic space.* (i.e. $Y \underset{\mathfrak{S}}{\times} Z \to Z$ is smooth for any algebraic space Z provided with a map $Z \longrightarrow \mathfrak{S}$). A morphism from $Y \xrightarrow{y} \mathfrak{S}$ to $Y' \xrightarrow{y'} \mathfrak{S}$ is a commutative triangle (of stacks: see [L-MB1] (6.1))

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle y'} & \\ & \mathfrak{S} & \end{array}$$

Covering families of an object $u : U \to \mathfrak{S}$ are defined as families $\{(U_i, u_i) | u_i : U_i \to U \quad in \quad \mathfrak{S}_{\underline{smt}} | i\}$ so that $\sqcup_i u_i : \sqcup U_i \to U$ is surjective.

(b) We also define the sites: $\mathfrak{S}_{smt}$ whose objects are smooth maps $\mathfrak{S}' \to \mathfrak{S}$ of algebraic stacks and morphisms defined as in (a). Observe that a map $f : \mathfrak{S}' \to \mathfrak{S}$ of algebraic stacks (which may in general be not representable) is defined to be smooth if for each map $x : X \to \mathfrak{S}$ from an algebraic space, the composite map $f' \circ \tilde{x}'$ defined below is *smooth.* One defines $X' = X \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ with $f' : X' \to X$ the obvious induced map and $\tilde{x}' : \tilde{X}' \to X'$ an atlas for the stack X'. Covering families of an object $u : U \to \mathfrak{S}$ are defined as families $\{(U_i, u_i) | u_i : U_i \to U \quad in \quad \mathfrak{S}_{smt} | i\}$ so that $\sqcup_i u_i : \sqcup U_i \to U$ is surjective.

In the above situation $\mathfrak{S}_{res.smt}$ ($\mathfrak{S}_{res.\underline{smt}}$) will denote the site with the same objects and coverings as $\mathfrak{S}_{smt}$ ($\mathfrak{S}_{\underline{smt}}$, respectively) but where the morphisms are required to be *flat maps.* All flat maps will be required to be of some fixed relative dimension. i.e. if $f : \mathfrak{S}' \to \mathfrak{S}$ is a flat map of algebraic stacks we will also require that for each irreducible component T of $\mathfrak{S}$, every irreducible component of $f^{-1}(T) = T \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ is of the same dimension ($=$ dimension $= dim\mathfrak{S}' - dim\mathfrak{S}$, in case both $\mathfrak{S}'$ and $\mathfrak{S}$ are of finite dimension).

(d) We define the *étale site* $\mathfrak{S}_{et}$ of an algebraic stack as follows. The objects of $\mathfrak{S}_{et}$ are representable étale maps $\alpha : \mathfrak{S}' \to \mathfrak{S}$ of algebraic stacks. Given two such objects $\alpha : \mathfrak{S}' \to \mathfrak{S}$ and $\beta : \mathfrak{S}'' \to \mathfrak{S}$, a morphism $\alpha \to \beta$ is a *representable* map $\gamma : \mathfrak{S}'' \to \mathfrak{S}'$ of algebraic stacks over $\mathfrak{S}$. Covering families are defined as above. (One may readily verify that in the above situation the map γ is also étale.) In the case of Deligne-Mumford stacks, one may also define an étale site $\mathfrak{S}_{\underline{et}}$ whose objects are maps $x : X \to \mathfrak{S}$ that are étale with X a scheme. The morphisms in this site and coverings are defined as before.

Remark. Observe that if U is connected, every smooth map $U \to \mathfrak{S}$ is of constant relative dimension. However, in general a smooth map $U \to \mathfrak{S}$ may have different relative dimensions on various *connected components* of U.

(2.2) Observe that all of the above sites except the sites $\mathfrak{S}_{res.smt}$ and $\mathfrak{S}_{res.\underline{smt}}$ are *closed under finite inverse limits.* For this it suffices to show they are closed under products and difference kernels. (To see that the latter exists in all of the above categories, see [Mi] p. 55.) The above sites we considered are what are commonly called small, even-though as categories they are only skeletally small.

(2.3) Most of the work in this paper will take place in the category of algebraic stacks locally of finite type over the given field k and where morphisms are required to be *flat maps.* This category will be denoted $(alg.stacks/k)_{Res}$. (Observe that we have not specified the coverings, so this is not a site, but only a category.) We will also consider its full sub-category consisting of all algebraic spaces locally of finite type over k: this category will be denoted $(alg.spaces/k)_{Res}$.

(2.4) **Definition**. Let $\mathfrak{S}$ denote a stack as before with $x : X \to \mathfrak{S}$ its atlas. Then one defines a simplicial object $B_x\mathfrak{S}.$ in the category of algebraic spaces, called the *classifying simplicial groupoid*, by letting $B_x\mathfrak{S}_0 = X$, $B_x\mathfrak{S}_1 = X \underset{\mathfrak{S}}{\times} X$,..., $(B_x\mathfrak{S})_n = \overbrace{X \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X ... \underset{\mathfrak{S}}{\times} X}^{(n+1)-times}$, with the structure maps induced from the two projections pr_1, $pr_2 : X \underset{\mathfrak{S}}{\times} X \to X$ and the diagonal $X \to X \underset{\mathfrak{S}}{\times} X$. (This is merely $cosk_{0\mathfrak{S}}(x)$.) Observe that all the face maps $\{d_i\}$ of this simplicial object, being induced by pr_1 or pr_2 are *smooth* maps. (If the choice of the atlas $x : X \to \mathfrak{S}$ is not important, we will often omit the subscript x in $B_x\mathfrak{S}$.) The étale topology (the smooth topology) of such a simplicial object $X_\bullet$ may be defined in the usual manner. (See [De].) Recall that an object in this topology will be an object $u : U \to X_n$ in $(X_n)_{et}$ ($(X_n)_{smt}$, respectively) and a map between two such objects will be a map lying over some structure map of the simplicial space $X_\bullet$. The étale (smooth) topology of $B_x\mathfrak{S}_\bullet$ will be denoted $Et(B_x\mathfrak{S}_\bullet)$ ($Smt(B_x\mathfrak{S}_\bullet)$, respectively.) In fact given any topology defined for all algebraic spaces, one may define a corresponding topology on the simplicial algebraic space $B_x\mathfrak{S}$ in a similar manner.

(2.4.*) Let $\bar{x}. : B_x\mathfrak{S}_\bullet \longrightarrow \mathfrak{S}$ denote the map given in degree n as $\bar{x}_n = x \circ d_0 \circ ... \circ d_0 : B_x\mathfrak{S}_n \longrightarrow \mathfrak{S}$.

(2.4.1). Observe that an algebraic space Y may be regarded as an algebraic stack in the obvious manner; any representable étale cover $\tilde{Y} \to Y$ (with $\tilde{Y}$ a scheme) now provides an atlas for the associated stack.

(2.4.2) If $f : \mathfrak{S}' \to \mathfrak{S}$ is a representable map of algebraic stacks, $x : X \to \mathfrak{S}$ is an atlas for $\mathfrak{S}$ and $x' : X \underset{\mathfrak{S}}{\times} \mathfrak{S}' \to \mathfrak{S}'$ is the induced atlas, one obtains an induced map $Bf. : B_{x'}\mathfrak{S}' \to B_x\mathfrak{S}$ of simplicial algebraic spaces.

(2.5.1)**Definition**. *Geometric points.* To avoid set-theoretic problems, for each characteristic $p \geq 0$, we will fix an algebraically closed field Ω that is large enough to contain the residue fields of characteristic p of all the algebraic spaces we consider. (i) A geometric point of an algebraic space X will mean a map $Spec \quad \Omega \to X$. (ii) A geometric point of an algebraic stack $\mathfrak{S}$ will also be defined as a map $Spec \quad \Omega \to \mathfrak{S}$. (iii) A geometric point of a simplicial algebraic space $X_\bullet$ will be simply defined as a geometric point of X_n for some n.

(2.5.2) **Definition**. *Neighborhoods of a geometric point.* Let $\mathfrak{S}$ denote an algebraic stack and let $\mathcal{C}$ denote any one of the topologies on $\mathfrak{S}$ defined in (2.1). If $\bar{x}$ is any fixed geometric point of $\mathfrak{S}$, a neighborhood of $\bar{x}$ in the given topology is a map $u : U \to \mathfrak{S}$ in the given topology along with a lift of $\bar{x}$ to a geometric point $\bar{u}$ of U. If $X_\bullet$ is a simplicial algebraic space and $\bar{x}_n$ is a fixed geometric point of X_n, a neighborhood of $\bar{x}_n$ in $Et(X_\bullet)$ ($Smt(X_\bullet)$) is a neighborhood of $\bar{x}_n$ in $(X_n)_{et}$ ($(X_n)_{smt}$, respectively).

(2.5.3) **Definition**. *Weakly cofinal system of coverings of algebraic spaces.* Let X denote an algebraic space of finite type over a field k. A system, $\{\mathcal{U}_\alpha|\alpha\}$, of *étale coverings* of X is weakly cofinal in the system of all étale coverings of X, if each étale covering has a refinement in the given system.

(2.5.4) **Definition**. If $\mathcal{C}$ is any site and R is any commutative ring, the category of abelian presheaves on $\mathcal{C}$ with values in the category of R-modules will be denoted $Presh(\mathcal{C}, Mod(R))$. The corresponding category of sheaves will be denoted $Sh(\mathcal{C}, Mod(R))$. In case the site $\mathcal{C}$ is the site $Et(X_\bullet)$ associated to a simplicial algebraic space, observe that a presheaf F on $Et(X_\bullet)$ corresponds to a collection $\{F_n \in Presh(X_{n_{et}})|n\}$ so that for each structure map $\alpha : X_n \to X_m$ of the simplicial space $X_\bullet$, there is given a map $\Phi_\alpha : \alpha^*(F_m) \to F_n$ satisfying certain obvious compatibility conditions. Presheaves on $Smt(X_\bullet)$ are defined similarly.

(2.5.5)**Lemma**. Let $\mathfrak{S}$ denote an algebraic stack. Then the sites $\mathfrak{S}_{smt}$, $\mathfrak{S}_{\underline{smt}}$, $Smt(B\mathfrak{S})$ and $Et(B\mathfrak{S})$ all have enough geometric points. The same holds for $\mathfrak{S}_{et}$ and $\mathfrak{S}_{\underline{et}}$ if $\mathfrak{S}$ is a Deligne-Mumford stack.

Proof. Since the smooth and étale sites of algebraic spaces clearly have enough points, it suffices to consider $\mathfrak{S}_{smt}$, $\mathfrak{S}_{\underline{smt}}$, $\mathfrak{S}_{et}$ and $\mathfrak{S}_{\underline{et}}$. For this it suffices to observe the following. Let $x : X \to \mathfrak{S}$ denote an atlas. Then the obvious map induced by x on the geometric points is *surjective.* If $\bar{p} : Spec \quad \Omega \to \mathfrak{S}$ is a geometric point and $\bar{q} : Spec \quad \Omega \to X$ is a lift of $\bar{p}$, one may observe the isomorphism $\bar{p}^*(F) \cong \bar{q}^*(x^*(F))$ for any sheaf F on $\mathfrak{S}$. □

3. Hypercohomology of algebraic stacks: cohomological descent

As one may see from the next section, we define the higher Chow groups of a stack in terms of the hypercohomology with respect to a certain complex of sheaves. In preparation for this, we define and establish the key properties of the hypercohomology of a stack in this section. For Riemann-Roch theorems on stacks, one needs to consider hypercohomology with respect to presheaves of K-theory spectra on the above site (and also on a related site). Therefore we also need to consider the hypercohomology with respect to presheaves of spectra. Furthermore, since the complexes we consider may in fact be unbounded complexes (see section 4), the total complex construction is not well-behaved. A technique that provides a solution to all these problems simultaneously is to consider presheaves with values in a complete pointed simplicial category and use homotopy inverse limits as a substitute for the total complex construction. (See section 6 for a detailed self-contained discussion of these.)

(3.0) Recall that if $\mathfrak{S}$ is an algebraic stack with $x : X \to \mathfrak{S}$ an atlas, we have the following sites associated to $\mathfrak{S}$: $\mathfrak{S}_{smt}$, $\mathfrak{S}_{\underline{smt}}$, $\mathfrak{S}_{et}$, $\mathfrak{S}_{\underline{et}}$ (for Deligne-Mumford stacks), $\mathfrak{S}_{res.smt}$, $\mathfrak{S}_{res.\underline{smt}}$, $Et(B_x\mathfrak{S})$ and $Smt(B_x\mathfrak{S})$. If $\mathcal{C}$ denotes any of the above sites and R denotes a commutative ring with 1, we let $Presh(\mathcal{C}, Mod(R))$ ($Sh(\mathcal{C}, Mod(R))$) denote the category of presheaves (sheaves, respectively) of R-modules on $\mathcal{C}$. Since we need to work with unbounded complexes it is preferable to use the canonical resolutions of Godement throughout. In fact we will let $\mathcal{C}$ denote, *throughout this paper*, any site with enough points and $Presh(\mathcal{C}, \mathbf{S})$ will denote the category of presheaves on $\mathcal{C}$ with values in a complete pointed simplicial category as in section 6. We will further assume such complete pointed simplicial categories will be *either one of the two categories: the category of fibrant spectra or the category $C(Mod(R))$ of complexes of modules over a fixed commutative Noetherian ring R.* Then a map of presheaves $f : P \to P'$ in $Presh(\mathcal{C}, \mathbf{S})$ will be called a *quasi-isomorphism* if it is one in the sense of section 6. (See (6.2.3) - (6.2.4). Recall this means the map f *induces a quasi-isomorphism at each stalk.* (*Throughout we will denote quasi-isomorphisms by $\simeq$ while isomorphisms will be denoted by $\cong$.*) If $\mathbf{S}$ denotes the category of fibrant spectra and $P \in Presh(\mathcal{C}, \mathbf{S})$, $\pi_n(P)\tilde{}$ will denote the sheaf associated to the abelian presheaf on $\mathcal{C} : U \to \pi_n(\Gamma(U, P))$ where π_n is the n-th (stable) homotopy group. If $\mathbf{S} = C(Mod(R))$ and $P \in Presh(\mathcal{C}, \mathbf{S})$, $\pi_n(P)$ will denote the sheaf associated to the presheaf $U \to \mathcal{H}^{-n}(P)$. (We adopt this notation so that many of the statements below can be stated in a uniform manner.)

(3.1) In either case $\tau_{\leq n}P$ will denote an object in $Presh(\mathcal{C}, \mathbf{S})$ defined by $\pi_i(\Gamma(U, \tau_{\leq n}P)) \cong \pi_i(\Gamma(U, P))$ if $i \leq n$ and $\cong 0$ otherwise for any U in the site $\mathcal{C}$. (In the case of fibrant spectra, the above truncation functors are defined by the canonical Postnikov truncation functors (see [T] Lemma (5.51) for example), while in the case of $C(Mod(R))$, they are the usual cohomology truncation functors that kill off the cohomology in degrees less than $-n$.) One may observe that $\{\Gamma(U, \tau_{\leq n}P)|n\}$ is an inverse system of fibrations for each U in the former case. In the latter case, one may define $\tau_{\leq n}$ in such a manner so that the obvious maps $\Gamma(U, \tau_{\leq n}P) \to \Gamma(U, \tau_{\leq n-1}P)$ are surjections for each U. It follows that the inverse limit is actually a homotopy inverse limit in the former case and may be identified with the derived functor of the inverse limit in the latter case. Moreover the natural map $P \to \lim_{\infty \leftarrow n} \tau_{\leq n}P$ is an *isomorphism* of presheaves in both of the above cases.

(3.2.1) In both of these cases there exists a bi-functor:

$\otimes$: (pointed simplicial sets) $\times Presh(\mathcal{C}, \mathbf{S}) \to Presh(\mathcal{C}, \mathbf{S})$

(The functor $\otimes$ is defined in section 6 as a colimit and therefore commutes with colimits in either argument.) Let $L(Const) : Presh(\mathcal{C}, \mathbf{S}) \to Presh(\mathcal{C}, \mathbf{S})^{\Delta}$ denote the functor sending an object $M \in \mathbf{S}$ to the cosimplicial object $n \mapsto \Delta[n]_+ \otimes M$.

The above functor has a right adjoint which is called the homotopy inverse limit along Δ and denoted $\underset{\Delta}{\text{holim}}$. This will be defined as an *end* and therefore will commute with inverse limits. (See section 6 for details on the homotopy inverse limit.)

(3.2.2) In the above situation, a map $f : X^\bullet \to Y^\bullet$ between two cosimplicial objects in $Presh(\mathcal{C}, \mathbf{S})$ will be called a quasi-isomorphism (weak-equivalence) if for each n, the map $f^n : X^n \to Y^n$ is a quasi-isomorphism (weak-equivalence). In both of the above situations, the functor $\underset{\Delta}{\text{holim}}$ preserves quasi-isomorphisms (and therefore defines a functor at the level of the associated derived categories). (See (6.3.4) for a discussion of these.)

The canonical resolutions of Godement. Let $\mathcal{C}$ denote a site and let $\bar{\mathcal{C}}$ denote a set, let $(sets)$ denote the category of all small sets and let $(sets)^{\bar{\mathcal{C}}}$ denote the product of the category (sets) indexed by $\bar{\mathcal{C}}$. Assume that we are given a conservative family of points of $\mathcal{C}$ indexed by $\bar{\mathcal{C}}$: recall this means we are given a morphism $\bar{\pi} : (sets)^{\bar{\mathcal{C}}} \to \mathcal{C}$ of sites so that a sequence of sheaves $F' \to F \to F''$ (with values in any abelian category) is short-exact if and only if $0 \to \bar{\pi}^*(F') \to \bar{\pi}^*(F) \to \bar{\pi}^*(F'') \to 0$ is exact. In case $\mathcal{C}$ is one of the sites considered in (3.0), observe that one may use the geometric points in (2.5.1) as the points. This follows from the fact that $Spec \quad \Omega$ is acyclic with respect to any abelian sheaf on the smooth or étale sites as well as (2.5.5) which shows there are enough such geometric points. (For each point p of $\bar{\mathcal{C}}$ is associated a point of the site $\mathcal{C}$ indexed by p itself.) Let a denote the functor sending a presheaf on $(sets)^{\bar{\mathcal{C}}}$ to its associated sheaf and let U denote the forgetful functor sending a sheaf on the site $\mathcal{C}$ to its underlying presheaf. Now the functors $U \circ \bar{\pi}_*$ and $a \circ \bar{\pi}^*$ define a triple; let $G = U \circ \bar{\pi}_* \circ a \circ \bar{\pi}^* = \bar{\pi}_* \circ U \circ a \circ \bar{\pi}^*$. Observe that $G = \prod_{p \in \bar{\mathcal{C}}} p_* \circ U \circ a \circ p^*$ where, for each point p of $\bar{\mathcal{C}}$ is the associated map of sites $p : (sets) \to \mathcal{C}$. Let $P \in Presh(\mathcal{C}; \mathbf{S})$.

(3.3.1) The above triple defines an augmented cosimplicial object

$$G^\bullet P : P \xrightarrow{d^{-1}} GP ... G^{n+1}P$$

in $Presh(\mathcal{C};\mathbf{S})$. We define $\mathcal{G}P = \underset{\Delta}{\text{holim}}\{G^nP|n\}$ i.e. $\Gamma(U,\mathcal{G}P) = \underset{\Delta}{\text{holim}}\{\Gamma(U,G^nP)|n\}$ for any U in the site $\mathcal{C}$.

Let $\mathcal{C}$, $\mathcal{C}'$ denote two sites and let $\phi_* : Presh(\mathcal{C}';\mathbf{S}) \to Presh(\mathcal{C};\mathbf{S})$ denote a *left-exact functor*. Now we define *the right-derived functor* $R\phi_* : Presh(\mathcal{C}';\mathbf{S}) \to Presh(\mathcal{C};\mathbf{S})$ by

(3.3.2) $R\phi_*(P) = \underset{\Delta}{\text{holim}}\{\phi(G^nP)|n\}$

This is the presheaf defined by $U \to \Gamma(U, R\phi_*(P)) = \underset{\Delta}{\text{holim}}\{\Gamma(U,\phi_*(G^nP))|n\}$.

The spectral sequence of (6.3.6) provides a spectral sequence

(3.3.3) $E_2^{s,t} = R^s\phi_*(\pi_t(P)) \Rightarrow R^{s-t}\phi_*(P)$.

(3.4.1) **Proposition**. Assume in addition to the above situation that there exists a functor ϕ^* left adjoint to ϕ_*. Then the obvious map $R\phi_*(P) \to \underset{\infty\leftarrow n}{\lim} R\phi_*(\tau_{\le n}P)$ is a quasi-isomorphism for any $P \in Pres(\mathcal{C}',\mathbf{S})$.

Proof. Recall $R\phi_*(K) = \underset{\Delta}{\text{holim}}\{\phi_*(G^mK)|m\}$ for any presheaf K as above. Since homotopy inverse limits commute with limits we obtain:

$$\underset{\infty\leftarrow n}{\lim} R\phi_*(\tau_{\le n}P) = \underset{\infty\leftarrow n}{\lim}\,\underset{\Delta}{\text{holim}}\{\phi_*(G^m\tau_{\le n}P)|m,n\} = \underset{\Delta}{\text{holim}}\,\underset{\infty\leftarrow n}{\lim}\{\phi_*(G^m\tau_{\le n}P)|n,m\}$$

Since ϕ_* has a left adjoint, it commutes with inverse limits. Therefore we may identify the last term with $\underset{\Delta}{\text{holim}}\{\phi_* \underset{\infty\leftarrow n}{\lim}(G^m\tau_{\le n}P)|n,m\}$. Now it suffices to show that the natural map $\underset{\infty\leftarrow n}{\lim} G^m(\tau_{\le n}P) \to G^m(\underset{\infty\leftarrow n}{\lim}\tau_{\le n}P)$ is an isomorphism for each fixed m. This is clear since $G = \prod_{p\,\epsilon\,\mathcal{C}} p_* \circ U \circ a \circ p^*$ and since the natural map $P \to \underset{\infty\leftarrow n}{\lim}\tau_{\le n}P$ is assumed to be an isomorphism (see (3.1)). □

(3.4.2) **Corollary**. Assume that both the sites in (3.4.1) are closed under finite inverse limits.

(i) Next assume the following in addition to the hypothesis of (3.4.1). Let $\mathcal{C}$ be a full sub-category of $\mathcal{C}'$, let $\phi : \mathcal{C}' \to \mathcal{C}$ be the map of sites associated to a fully-faithful functor $\bar{\phi} : \mathcal{C} \to \mathcal{C}'$ and let ϕ_* be the direct image functor of presheaves associated to ϕ. Assume that every $\mathcal{C}$-covering of any object U in $\mathcal{C}$ is a $\mathcal{C}'$-covering and that every $\mathcal{C}'$-covering of such an object is dominated by a $\mathcal{C}$-covering. If $P \in Presh(\mathcal{C}',\mathbf{S})$, the natural map $\phi_*(P) \to R\phi_*(P)$ is a quasi-isomorphism.

(ii) Assume the following in addition to the hypotheses of (3.4.1). There exists a map of sites $\phi : \mathcal{C}' \to \mathcal{C}$ so that ϕ^* is the inverse image functor of presheaves associated to ϕ. If $P \in Presh(\mathcal{C},\mathbf{S})$, the obvious map $P \to R\phi_*\phi^*(P)$ is a quasi-isomorphism if the corresponding map $F \to R\phi_*\phi^*(F)$ is a quasi-isomorphism for any abelian sheaf F on the site $\mathcal{C}$.

Proof. We consider (i) first. The hypotheses readily imply that the functor ϕ_* on abelian sheaves is exact. (See [Mi] p. 111.) It follows also that the spectral sequence in (3.3.3) degenerates identifying $\pi_k(R\phi_*(P))\tilde{}$ with $\phi_*(\pi_k(P))\tilde{}$. Since the sites are all closed under finite inverse limits, the direct limits involved in the definition of the stalks are all filtered direct limits and commute with taking π_k. Now the hypotheses imply that the stalks of $\pi_k(\phi_*(P))$ and $\phi_*\pi_k(P)$ are both isomorphic to the stalks of the presheaf $\pi_*(P)$. It follows that the natural map $\pi_k(\phi_*(P))\tilde{} \to \phi_*(\pi_k(P))\tilde{}$ is an isomorphism. This proves (i).

First we show that (ii) holds when P is replaced by $\tau_{\le n}P$ for any fixed integer n. Recall ϕ^* is exact in the sense it commutes with finite direct and inverse limits. (This follows from the hypothesis that the sites are closed under finite inverse limits.) It follows that the spectral sequence in (6.3.6) for $R\phi_* \circ \phi^*(P)$ now reduces to the spectral sequence in (3.3.3) for $R\phi_*$ applied to $\phi^*(P)$. The hypothesis on P ensures that this spectral sequence converges strongly. Therefore we reduce to showing that the map $\pi_t(P)\tilde{} \to R\phi_*\phi^*(\pi_t(P)\tilde{})$ is a quasi-isomorphism for all t. This proves (ii) holds when P is replaced by any $\tau_{\le n}P$.

Then $P \cong \underset{\infty\leftarrow n}{\lim}\tau_{\le n}P$. Applying (3.4.1) to P replaced by $\phi^*(P)$, it suffices to show that $\phi^*(\tau_{\le n}P) \simeq \tau_{\le n}(\phi^*(P))$ as presheaves. Since the functor $\tau_{\le n}$ is characterized by $\pi_k(\tau_{\le n}P) \cong \pi_k(P)$ if $k \le n$ and $\cong 0$ otherwise, it suffices to show $\pi_k(\phi^*(P))\tilde{} \cong \phi^*(\pi_k(P))\tilde{}$ as abelian presheaves. Since ϕ^* is assumed to be the inverse image functor associated to a map of sites, it is defined by a filtered direct limit which commutes with taking π_k. □

(3.5.1) **Definition**. (i) If $\mathfrak{S}$ is any algebraic stack, the map of sites $\mathfrak{S}_{smt} \to \mathfrak{S}_{\underline{smt}}$ in (2.1) will be denoted α. (In case $\mathfrak{S}$ is a Deligne-Mumford stack, the corresponding map $\mathfrak{S}_{et} \to \mathfrak{S}_{\underline{et}}$ will also be denoted by α. (ii) If $\mathfrak{S}$ is any algebraic stack and $B_x\mathfrak{S}$ is the classifying simplicial space associated to an atlas $x : X \to \mathfrak{S}$, the induced map $B_x\mathfrak{S} \to \mathfrak{S}$ of simplicial stacks will be denoted $\bar{x}$. $(\bar{x})_n : (B_x\mathfrak{S})_n \to \mathfrak{S}$ will denote the corresponding map in degree n. If $F = \{F_n|n\}$ is a presheaf on $Smt(B\mathfrak{S})$, we define ${}_s\bar{x}_*(F) =$ the inverse limit of the cosimplicial presheaf on $\mathfrak{S}_{smt}$ defined by $\{x_{n*}(F_n)|n\}$. The functor ${}_s\bar{x}_*$ has a left adjoint which is denoted ${}_s\bar{x}^*$ and is defined by $F =$ a sheaf on $\mathfrak{S}_{smt} \mapsto \{x_n^*(F)|n\}$. (iii) If $\mathfrak{S}$ is a Deligne-Mumford stack, the obvious map of sites of $\mathfrak{S}_{\underline{smt}} \to \mathfrak{S}_{\underline{et}}$ will be denoted β. If $X_\bullet$ is a simplicial algebraic space, β will also denote the corresponding map of sites $Smt(X_\bullet) \to Et(X_\bullet)$. (iv) If $F = \{F_n|n\}$ is a presheaf on $Et(B\mathfrak{S})$, we let ${}_{et}\bar{x}_*(F) =$ the inverse limit of the cosimplicial presheaf on $\mathfrak{S}_{et}$ defined by $\{x_{n*}(F_n)|n\}$. In this case the functor ${}_{et}\bar{x}_*$ has a left adjoint which is denoted ${}_{et}\bar{x}^*$ and is defined by $F =$ a sheaf on $\mathfrak{S}_{et} \mapsto \{x_n^*(F)|n\}$.

(v) We also define the global section functor for presheaves. For this purpose let pt denote the site with one object, pt, and one morphism which is the identity map of pt. (This category is made into a site in the obvious trivial manner.) Now one may identify presheaves on pt with values in a category $\mathbf{S}$ with the category $\mathbf{S}$ itself. If $\mathcal{C}$ is a site with a terminal object X, we define a map of sites $\pi : \mathcal{C} \to pt$ by sending pt to X. Now we let $\Gamma(\mathcal{C}, P) = \Gamma(X, P) = \pi_*(P)$ for any $P \in Presh(\mathcal{C}, \mathbf{S})$. This defines the global section functor on the sites $\mathfrak{S}_{smt}$ and $\mathfrak{S}_{et}$ for any algebraic stack $\mathfrak{S}$. If $P \in Presh(\mathfrak{S}_{\underline{smt}}, \mathbf{S})$, we let $\Gamma(\mathfrak{S}, P) = \Gamma(\mathfrak{S}, \alpha^*(P))$. If $P = \{P_n|n\} \in Presh(Et(B\mathfrak{S}))$, we let $\Gamma(B\mathfrak{S}, P) = \underset{\Delta}{\text{holim}}\{\Gamma((B\mathfrak{S})_n, P_n)|n\}$. A similar definition applies to presheaves on $Smt(B\mathfrak{S})$.

(vi) If $\mathcal{C}$ is any of the above sites and $P \in Presh(\mathcal{C}, \mathbf{S})$, we let $\mathbb{H}(\mathcal{C}, P) = R\Gamma(\mathcal{C}, P)$. If $U \in \mathcal{C}$, we let $\mathbb{H}_{\mathcal{C}}(U, P) = \mathbb{H}(\mathcal{C}/U, P)$ where $\mathcal{C}/U$ is the full sub-category of $\mathcal{C}$ consisting objects above U. $\mathbb{H}^n(\mathcal{C}, P)$ will denote $\pi_{-n}(\mathbb{H}(\mathcal{C}, P))$.

(3.5.2) **Proposition**. (i) Let $\phi : \mathcal{C}' \to \mathcal{C}$ denote any one of the following maps of sites: the map of sites $\mathfrak{S}_{smt} \overset{\alpha}{\to} \mathfrak{S}_{\underline{smt}}$ (or in the case of a Deligne-Mumford stack $\mathfrak{S}$, $\mathfrak{S}_{et} \overset{\alpha}{\to} \mathfrak{S}_{\underline{et}}$), the map $Smt(B\mathfrak{S}) \overset{\beta}{\to} Et(B\mathfrak{S})$ if $\mathfrak{S}$ is any algebraic stack and the map $\mathfrak{S}_{smt} \overset{\beta}{\to} \mathfrak{S}_{et}$ in case $\mathfrak{S}$ is a Deligne-Mumford stack. If $P \in Presh(\mathcal{C}', \mathbf{S})$, the natural map $\phi_*(P) \to R\phi_*(P)$ is a quasi-isomorphism.

(ii) Let $\mathfrak{S}$ denote any algebraic stack and let $P \in Presh(\mathfrak{S}_{smt}, \mathbf{S})$, the natural map $P \to R_s\bar{x}_*({}_s\bar{x}^*(P))$ is a quasi-isomorphism. The corresponding assertion also holds with $R_s\bar{x}_*$ $({}_s\bar{x}^*)$ replaced by $R_{et}\bar{x}_*$ $({}_{et}\bar{x}^*)$ if $\mathfrak{S}$ is any Deligne-Mumford stack and $P \in Presh(\mathfrak{S}_{et}, \mathbf{S})$.

Proof. (3.4.2)(i) shows the first assertion is true for the functor $\mathfrak{S}_{smt} \to \mathfrak{S}_{\underline{smt}}$ by observing that any covering of an object $u : U \to \mathfrak{S}$ in $\mathfrak{S}_{smt}$ is dominated by a covering in $\mathfrak{S}_{\underline{smt}}$. An entirely similar argument applies to the remaining functors in (i).

Now we consider (ii). By (3.4.2)(ii) it suffices to consider the case when P is replaced by an abelian sheaf F. The proof for the first functor in (ii) follows readily as follows. Let $U \in \mathfrak{S}_{smt}$. Then $\Gamma(U,\, F) \cong \ker(\delta^0 - \delta^1 : \Gamma(U \underset{\mathfrak{S}}{\times} X, F^0) \to \Gamma(U \underset{\mathfrak{S}}{\times} X \underset{\mathfrak{S}}{\times} X, F^1))$, $U \in \mathfrak{S}_{smt}$ by the sheaf-axiom for F on $\mathfrak{S}_{smt}$. It follows that $R^i{}_s\bar{x}_{*s}\bar{x}^*(F) \cong 0$ if $i > 0$ and $\cong F$ if $i = 0$. Moreover observe that the functor ${}_s\bar{x}^*$ is defined as a filtered colimit and is therefore exact. The proof for the second functor in (ii) is similar. □

(3.6.1) **Proposition** (Comparison of hypercohomology). (i) Let $\mathfrak{S}$ denote any algebraic stack. If $P \in Presh(\mathfrak{S}_{smt}, \mathbf{S})$, one obtains a quasi-isomorphism $\mathbb{H}(\mathfrak{S}_{smt}, P) \simeq \mathbb{H}(\mathfrak{S}_{\underline{smt}}, P)$. Moreover, in this case we also obtain the quasi-isomorphism $\mathbb{H}(\mathfrak{S}_{smt}, P) \simeq \mathbb{H}((B\mathfrak{S})_{smt}, {}_s\bar{x}^*(P)) \simeq \mathbb{H}((B\mathfrak{S})_{et}, {}_s\bar{x}^*(P))$.

(ii) If $\mathfrak{S}$ is a Deligne-Mumford stack and $P \in Presh(\mathfrak{S}_{smt}, \mathbf{S})$, we also obtain a quasi-isomorphism $\mathbb{H}(\mathfrak{S}_{smt}, P) \simeq \mathbb{H}(\mathfrak{S}_{et}, P)$.

(iii) Let $f : \mathfrak{S}' \to \mathfrak{S}$ denote a purely inseparable representable map of algebraic stacks. Let $x : X \to \mathfrak{S}$ denote an atlas for the stack $\mathfrak{S}$ and let $x' = x \underset{\mathfrak{S}}{\times} \mathfrak{S}'$. If $P \in Presh(B_x\mathfrak{S}_{et}, \mathbf{S})$, one obtains a natural quasi-isomorphism: $\mathbb{H}(B_{x'}\mathfrak{S}'_{et}, f^*(P)) \simeq \mathbb{H}(B_x\mathfrak{S}_{et}, P)$. In particular the conclusion above holds for the obvious map $\mathfrak{S}_{red} \to \mathfrak{S}$ where $\mathfrak{S}_{red}$ denotes the reduced stack associated to $\mathfrak{S}$.

Proof. Let $P \in Presh(\mathfrak{S}_{smt}, \mathbf{S})$. Now (3.5.2)(i) shows that the natural map $\alpha_*(P) \to R\alpha_*(P)$ is a quasi-isomorphism. Observe that α_* is merely the restriction of P to $\mathfrak{S}_{\underline{smt}}$. This proves the first assertion in (i). The second assertion in (i) follows by applying (3.5.2)(ii) to $P \to R_s\bar{x}_{*s}x^*(P)$, $P \in Presh(\mathfrak{S}_{smt}, \mathbf{S})$. The remaining statements in (i) and (ii) are proven similarly. Observe, in view of (3.4.2)(ii), that the natural map $P \to Rf_*f^*(P)$ is a quasi-isomorphism. Therefore (iii) follows readily. □

(3.6.2) Let $\mathfrak{S}$ denote an algebraic stack and $\bar{x} : B_x\mathfrak{S} \to \mathfrak{S}$ denote the map in (3.5.1) with respect to a fixed atlas. Let $P \in Presh(\mathfrak{S}_{res.\underline{smt}}, \mathbf{S})$, where $\mathbf{S}$ now denotes the category of all (unbounded) complexes of abelian groups or modules over a Noetherian ring. We assume further that P is *additive*: i.e. for any two U, $V \in \mathcal{C}$, $\Gamma(U \sqcup V, P) = \Gamma(U, P) \times \Gamma(V, P)$. We proceed to define the hypercohomology of $B_x\mathfrak{S}$ with respect to P on the étale and smooth sites of $B_x\mathfrak{S}$. Observe even-though each of the maps $\bar{x}_n : (B_x\mathfrak{S})_n \to \mathfrak{S}$ belongs to the restricted smooth site $\mathfrak{S}_{res.smt}$, since we are considering the restricted sites, *one cannot in general* conclude that $(\bar{x})^*_n(P)$ is simply the restriction of P to site $((B_x\mathfrak{S})_n)_{res.smt}$. This makes it necessary to adopt the following approach.

(3.6.3) What is crucial for us is the observation that all the face maps (but not necessarily the degeneracies) of $B_x\mathfrak{S}$ are *flat* (in fact smooth) maps. More generally let $X_\bullet$ denote a simplicial algebraic space with all the *face maps smooth* and let P denote an *additive* presheaf on $(alg.spaces/k)_{Res}$ with values in $\mathbf{S}$. Observe that smooth maps have constant relative dimension on each connected component of the domain: in particular smooth maps are flat (in the sense of (2.1)(b)) on restriction to each connected component of the domain. Therefore one observes that if $\delta = \Sigma_i(-1)^i(d^i)^*$, then $\{\mathbb{H}_{et}(X_n; P)|n\}$ forms a co-chain complex in the category $\mathbf{S}$ trivial in negative degrees. Recall the equivalence of categories between co-chain complexes trivial in negative degrees with values in an abelian category and the category of cosimplicial objects in the same abelian category. This equivalence is provided by a functor DN that sends a co-chain complex $K^\bullet$, trivial in negative degrees, to $(DN(K^\bullet))^n = \prod_s(K^m, s)$ where $s : [m] \to [n]$ in Δ is a surjective map and $0 \leq m \leq n$. Therefore $DN(\{\mathbb{H}_{et}(X_n; P)|n\})$ is a cosimplicial object in $\mathbf{S}$. We let

(3.6.4) $\mathbb{H}_{et}(X_\bullet; P) = \underset{\Delta}{\text{holim}} DN(\{\mathbb{H}_{et}(X_n; P)|n\})$

For each n, let $\beta_n : X_{n,smt} \to X_{n,et}$ denote the obvious map of sites. Now one may define the *smooth hypercohomology* of $X_\bullet$ with respect to P in a similar manner to be $\mathbb{H}_{smt}(X_\bullet; P) = \underset{\Delta}{\text{holim}} DN(\{\mathbb{H}_{smt}(X_n; \beta_n^*(P))\})$.

(3.6.5) One may now observe the following properties readily from the properties of étale hypercohomology, the definition of the functor DN in each degree as a product and the homotopy inverse limit. (The first three evidently also hold for smooth hypercohomology.)

- If $P' \to P$ is a quasi-isomorphism, then the induced map $\mathbb{H}_{et}(X_\bullet, P') \to \mathbb{H}_{et}(X_\bullet, P)$ is also a quasi-isomorphism.
- If $P' \to P \to P'' \to P'[1]$ is a distinguished triangle, so is the induced diagram $\mathbb{H}_{et}(X_\bullet; P') \to \mathbb{H}_{et}(X_\bullet; P) \to \mathbb{H}_{et}(X_\bullet; P'') \to \mathbb{H}_{et}(X_\bullet; P')[1]$
- $\mathbb{H}_{et}(X_\bullet; P) \simeq \lim_{\infty \leftarrow n} \mathbb{H}_{et}(X_\bullet; \tau_{\leq n}P)$
- There exists a spectral sequence $E_1^{s,t} = \mathbb{H}^t_{et}((X_\bullet)_s; P) \Rightarrow \mathbb{H}^{s+t}_{et}(X_\bullet; P)$.
- There exists a quasi-isomorphism $\mathbb{H}_{et}(X_\bullet; P) \simeq \mathbb{H}_{smt}(X_\bullet; \{\beta_n^*(P)|n\})$
- In case P is an additive presheaf on $(alg.spaces/k)$ (or on $Et(X_\bullet)$) with values in $\mathbf{S}$, $\mathbb{H}_{et}(X_\bullet, P)$ defined above is quasi-isomorphic to $\underset{\Delta}{\text{holim}} \Delta\{\mathbb{H}_{et}(X_n, P)|n\}$.

(3.6.6) If $\mathcal{C}$ is a general site and X is an object of $\mathcal{C}$, a hypercovering $U_\bullet \to X$ will denote a simplicial object in $\mathcal{C}$ provided with a covering map $U_0 \to X$ in $\mathcal{C}$ so that the induced map $U_n \to cosk^X_{n-1}(U_\bullet)_n$ is a covering for all $n \geq 0$. Let $X_\bullet$ denote a simplicial algebraic space. An *étale hypercovering* (or simply hypercovering) of $X_\bullet$ is a bi-simplicial algebraic space $U_{\bullet,\bullet}$ provided with an étale map $d_{-1} : U_{\bullet,\bullet} \to X_\bullet$ (of bi-simplicial algebraic spaces) so that the induced map $U_{s,t} \to cosk^{X_s}_{t-1}(U_{s,\bullet})_t$ is étale surjective for all $t \geq 0$. (Here $cosk^{X_s}_{-1}(U_{s,\bullet}) = X_s$.) A *Zariski* (*smooth*) hypercovering will be a hypercovering where the corresponding maps are assumed to be open immersions (smooth maps, respectively) instead of being étale. A *rigid* hypercovering of $X_\bullet$ will be a hypercovering $U_{\bullet,\bullet}$ satisfying the conditions in [Fr] p. 34. The category of rigid hypercoverings of $X_\bullet$ will be denoted $HRR(X_\bullet)$. This is a *left directed category* as shown in [Fr] p. 35. The homotopy category of hypercoverings of $X_\bullet$ will be denoted $\overline{HR}(X_\bullet)$: this is a left filtering category as in [Fr] p. 24. Since there is at-most one map between two Zariski coverings, the category of Zariski coverings (and hence the category of Zariski hypercoverings) of $X_\bullet$ is a left directed category.

(3.6.7) **Proposition**. Let $X_\bullet$ denote a simplicial algebraic space so that each X_n is of finite type over k. Let P denote a presheaf on $(alg.spaces/k)_{Res}$ with values in a complete pointed simplicial category $\mathbf{S}$ so that P is additive. Assume that there exists an integer N so that $\pi_i(\Gamma(U, P)) = 0$ for all $i < N$ and all algebraic spaces U of finite type over k. Assume, in addition, that for each algebraic space V of finite type over k and of dimension d_V, $H^i_{et}(V, \pi_t(\tilde{P})) \cong 0$ for all $i > d_V$ and all t. (For example, this holds if $\pi_t(P)$ are all presheaves of $\mathbb{Q}$-vector spaces.) Let $\{U^\alpha_{\bullet,\bullet}|\alpha\}$ denote $HRR(X_\bullet)$. Then the obvious maps

(3.6.7.*) $\mathbb{H}_{et}(X, P) \to \underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}\{\mathbb{H}_{et}(U^\alpha_{m,m}, P)|m\} \leftarrow \underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}DN\{\Gamma(U^\alpha_{m,m}, P)|m\}$

are quasi-isomorphisms.

Proof. Let V be an algebraic space of finite type over k and of dimension d_V. Now observe that $\pi_m(\mathbb{H}_{et}(V, \tau_{\le n}P)) = \pi_m(\mathbb{H}_{et}(V, P))$ if $n \ge m + d_V$. Therefore, the inverse limits stabilize and one obtains the quasi-isomorphism for each fixed m:

$$\lim_{\infty \leftarrow n}\{\underset{\alpha}{\text{colim}}\mathbb{H}_{et}(U^\alpha_{m,m}, \tau_{\le n}P)|m\} \simeq \underset{\alpha}{\text{colim}}\lim_{\infty \leftarrow n}\{\mathbb{H}_{et}(U^\alpha_{m,m}, \tau_{\le n}P)|m\}.$$

(Observe that each $U^\alpha_{m,m}$ being étale over X_m is of the same dimension as X_m. Observe also that the inverse limit above is a homotopy inverse limit since the corresponding lim^1 -terms are trivial; the colimit above is a homotopy colimit since it is taken over a left filtering category.) Moreover, clearly $\pi_k(\Gamma(V, \tau_{\le n}P)) = \pi_k(\tau_{\le n}\Gamma(V, P)) \cong \pi_k(\Gamma(V, P))$ if $n \ge k$ and the homotopy inverse limit, $\underset{\Delta}{\text{holim}}$, commutes with the above inverse limit over $n \to \infty$. In view of these observations, it suffices to prove the maps considered in the proposition is a quasi-isomorphism when P is replaced by $\tau_{\le n}P$ for any n. At this point one may observe that $\mathbb{H}_{et}(X, P)$, $\underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}\{\mathbb{H}_{et}(U^\alpha_{m,m}, P)|m\}$ and $\underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}DN\{\Gamma(U^\alpha_{m,m}, P)|m\}$ all preserve distinguished triangles in P; therefore one may readily reduce to the case when $\pi_i(\Gamma(U, P)) \ne 0$ in only one degree n for any algebraic space U of finite type over k. One may identify the second term in (3.6.7.*) with $\underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}DN\{\Gamma(U^\alpha_{m,m}, \mathcal{G}P)|m\}$ where $\mathcal{G}P = \underset{\Delta}{\text{holim}}\mathcal{G}^\bullet P$. Observe that the natural augmentation $P \to \mathcal{G}P$ is a quasi-isomorphism (i.e. stalk-wise) and that on taking the colimit over all rigid hypercoverings, one runs over all étale neighborhoods of each geometric point of each X_n (see [Fr] p. 12). Therefore, the last map in (3.6.7.*) is a quasi-isomorphism. The spectral sequence in (6.3.6)(iii) when applied to the last term in (3.6.7.*) degenerates so that one may in fact assume P is an additive abelian presheaf and the last term of (3.6.7.*) may be replaced by $\underset{\alpha}{\text{colim}}\{\Gamma(U^\alpha_{m,m}, P)|m\}$. At this point, one may invoke Corollary 4.6 of [Fr] which shows that we may in fact replace the category $HRR(X_\bullet)$ with $\overline{HR}(X_\bullet)$. In view of the hypothesis that each X_n is quasi-compact, it suffices to consider hypercoverings $U_{\bullet,\bullet}$ so that each $U_{s,t}$ is quasi-compact. (To see this, observe that quasi-compact objects on the étale site of each X_n are closed under finite fibered products; this follows from the hypothesis that all objects we consider are quasi-separated. Therefore, the proof in [SGA] 4, Exposé V, Théorème 7.3.2 and Proposition 3.4 of [Fr], apply verbatim to show that any (quasi-compact) étale covering of any X_n is dominated by $U_{n,n}$, where $U_{\bullet,\bullet}$ is a hypercovering of $X_\bullet$ so that each $U_{m,n}$ is quasi-compact.) Therefore, the hypothesis that P is additive together with Corollary (3.10) of [Fr] completes the proof. (Observe that all the arguments invoked from [Fr] in this proof are stated in [Fr] only for simplicial schemes; however, these are not scheme-theoretic and extend to algebraic spaces readily.) □

(3.6.7)' *Remark.* Let $X_\bullet$ denote a simplicial scheme where each X_n is of finite type over k. Now one obtains a similar isomorphism between Zariski hypercohomology with respect to an additive abelian presheaf and $\underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}\{\Gamma(U^\alpha_{m,m}, P)|m\}$ where $U^\alpha_{\bullet,\bullet}$ varies over all Zariski hypercoverings of the simplicial scheme $X_\bullet$.

The rest of this section will be devoted to the notion of *cohomological descent.*

(3.7.1) **Definition**. Let P denote an *additive* presheaf on a site $\mathcal{C}$ with values in a complete pointed simplicial category $\mathbf{S}$. We say that an additive presheaf P has *cohomological descent* if the obvious augmentation $\Gamma(U, P) \to \mathbb{H}_{\mathcal{C}}(U, P)$ is a quasi-isomorphism for all $U \in \mathcal{C}$.

(3.7.2) **Examples**. (i) The sheaf $U \to z^*(U, .)$ on the Zariski site of any quasi-projective scheme has cohomological descent. Here $z^*(U, .)$ is the higher cycle complex of Bloch - see [Bl-1]. This follows from the localization theorem established in [Bl-1] and [Bl-2] as follows. Consider the sheaf $U \to z^*(X, .)/z^*(X - U, .)$ defined on the Zariski site of a quasi-projective scheme X. This sheaf is evidently flabby; by the localization theorem, it is quasi-isomorphic to the sheaf above. Moreover they both have the same global sections.

(ii) It follows from (3.8.1) below that if P is a presheaf on the étale site of a scheme satisfying the hypotheses in (3.8.1) (in particular $\pi_i(P)$ are all assumed to be presheaves of $\mathbb{Q}$-vector spaces), then P has cohomological descent.

(3.7.3) **Proposition**. Let $\mathcal{C}$ denote any site, let $X \in \mathcal{C}$ and let P denote a presheaf on $\mathcal{C}$ as before. (i) If P has cohomological descent on the above site and $u : U_\bullet \to U$ is a hypercovering of U in the same site, the obvious map $\Gamma(U, P) \to \check{\mathbb{H}}(U_\bullet; P) = \underset{\Delta}{\text{holim}}\{\Gamma(U_n, P)|n\}$ is a quasi-isomorphism. (ii) Assume in addition that there exists a uniform finite cohomological dimension for the cohomology of objects in the site $\mathcal{C}$ with respect to all the abelian sheaves $\pi_t(P)\tilde{}$. If $\{U^\alpha_\bullet|\alpha\}$ is a filtered direct system of hypercoverings of the the given object U in the site $\mathcal{C}$, the obvious map $\Gamma(U, P) \to \underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}\Gamma(U^\alpha_\bullet, P)$ is a quasi-isomorphism.

Proof. (i) Observe that since P has cohomological descent, $\Gamma(U, P) \simeq \mathbb{H}_{\mathcal{C}}(U, P)$ for all U in the site; therefore $\check{\mathbb{H}}(U_\bullet, P) = \underset{\Delta}{\text{holim}}\{\Gamma(U_n, P)|n\} \simeq \underset{\Delta}{\text{holim}}\{\mathbb{H}_{\mathcal{C}}(U_n, P)|n\}$. (Observe from (6.3.6) that $\underset{\Delta}{\text{holim}}$ preserves quasi-isomorphisms.) By definition, the latter $= \check{\mathbb{H}}(U_\bullet, \mathbb{H}_{\mathcal{C}}(\quad, P))$. Therefore it suffices to show the natural map

(3.7.3.*) $\mathbb{H}_{\mathcal{C}}(U, P) \to \check{\mathbb{H}}(U_\bullet, \mathbb{H}_{\mathcal{C}}(\quad, P))$

is a quasi-isomorphism. Since both sides commute with inverse limits, we may therefore reduce to proving the above map is a quasi-isomorphism when P is replaced by $\tau_{\le n}P$ as in (3.4.1).

Observe also that both sides of (3.7.3.*) send distinguished triangles in P to distinguished triangles. Observe also that $\check{H}(U_\bullet; \mathbb{H}_{\mathcal{C}}(\quad, P)) = \underset{\Delta}{\text{holim}}\{\mathbb{H}_{\mathcal{C}}(U_n; P)|n\}$ which we may denote by $\mathbb{H}_{\mathcal{C}}(U_\bullet; P)$. Then the same observations as above shows we obtain a map of the spectral sequences:

$E_2^{s,t} = \mathbb{H}^s_{\mathcal{C}}(U; \pi_t(P)) \to \pi_{-s+t}(\mathbb{H}_{\mathcal{C}}(U; P))$ and

$E_2^{s,t} = \mathbb{H}^s_{\mathcal{C}}(U_\bullet; \pi_t(P)) \to \pi_{-s+t}(\mathbb{H}_{\mathcal{C}}(U_\bullet; P))$

Both these spectral sequences converge strongly when P is replaced by $\tau_{\le n}P$. Therefore, in this case, it suffices to show that one obtains an isomorphism at the E_2-terms. In this case we reduce to proving the obvious map $\mathbb{H}_{\mathcal{C}}(U, P) \to \mathbb{H}_{\mathcal{C}}(U_\bullet, P)$ is a quasi-isomorphism when P is an abelian presheaf. (The right hand side is the cohomology of the simplicial object $U_\bullet$ with respect to the restriction of P.) Clearly we may replace P by the associated abelian sheaf. Now observe that $\underline{\mathbb{Z}}_{U_\bullet} \to \underline{\mathbb{Z}}_U$ is a resolution and $\mathbb{H}_{\mathcal{C}}(U, P) = RHom(\underline{\mathbb{Z}}_U, P)$ while $\mathbb{H}_{\mathcal{C}}(U_\bullet, P) = RHom(\underline{\mathbb{Z}}_{U_\bullet}, P)$. Therefore this completes the proof of the proposition. This proves (i).

Now we consider (ii). By cohomological descent, one may identify $\Gamma(U, P)$ ($\underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}\{\Gamma(U^\alpha_m, P)|n\}$) with $\mathbb{H}_{\mathcal{C}}(U, P)$ ($\underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}\{\mathbb{H}_{\mathcal{C}}(U^\alpha_m, P)|m\}$, respectively). Therefore it suffices to show that the natural map

(3.7.3.**) $\mathbb{H}_{\mathcal{C}}(U, P) \to \underset{\Delta}{\text{holim}}\underset{\alpha}{\text{colim}}\{\mathbb{H}_{\mathcal{C}}(U^\alpha_m, P)|m\}$

is a quasi-isomorphism. Let d denote the uniform cohomological dimension assumed in (ii). Now observe that $\pi_k(\mathbb{H}_{\mathcal{C}}(V, \tau_{\le n}P)) = \pi_k(\mathbb{H}_{\mathcal{C}}(V, P))$ if $n \ge k + d$, for all V in the site $\mathcal{C}$. Therefore one obtains the quasi-isomorphism $\underset{\infty \leftarrow n}{\lim}\{\underset{\alpha}{\text{colim}}\mathbb{H}_{\mathcal{C}}(U^\alpha_m, \tau_{\le n}P)|m\} \simeq \underset{\alpha}{\text{colim}}\underset{\infty \leftarrow n}{\lim}\{\mathbb{H}_{\mathcal{C}}(U^\alpha_m, \tau_{\le n}P)|m\}$. (In fact the inverse limit above is a homotopy inverse limit.) In view of this observation, it suffices to prove the map in (3.7.3.**) is a quasi-isomorphism when P is replaced by $\tau_{\le n}P$ for any n. At this point one may apply the same arguments as in the proof of (i) to reduce to the case P is replaced by an abelian sheaf. In this case the assertion is clear. $\square$

(3.7.4) **Proposition**. Assume in addition to the hypotheses of (3.7.4) that the the site $\mathcal{C}$ is the étale site of a scheme X quasi-projective over a Noetherian ring and that the presheaf P is *additive*. Let $\check{\mathbb{H}}_{et}(X, P) = \underset{\Delta}{\text{holim}}\underset{\rightarrow}{\lim}\Gamma(cosk_0(u), P)$, where the colimit is over a weakly cofinal system of étale coverings $u : U \to X$ of X. Then there exists a quasi-isomorphism $\mathbb{H}_{et}(X, P) \simeq \check{\mathbb{H}}_{et}(X, P)$

Proof. Let $\mathbb{H}_{et}(\quad, P)$ denote the presheaf $U \to \mathbb{H}_{et}(U, P)$, U in the étale site of X. Let $\{\mathcal{U}_\alpha|\alpha\}$ denote a weakly cofinal system of étale covers of X. Then one obtains natural maps

(3.7.4.1) $\underset{\Delta}{\text{holim}}\,colim\Gamma(cosk_0(u), P) \to \underset{\Delta}{\text{holim}}\,colim\Gamma(cosk_0(u), \mathbb{H}_{et}(\quad, P)) \leftarrow \mathbb{H}_{et}(X, P)$

natural in P where the colimit is over the given system of covers. (3.4.1) shows that $\mathbb{H}_{et}(U, P) \xrightarrow{\simeq} \underset{\infty \leftarrow n}{\lim}\mathbb{H}_{et}(U, \tau_{\le n}P)$ for any U in the site $\mathcal{C}$. Next observe that $\Gamma(cosk_0(u), \tau_{\le n}P) = \tau_{\le n}\Gamma(cosk_0(u), P)$ and the inverse system

$\{\pi_k(\tau_{\leq n}\Gamma(V,P))|k\}$ stabilizes as $n \to \infty$ independent of V in the site; therefore one obtains the isomorphism $\pi_k \lim_{\infty \leftarrow n} colim\Gamma(cosk_0(u), \tau_{\leq n}P) \cong \pi_k colim \lim_{\infty \leftarrow n} \Gamma(cosk_0(u), \tau_{\leq n}P)$. Moreover since inverse limits commute with homotopy inverse limits, it follows that we may replace P by any $\tau_{\leq n}P$. See (6.1.6) that shows filtered colimits preserve distinguished triangles and quasi-isomorphisms. Therefore all the three terms in (3.7.4.1) preserve distinguished triangles and quasi-isomorphisms in P and we obtain spectral sequences

$$E_2^{s,t} = colimH^s(cosk_0(u), \pi_t(P)) \Rightarrow \pi_{-s+t}(\underset{\Delta}{\text{holim}}\underset{\rightarrow}{\lim}\Gamma(cosk_0(u), P)),$$

$$E_2^{s,t} = colimH^s(cosk_0(u), \pi_t(\mathbb{H}_{et}(\quad, P))) \Rightarrow \pi_{-s+t}(\underset{\Delta}{\text{holim}}\underset{\rightarrow}{\lim}\Gamma(cosk_0(u), \mathbb{H}_{et}(\quad, P))) \text{ and}$$

$$E_2^{s,t} = H^s_{et}(X, \pi_t(P)) \to \pi_{-s+t}(\mathbb{H}_{et}(X, P))$$

These spectral sequences all converge strongly since we have assumed P has been replaced by $\tau_{\leq n}P$. Therefore it suffices to show we obtain an isomorphism at the E_2-terms. Now consider the map $P \to \mathbb{H}_{et}(\quad, P)$ of presheaves. The spectral sequence in (3.3.3) with $\phi =$ the identity applied to P and $\mathbb{H}_{et}(\quad, P)$, shows the above map is a quasi-isomorphism. Therefore it suffices to show the E_2-terms of the first and last spectral sequences are isomorphic. i.e. One reduces to proving the proposition when P is replaced by an abelian presheaf: this is Artin's theorem - see [Ar-3]. □

Remark. For the purposes of (3.7.8) and (3.8.1) one needs to consider presheaves that are defined not only on schemes locally of finite type over the field k, but also on localizations at ideals of such schemes. Therefore we consider presheaves that are defined not only on stacks locally of finite type over k, but on all locally Noetherian stacks over k.

Definitions. (3.7.5.1) Recall that $(alg.stacks/k)_{Res}$ denotes the category of algebraic stacks locally of finite type over k and flat maps. Let $(alg.stacks/k)^{l.Noeth}$ denote the category of all *locally Noetherian* stacks over k. $(alg.stacks/k)^{l.Noeth}_{Res}$ will denote the subcategory where the morphisms are allowed to be only flat maps. Let $P : (alg.stacks/k)^{l.Noeth}_{Res} \to \mathbf{S}$ denote a contravariant functor *fixed throughout the remainder of this section* i.e. a presheaf on the category $(alg.stacks/k)^{l.Noeth}_{Res}$. We will always assume that P is *additive*. We let $\mathcal{C}$ denote a full sub-category of $(alg.stacks/k)^{l.Noeth}_{Res}$.

(3.7.5.2) We say that P is *covariant with respect to closed immersions in* $\mathcal{C}$, if for any Y in $\mathcal{C}$ and $i : Y' \to Y$ a closed immersion in $\mathcal{C}$ there exists a map $\Gamma(Y', P) \to \Gamma(Y, P)$ natural in Y.

(3.7.5.3) We say P has the *localization property on* $\mathcal{C}$, if P is covariant with respect to closed immersions and if if for any Y in $\mathcal{C}$, $i : Y' \to Y$ a closed immersion with $j : U \to Y$ its open complement, one obtains a distinguished triangle

$$\Gamma(Y', P) \to \Gamma(Y, P) \to \Gamma(U, P) \to \Gamma(Y', P)[1]$$

which is natural in i.

(3.7.5.4) Let P be as before. We say that P has the *Mayer-Vietoris property on* $\mathcal{C}$ if for any $Y =$ an object in the above site, $u : U \to Y$ and $v : V \to Y$ are two open immersions so that $Y = U \cup V$, one obtains a distinguished triangle:

$$\Gamma(Y, P) \to \Gamma(U, P) \oplus \Gamma(V, P) \to \Gamma(U \cap V, P) \to \Gamma(Y, P)[1]$$

(3.7.5.5) Let P be as before. We say that P has the *continuity property on* $\mathcal{C}$ if $\{Y_\alpha \xleftarrow{\phi_{\alpha,\beta}} Y_\beta | \alpha, \beta \in I\}$ is an inverse system in $\mathcal{C}$ with the maps $\phi_{\alpha,\beta}$ affine and flat and $Y = \lim_{i \in I} Y_\alpha \in (alg.stacks/k)^{l.Noeth}$, then there is a natural map $\underset{\underset{I}{\rightarrow}}{\lim}\Gamma(Y_\alpha, P) \to \Gamma(Y, P)$ which is a quasi-isomorphism. (For example, P has the continuity property on the Zariski (étale site) of a scheme, if the conclusion holds for all inverse systems as above in the Zariski site (étale site, respectively).

(3.7.5.6) Let P be as before. We say that P has the *weak transfer property on* $\mathcal{C}$ if for every finite étale cover $Y_2 \xrightarrow{\lambda} Y_1$ over $\mathfrak{S}$ with both Y_2 and Y_1 in $\mathcal{C}$, there is an induced map $\lambda_* : \Gamma(Y_2, P) \to \Gamma(Y_1, P)$ in addition to the map $\lambda^* : \Gamma(Y_1, P) \to \Gamma(Y_2, P)$ so that the following properties hold:

(a) If $Y_2 = Y_2' \sqcup Y_2''$ (each summand in $\mathcal{C}$) and $\lambda : Y_2 \to Y_1$ decomposes as $\lambda' \times \lambda''$, then the following diagram commutes in the derived category (see (6.2.6) for the definition of these derived categories):

$$\begin{array}{ccc}
\Gamma(Y_2, P) & \xrightarrow{\lambda_*} & \Gamma(Y_1, P) \\
\downarrow & \nearrow_{\lambda'_* \times \lambda''_*} & \\
\Gamma(Y'_2, P) \times \Gamma(Y''_2, P) & &
\end{array}$$

(b) Given $\lambda : Y_2 \to Y_1$ and $\mu : Y_3 \to Y_1$ both finite étale and in $\mathcal{C}$, the diagram

$$\begin{array}{ccc}
\Gamma(Y_2 \underset{Y_1}{\times} Y_3, P) & \xrightarrow{(\lambda \times 1)_*} & \Gamma(Y_3, P) \\
{\scriptstyle (1\times\mu)^*} \uparrow & & \uparrow {\scriptstyle \mu^*} \\
\Gamma(Y_2, P) & \xrightarrow{\lambda_*} & \Gamma(Y_1, P)
\end{array}$$

commutes in the derived category .

(c) If $\lambda : Y_2 \to Y_1$ is an isomorphism in $\mathcal{C}$, then $\lambda_* = (\lambda^{-1})^*$

(d) If $\lambda : Y_2 \to Y_1$ is étale and finite of degree n in $\mathcal{C}$, then the composition $\lambda_* \circ \lambda^* : \Gamma(Y_1, P) \to \Gamma(Y_2, P)$ is multiplication by n.

(3.7.6) **Examples**. (i) Let $z_*(\ \ ,.)$ denote the functor sending $U \to \underset{i}{\oplus} z_i(U, .)$ where $z_i(U, .)$ denotes the higher cycle complex of Bloch (see section 4 for its extension to algebraic stacks) and U belongs to the site $(algstacks/k)_{Res}$. The above functor is contravariant for flat maps and therefore defines an additive presheaf we denote by $\mathcal{Z}_*(\ \ ,.)$.

(ii) The presheaf $\mathcal{Z}_*(\ \ ,.)$ on the Zariski site of any quasi-projective scheme over k has the localization property: this is the content of Bloch's localization theorem - see [Bl-1](3.1). The presheaf of G-theory spectra (i.e. the presheaf $U \to K(Mod_{coh}(U))$ = the K-theory spectrum of the symmetric monoidal category of coherent sheaves on U) has the localization property for all schemes.

(iii) Both of the presheaves in (ii) have the continuity property at least if $\{X_\alpha \xleftarrow{\phi_{\alpha,\beta}} X_\beta | \alpha, \beta \in I\}$ is an inverse system of of separated schemes that are locally of finite type over k with $\phi_{\alpha,\beta}$ affine and flat. For the first (second) this follows from [EGA] IV, sections 8 and 9 ([Qu] section 7, (2.2), respectively).

(iv)Let $\mathbf{G}$ denote the presheaf $U \to G(U) = K(Mod_{coh}(U))$ where $K(Mod_{coh}(U))$ is the spectrum of algebraic K-theory of coherent sheaves on U. Let $\mathbf{G}_{\mathbb{Q}}$ denotes its localization at $\mathbb{Q}$. It is shown (in [T], Remarks following Definition (2.12)) that this presheaf of spectra has the weak-transfer property on the étale site of any scheme X. Let $\mathcal{Z}_*(\ \ ,.)$ denote the presheaf considered in (i). If the stack $\mathfrak{S}$ is in fact a scheme, all the properties in (3.7.5.6) are well-known for this presheaf. If $\mathfrak{S}$ is a Deligne-Mumford stack it is shown in (4.3.2) (see [Vis] (1.16)) that the functor $\mathcal{Z}_*(\ \ ,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is covariant for representable proper maps. Since this is already contravariant for flat maps, we obtain the maps λ^* and λ_* as in (3.7.5.6). All but the condition in (3.7.5.6)(d) may be readily verified. One may verify the latter using the observation that $\mathcal{Z}_*(\ \ ,.)$ is in fact a sheaf on the étale site.

(3.7.7) **Proposition**. (i) The localization property implies the Mayer-Vietoris property. (ii) Suppose X is a scheme of finite type over k and the site $\mathcal{C}$ contains all $U \to X$ which are open immersions of Zariski open subschemes. Suppose P is a presheaf as above, having the Mayer-Vietoris property on the Zariski site of X. Then P has cohomological descent on the Zariski site of X and there exists a Brown-Gersten type spectral sequence:

$$E_2^{s,t} = H^s_{Zar}(Z, \pi_t(P)) \Rightarrow \mathbb{H}^{s-t}_{Zar}(Z, P)$$

where the right-hand-side is the hyper-cohomology of Z with respect to P computed on the Zariski site of Z.

Proof. The first statement follows by considering the commutative diagram of distinguished triangles

$$\begin{array}{ccccc}
\Gamma(Z, P) & \longrightarrow & \Gamma(Y, P) & \longrightarrow & \Gamma(Y - Z, P) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(Z', P) & \longrightarrow & \Gamma(Y', P) & \longrightarrow & \Gamma(Y' - Z', P)
\end{array}$$

with $Y = U \cup V$, $Z = Z' = U \cup V - U$ and $Y' = V$. The second statement is discussed in detail in [BG] section 2, Theorem 4 in the context of presheaves of spectra. The same proof carries over to our setting. □

For our purposes (see the proof of (3.8.1)) it is necessary to extend the localization property and cohomological descent to the Zariski and étale sites of schemes that are localizations of quasi-projective schemes over k. This is discussed in the following theorem.

(3.7.8) **Proposition**. Let $\{X_\alpha \xleftarrow{\phi_{\alpha,\beta}} X_\beta | \alpha, \beta \in I\}$ denote an inverse system of schemes of finite type over k with the structure maps $\phi_{\alpha,\beta}$ all *affine and flat*. Let $X = \lim_{i \in I} X_\alpha$ and assume this is Noetherian and of finite dimension over k. Assume that for any such inverse system the obvious map $\Gamma(X, P) \to \lim_{\vec{I}} \{\Gamma(X_\alpha, P)|\alpha\}$ is a quasi-isomorphism, i.e. P has the continuity property. If P has the localization property (cohomological descent) on the étale site of each X_α, then P has the localization property (cohomological descent, respectively) on the étale site of X. The corresponding statement also holds for the étale sites everywhere replaced by the Zariski sites.

Proof. Since the proofs of the corresponding statements for the Zariski site are similar to those of the étale site, we will only consider the latter. Observe first that each X_α is quasi-separated and quasi-compact (and hence coherent in the sense of [SGA] 4, VI). Let $X' \to X$ belong to the étale site of X. Let $i' : Z' \to X'$ denote a closed immersion with $j' : X' - Z' = U' \to X'$ the open immersion of its complement. Now Z' (U') corresponds to a compatible collection of closed (open) subschemes $\{Z'_\alpha \subseteq X'_\alpha|\alpha\}$ ($\{U'_\alpha \subseteq X'_\alpha|\alpha\}$, respectively) so that $Z' = \lim_{i \in I} Z'_\alpha$ and $U' = \lim_{i \in I} U'_\alpha$. Moreover $Z'_\alpha = Z' \underset{X'}{\times} X'_\alpha$ and $U'_\alpha = U' \underset{X'}{\times} X'_\alpha$. The inverse systems $\{X'_\alpha|\alpha\}$, $\{Z'_\alpha|\alpha\}$ and $\{U'_\alpha|\alpha\}$ all satisfy the same hypotheses as the original inverse system $\{X_\alpha|\alpha\}$. Now

$$\{\Gamma(Z'_\alpha, P) \to \Gamma(X'_\alpha, P) \to \Gamma(U'_\alpha, P)|\alpha\}$$

is a compatible filtered direct system of distinguished triangles. Since filtered direct limits preserve distinguished triangles, we obtain a distinguished triangle on taking the direct limits over I. The continuity property of P implies one may identify the direct limit of the first term (second term, last term) with $\Gamma(Z', P)$ ($\Gamma(X', P)$, $\Gamma(U', P)$, respectively). This proves that P has the localization property on the étale site of X (and therefore the Zariski site of X as well).

Given any U in the étale site of X, one may find a compatible system $\{U_\alpha \xrightarrow{u_\alpha} X_\alpha|\alpha\}$ so that each u_α is étale and $U = \lim_{i \in I} U_\alpha$. Now $\Gamma(U, P) = \lim_{\vec{I}} \{\Gamma(U_\alpha, P)|\alpha\}$ by the continuity property of P. In this case it is well known that $\mathbb{H}(U, P) \simeq \lim_{\vec{I}} \{\mathbb{H}(U_\alpha, P)|\alpha\}$ where $\mathbb{H}$ denotes hypercohomology computed on the Zariski or étale sites at least if P is replaced by an abelian sheaf. (See [SGA]4 VI, 8.7.4.) Now the extension to a general presheaf holds by (3.4.1) as in the proof of (3.4.2). Observe that filtered colimits preserve quasi-isomorphism to complete the proof. □

(3.7.9) **Proposition**. Let P denote a presheaf as above with $\mathcal{C}$ the étale site of a separated scheme X of finite type over k. Suppose there exists a large enough integer N so that all the objects in the site $\mathcal{C}$ have cohomological dimension $\leq N$ with respect to all the presheaves $\pi_n(P)$, $n \in \mathbb{Z}$.

(i) Then the presheaf $U \to \mathbb{H}_{et}(U, P)$ (= the hypercohomology spectrum of U computed on the site $\mathcal{C}/U$), $U \in \mathcal{C}$ has the Mayer-Vietoris property.

(ii) If P has the continuity property on the étale site (the Zariski site) of X, then so does the presheaf $U \to \mathbb{H}_{et}(U, P)$.

Proof. (i). The key point is the following quasi-isomorphism: if U and V are two Zariski open subschemes of X, $\mathbb{H}_{et}(U \cup V, P) \xrightarrow{\simeq} \check{\mathbb{H}}(\mathcal{U}, \mathbb{H}_{et}(\quad, P))$ where $\mathcal{U} = \{U, V\}$. As $U \cap U = U$ and $V \cap V = V$, the Čech complex $\pi_n(\check{H}(\mathcal{U}, \mathbb{H}_{et}(\quad, P)))$ is highly degenerate and the corresponding spectral sequence degenerates providing the required Mayer-Vietoris sequence. (See [T] Theorem (1.46), Corollary (1.48) and example (1.49) for more details.)

(ii) Let $\{Y_\alpha \xleftarrow{\phi_{\alpha,\beta}} Y_\beta | \alpha, \beta \in I\}$ be an inverse system in the Zariski or étale sites of X with the maps $\phi_{\alpha,\beta}$ affine. The hypotheses imply that we obtain a map of the spectral sequence in (3.3.3) (with $\phi = \Gamma$ = the global section functor)

$$\lim_{\substack{\to\\ I}} E_2^{u,v}(\alpha) = \lim_{\substack{\to\\ I}} H^u_{et}(Y_\alpha, \pi_v(P)) \to E_2^{u,v} = H^u_{et}(Y, \pi_v(P)).$$

The spectral sequence on the left converges strongly to $\lim_{\substack{\to\\ I}} \mathbb{H}^{u-v}(Y_\alpha, P)$ while the spectral sequence on the right converges strongly to $\mathbb{H}^{u-v}(Y, P)$. (The strong convergence follows by the hypotheses on the uniform cohomological dimension.) Therefore, it suffices to show we obtain an isomorphism of the above E_2-terms. This is clear by the continuity property of étale cohomology. (See [SGA] 4, VII (5.7).) □

(3.7.10) **Definition**. Let $f : \mathfrak{S}' \to \mathfrak{S}$ denote map of algebraic stacks over k. Let G denote a finite group acting on $\mathfrak{S}'$ so that the map f is G-equivariant with respect to the given action of G on $\mathfrak{S}'$ and the trivial action on $\mathfrak{S}$. We say f is *Galois* with Galois group G if f is finite étale and the morphism $\psi : G_{\mathfrak{S}'} = \underset{g\,\epsilon\, G}{\sqcup} \mathfrak{S}' \to \mathfrak{S}' \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ given by $\psi_{|\mathfrak{S}'_g} = (id, g)$ is an isomorphism. Here $\psi_{|\mathfrak{S}'_g}$ denotes the restriction of ψ to the summand indexed by g and (id, g) denotes the map $\mathfrak{S}' \to \mathfrak{S}' \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ induced by the identity onto the first factor and by $g : \mathfrak{S}' \to \mathfrak{S}'$ onto the second factor.

(3.7.11) **Proposition**. Let $\mathfrak{S}$ denote an algebraic stack locally of finite type over k and let P denote a presheaf as before on the étale site, $\mathfrak{S}_{et}$. Suppose P has the weak-transfer property. (i) If $\lambda : Y_2 \to Y_1$ is a *Galois* extension of algebraic stacks Y_i that are étale and of finite type over $\mathfrak{S}$ with Galois group $G = Gal(Y_2/Y_1)$, $\lambda^* \circ \lambda_* = \Sigma_{g\,\epsilon\, G} g^* : \Gamma(Y_2, P) \to \Gamma(Y_2, P)$.

(ii) Assume in addition, that each of the presheaves $\pi_n(P)$ is a presheaf of $\mathbb{Q}$-vector spaces. If $\lambda : Y_2 \to Y_1$ is as above, then $\lambda^* : \pi_n(\Gamma(Y_1, P)) \to \pi_n(\Gamma(Y_2, P))^G$ is an isomorphism for all n.

(iii) If each of the presheaves $\pi_n(P)$ is a presheaf of $\mathbb{Q}$-vector spaces and $\mathfrak{S}$ is the spectrum of an Artin local ring, then P has étale cohomological descent and therefore the natural map $\Gamma(Spec \quad L, P) \to \mathbb{H}_{et}((Spec \quad L), P)$ is a quasi-isomorphism.

Proof. One first recalls the isomorphism $\psi : \underset{g\,\epsilon\, G}{\sqcup} Y_2 \overset{\cong}{\to} Y_2 \underset{Y_1}{\times} Y_2$. It follows from the definition of ψ that the composition $(1 \times \lambda) \circ \psi = \bar{\Delta} =$ the map $\underset{g\,\epsilon\, G}{\sqcup} Y_2 \to Y_2$ sending each summand by the identity to Y_2. Now $\bar{\Delta}^* : \Gamma(Y_2, P) \to \underset{g\,\epsilon\, G}{\Pi} \Gamma(Y_2, P) = \Gamma(\underset{g\,\epsilon\, G}{\sqcup} Y_2, P)$ is the diagonal map. By (3.7.5.6)(a) the composition $(\lambda \times 1)_* \circ \psi_*$ breaks up as the sum $\oplus_{g\,\epsilon\, G} g_*$. By (3.7.5.6)(c) one may identify this with $\oplus_{g\,\epsilon\, G} g^*$. Finally we pre-compose this with the diagonal $\bar{\Delta}^* : \Gamma(Y_2, P) \to \underset{g\,\epsilon\, G}{\Pi} \Gamma(Y_2, P)$ to obtain $(\lambda \times 1)_* \circ \psi_* \circ \bar{\Delta}^* = \Sigma_{g\,\epsilon\, G} g^*$. Then the observation that $\bar{\Delta}^* = \psi^* \circ (1 \times \lambda)^*$ shows $(\lambda \times 1)_* \circ (1 \times \lambda)^* = (\lambda \times 1)_* \circ \psi_* \circ \psi^* \circ (1 \times \lambda)^* = (\underset{g\,\epsilon\, G}{\oplus} g_*) \circ \bar{\Delta}^* = \underset{g\,\epsilon\, G}{\Sigma} g_*$. Finally an application of (3.7.5.6)(b) with $\mu = \lambda$ shows $\lambda^* \circ \lambda_* = (\lambda \times 1)_* \circ (1 \times \lambda)^*$. This completes the proof of the first assertion.

Observe that the composition $= \lambda_* \circ \lambda^* : \pi_n(\Gamma(Y_1, P)) \to \pi_n(\Gamma(Y_1, P))$ is multiplication by the degree of λ. Therefore $\lambda^* : \pi_n(\Gamma(Y_1, P)) \to \pi_n(\Gamma(Y_2, P))$ is a split monomorphism. Using the first assertion, we see that the composition $1/|G|.\lambda^* \circ \lambda_*$ induces a projection of $\pi_n(\Gamma(Y_2, P))$ onto the summand fixed by the Galois group G. (Here $|G|$ denotes the order of the group G.) It follows that $\lambda^* : \pi_n(\Gamma(Y_1, P)) \to \pi_n(\Gamma(Y_2, P))^G$ is an isomorphism for all n with inverse induced by the restriction of $1/|G|\lambda_*$. This proves the second statement.

Let $\tilde{L}$ denote a strict Henselization of L and let $\lambda : L \to L_\alpha$ denote a subring of $\tilde{L}$ which is a finite Galois extension of L. By (ii) applied to $\mathfrak{S} = Y_1 = Spec \quad L$ and $Y_2 = Spec \quad L_\alpha$, it follows that $\lambda^* : \pi_n(\Gamma(Spec \quad L, P)) \to \pi_n(\Gamma(Spec \quad L_\alpha, P))^G$ is an isomorphism with inverse induced by the restriction of $1/|G|\lambda_*$. Next observe that the spectral sequence in (3.3.3) (for $\phi = \Gamma$), $E_2^{s,t} = H^s(BG, \pi_t(\Gamma(Spec \quad L_\alpha, P))) \Rightarrow \pi_{-s+t}(R\Gamma(Spec \quad L, P))$, degenerates providing the identification $\pi_n(\Gamma(Spec \quad L, P)) \cong \pi_n(R\Gamma(Spec \quad L, P)) \cong \pi_n(\Gamma(Spec \quad L_\alpha, P))^G \cong \pi_n(\Gamma(Spec \quad L, P))$. One may now complete the proof of (iii) by taking the direct limit over all α. □

The following result shows that under reasonable conditions, any presheaf of $\mathbb{Q}$-vector spaces has étale cohomological descent. Throughout the rest of this section P will denote a presheaf on $(alg.stacks/k)_{Res.}$ taking values in a category **S** as in (3.7.5.1).

(3.8.1) **Proposition**. Let X denote a separated scheme of finite type over k. Assume that the presheaves $U \to \pi_i(\Gamma(U, P))$ are all presheaves of $\mathbb{Q}$-vector spaces and that P has the following properties:

(i) the localization property on restriction to the étale site of X and on restriction to the étale sites of the stalks of the structure sheaf of X (in the sense of (3.7.5.3))

(ii) the continuity property on the Zariski site of X and

(iii) the weak-transfer property for the restriction of P to the étale site of any Artin local ring whose residue field is an extension of k.

Then the restriction of P to the étale site of X has cohomological descent.

Proof. This will follow from the results in (3.7.8) through (3.7.11) in the following steps.

Step 1. Under the above hypothesis, (3.7.8) shows that one has cohomological descent on the Zariski site of X. In particular one obtains a Brown-Gersten type strongly convergent spectral sequence: $E_2^{s,t} = \mathbb{H}^s_{Zar}(X, \pi_t(P)) \Rightarrow \mathbb{H}^{s-t}_{Zar}(X, P)$.

Step 2. Next one observes that all objects in the étale site of any scheme have a uniform finite cohomological dimension for any of the presheaves $\pi_t(P)$. (In fact one may take this to be the dimension of X.) By (3.7.9), it follows that the presheaf $U \to \mathbb{H}_{et}(U, P)$ has the continuity property as well as the Mayer-Vietoris property considered above. In particular, this presheaf also has cohomological descent on the Zariski site of X. Now, a comparison of the spectral sequences in Step 1 for the two presheaves $U \mapsto \Gamma(U, P)$ and $U \mapsto \mathbb{H}_{et}(U, P)$ shows that we reduce to proving cohomological descent for local rings.

Step 3. One now reduces to establishing cohomological descent for the case X is replaced by the spectrum of an Artin local ring. Assume that N is a fixed integer so that one has cohomological descent for all local rings of dimension $< N$. Let R denote a local ring of Krull dimension N which is the one of the stalks of the structure sheaf of X. We may now replace the original scheme X by $Spec \quad R$ and denote this by X itself. Let m denote the unique closed point of the ring and let $U = Spec \quad R - m = Spec \quad R - Spec \quad R/m$. Then U has Krull dimension $< N$ (since m is the unique closed point of $X = Spec \quad R$). (In case U has dimension 0, all the stalks of its structure sheaf are Artin local rings whose residue fields are extensions of k. Therefore, by steps 1 and 2, the proposition holds for for U if it holds for all Artin local rings whose residue fields are extensions of k and if U has Krull dimension 0.)

R is a Noetherian ring so that $Z = Spec \quad R/m$, U and X are all quasi-projective over a Noetherian ring. Therefore, by (3.7.5), $\check{\mathbb{H}}_{et}(X, P) \simeq \mathbb{H}_{et}(X, P)$.

Then the localization property for P provides a distinguished triangle $i_*P \to P \to j_*P$ on the Zariski site of X where $i : Z \to X$ and $j : U \to X$ are the obvious maps. Let $\mathcal{U}$ denote an étale cover of X. Then one obtains the commutative diagram:

$$\begin{array}{ccccc}
\Gamma(Z,P) & \longrightarrow & \Gamma(X,P) & \longrightarrow & \Gamma(U,P) \\
\cong\big\downarrow & & \cong\big\downarrow & & \big\downarrow\cong \\
\Gamma(X,i_*P) & \longrightarrow & \Gamma(X,P) & \longrightarrow & \Gamma(X,j_*P) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\check{\mathbb{H}}(\mathcal{U},i_*P) & \longrightarrow & \check{\mathbb{H}}(\mathcal{U},P) & \longrightarrow & \check{\mathbb{H}}(\mathcal{U},j_*P) \\
\cong\big\downarrow & & \cong\big\downarrow & & \big\downarrow\cong \\
\check{\mathbb{H}}(i^{-1}(\mathcal{U}),P) & \longrightarrow & \check{\mathbb{H}}(\mathcal{U},P) & \longrightarrow & \check{\mathbb{H}}(j^{-1}(\mathcal{U}),P)
\end{array}$$

where $i^{-1}(\mathcal{U})$ ($j^{-1}(\mathcal{U})$) is the induced étale cover of Z (U) induced from $\mathcal{U}$. The top two rows are distinguished triangles by the localization property of P on the Zariski site of X while the lower two rows are distinguished triangles by the localization property of P on the étale site of X. (Recall that if $u : U \to Z$ is an étale cover, $\check{\mathbb{H}}(U, P) = \underset{\Delta}{\mathrm{holim}}\,\Gamma(cosk_0(u), P)$.) Then (3.7.4) and ascending induction on the dimension shows that the composition of the maps at the two end-columns are quasi-isomorphisms. Therefore so is the composition of the maps in the middle column. Take the direct limit over all étale covers of X and apply (3.7.5) to conclude the augmentation $\Gamma(X, P) \to \mathbb{H}_{et}(X, P)$ is a quasi-isomorphism. This completes the proof of this step.

The final step. Here the weak transfer property and the observation that the presheaves $\pi_t(P)$ are all sheaves of $\mathbb{Q}$-vector spaces enables us to conclude the proof - see (3.7.11)(iii). $\square$

(3.8.2) **Corollary**. Suppose P is a presheaf having the following properties:

i) the presheaves $U \to \pi_i(\Gamma(U, P))$ are all presheaves of $\mathbb{Q}$-vector spaces

ii) P has the continuity property and the localization property on restriction to the étale site of any quasi-projective scheme over k

iii) P has the weak-transfer property on the étale site of any Artin local ring whose residue field is an extension of k.

Then P has cohomological descent on the étale site of any quasi-projective scheme. In particular the presheaf $\mathcal{Z}_*(\ \ ,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ has cohomological descent on the étale site of any quasi-projective scheme.

Proof. Observe from (3.7.8) that the localization property extends to the étale sites of the local rings that are stalks of the structure sheaves on quasi-projective schemes. Therefore the hypotheses of (3.8.1) are satisfied. $\square$

(3.8.3) **Theorem**. Assume that P is an additive presheaf $(alg.stacks/k)^{l.Noeth,o}_{Res} \to \mathbf{S}$ having the localization property on the Zariski site of any quasi-projective scheme over k.

(i) Assume further it is covariant with respect to closed immersions for all closed immersions of schemes (as in (3.7.5.2)) and that if $Z \to X$ is a closed immersion of schemes with U the open complement of Z, then the composition $\Gamma(Z, P) \to \Gamma(X, P) \to \Gamma(U, P)$ is strictly trivial. Then the presheaf of hypercohomology spectra $U \to \mathbb{H}_{Zar}(U, P)$ has the localization property on any scheme locally of finite type over k.

(ii) Assume that P has the continuity property on the Zariski site of any quasi-projective scheme, the weak-transfer property on restriction to the étale site of any Artin local ring whose residue field is a finite extension of k and the presheaves $U \to \pi_i(\Gamma(U, P))$, are presheaves of $\mathbb{Q}$-vector spaces. We further assume that P is covariant with respect to closed immersions of all algebraic spaces and that if $i : \mathfrak{S}' \to \mathfrak{S}$ is a closed immersion of algebraic spaces with $\mathfrak{S}'' = \mathfrak{S} - \mathfrak{S}'$, the composition $\Gamma(\mathfrak{S}', P) \to \Gamma(\mathfrak{S}, P) \to \Gamma(\mathfrak{S}'', P)$ is strictly trivial.

Let $\mathfrak{S}$ denote any algebraic stack locally of finite type over k, $i : \mathfrak{S}' \to \mathfrak{S}$ a closed immersion of algebraic stacks and $j : \mathfrak{S}'' = \mathfrak{S} - \mathfrak{S}' \to \mathfrak{S}$ the open immersion of its complement. Let $x : X \to \mathfrak{S}$ denote an atlas, $x' = x\underset{\mathfrak{S}}{\times}\mathfrak{S}'$ and $x'' = x\underset{\mathfrak{S}}{\times}\mathfrak{S}''$. Then one obtains a distinguished triangle:

$$... \to \mathbb{H}_{et}(B_{x'}\mathfrak{S}'; P) \to \mathbb{H}_{et}(B_x\mathfrak{S}, P) \to \mathbb{H}_{et}(B_{x''}\mathfrak{S}'', P) \to ...$$

(iii) More generally let $Z_\bullet$, $X_\bullet$ and $U_\bullet$ be simplicial algebraic spaces with *smooth face maps* and let $i_\bullet : Y_\bullet \to X_\bullet$ be a closed immersion in each degree with $j_\bullet : U_\bullet \to X_\bullet$ its complement. Let P denote a presheaf as in (ii) and let P_X (P_Z, P_U) denote the collection of sheaves $\{P_{|X_{n,et}}|n\}$ ($\{P_{|Z_{n,et}}|n\}$, $\{P_{U_{n,et}}|n\}$). Then one obtains a distinguished triangle:

$$... \to \mathbb{H}_{et}(Z_\bullet, P_Z) \to \mathbb{H}_{et}(X_\bullet, P_X) \to \mathbb{H}_{et}(U_\bullet, P_U) \to ...$$

Proof (i). Let X denote a scheme locally of finite type over k and let $i : Z \to X$ denote a closed immersion with $j : U = X - Z \to X$ the open immersion of its complement. Let $Rj_* : Presh(U_{Zar}, \mathbf{S}) \to Presh(X_{Zar}, \mathbf{S})$ and $Ri^! : Presh(X_{Zar}, \mathbf{S}) \to Presh(Z_{Zar}, \mathbf{S})$ denote the obvious functors. Now we obtain the distinguished triangle $i_*Ri^!P_X \to P_X \to Rj_*j^*(P_X)$ where P_X denotes the restriction of the presheaf P (from the category $(alg.stacks/k)_{Res}$) to the Zariski site of X. This provides the distinguished triangle:

$$\mathbb{H}_{Zar}(X, i_*Ri^!P_X) \to \mathbb{H}_{Zar}(X, P_X) \to \mathbb{H}_{Zar}(X, Rj_*j^*P_X) \simeq \mathbb{H}(U_{Zar}, P)$$

Therefore it suffices to show the first term is quasi-isomorphic to $\mathbb{H}_{Zar}(Z, P)$. This will follow by showing there is a quasi-isomorphism of presheaves $i_*P_Z \simeq i_*Ri^!P_X$ on X_{Zar}. (Once again P_Z is the restriction of the presheaf P from $(alg.stacks/k)_{Res}$ to the Zariski site of Z.) Let $W \to X$ be a Zariski open sub-scheme of X. Then one obtains the commutative diagram

$$\begin{array}{ccccc}
\mathbb{H}_{Zar}(W, i_*Ri^!P_X) & \longrightarrow & \mathbb{H}_{Zar}(W, P_X) & \longrightarrow & \mathbb{H}_{Zar}(W, Rj_*j^*P_X) \cong \mathbb{H}_{Zar}((W \cap U), P) \\
 & & \uparrow & & \uparrow \\
\Gamma(W \cap Z, P_Z) & \longrightarrow & \Gamma(W, P_X) & \longrightarrow & \Gamma(W \cap U, P_X)
\end{array}$$

Observe that the above diagram is compatible with restriction to smaller Zariski open subsets of W i.e. if $W' \subseteq W$ is an open immersion, the diagram for W maps into the diagram for W' providing a 3-dimensional commutative diagram whose two faces are the above diagrams for W and W'. By the hypothesis, the composition of maps in the bottom row is strictly trivial. One may also identify $\Gamma(W \cap Z, P_Z)$ with $\Gamma(W, i_*(P_Z))$: we now obtain a map $\Gamma(W, i_*(P_Z)) \to \mathbb{H}_{Zar}(W, i_*Ri^!P_X) \simeq \Gamma(W, i_*Ri^!P_X)$. Since these hold for all W in the Zariski site of X, it follows that we obtain a map of presheaves $i_*(P_Z) \to i_*Ri^!P_X$. In order to show this is a quasi-isomorphism, one may take W to be an affine open sub-scheme of X. (Since X is locally of finite type over k, each point has a Zariski open neighborhood which is an affine scheme.) Our hypotheses imply that the last two vertical maps are quasi-isomorphisms while the two rows are distinguished triangles. Therefore the natural map $\Gamma(W \cap Z, P_Z) = \Gamma(W, i_*(P_Z)) \to \mathbb{H}(W_{Zar}, i_*Ri^!P_X)$ is also a quasi-isomorphism. Since this holds for all open neighborhoods W of any point of X, it follows that one obtains a quasi-isomorphism $i_*(P_Z) \simeq i_*Ri^!P_X$ as required. This proves (i).

Since (ii) is a special case of (iii), we will only prove (iii). Let $Ri^!_\bullet = \{Ri^!_n|n\} : Presh(X_{\bullet,et}, \mathbf{S}) \to Presh(Z_{\bullet,et}, \mathbf{S})$ and $Rj_* = \{Rj_{n*}|n\} : Presh(U_{\bullet,et}, \mathbf{S}) \to Presh(X_{\bullet,et}, \mathbf{S})$ denote the obvious functors. These functors provide us with a collection $\{T_n = i_{n*}Ri^!_n(P_{X_n,et}) \to P_{X_n,et} \to Rj_{n*}P_{U_n,et}|n\}$ of distinguished triangles so that T_n is a distinguished triangle in $Presh(X_{n,et}, \mathbf{S})$ and they are compatible with the face maps as n varies. By (3.6.5), we obtain the distinguished triangle:

$$\mathbb{H}_{et}(X_n, i_{n*}Ri^!_nP_{|X_{n,et}}) \to \mathbb{H}_{et}(X_n, P_{X_{n,et}}) \to \mathbb{H}_{et}(U_n, P_{U_{n,et}}) \cong \mathbb{H}_{et}(X_n, Rj_{n*}j^*_nP_{X_{n,et}})$$

Therefore it suffices to show that the first term is quasi-isomorphic to $\mathbb{H}_{et}(Z_n, P_{Z_{n,et}})$. This will follow by showing there is a quasi-isomorphism of presheaves $i_{n*}P_{Z_{n,et}} \simeq i_{n*}Ri^!_nP_{X_{n,et}}$ on $X_{n,et}$ for all $n \geq 0$. Let $W \to X_{n,et}$ denote any object in the site $X_{n,et}$. Then one obtains the commutative diagram:

$$\begin{array}{ccccc}
\mathbb{H}_{et}(W, i_*Ri^!P_{X_{n,et}}) & \longrightarrow & \mathbb{H}_{et}(W, P_{X_{n,et}}) & \longrightarrow & \mathbb{H}_{et}(W, Rj_*j^*P_{X_{n,et}}) \cong \mathbb{H}_{et}((W \underset{X_n}{\times} U_n), P_{U_{n,et}}) \\
 & & \uparrow & & \uparrow \\
\Gamma(W \underset{X_n}{\times}, P_{Z_{n,et}}) & \longrightarrow & \Gamma(W, P_{X_{n,et}}) & \longrightarrow & \Gamma(W \underset{X_n}{\times} U_n, P_{U_{n,et}})
\end{array}$$

Observe that the above diagram is compatible with restriction to maps $W' \to W$ in the site $X_{n,et}$ i.e. if $W' \to W$ is a map in $X_{n,et}$, the diagram for W maps into the diagram for W' providing a 3-dimensional commutative diagram whose two faces are the above diagrams for W and W'. The hypothesis implies the composition of the maps in the bottom row is strictly trivial. Therefore one obtains an induced map $\Gamma(W, i_*(P_{Z_{n,et}})) = \Gamma(W \underset{X_n}{\times} Z_n, P_{Z_{n,et}}) \to \mathbb{H}_{et}(W, i_*Ri^!P_{X_{n,et}}) \simeq \Gamma(W, i_*Ri^!P_{X_{n,et}})$. Since this holds for all W, we obtain a map of presheaves $i_*(P_{Z_{n,et}}) \to i_*Ri^!P_{X_{n,et}}$. We proceed to show this is a quasi-isomorphism. Since each X_n is locally of finite type over k, each geometric point of X_n has a cofinal system of étale neighborhoods which are affine schemes. Let W denote such an affine neighborhood. (3.8.2) shows the hypotheses imply one has cohomological descent on the étale site of any quasi-projective scheme: therefore the last two vertical maps are quasi-isomorphisms. Observe also that the two rows are distinguished triangles. (The bottom row is one by the localization property of P on the quasi-projective scheme W.) Therefore the induced map $\Gamma(W, i_*(P_{Z_{n,et}})) = \Gamma(W \underset{X_n}{\times} Z_n, P_{Z_{n,et}}) \to \mathbb{H}_{et}(W, i_*Ri^!P_{X_{n,et}})$ is a quasi-isomorphism. Since this holds for a cofinal system of neighborhoods W of any geometric point of X_n, it follows that one obtains a quasi-isomorphism $i_*P_{Z_{n,et}} \simeq i_*Ri^!P_{X_{n,et}}$ as required. This proves (iii). □

(3.8.4) **Proposition**. Let $X_\bullet$ denote a simplicial algebraic space, so that each X_n is of finite type over k. Let P denote a complex of additive abelian presheaves (or equivalently a simplicial abelian presheaf) on $Et(X_\bullet)$. Assume further that there exists an integer $N >> 0$, so that $H^s_{et}(X_\bullet, \pi_t(P) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) = 0$ for all $s > N$ and all t. Then the natural map $\mathbb{H}_{et}(X_\bullet, P) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathbb{H}_{et}(X_\bullet, P \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ is a quasi-isomorphism.

Proof. Consider the spectral sequences:

$$E_2^{s,t} = \mathbb{H}^s_{et}(X_\bullet, \pi_t(P)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \Rightarrow \pi_{-s+t}(\mathbb{H}_{et}(X_\bullet, P) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \cong \pi_{-s+t}(\mathbb{H}_{et}(X_\bullet, P)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \text{ and}$$

$$E_2^{s,t} = \mathbb{H}^s_{et}(X_\bullet, \pi_t(P \underset{\mathbb{Z}}{\otimes} \mathbb{Q})) \cong \mathbb{H}^s_{et}(X_\bullet, \pi_t(P) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \Rightarrow \pi_{-s+t}(\mathbb{H}_{et}(X_\bullet, P \underset{\mathbb{Z}}{\otimes} \mathbb{Q})).$$

While the second spectral sequence is clearly strongly convergent, a priori, the first spectral sequence is only *conditionally convergent* (as in [Board] p. 67). Nevertheless, the natural map from the first to the second is an isomorphism at the E_2-level by Proposition (3.6.7). To see this, one may argue as follows: since tensoring with $\mathbb{Q}$ commutes with finite products, it follows that the complex of presheaves $P \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is also additive. Observe next that hypercohomology groups forming the E_2-terms may be computed using hypercoverings as in Proposition (3.6.7) (see also Corollary (3.10) of [Fr]); since each X_n is quasi-compact we only need to consider hypercoverings that are quasi-compact in each degree. Therefore both the $E_2^{s,t}$-terms get identified with $\underset{\alpha}{\text{colim}} H^s(\pi_t \Delta\Gamma(U^{\alpha}_{\bullet,\bullet}, P) \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ where the colimit is over the homotopy category of hypercoverings of $X_\bullet$. Therefore both spectral sequences are strongly convergent; this implies the induced map on the abutments is also an isomorphism. □

We complete this section with the following two results that extend localization sequences and cohomological descent. (The main application will be to the K-theory of coherent sheaves, both rational and mod-l^ν with the Bott element inverted.)

(3.8.5) **Proposition**. Assume the following:

(i) P is a presheaf on $(alg.stacks/k)_{Res}$ taking values in **S**.

(ii) P has the localization property on $(alg.stacks/k)$

(ii) P has cohomological descent on the étale site of any scheme of finite type over k.

Then the presheaf of hypercohomology $U \to \mathbb{H}_{smt}(U, P)$ also has the localization property on the category $(alg.stacks/k)$.

Proof. Observe first that P defines by restriction a presheaf on the smooth site of any algebraic stack $\mathfrak{S}$. We will denote this presheaf by $P_{\mathfrak{S}}$. Let $i : \mathfrak{S}' \to \mathfrak{S}$ denote the closed immersion of algebraic stacks with $j : \mathfrak{S}'' \to \mathfrak{S}$ its open complement. For each $W \to \mathfrak{S}$ in the site $\mathfrak{S}_{smt}$, one obtains a compatible system of commutative diagrams (as W varies in the site $\mathfrak{S}_{smt}$)

$$\begin{array}{ccccc}
\mathbb{H}_{smt}(W, i_*Ri^!P_{\mathfrak{S}}) & \longrightarrow & \mathbb{H}_{smt}(W, P_{\mathfrak{S}}) & \longrightarrow & \mathbb{H}_{smt}(W, Rj_*j^*P_{\mathfrak{S}}) \simeq \mathbb{H}_{smt}(\mathfrak{S}'' \underset{\mathfrak{S}}{\times} W, P_{\mathfrak{S}''}) \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma(\mathfrak{S}' \underset{\mathfrak{S}}{\times} W, P_{\mathfrak{S}'}) & \longrightarrow & \Gamma(W, P_{\mathfrak{S}}) & \longrightarrow & \Gamma(\mathfrak{S}'' \underset{\mathfrak{S}}{\times} W, P_{\mathfrak{S}''})
\end{array}$$

where each row is a distinguished triangle. It follows that we obtain a map of presheaves $i_*(P_{\mathfrak{S}'}) \to i_*Ri^!(P_{\mathfrak{S}})$. In order to show this is a quasi-isomorphism, it suffices to show that the vertical maps are all quasi-isomorphisms when W is a scheme of finite type over k. In this case, the last two vertical maps are quasi-isomorphisms and therefore so is the first completing the proof. □

(3.8.6) **Corollary**. Assume in addition to the hypotheses of (3.8.3) that P is a presheaf satisfying the hypotheses of (3.8.1) on restriction to the étale site of any separated scheme of finite type over k (instead of on all of $(alg.stacks/k)$). If X is any separated algebraic space of finite type over k, the restriction of P to the étale and smooth sites of X have cohomological descent.

Proof. Once again (3.5.2) shows that it suffices to consider the étale site. Observe from [Kn] p. 131 that there exists a dense open affine subscheme U of X. Let Y denote the complement of U in X; by Noetherian induction we may assume that the presheaf P has cohomological descent on the étale site of Y. Moreover by (3.8.1), P has cohomological descent on the étale site of U. Then we obtain the commutative diagram:

$$\begin{array}{ccccc}
\mathbb{H}_{et}(Y, P) & \longrightarrow & \mathbb{H}_{et}(X, P) & \longrightarrow & \mathbb{H}_{et}(U, P) \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma(Y, P) & \longrightarrow & \Gamma(X, P) & \longrightarrow & \Gamma(U, P)
\end{array}$$

where the top row is a distinguished triangle as shown by (3.8.3); the bottom row is also a distinguished triangle by the hypotheses. Since the first and last maps are quasi-isomorphisms, it follows that so is the middle map. □

4. The higher Chow groups of an algebraic stack

We begin by defining the *naive higher Chow groups for algebraic stacks over k*.

(4.1.1) **Definition** *The naive higher Chow groups.* Let $\mathfrak{S}$ denote an algebraic stack. For each integer $n \geq 0$, we let $\Delta_k[n]$ denote the standard n-simplex and let $\mathfrak{S} \times \Delta_k[n]$ denote the obvious algebraic stack that is the product of $\mathfrak{S}$ and $\Delta_k[n]$. For each integer d, a *dimension d* cycle on $\mathfrak{S} \times \Delta_k[n]$ is an element of the free abelian group on dimension d integral sub-stacks of $\mathfrak{S} \times \Delta_k[n]$. (See (1.3.4)(v) and (1.3.4)(vi) for the definition of the dimension of an algebraic stack. Observe that this could be a negative integer.)

(4.1.2) As in [Bl-1] we will only consider those cycles that intersect all the faces of $\mathfrak{S} \times \Delta_k[n]$ properly. This defines a simplicial abelian group which will be denoted $z_d(\mathfrak{S}, n)$.

We proceed to associate to any closed sub-stack of a given algebraic stack, a cycle i.e. an integral linear combination of its irreducible components. This will be first done for algebraic spaces.

(4.1.3) Let $\mathfrak{S}$ denote an algebraic space and let T denote a closed algebraic subspace of pure dimension d, i.e. all the irreducible components of T have the same dimension. Let $x : X \to \mathfrak{S}$ denote an atlas for $\mathfrak{S}$ with X a scheme. Now $\tilde{T} = x^{-1}(T) = T \underset{\mathfrak{S}}{\times} X$ is a closed subscheme of X of pure dimension. Let the irreducible components of T ($\tilde{T}$) be denoted $\{T_i|i\}$ ($\{\tilde{T}_i|i\}$, respectively). Now $x(\tilde{T}_i)$ is contained in some irreducible component T_i of T; moreover since x is flat, $x(\tilde{T}_i)$ is dense in T_i so that $T_i =$ the closure of $(x(\tilde{T}_i))$. We let

(4.1.4) $[T] = \Sigma n_i T_i$ if $[\tilde{T}] = \Sigma_i n_i \tilde{T}_i$

(We will prove below (see (4.1.6)) that the cycle $[T]$ is independent of the choice of the atlas $x : X \to \mathfrak{S}$. Recall that $n_i =$ the length of the Artin local ring $\mathcal{O}_{\tilde{T},\tilde{T}_i}$ where $\mathcal{O}_{\tilde{T},\tilde{T}_i}$ is the localization of $\mathcal{O}_{\tilde{T}}$ at the generic point of $\tilde{T}_i$. See [F] (1.5).)

Next let $\mathfrak{S}$ denote an algebraic stack and let T denote a closed algebraic sub-stack of pure dimension d. Let $x : X \to \mathfrak{S}$ denote an atlas for $\mathfrak{S}$: recall the map x is smooth with X an algebraic space. Let $\tilde{T} = x^{-1}(T) = T \underset{\mathfrak{S}}{\times} X$ and let $\{\tilde{T}_i|i\}$ ($\{T_i|i\}$) denote the irreducible components of $\tilde{T}$ (T, respectively). We may assume that the closure of $x(\tilde{T}_i) = T_i$. (i.e. Recall that T_i corresponds to the sub-algebraic space $x^{-1}(T_i) = T_i \underset{\mathfrak{S}}{\times} X$ of X so that $\pi_1^{-1}(x^{-1}(T_i)) = \pi_2^{-1}(x^{-1}(T_i))$ where $\pi_i : X \underset{\mathfrak{S}}{\times} X \to X$, $i = 1, 2$ are the two projections. Observe that $x^{-1}(T_i)$ is of pure dimension. If $\tilde{T}_i$ is an irreducible component of $x^{-1}(T_i)$, (using the above observations) one can now readily show the closure of $x(\tilde{T}_i) = T_i$.) We let

(4.1.5) $[T] = \Sigma_i n_i T_i$ if $[\tilde{T}] = \Sigma n_i \tilde{T}_i$.

(4.1.6) **Proposition**. The above definitions are independent of the choice of atlases.

Proof. First we will prove this for algebraic spaces. Let $\mathfrak{S}$ denote an algebraic space and let $x : X \to \mathfrak{S}$, $x' : X' \to \mathfrak{S}$ denote two atlases for $\mathfrak{S}$ with X and X' schemes. Without loss of generality we may assume there exists an étale surjective map $X' \overset{\epsilon}{\to} X$ so that $x \circ \epsilon = x'$. Let $\tilde{T} = T \underset{\mathfrak{S}}{\times} X$, $\tilde{T}' = T \underset{\mathfrak{S}}{\times} X' = \tilde{T} \underset{X}{\times} X'$. We may assume without loss of generality that $\epsilon : \tilde{T}' \to \tilde{T}$ is an étale surjective map between affine schemes.

Let $\tilde{T}_i'$ ($\tilde{T}_i$) denote an irreducible component of $\tilde{T}'$ ($\tilde{T}$, respectively) so that the closure of $\epsilon(\tilde{T}_i') = \tilde{T}_i$. Now $\tilde{T}_i'$ denotes a point of the scheme $\tilde{T}'$ and $\tilde{T}_i = \epsilon(\tilde{T}_i')$ is the image of this point in the scheme $\tilde{T}$. Since the map ϵ is étale, one obtains an isomorphism $\hat{\mathcal{O}}_{\tilde{T}',\tilde{T}_i'} \cong \hat{\mathcal{O}}_{\tilde{T},\tilde{T}_i} \underset{k(\tilde{T}_i)}{\otimes} k(\tilde{T}_i')$. (See [H] p. 275.) Therefore $length(\mathcal{O}_{\tilde{T}',\tilde{T}_i'}) = length(\hat{\mathcal{O}}_{\tilde{T}',\tilde{T}_i'}) = length(\hat{\mathcal{O}}_{\tilde{T},\tilde{T}_i}) = length(\mathcal{O}_{\tilde{T},\tilde{T}_i})$. This proves the proposition in the case of algebraic spaces.

Next let $\mathfrak{S}$ denote an algebraic stack and let $x : X \to \mathfrak{S}$, $x' : X' \to \mathfrak{S}$ denote two atlases for $\mathfrak{S}$. Let $\tilde{T} = T \underset{\mathfrak{S}}{\times} X$, $\tilde{T}' = T \underset{\mathfrak{S}}{\times} X'$. Without loss of generality we may assume there exists a smooth surjective map $X' \overset{\epsilon}{\to} X$ so that $x \circ \epsilon = x'$. We may assume now that $\epsilon : \tilde{T}' \to \tilde{T}$ is a smooth surjective map between two algebraic spaces. By considering atlases for these algebraic spaces (and applying the proposition for algebraic spaces, which we have just established), now one may assume that $\epsilon : \tilde{T}' \to \tilde{T}$ is a smooth surjective map of schemes. Further one may assume these are affine schemes and that ϵ factors as the composition of an étale map $\tilde{T}' \to \tilde{T} \underset{k}{\times} \mathbb{A}^n$ (for some $n \geq 0$) and the obvious projection $\tilde{T} \underset{k}{\times} \mathbb{A}^n \overset{\pi}{\to} \tilde{T}$. Let $\tilde{T}_i'$ ($\tilde{T}_i$) denote an irreducible component of $\tilde{T}'$ ($\tilde{T}$, respectively) so that the closure of $\epsilon(\tilde{T}_i') = \tilde{T}_i$. Now $\tilde{T}_i'$ denotes a point of the scheme $\tilde{T}'$ and $\tilde{T}_i = \epsilon(\tilde{T}_i')$ is the image of this point in the scheme $\tilde{T}$. Since π^{-1} of any irreducible closed sub-scheme is an irreducible closed sub-scheme of $\tilde{T} \underset{k}{\times} \mathbb{A}^n$, it suffices to assume the map ϵ itself is an étale map between two schemes. We have already proved the proposition in this case in the above paragraph. $\square$

(4.2.1) Let $f : \mathfrak{S}' \to \mathfrak{S}$ denote a flat map of algebraic stacks so that it is of relative dimension n i.e. for each irreducible component T of $\mathfrak{S}$, every irreducible component of $f^{-1}(T) = T\underset{\mathfrak{S}}{\times}\mathfrak{S}'$ is of dimension $= n+$ the dimension of T. *We will assume by default that all flat maps are of relative dimension n for some integer* $n \geq 0$. Observe that now $f \times id : \mathfrak{S}' \times \Delta_k[n] \to \mathfrak{S} \times \Delta_k[n]$ is also flat of the same relative dimension for all n.

(4.2.2) If $T \subseteq \mathfrak{S} \times \Delta_k[n]$ is a closed integral sub-stack of dimension c, observe that $f^{-1}(T) = T \underset{\mathfrak{S}\times\Delta_k[n]}{\times} \mathfrak{S}' \times \Delta_k[n]$ is a closed sub-stack of $\mathfrak{S}' \times \Delta_k[n]$ with dimension $= d+n$. One may readily verify that the each irreducible component of $f^{-1}(T)$ intersects all the faces of $\mathfrak{S}' \times \Delta[n]$ properly. Therefore we will let $[f^{-1}(T)] \,\epsilon\, z_*(\mathfrak{S}', n)$ denote the class of $f^{-1}(T)$. Now one defines a map $f^* : z_*(\mathfrak{S}, n) \to z_*(\mathfrak{S}', n)$ for each $n \geq 0$ by extending the above definition using linearity to all cycles in $z_*(\mathfrak{S}, n)$. This makes $\mathfrak{S} \to z_*(\mathfrak{S}, n)$, for each fixed integer n, a *contravariant* functor for flat maps of algebraic stacks. Therefore, this defines an *additive presheaf* on $(alg.stacks/k)^{l.Noeth}_{Res}$. We will denote this *presheaf* by

(4.2.3) $\mathcal{Z}_*(\quad, n)$.

(4.2.4) Next assume that $\mathfrak{S}$ is an algebraic stack. If $x : X \to \mathfrak{S}$ is an object of the site $\mathfrak{S}_{res.smt}$, with X connected, observe that it is also of a fixed relative dimension. Therefore the restriction of presheaf $\mathcal{Z}_*(\quad, n)$ to $\mathfrak{S}_{res.smt}$ defines an *additive* presheaf which will be denoted $\mathcal{Z}_*^{\mathfrak{S}_{res.smt}}(\quad, .)$. As shown in (3.6.2) through (3.6.5), one may therefore consider the étale and smooth hypercohomology of $B_x\mathfrak{S}$ with respect to the above complex. The restriction of $\mathcal{Z}_*^{\mathfrak{S}_{res.smt}}(\quad, .)$ to the étale site of $\mathfrak{S}$ will be denoted $\mathcal{Z}_*^{\mathfrak{S}_{et}}(\quad, .)$.

(4.2.5) To understand the grading of the presheaf $\mathcal{Z}_*(\quad, .)$ one may proceed as follows. Let $\mathfrak{S}$ denote an algebraic stack as before. For each fixed integers d and $n \geq 0$, we also define a presheaf $\mathcal{Z}_d^{\mathfrak{S}_{res.smt}}(\quad, n)$ on $\mathfrak{S}_{res.smt}$ as follows. If $u : U \to \mathfrak{S}$ belongs to $\mathfrak{S}_{res.smt}$ and U is *connected*, it is of a fixed relative dimension m, for some integer $m \geq 0$; therefore we let $\Gamma(U, \mathcal{Z}_d^{\mathfrak{S}_{res.smt}}(\quad, n)) = \mathcal{Z}_{d+m}(U, m)$. We extend the definition of the presheaf $\mathcal{Z}_d^{\mathfrak{S}_{res.smt}}(\quad, n)$ by additivity to all of $\mathfrak{S}_{res.smt}$, not necessarily connected. Observe that if $f : \mathfrak{S}' \to \mathfrak{S}$ is a representable *flat* map of relative dimension m, then f induces a pull-back $f^* : \mathcal{Z}_d^{\mathfrak{S}_{res.smt}}(U, n) \to \mathcal{Z}_{d+m}^{\mathfrak{S}_{res.smt}}(f^{-1}(U), n)$ for any $U \,\epsilon\, \mathfrak{S}_{res.smt}$ with $f^{-1}(U) = U\underset{\mathfrak{S}}{\times}\mathfrak{S}'$.

(4.2.6) Next let $p : \mathfrak{S}' \to \mathfrak{S}$ denote a *proper representable* map of algebraic stacks and let $t' : T' \to \mathfrak{S}'$ denote a closed immersion of an integral sub-stack. Let $\mathcal{O}_{T'}$ denote the structure sheaf of T'. Now $p_*(\mathcal{O}_{T'})$ is a coherent $\mathcal{O}_{\mathfrak{S}}$-module. We define the *stack-theoretic image* of T' to be the closed reduced algebraic sub-stack of $\mathfrak{S}$ defined by the *support of* $p_*(\mathcal{O}_{T'})$. This will be denoted $p(T')$. (Since p is representable, one may define the support of a coherent $\mathcal{O}_{\mathfrak{S}'}$-module, $\mathcal{M}$, as follows. Let $x : X \to \mathfrak{S}$ denote an atlas for $\mathfrak{S}$, let $x' : X' = X\underset{\mathfrak{S}}{\times}\mathfrak{S}' \to \mathfrak{S}'$ denote the induced atlas for $\mathfrak{S}'$ and let $\tilde{p} : X' \to X$ denote the induced map. The support of $\tilde{p}_*(x'^*(\mathcal{M}))$ is a closed algebraic sub-space of X. One may show that if $\pi_i : X\underset{\mathfrak{S}}{\times}X \to X$ is the projection to the i-th factor, $\pi_1^{-1}(\tilde{p}_*(x'^*(\mathcal{M}))) = \pi_2^{-1}(\tilde{p}_*(x'^*(\mathcal{M})))$ so that we obtain a closed algebraic sub-stack of $\mathfrak{S}'$. In case p is *not necessarily* proper, we may define the image of the sub-stack T' of $\mathfrak{S}'$ to be smallest closed algebraic sub-stack T of $\mathfrak{S}$ through which the morphism $p : \mathfrak{S}' \to \mathfrak{S}$ factors.)

(4.2.7) Assume the above situation. Then $p^{-1}(p(T')) = p(T')\underset{\mathfrak{S}}{\times}\mathfrak{S}'$ is a closed algebraic sub-stack of $\mathfrak{S}'$, the morphism $T' \to p^{-1}(p(T'))$ is a closed immersion and therefore the induced map $T' \to p(T')$ is a representable proper map. Let $y : Y \to p(T')$ be an atlas, let $y' : Y' = Y \underset{p(T')}{\times} T'$ and let $\tilde{p}_{T'} : Y' \to Y$ denote the induced map. Now we define the *degree of* $p_{|T'}$ to be the degree of the map $\tilde{p}_{|T'}$. (It is shown in [Vis] (1.16) that this definition is independent of the choice of an atlas.) Observe that the degree is 0 unless $p_{|T'}$ is generically finite.

(4.2.8) Next we define the direct image $p_* : \mathcal{Z}_*(\mathfrak{S}', n) \to \mathcal{Z}_*(\mathfrak{S}, n)$ for each fixed $n \geq 0$ to be given by $p_*([T']) = degree(p_{|T'}).[p(T')]$ for any integral sub-stack T' of $\mathfrak{S}' \times \Delta_k[n]$ that belongs to $\mathcal{Z}_*(\mathfrak{S}', n)$. (Observe that the same definition defines $p_* : \mathcal{Z}_d(\mathfrak{S}', n) \to \mathcal{Z}_d(\mathfrak{S}, n)$ for each fixed integers d and $n \geq 0$.) Observe that this direct image is strictly compatible with flat pull-backs as in [F] Proposition 1.7. (See (4.3.2.*), Part II for more details.)

(4.3.0) In (4.3.1) through (4.3.4) we will restrict to Deligne-Mumford stacks of finite type over k. *Since any algebraic space may be viewed as a Deligne-Mumford stack, these results all hold for algebraic spaces.*

(4.3.1) Observe that, in case $\mathfrak{S}$ is a Deligne-Mumford stack, $\mathfrak{S} \times \Delta_k[n]$ is also a Deligne-Mumford stack. Therefore the arguments in [Gi] (4.1) apply to show that the functor $\mathfrak{S} \mapsto z_d(\mathfrak{S} \times \Delta[n])$ is in fact a *sheaf* on $\mathfrak{S}_{\underline{et}}$. Since dimension is stable under étale localization, one may show that for each fixed integers d and n, the presheaf $\mathcal{Z}_d(\quad, .)$ restricted to $\mathfrak{S}_{\underline{et}}$ is a *sheaf.*

(4.3.2) Next let $f : \mathfrak{S}' \to \mathfrak{S}$ denote a proper map of Deligne-Mumford stacks, not necessarily representable. In case f is not representable, one defines the degree $deg(f) = deg(f \circ x)/deg(x)$ where $x : X \to \mathfrak{S}$ is an atlas. It is shown in [Vis] (1.16) that this degree is independent of the choice of an atlas x. Now one obtains a proper push-forward $f_* : \mathcal{Z}_d(\mathfrak{S}, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \to \mathcal{Z}_d(\mathfrak{S}',)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$.

(4.3.3) Assume $\mathfrak{S}$ is a Deligne-Mumford stack. A rational function on $\mathfrak{S}$ may be defined to be a dominant map $f : U \to \mathbb{P}^1$ where U is an open sub-stack of $\mathfrak{S}$. If Z is an integral sub-stack of $\mathfrak{S}$, let $k(Z)^*$ denote the multiplicative group of rational functions on Z. Now we let $W_*(\mathfrak{S})$ denote the *rational equivalences* on $\mathfrak{S}$, namely $\underset{j}{\oplus} W_j(\mathfrak{S})$ where $W_j(\mathfrak{S})$ is the direct sum of $k(Z)^*$ over integral closed sub-stacks Z of dimension $j+1$. It is shown in [Gi] section 4, that these are sheaves on $\mathfrak{S}_{\underline{et}}$ and that one obtains a homomorphism $\delta : W_*(\mathfrak{S}) \to \mathcal{Z}_*(\mathfrak{S}, 0)$. The *naive Chow group* $CH_q^{naive}(\mathfrak{S})$ is defined to be the cokernel of $\delta : W_q(\mathfrak{S}) \to \mathcal{Z}_q(\mathfrak{S}, 0)$.

(4.3.4) **Proposition**. If $\mathfrak{S}$ is a Deligne-Mumford stack of finite type over k, one obtains the isomorphism:

$$CH_q^{naive}(\mathfrak{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \cong \pi_0(\mathcal{Z}_q(\mathfrak{S}, .))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$$

Proof. Let $\alpha = \delta(r)$, $r \,\epsilon\, k(Z)^*$ where Z is a dimension $q+1$ closed integral sub-stack of $\mathfrak{S}$. Then r corresponds to rational function $\tilde{r}$ on the atlas $\tilde{Z}$ of Z; i.e. a map $\tilde{U} \to \mathbb{P}^1$ where $\tilde{U}$ is an open subscheme of $\tilde{Z}$. Since $\tilde{r}$ is obtained from the rational function r, $\tilde{U}$ descends to an open sub-stack U of Z. Let $\tilde{V}$ be the closure of the graph of the map $\tilde{U} \to \mathbb{P}^1$ in $X \times \mathbb{P}^1$ where X is an atlas of $\mathfrak{S}$. (We may assume $\tilde{Z} = Z\underset{\mathfrak{S}}{\times}X$.) Then $\tilde{V}$ descends to a closed integral sub-stack V of $\mathfrak{S} \times \mathbb{P}^1$ which has the dimension $q+1$. V defines by restriction to $\mathfrak{S} \times \Delta_k[1]$ a closed sub-stack of dimension $q+1$. This defines an element in $\mathcal{Z}_{q+1}(\mathfrak{S}, 1)$.

If $f : V \to \mathbb{P}^1$ is the composition $V \to \mathfrak{S} \times \mathbb{P}^1 \to \mathbb{P}^1$, one observes that $\alpha = \delta(r) = p_*(\delta(f))$ where $p : V \to Z$ is induced by the composition $V \to \mathfrak{S} \times \mathbb{P}^1 \to \mathfrak{S}$. Observe that V and Z are of the same dimension. The push-forward $p_* : \mathcal{Z}_*(V, 0)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \to \mathcal{Z}_*(Z, 0)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is defined in [Vis] (3.6). Recall that $\mathcal{Z}_*(\quad, 0)$ is a sheaf on the étale topology of the stacks V and Z. Therefore, in order to obtain the equality $p_*(\delta(f)) = \delta(r)$, it suffices to work locally in the étale topology where this is known - see [F] p.16. Observe that for any point P in $\mathbb{P}^1$, the fiber $f^{-1}(P)$ is a sub-stack of $\mathfrak{S} \times P$ which p maps isomorphically onto a closed sub-stack of Z and hence of $\mathfrak{S}$; we denote this sub-stack by $V(P)$. Then $p_*[f^{-1}(P)] = V(P)$ and therefore
$p_*(\delta(f)) = p_*(f^{-1}(0) - f^{-1}(1)) = V(0) - V(1)$. These show that if a cycle in $\mathcal{Z}_q(\mathfrak{S}, 0)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is rationally equivalent to 0, then the class of the cycle in $\pi_0(\mathcal{Z}_q(\mathfrak{S}, .))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is trivial.

Now it suffices to show the converse of the last assertion. For this one begins with a class $V \,\epsilon\, \mathcal{Z}_{q+1}(\mathfrak{S}, 1)$. Let $f : V \to \mathfrak{S} \times \Delta_k[1] \cong \mathfrak{S} \times \mathbb{A}^1 \to \mathfrak{S} \times \mathbb{P}^1 \xrightarrow{pr_2} \mathbb{P}^1$ denote the obvious map. Now the same arguments as above show $V(0) - V(1) = p_*(\delta(f))$. Since $\delta(f)$ is rationally equivalent to 0, it suffices to show that W_* pushes forward: since this was observed to be a sheaf on the étale topology of the stacks, it suffices to prove this locally, where it is well-known. (See for example, Theorem 1.4, Chapter I, [F].) Observe that push-forward is defined for non-representable proper maps only after tensoring with $\mathbb{Q}$. (See [Vis] (3.7).) □.

We proceed to define the higher Chow groups for algebraic stacks as suitable hypercohomology with respect to the cycle complex. We begin by showing that even Zariski hypercohomology with respect to the above presheaf has reasonable properties for all schemes locally of finite type over k.

(4.4.1) **Definition**. Let X denote a scheme locally of finite type over k. For each integer d, let $\mathcal{Z}_d^{X_{Zar}}(\quad, .)$ denote the restriction of the presheaf $\mathcal{Z}_d^{X_{res.smt}}(\quad, .)$ to the Zariski site of X. Now we let $\mathbf{CH}_d(X, .) = \mathbb{H}_{Zar}(X, \mathcal{Z}_d^{X_{Zar}}(\quad, .))$ and $CH_d(X, n) = \mathbb{H}_{Zar}^{-n}(X, \mathcal{Z}_d^{X_{Zar}}(\quad, .))$ = the hypercohomology with respect to the above presheaf on the Zariski site of X. (Observe that we are computing hypercohomology in the sense of (3.5.1) where the complete pointed simplicial category is the category of all unbounded co-chain complexes of abelian sheaves as in (6.2.4).) Similarly we let $\mathbf{CH}_d(X, .; \mathbb{Q}) = \mathbb{H}_{Zar}(X, \mathcal{Z}_d^{X_{Zar}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$.

(4.4.2) **Remark** If X is a quasi-projective scheme over k, the above *integral higher Chow groups* are isomorphic to the naive Chow group defined as $\pi_0(z_*(X,.))$. (This follows from the localization theorem of Bloch.)

(4.4.3) **Proposition**. Let X denote a scheme locally of finite type over k. Let $Z \to X$ denote the closed immersion of a closed subscheme with $U = X - Z$. Then one obtains a distinguished triangle:

$$\mathbf{CH}_m(Z,.) \to \mathbf{CH}_m(X,.) \to \mathbf{CH}_m(U,.) \to \mathbf{CH}_m(Z,.)[1]$$

and therefore a long-exact sequence:

$$... \to CH_m(Z,n) \to CH_m(X,n) \to CH_m(X-Z,n) \to CH_m(Z,n-1) \to ...$$

Proof. Take $P = \mathcal{Z}_*(\quad,.)$ in (3.8.3)(i). Now observe that the map $i_*\mathcal{Z}_*^{Z_{Zar}}(\quad,.) \to \mathcal{Z}_*^{X_{Zar}}(\quad,.)$ preserves the dimension of the cycles. □

(4.4.4) In the above situation, it is not clear that $CH_*(X,n) = 0$ for $n < 0$. However we can readily show this is the case modulo torsion provided X is of finite type over k as follows. Since X is assumed to be of finite type, we may use ascending induction on the Krull dimension of X. By the localization sequence, now it suffices to assume X is regular. Let $u : U_\bullet \to X$ denote a Zariski hypercovering by schemes that are quasi-projective over k i.e. the simplicial scheme $U_\bullet$ is quasi-projective over k in each degree. Therefore, the strengthening of the Riemann-Roch theorem of [Bl-1](9.1) as in (4.4.5) below provides the quasi-isomorphism: $\underset{\Delta}{\text{holim}}\underset{\to}{\lim}\{\Gamma(U_n,\mathcal{Z}_*(\quad,.))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}|n\} \simeq \underset{\Delta}{\text{holim}}\underset{\to}{\lim}\{\Gamma(U_n,\mathbf{G}_{\mathbb{Q}})|n\}$ where the colimit is over the category of all Zariski hypercoverings $U_\bullet$ of U with each U_n a quasi-projective scheme over k. By (4.4.5) and (4.4.6) below one may identify the former (latter) with $\mathbb{H}_{Zar}(X,\mathcal{Z}_*(\quad,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$ $(\mathbb{H}_{Zar}(X,\mathbf{G}_{\mathbb{Q}}) \simeq G(X)_Q)$. Clearly this has trivial homotopy groups in negative degrees.

(4.4.5) **Lemma**. Let X denote a regular quasi-projective scheme. Then there exists a quasi-isomorphism of presheaves $G(\quad)_{\mathbb{Q}}\overset{\simeq}{\to}\mathcal{Z}^*(\quad;.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ on the Zariski or étale site of X that is natural in X.

Proof. Observe that the Riemann-Roch theorem of Bloch in [Bl-1] is established only as an isomorphism of the rational homotopy groups of the cycle complex with rational G-theory. However the higher Chern classes constructed in [Bl-1] can be made functorial for smooth schemes as follows. As shown in [J-1] Theorem (3.1), for smooth schemes, one may replace the cycle complex canonically upto natural quasi-isomorphism by the motivic complex. Now the higher Chern classes constructed in [J-1] section 4 apply to provide a Chern-character map $K(\quad)_{\mathbb{Q}} \to \mathcal{Z}^*(\quad;.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ as a map of presheaves on the small Zariski or étale topology of a given scheme. The Riemann-Roch theorem of Bloch shows this is a quasi-isomorphism (i.e. induces an isomorphism on taking sheaves of homotopy groups). Since X is assumed to be regular, the presheaf $K(\quad)$ identifies with $G(\quad)$ and proves the lemma.

(4.4.6) **Lemma**. Let X denote a scheme of finite type over k and let P denote an additive presheaf on X_{Zar} with values in a category $\mathbf{S}$ as before. Assume further that $\pi_n(P) = 0$ for $n < 0$ and that $\pi_i(P)$ is a presheaf of $\mathbb{Q}$-vector spaces for each i. Let $HR(X)$ denote a cofinal sub-category of the (left-filtered) category of Zariski hypercoverings of X. Then $\mathbb{H}_{Zar}(X,P) \simeq \underset{\Delta}{\text{holim}}\underset{\to}{\lim}\{\Gamma(U_\bullet,P)|n\}$ where the colimit is over $HR(X)$.

Proof. One observes that there exist maps

$$\mathbb{H}_{Zar}(X,P) \to \underset{\Delta}{\text{holim}}\underset{\to}{\lim}\mathbb{H}_{Zar}(U_\bullet,P) \leftarrow \underset{\Delta}{\text{holim}}\underset{\to}{\lim}\{\Gamma(U_\bullet,P)\}$$

natural in P. Now we will first restrict to the case when P has been replaced by a $\tau_{\leq n}P$ for some $n \geq 0$. Since the distinguished triangle $\tau_{\leq n-1}\Gamma(V,P) \to \tau_{\leq n}\Gamma(V,P) \to \tau_{\geq n}(\tau_{\leq n}\Gamma(,P))$ exists for all V in the Zariski site of X and is natural in V, and since the homotopy inverse limits and filtered colimits preserve distinguished triangles, one may reduce to the case when P has cohomology sheaves only in one degree. i.e. We reduce to the case when P is replaced by an abelian presheaf. This is well-known.

Now we consider the general case. Observe that inverse limits commute with homotopy inverse limits and $\underset{\infty\leftarrow n}{\lim}\mathbb{H}_{Zar}(X,\tau_{\leq n}P) \simeq \mathbb{H}_{Zar}(X,P)$ (by (3.4.1)). Therefore it suffices to show $\underset{\infty\leftarrow n}{\lim}\underset{\to}{\lim}\mathbb{H}_{Zar}(U_\bullet,\tau_{\leq n}P) \simeq \underset{\to}{\lim}\underset{\infty\leftarrow n}{\lim}\mathbb{H}_{Zar}(U_\bullet,\tau_{\leq n}P)$ and $\underset{\infty\leftarrow n}{\lim}\underset{\to}{\lim}\Gamma(U_\bullet,\tau_{\leq n}P) \simeq \underset{\to}{\lim}\underset{\infty\leftarrow n}{\lim}\Gamma(U_\bullet,\tau_{\leq n}P)$. Both follow from the observation that on taking each fixed π_k, the inverse limits all stabilize. (This, in turn, follows from the observation that all schemes considered have uniform finite Zariski-cohomological dimension with respect to sheaves of $\mathbb{Q}$-vector spaces.) □

(4.5.1) **Definition**. Let $\mathfrak{S}$ denote an algebraic stack and let $x : X \to \mathfrak{S}$ denote a fixed atlas and let $B_x\mathfrak{S}$ denote the corresponding classifying simplicial space and let d denote a fixed integer. (i) We define $\mathbf{CH}_d(\mathfrak{S}, x, .) = \mathbb{H}_{et}(B_x\mathfrak{S}, \mathcal{Z}_d^{\mathfrak{S}_{res.smt}}(\quad, .))$. (See (3.6.3).)

(ii) If R denotes either $\mathbb{Q}$, or $\mathbb{Z}/l^\nu$, $\nu > 0$, one defines $\mathbf{CH}_d(\mathfrak{S}, x, .; R) = \mathbb{H}_{et}(\mathfrak{S}, \mathcal{Z}_d^{\mathfrak{S}_{res.smt}}(\quad, .)\underset{\mathbb{Z}}{\otimes} R)$.

(iii) If n is an integer, we will let $CH_d(\mathfrak{S}, x, n) = \pi_n(\mathbf{CH}_d(\mathfrak{S}, x, .))$ while $CH_d(\mathfrak{S}, x, n; R) = \pi_n(\mathbf{CH}_d(\mathfrak{S}, x, .; R))$

(Observe that we are computing hypercohomology in the sense of (3.6.4) in general, where the complete pointed simplicial category is the category of all unbounded co-chain complexes of sheaves of R-modules as in (6.2.4). We may replace this by hypercohomology as in (3.5.1) for Deligne-Mumford stacks.)

Next we consider the *basic properties of the above higher Chow groups.*

(4.5.2) **Theorem**. (Independence on the choice of the atlas). (i) If $\mathfrak{S}$ is a Deligne-Mumford stack and $x : X \to \mathfrak{S}$ is an *étale* atlas, $CH_d(\mathfrak{S}, x, .; R) \simeq \mathbb{H}_{et}(\mathfrak{S}, \mathcal{Z}_d^{\mathfrak{S}_{et}}(\quad, .)\underset{\mathbb{Z}}{\otimes} R)$ and is therefore intrinsic to the stack. (ii) Suppose $\mathfrak{S}$ is a general Artin stack that is *smooth* and $R = \mathbb{Z}/l^\nu$, *l different from the characteristic of k*. If $x : X \to \mathfrak{S}$ and $y : Y \to \mathfrak{S}$ are two atlases for $\mathfrak{S}$, $CH_*(\mathfrak{S}, x, n; R) \simeq CH_*(\mathfrak{S}, y, n; R)$. (iii) If $\mathfrak{S}$ is *smooth* and of dimension d, $CH_{d-i}(\mathfrak{S}, n; \mathbb{Z}/l^\nu) \cong H^{2i-n}_{et}(\mathfrak{S}; \mu_{l^\nu}(i))$ □

Proof. (i) follows immediately from (3.6.1). Now we consider (ii). Let $x : X \to \mathfrak{S}$ and $y : Y \to \mathfrak{S}$ denote two atlases for the stack $\mathfrak{S}$. Now $z : X \underset{\mathfrak{S}}{\times} Y \to \mathfrak{S}$ is also an atlas for $\mathfrak{S}$. One may identify the simplicial space $B_z\mathfrak{S}$ with the diagonal of the bi-simplicial space: $(n, m) \mapsto X^n \underset{\mathfrak{S}}{\times} Y^m$ where X^n (Y^m) denotes the fibered product of X over $\mathfrak{S}$ n-times (Y over $\mathfrak{S}$ m-times, respectively). Let $B_{x,y}\mathfrak{S}$ denote this bi-simplicial space.

(4.5.2.*) Observe that for each fixed integer n (m), $B_{x,y}\mathfrak{S}_{n,\bullet}$ ($B_{x,y}\mathfrak{S}_{\bullet,m}$) is a smooth hypercovering of $(B_x\mathfrak{S})_n$ ($(B_y\mathfrak{S})_m$, respectively).

Next let $\mathcal{Z}_\bullet^{B_x\mathfrak{S}_{n,et}}(\quad, .; \mathbb{Z}/l^\nu)$ ($\mathcal{Z}_\bullet^{B_y\mathfrak{S}_{m,et}}(\quad, .; \mathbb{Z}/l^\nu)$, $\mathcal{Z}_\bullet^{B_{x,y}\mathfrak{S}_{n,m,et}}(\quad, .; \mathbb{Z}/l^\nu)$) denote the mod-$l^\nu$ higher cycle complex on the étale site of $(B_x\mathfrak{S})_n$ ($(B_y\mathfrak{S})_m$, $(B_{x,y}\mathfrak{S})_{n,m}$, respectively). The first two pull back to a complex that is quasi-isomorphic to the third. To see this it suffices to consider the following.

Let $f : Z' \to Z$ denote a smooth map of smooth schemes locally of finite type over k. Let $\mathcal{Z}_\bullet^{Z_{et}}(\quad, .)$ and $\mathcal{Z}_\bullet^{Z'_{et}}(\quad, .)$ denote the pre-sheafification of the cycle complexes on Z and Z' on their étale sites. Then the induced map $f^*\mathcal{Z}_\bullet^{Z_{et}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Z}/l^\nu \to \mathcal{Z}_\bullet^{Z'_{et}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Z}/l^\nu$ is a quasi-isomorphism. This follows immediately from the rigidity property of the mod$-l^\nu$ higher cycle complexes established in [Bl-1](11.1).

Now consider the étale hypercohomology of $B_x\mathfrak{S}$, $B_y\mathfrak{S}$ and $B_z\mathfrak{S}$ with respect to the above mod-l^ν higher cycle complex. (3.6.5) shows that this étale hypercohomology may be identified with the smooth hypercohomology with respect to corresponding induced complexes of presheaves on the smooth-sites. (4.5.2.*), (3.6.1) and (3.6.5) show that the smooth hypercohomology of $B_x\mathfrak{S}$, $B_y\mathfrak{S}$ and $B_z\mathfrak{S}$ with respect to these complexes are in fact isomorphic.

Next we consider (iii). First, the comparison theorem, [J-1] (3.1) Theorem, identifies our higher Chow-groups with finite coefficients with the (Lichtenbaum) motivic cohomology with respect to $\mathbb{Z}/l^\nu(i)$. Next, we observe from [Voev-2] Theorem 6.8, that the motivic complex $\mathbb{Z}/l^\nu(i)$ (on the étale (or smooth) site of any smooth scheme and hence on the smooth site of the stack $\mathfrak{S}$) identifies naturally with sheaf $\mu_{l^\nu}(i)$. □

Remarks. i). In part II of this paper we will extend the independence on the choice of the atlas to all smooth stacks integrally, and to all stacks of finite type over the given field modulo torsion, thereby proving our theory is intrinsic to algebraic stacks. This involves an extension of recent comparison results between motivic cohomology and the higher Chow groups to algebraic stacks.

ii). We thank the referee for suggesting that we add the computation in (iii). As a corollary, we are able to compute the higher Chow groups (with finite coefficients) of the stack BE, where E is an elliptic curve. (See (5.3).)

(4.6.1) **Theorem** (Localization sequence). Assume that $i : \mathfrak{S}' \to \mathfrak{S}$ is a closed immersion of algebraic stacks with $\mathfrak{S}''$ = the complement of $\mathfrak{S}'$ in $\mathfrak{S}$. Assume further that the stack $\mathfrak{S}$ is locally of finite type over k. Let $x : X \to \mathfrak{S}$ denote a fixed atlas, let $x' = x \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ and $x'' = x \underset{\mathfrak{S}}{\times} \mathfrak{S}''$. Then one obtains a long exact sequence:

$$... \to CH_*(\mathfrak{S}', x', n; \mathbb{Q}) \to CH_*(\mathfrak{S}, x, n; \mathbb{Q}) \to CH_*(\mathfrak{S}'', x'', n; \mathbb{Q}) \to ...$$

Proof. This follows immediately from (3.8.2) and (3.8.3)(ii). □.

(4.6.2) **Corollary** (Mayer-Vietoris). Let $\mathfrak{S}$ denote an algebraic stack with $\mathfrak{S}_0$ and $\mathfrak{S}_1$ two *open* algebraic sub-stacks so that $\mathfrak{S}$ is isomorphic to $\mathfrak{S}_0 \cup \mathfrak{S}_1$. Let $x : X \to \mathfrak{S}$ denote a fixed atlas for $\mathfrak{S}$. Let x_0, x_1 and x_{01} denote the induced atlases for $\mathfrak{S}_0$, $\mathfrak{S}_1$ and $\mathfrak{S}_0 \cap \mathfrak{S}_1$. Then one obtains a long-exact sequence:

$$... \to CH_*(\mathfrak{S}, x, n; \mathbb{Q}) \to CH_*(\mathfrak{S}_0, x_0, n; \mathbb{Q}) \oplus CH_*(\mathfrak{S}_1, x_1, n; \mathbb{Q}) \to CH_*(\mathfrak{S}_0 \cap \mathfrak{S}_1, x_{01}, n; \mathbb{Q}) \to ...$$

Proof. This follows in the usual manner from (4.6.1). (See (3.7.8)(i) for example.) □

(4.6.3) **Theorem.** (i) Let $f : \mathfrak{S}' \to \mathfrak{S}$ denote a representable map of algebraic stacks. Let $x : X \to \mathfrak{S}$ denote a fixed atlas and let $x' : x \underset{\mathfrak{S}}{\times} \mathfrak{S}'$. If f is *flat* of relative dimension m, it induces a map $f^* : CH_*(\mathfrak{S}, x, n) \to CH_{*+m}(\mathfrak{S}', x', n)$ for every $n \geq 0$.

(ii) If $f : \mathfrak{S}' \to \mathfrak{S}$ is any *proper representable* map between algebraic stacks of finite type, $x : X \to \mathfrak{S}$ and $x' : X' \to \mathfrak{S}'$ denote atlases so that $x' = x \underset{\mathfrak{S}}{\times} \mathfrak{S}'$ with X of finite type over *Spec* k, then f induces a map $f_* : CH_*(\mathfrak{S}', x', n; \mathbb{Q}) \to CH_*(\mathfrak{S}, x, n; \mathbb{Q})$.

(iii) Let $f : \mathfrak{S}' \to \mathfrak{S}$ denote any finite map of Deligne-Mumford stacks. Now f induces a map $f_* : CH_*(\mathfrak{S}', .; \mathbb{Q}) \to CH_*(\mathfrak{S}, .; \mathbb{Q})$.

Proof. In order to establish (i), observe the existence of a natural map $\mathcal{Z}_d^{B_{x'}\mathfrak{S}'_{n,et}}(\quad, .) \to Rf_* f^{-1} \mathcal{Z}_d^{B_{x'}\mathfrak{S}'_{n,et}}(\quad, .)$ for each fixed $n \geq 0$. Therefore it suffices to prove the existence of a natural map $f^{-1}\mathcal{Z}_d^{B_{x'}\mathfrak{S}'_{n,et}}(\quad, .) \to \mathcal{Z}_{d+m}^{B_x\mathfrak{S}_{n,et}}(\quad, .)$ compatible with the face maps of the simplicial space $B_x\mathfrak{S}$. This exists since f is a flat map. (See (4.2.2) and (4.2.5).)

We will next consider (ii). The covariance property in (4.2.8) shows the existence of a map $f_*\mathcal{Z}_*^{B_{x'}\mathfrak{S}'_{n,et}}(\quad, .) \to \mathcal{Z}_*^{B_x\mathfrak{S}_{n,et}}(\quad, .)$ of presheaves on $B_x\mathfrak{S}_{n,et}$. At this point we will make use of Proposition (3.6.7). Observe that the presheaves $\mathcal{Z}_*(\quad, .)$ and $\mathcal{Z}_*(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ are additive. This suffices to apply Proposition (3.6.7), since the hypothesis that X is of finite type implies that so is X' as well as $B_x\mathfrak{S}_n$ and $B_{x'}\mathfrak{S}'$ for each $n \geq 0$. Let $HRR(B_x\mathfrak{S})$ ($HRR(B_{x'}\mathfrak{S}')$) denote the left directed category of rigid hypercoverings of $B_x\mathfrak{S}$ ($B_{x'}\mathfrak{S}'$, respectively). The given map f induces a map of these left directed categories, sending a rigid hypercovering $V_{\bullet,\bullet}$ of $B_x\mathfrak{S}$ to $V_{\bullet,\bullet} \underset{B_x\mathfrak{S}}{\times} B_{x'}\mathfrak{S}'$. Proposition (3.6.7) shows that $\mathbb{H}_{et}(B_x\mathfrak{S}, \mathcal{Z}_\bullet^{B_x\mathfrak{S}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \simeq \underset{\Delta}{\text{holim}} DN \underset{\alpha}{\text{colim}} \Delta\Gamma(V_{\bullet,\bullet}^\alpha, \mathcal{Z}_\bullet^{B_x\mathfrak{S}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$ and that $\mathbb{H}_{et}(B_{x'}\mathfrak{S}', \mathcal{Z}_\bullet^{B_{x'}\mathfrak{S}'}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \simeq \underset{\Delta}{\text{holim}} DN \underset{\beta}{\text{colim}} \Delta\Gamma(U_{\bullet,\bullet}^\beta, \mathcal{Z}_\bullet^{B_{x'}\mathfrak{S}'}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$ where the direct limit is over the left directed category of rigid hypercoverings. Moreover the hypercohomology of the simplicial spaces $B_x\mathfrak{S}$ and $B_{x'}\mathfrak{S}'$ are computed as in (3.6.4). The covariance property shows that f induces a map $\Delta\Gamma(V_{\bullet,\bullet}^\alpha \underset{B_x\mathfrak{S}}{\times} B_{x'}\mathfrak{S}', \mathcal{Z}_\bullet^{B_{x'}\mathfrak{S}'}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \to \Delta\Gamma(V_{\bullet,\bullet}^\alpha, \mathcal{Z}_\bullet^{B_x\mathfrak{S}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$ and hence $f_* : \mathbb{H}_{et}(B_{x'}\mathfrak{S}', \mathcal{Z}_\bullet^{B_{x'}\mathfrak{S}'}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \to \mathbb{H}_{et}(B_x\mathfrak{S}, \mathcal{Z}_\bullet^{B_x\mathfrak{S}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$. This proves (ii).

To obtain (iii), observe that the push-forward in (4.3.2) defines a map $f_*\mathcal{Z}_{\mathfrak{S}'_{et}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \to \mathcal{Z}_{\mathfrak{S}_{et}}(\quad, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ of presheaves on the site $\mathfrak{S}_{et}$. Since f is finite and we are using rational coefficients, one may identify Rf_* with f_*. Moreover hypercohomology on the two sites $\mathfrak{S}_{et}$ and $\mathfrak{S}_{\underline{et}}$ are isomorphic by (3.5.2)(i). These observations readily prove (iii). □

(4.6.4) *Remark.* We will show in [J-1] (4.2.3) that one can in fact define push-forward integrally for projective maps that factor as the composition $\mathfrak{S}' \xrightarrow{i} \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathfrak{S}$ where $\mathcal{E}$ is vector bundle over $\mathfrak{S}$, i is a closed immersion and π is the obvious projection and that this extends to stacks that are locally of finite type.

(4.6.5) **Proposition.** (i) If X is a quasi-projective scheme, the definitions in (4.4.1) and (4.5.1) using any atlas agree modulo torsion with $\pi_*(z_*(X, .))$.

(ii) If X is a separated scheme of finite type over k, the definitions in (4.4.1) and (4.5.1) with $\mathbb{Q}$-coefficients agree modulo torsion.

Proof. (i) The sheaf $\mathcal{Z}_*(\quad,.)$ is a flabby sheaf on X_{Zar} which implies the object in (4.4.1) is isomorphic to $\pi_*(z_*(X,.))$. (3.8.2) shows that the two definitions in (4.4.1) and (4.5.1) agree modulo torsion. This completes the proof of (i).

To obtain (ii), observe that the definition in (4.4.1) and the definition in (4.5.1) both have localization sequences and therefore Mayer-Vietoris sequences (at least with $\mathbb{Q}$-coefficients). Since the scheme is of finite type, it has a finite covering by affine open sub-schemes. Therefore (i) and an induction on the number of such affine open sub-schemes forming a covering completes the proof. $\square$

(4.6.6)**Definition**. (Higher Chow groups of algebraic spaces). If $\mathfrak{S}$ is an algebraic space, we may view it as a Deligne-Mumford stack in the obvious manner. Therefore, we define $\mathbf{CH}_d(\mathfrak{S},.;R)$ to be as in (4.5.1). We let $CH_d(\mathfrak{S},n;R) = \pi_n(\mathbf{CH}_d(\mathfrak{S},.;R))$.

5. Applications.

We proceed to compare the Chow groups of a Deligne-Mumford stack with the Chow groups of its coarse moduli space which will be as in (1.3.4)(vii). This is made possible by the long-exact localization sequences in Theorem 1. We begin with the following result on quotient stacks.

(5.1.1) **Proposition**. Suppose G is a finite group acting on a separated scheme X of finite type over k so that $\mathfrak{M}_{[X/G]}$ is a coarse moduli space for the stack $[X/G]$. Assume that the map $\pi : [X/G] \to \mathfrak{M}_{[X/G]}$ is *flat.* Let $\mathcal{Z}_*^{[X/G]_{et}}(\quad,.)$ and $\mathcal{Z}_*^{\mathfrak{M}_{[X/G]_{et}}}(\quad,.)$ denote the presheaves on $[X/G]_{et}$ and $\mathfrak{M}_{[X/G]_{et}}$ as in (4.2.4). Then

$$R\pi_*\mathcal{Z}_*^{[X/G]}(\quad,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \simeq \pi_*\mathcal{Z}_*^{[X/G]}(\quad,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \simeq \mathcal{Z}_*^{\mathfrak{M}_{[X/G]}}(\quad,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$$

Proof. The first quasi-isomorphism is clear since π is finite. The fact that π is flat as well as finite shows first that $\pi_* : \mathcal{Z}_*^{[X/G]}(\pi^{-1}(U),.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \to \mathcal{Z}_*^{\mathfrak{M}_{[X/G]}}(U,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is surjective for any $U \to \mathfrak{M}_{[X/G]}$ étale and where $\pi^{-1}(U) = U\underset{\mathfrak{M}}{\times}[X/G]$. (Now π^* is also defined and the composition $\pi_* \circ \pi^* =$ multiplication by the degree of π.) The latter is easily seen to be the quotient stack associated to an algebraic space V with a G-equivariant étale map to X. Let $z \in \mathcal{Z}_*(\pi^{-1}(U),.) = \mathcal{Z}_*([V/G],.)$ be the class of an integral sub-stack Z of $[V/G] \times \Delta_k[n]$. Now the map $Z \to [V/G] \times \Delta_k[n]$ is a closed immersion, which implies that Z is the quotient stack $[\tilde{Z}/G]$ associated to an integral G-stable sub-algebraic space $\tilde{Z}$ of $V \times \Delta_k[n]$. (Both of these assertions follow from [L-MB1] Théoreme (10.2).) By considering geometric points one may now observe that $Z = \pi^*(\pi_*(Z))_{red}$. Therefore $\pi_* : \mathcal{Z}_*^{[X/G]}(\pi^{-1}(U),.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \to \mathcal{Z}_*^{\mathfrak{M}_{[X/G]}}(U,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is an isomorphism. $\square$

(5.1.2) **Proposition**. Suppose $\mathfrak{S}$ is a separated Deligne-Mumford stack which is of finite type over k and where $\mathfrak{M}_{\mathfrak{S}}$ is a coarse moduli space for the stack $\mathfrak{S}$ (as in (1.3.4)(vii)). Then the map $\pi : \mathfrak{S} \to \mathfrak{M}_{\mathfrak{S}}$ induces an isomorphism $\pi_* : CH_*(\mathfrak{S},n;\mathbb{Q}) \simeq CH_*(\mathfrak{M}_{\mathfrak{S}},n;\mathbb{Q})$.

Proof. Let $\mathfrak{M}$ denote the coarse moduli space from now onwards. Let $i : Z \to \mathfrak{M}$ denote the closed immersion of a sub-algebraic space with U its open complement. Recall that the map π is finite.

Since π is finite, by (4.6.1) and (4.6.3)(iii), we obtain a commutative diagram of long exact sequences where the vertical maps are π_*:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & CH_d(\mathfrak{S}_Z,n;\mathbb{Q}) & \longrightarrow & CH_d(\mathfrak{S},n;\mathbb{Q}) & \longrightarrow & CH_d(\mathfrak{S}_U,n;\mathbb{Q}) & \longrightarrow & CH_d(\mathfrak{S}_Z,n-1;\mathbb{Q}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & CH_d(Z,n;\mathbb{Q}) & \longrightarrow & CH_d(\mathfrak{M},n;\mathbb{Q}) & \longrightarrow & CH_d(U,n;\mathbb{Q}) & \longrightarrow & CH_d(Z,n-1;\mathbb{Q}) & \longrightarrow \cdots
\end{array}$$

Here $\mathfrak{S}_Z = Z\underset{\mathfrak{M}}{\times}\mathfrak{S}$ and $\mathfrak{S}_U = U\underset{\mathfrak{M}}{\times}\mathfrak{S}$. The Chow groups in the bottom row are the ones defined in (4.6.6) and the two rows are exact by (4.6.1). Now it suffices to show the vertical maps corresponding to Z and U are isomorphisms. Using ascending induction on the dimension of the coarse-moduli space $\mathfrak{M}_{\mathfrak{S}}$, the devissage technique above along-with (4.6.1), it follows that we only need prove the proposition in the following two cases:

(i) $\mathfrak{M}$ is the spectrum of a field K and

(ii) $\mathfrak{M} = \mathfrak{M}_{[X/G]}$, where X is a separated scheme with the action of a finite group G and the map $X \to \mathfrak{M}_{[X/G]}$ is flat.

The last case follows from (5.1.1) by taking hypercohomology on $\mathfrak{M}$. In the first case, observe that the stack $\mathfrak{S} \to \mathfrak{M}$ is a gerbe; the same devissage technique shows that after a finite separable extension of the field K, $\mathfrak{M}$ is as in (ii). Once again (5.1.1) applies to complete the proof. □

(5.1.3) **Corollary**. If $\mathfrak{S}$ is a separated Deligne-Mumford stack of finite type over k with a coarse moduli space which is a quasi-projective scheme, one obtains the isomorphisms:

$$CH_q^{naive}(\mathfrak{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \simeq \pi_0(\mathcal{Z}_q(\mathfrak{S},.))\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \simeq CH_q(\mathfrak{S},0;\mathbb{Q}), \text{ for each } q.$$

Proof. The first isomorphism follows from (4.3.4). In order to obtain the second, recall from [Gi] Theorem (6.8) the isomorphism $CH_q^{naive}(\mathfrak{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \simeq CH_q^{naive}(\mathfrak{M}_{\mathfrak{S}})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ which is the Chow group $\pi_0(z_*(\mathfrak{M}_{\mathfrak{S}}))\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. By (4.4.2) the latter is isomorphic to the group $CH_q(\mathfrak{M}_{\mathfrak{S}},0;\mathbb{Q})$ which is the group defined in (4.4.1). By (5.1.2), $CH_d(\mathfrak{M}_{\mathfrak{S}},0;\mathbb{Q}) \cong CH_d(\mathfrak{S},0,\mathbb{Q})$. □

(5.1.4) **Examples**. (i)Let G denote a finite group acting on a quasi-projective scheme so that a coarse moduli space $\mathfrak{M}_{[X/G]}$ exists as a quasi-projective scheme. Then $CH_*([X/G],.;\mathbb{Q}) \cong CH_*(\mathfrak{M}_{[X/G]},.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. In particular (taking $X = Spec \quad k$), $CH_*(BG,.;\mathbb{Q}) \cong CH_*(Spec \quad k,.)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$.

(ii) Let G denote a linear algebraic group acting on a separated scheme X *locally properly* so that the stack $[X/G]$ is separated Deligne-Mumford and a coarse moduli space $\mathfrak{M}_{[X/G]}$ exists as a quasi-projective scheme. (The stack $[X/G]$ is Deligne-Mumford if the stabilizers of geometric points are finite and reduced.) Then

$$CH_*([X/G],0;\mathbb{Q}) \cong CH_*^G(X)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$$

where the last term is the equivariant intersection theory defined in [EG]. (This follows from (5.1.3) and [EG] Proposition 14 where it is shown that the right-hand-side is isomorphic to the naive Chow groups of the stack $[X/G]$ tensored with $\mathbb{Q}$. By (3.5) of [J-1], the same result holds in much more generality as proved in Corollary 3 (ii).)

We will conclude by considering briefly proofs of the main results stated in the introduction.

(5.1.5) **Proof of Theorem 1** Statements in (i) follow from (4.5.2) Theorem along with [J-1], (3.2) Corollary. The statements in (ii) follow from (4.6.3) while those in (iii) will follow from the definition in (4.5.1) since the cycle complex is homotopy invariant. (See Part II, (4.1) for a more general result.) (iv) follows from (4.6.1) and (v) from (3.8.4). (Observe that the classifying simplicial groupoid of a Deligne-Mumford stack of finite type over k satisfies the hypothesis on finite cohomological dimension as in (3.8.4).) □

Proof of Theorem 2. The statements in (i) and (ii) follow from the definitions in (4.4.1), (4.5.1) as well as (4.3.4), (3.8.4) and (4.6.5). The remaining statements follow from (5.1.2) and (5.1.3). □

Proof of Corollary 3. (i) and (iv) are immediate consequences of the localization sequence in Theorem 1. It is known that the stacks in (iv) are locally of finite type over k - see [L-MB1] (4.14.2.1). It is shown in [B-M] that the stacks in (v) are in fact separated and Deligne-Mumford. Therefore (5.1.2) applies to prove (v). (iii) follows clearly from Theorem 2. It remains to establish the isomorphism with the Totaro-Edidin-Graham equivariant intersection theory discussed in (ii). We will presently provide a proof of this, modulo [J-1] (3.5). Let G denote a linear algebraic group acting on the scheme X so that the action is G-linearized. One may now choose a smooth quasi-projective scheme $\tilde{X}$ that contains X as a closed G-stable sub-scheme and with a linearized G-action. We consider the commutative diagram (with exact rows) for each integer $n \geq 0$. (In fact, the top row exists as a long exact sequence for all integers n and shows, in view of [J-1] (3.5), that $CH^*([X/G],n;\mathbb{Q}) = 0$ for all $n < 0$.)

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta_{n+1}} & CH_*([X/G],n;\mathbb{Q}) & \xrightarrow{\beta_n} & CH_*([\tilde{X}/G],n;\mathbb{Q}) & \xrightarrow{\alpha_n} & CH_*([(\tilde{X}-X)/G],n;\mathbb{Q}) & \longrightarrow & \cdots \\
 & & & & \downarrow\cong & & \downarrow\cong & & \\
\cdots & \xrightarrow{\delta'_{n+1}} & CH_*^G(X,n)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} & \xrightarrow{\beta'_n} & CH_*^G(\tilde{X},n)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} & \xrightarrow{\alpha'_n} & CH_*^G(\tilde{X}-X,n)\underset{\mathbb{Z}}{\otimes}\mathbb{Q} & \longrightarrow & \cdots
\end{array}$$

The top row is exact by the localization sequence in Corollary 3 (i) and the bottom row is exact by the corresponding localization sequence in [EG]. The last two vertical maps are isomorphisms and the last square commutes by [J-1] (3.5). By breaking up the long-exact sequences forming the two rows, one obtains the short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & coker(\alpha_{n+1}) \cong Im(\delta_{n+1}) & \longrightarrow & CH_*([X/G], n; \mathbb{Q}) & \longrightarrow & Im(\beta_n) \cong ker(\alpha_n) & \longrightarrow & 0 \\ & & \cong \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & coker(\alpha'_{n+1}) \cong Im(\delta'_{n+1}) & \longrightarrow & CH^G_*(X, n) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} & \longrightarrow & Im(\beta'_n) \cong ker(\alpha'_n) & \longrightarrow & 0 \end{array}$$

Using the observation all the terms above are $\mathbb{Q}$-vector spaces, one sees that it is possible to define an (abstract) isomorphism forming the middle vertical map in the last diagram. □

Remarks. (i) In case $\mathfrak{S}$ is also smooth we define an intersection product on $CH_*(\mathfrak{S}, 0)$ in [J-1](5.3.10). It is shown in [Gi] (using K-theoretic techniques) that, in this case, there exist a ring structure on $CH^{naive}_*(\mathfrak{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. We show in [J-1] (4.3.10) that for smooth separated Deligne-Mumford stacks one obtains an isomorphism between $CH_*(\mathfrak{S}, 0; \mathbb{Q})$ and $CH^{naive}_*(\mathfrak{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ preserving the above intersection products.

(ii) Recall Kresch (see [Kr]) defines Chow groups for stacks with affine stabilizer groups. When they are quotient stacks associated to linear algebraic groups acting linearly on quasi-projective schemes, his groups coincide with the equivariant Chow groups as in [EG] and hence (by Corollary 3(ii)) also with our Chow groups modulo torsion. It is not clear, however, whether Kresch's groups, whenever they are defined, agree with ours. (It may be worth pointing out that Kresch's groups are better behaved integrally than ours: for example, for schemes, his integral Chow groups are isomorphic to the usual (integral) Chow groups for schemes.)

(5.2) **Proof of Theorem 4**. Recall the higher Chow groups for all schemes locally of finite type over k are defined in (4.4.1). Contravariance for all flat maps follows as in (4.6.3)(i). The same proof as in (4.6.3)(ii) will prove the covariance for all proper maps between schemes of finite type: this may be seen by invoking the remarks in (3.6.7)'. Observe these hold integrally. The first statement of (ii) is clear from the definition in (4.4.1). To prove the second statement of (ii), observe that both sides have Mayer-Vietoris sequences. Therefore one may reduce to the case of quasi-projective varieties; in this case both sides are isomorphic to $\pi_*(z_*(X, .)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. (See (4.6.5)(i).) (iii) follows from (4.4.3) while (iv) is established in (4.4.4). Finally (v) follows from (4.4.2). □

(5.3) **Example: The higher Chow groups (with $\mathbb{Z}/l^\nu$-coefficients) of BE, where E is a complex elliptic curve**.

One of the advantages of the theory presented here is that it applies to algebraic stacks of the form BE, where E is an elliptic curve, or (more generally) an abelian variety. Though computing the Chow groups with rational coefficients in general, may be difficult, one has the following result with finite coefficients.

(5.3.1) **Proposition**. Let E denote a complex elliptic curve. Then there exists an isomorphism

$$CH_{-i}(BE, n; \mathbb{Z}/l^\nu) \cong H^{2i-n}_{et}(BE; \mu_{l^\nu}(i)) \cong H^{2i-n}_{sing}(B(S^1 \times S^1); \mathbb{Z}/l^\nu(i)).$$

(The last term is singular cohomology with $\mathbb{Z}/l^\nu$-coefficients.) Moreover this isomorphism preserves the ring structures, where the one on the left is defined as in [J-1] section 5.

Proof. The first isomorphism follows from (4.5.2)(iii). Moreover, the assertion that this isomorphism preserves the ring structures also follows from the same. Since $E(\mathbb{C})$ is isomorphic as a topological group to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong S^1 \times S^1$, the last isomorphism follows by standard comparison between étale cohomology and singular cohomology with finite coefficients. □

6. Appendix: Homotopy inverse limits.

In this section we provide a self-contained discussion of homotopy inverse limits. We show that the homotopy inverse limit represents the derived functor of the inverse limit functor from the category of cosimplicial objects in the category of unbounded complexes of sheaves of modules on any ringed site. It enables us to define hypercohomology with respect to any unbounded complex of sheaves on a site with reasonable properties. Moreover, for the purposes of Riemann-Roch theorems on stacks, we need to consider the homotopy inverse limits for presheaves of spectra. To handle all of these in a uniform manner (and without using sophisticated techniques from algebraic topology like closed model categories), we introduce the following terminology.

Let $\mathbf{S}$ denote a simplicial category, i.e. a category with the following properties:

(6.0.1) there exists a pairing $\otimes$:(pointed simplicial sets) $\times \mathbf{S} \rightarrow \mathbf{S}$ and a functor

(6.0.2) $Map : \mathbf{S}^{op} \times \mathbf{S} \rightarrow$(pointed simplicial sets) so that given a pointed simplicial set K and two objects X, $Y \,\epsilon\, \mathbf{S}$, one obtains the isomorphism:

$Hom_{\mathbf{S}}(K, Map(X,Y)) \cong Hom_{\mathbf{S}}(K \otimes X, Y)$.

($Hom_{\mathbf{S}}$ denotes the external Hom in the category $\mathbf{S}$.) Assume further that there exists a functor

(6.0.3) F :(pointed simplicial sets) $\rightarrow \mathbf{S}$.

(6.0.4) We will also assume that the category $\mathbf{S}$ is closed under all small limits and has a zero object. This object will be denoted $*$. Such simplicial categories will be called *complete pointed simplicial categories*. The internal Hom in the category $\mathbf{S}$ will be denoted $\mathcal{H}om$. We will further require that $\mathcal{H}om(F(\Delta[0]_+), Y) \cong Y$ for any object $Y \,\epsilon\, \mathbf{S}$.

Let $\mathbf{S}^{\Delta} = Cosimp(\mathbf{S})$ = the category of all cosimplicial objects in $\mathbf{S}$. We proceed to define a functor

$\underset{\Delta}{\text{holim}} : Cosimp(\mathbf{S}) \rightarrow \mathbf{S}$

This will be defined as an *end* following [B-K] chapter XI.

Let $\{C^m|m\} \,\epsilon\, \mathbf{S}^{\Delta}$. Now we define a functor:

(6.1.1) $\mathcal{H}om_{C^\bullet} : \Delta^{op} \times \Delta \rightarrow \mathbf{S}$ by

$\mathcal{H}om_{C^\bullet}(n,m) = \mathcal{H}om(F(\Delta[n]_+), C^m)$

We define $\underset{\Delta}{\text{holim}}\{C^m|m\}$ to be the *end* of this functor in the sense of [Mac] p. 218. Recall this means

(6.1.2) $\underset{\Delta}{\text{holim}}\{C^m|m\} = Equalizer(\prod_n \mathcal{H}om(F(\Delta[n]_+), C^n) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{\gamma:n \rightarrow m} \mathcal{H}om(F(\Delta[n]_+), C^m)$

where a (b) is the map sending $\mathcal{H}om(F(\Delta[n]_+), C^n)$ to $\mathcal{H}om(F(\Delta[n]_+), C^m)$ by the map $C(\gamma) : C^n \rightarrow C^m$ (to $\mathcal{H}om(F(\Delta[m]_+), C^n)$ by the map $\mathcal{H}om(\gamma_+ \otimes id, id)$, respectively).

Now let $Const : \mathbf{S} \rightarrow Cosimp(\mathbf{S})$ denote the obvious constant functor sending an object $M \,\epsilon\, \mathbf{S}$ to the obvious constant cosimplicial object associated to M. Let $L(Const) : \mathbf{S} \rightarrow Cosimp(\mathbf{S})$ denote the functor sending M to the cosimplicial object $L(Const(M))^n = \Delta[n]_+ \otimes M$ (with the obvious structure maps). So defined, one can readily see that

(6.1.3) $\underset{\Delta}{\text{holim}} : Cosimp(\mathbf{S}) \rightarrow \mathbf{S}$ is right adjoint to the functor $L(Const)$.

We will also briefly consider the dual notion of *homotopy colimits*. Assume now that $\mathbf{S}$ is also closed under all small colimits. Let $\mathbf{S}^{\Delta^{op}} = Simp(\mathbf{S})$ = the category of all simplicial objects in $\mathbf{S}$. Let $\{S_n|n\} \,\epsilon\, Simp(\mathbf{S})$. Now we define a functor $S. \otimes - : \Delta \times \Delta \rightarrow \mathbf{S}$ by $(S. \otimes -)(m,n) = S_m \otimes F(\Delta[n]_+)$. We define $\underset{\Delta}{\text{hocolim}}\{S_m|m\}$ to be the *co-end* of this functor. i.e.

(6.1.4) $\underset{\Delta}{\text{hocolim}}\{S_m|m\} = Coequalizer(\underset{\gamma:n \rightarrow m}{\sqcup} S_m \otimes (F(\Delta[n]_+)) \underset{b}{\overset{a}{\rightrightarrows}} \underset{n}{\sqcup} S_n \otimes F(\Delta[n]_+))$

where the map a (b) is the map sending $S_m \otimes F(\Delta[n]_+)$ to $S_n \otimes F(\Delta[n]_+)$ by the map $S(\gamma) \otimes id$ (to $S_n \otimes F(\Delta[m]_+)$ by the map $id \otimes F(\gamma_+)$, respectively).

Let $\{S_n|n\} \in \mathbf{S}^{\Delta^{op}}$ denote an object as above. Now one may show readily (using the definitions) that if $X \in \mathbf{S}$,

(6.1.5) $\underset{\Delta}{\text{holim}}\{\mathcal{H}om(S_n, X)|n\} \cong \mathcal{H}om(\underset{\Delta}{\text{hocolim}}\{S_n|n\}, X)$ and

(6.1.5') $\underset{\Delta}{\text{holim}}\{Map(S_n, X)|n\} \cong Map(\underset{\Delta}{\text{hocolim}}\{S_n|n\}, X)$

(6.1.6) We also make the following observation about filtered colimits: in Grothendieck categories they are exact and hence in the setting of (6.2.4) they preserve quasi-isomorphisms. In the setting of (6.2.1) through (6.2.3) also it is shown in [B-K] and [T] that they preserve quasi-isomorphisms. (Observe that the colimit of a diagram in the setting of (6.2.1) through (6.2.3) one needs to take colimit first and then apply a functor to convert to a quasi-isomorphic (presheaf of) fibrant objects.)

Examples.

(6.2.1) Let $\mathbf{S}$ denote the category of all pointed *fibrant* simplicial sets. Now the functor F is a functor that takes a pointed simplicial set to a weakly-equivalent pointed fibrant simplicial set. The pairing $\otimes$ in (6.0.1) is defined as $P \otimes M = F(P \wedge M)$ where $P \wedge M = P \times M/(* \times M \cup P \times *)$. One defines $Map(X, Y)$ by $Map(X, Y)_n = Hom_{\mathbf{S}}(\Delta[n]_+ \otimes X, Y)$. (One defines the functor $Map(X, Y)$ in the same manner for the examples in (6.2.2) through (6.2.4).) In this case, the homotopy inverse limit is the one defined in [B-K] XI, (3.1), (3.3). A map between pointed fibrant simplicial sets is a *quasi-isomorphism or a weak-equivalence* if it induces an isomorphism on all π_n, $n \geq 0$. The class of maps that are quasi-isomorphisms admit a calculus of left and right fractions in the homotopy category of pointed fibrant simplicial sets.

(6.2.2) Let $\mathbf{S}$ denote the category of all *fibrant* simplicial spectra. Now the functor F sends a pointed simplicial set to its suspension spectrum and then replaces it by a weakly-equivalent fibrant spectrum. If P is a pointed simplicial set and M is a fibrant spectrum, $P \otimes M$ is the fibrant spectrum obtained by first taking the smash product and then by converting it to a weakly-equivalent fibrant spectrum. The homotopy inverse limit now is the one considered in [T] (5.6). A map between fibrant spectra is a quasi-isomorphism if it induces an isomorphism on all π_n, $n \in \mathbb{Z}$. Once again the class of maps that are quasi-isomorphisms admit a calculus of left and right fractions in the homotopy category.

(6.2.3) Let $\mathcal{S}$ denote a site and let $\mathbf{S}$ denote the category of presheaves of fibrant pointed simplicial sets (or fibrant simplicial spectra) on the site $\mathcal{S}$. The functor F first sends a pointed simplicial set to a pointed fibrant simplicial set as in (6.2.1) (a fibrant spectrum as in (6.2.2), respectively) and then to its associated constant sheaf. The functor $\otimes$ is defined similar to the one in (6.2.2). We assume that the site $\mathcal{S}$ has a conservative family of points. A map between presheaves of fibrant spectra is *a quasi-isomorphism or weak-equivalence if it induces an isomorphism on all sheaves of homotopy groups.* The corresponding homotopy inverse limit is defined as follows: let P denote a presheaf in $\mathbf{S}^{\Delta}$. If $U \in \mathcal{S}$,

$$\Gamma(U, \underset{\Delta}{\text{holim}} P) = \underset{\Delta}{\text{holim}}\{\Gamma(U, P)\}$$

(6.2.4) Let $(\mathcal{S}, \underline{R})$ denote a ringed site where $\underline{R}$ is a sheaf of commutative rings with 1. Now $Mod(\mathcal{S}; \underline{R}))$ will denote the category of presheaves of modules over $(\mathcal{S}, \underline{R})$. Let $\mathbf{S} = C(Mod(\mathcal{S}; \underline{R}))$ = the category of all (unbounded) co-chain complexes in $Mod(\mathcal{S}; \underline{R}))$. (Observe that a co-chain complex simply denotes a complex where the differentials are of degree +1.) If P is a pointed simplicial set, we let $F(P)$ denote the co-chain complex in $Mod(\mathcal{S}; \underline{R})$ obtained the following way: first one forms the simplicial object $P \otimes \underline{R}$ in $Mod(\mathcal{S}, \underline{R})$ defined by $(P \otimes \underline{R})_n = \underset{P_n}{\oplus} \underline{R}$ with the summand $\underline{R}$ indexed by the base point $*$ identified to 0. The structure maps of this simplicial object are induced in the obvious manner by those of P. Now one applies the normalizing functor N that sends a simplicial object $S. \in Mod(\mathcal{S}, \underline{R})$ to $N(S.)$: this is defined by $(N(S.))_n = \underset{0<i\leq n}{\cap} ker(d_i : S_n \to S_{n-1})$ and with the boundary map $d : N(S.)_n \to N(S.)_{n-1}$ induced by d_0. Now $F(P) = N(P \otimes \underline{R})$ viewed as a co-chain complex trivial in positive degrees. Given a pointed simplicial set P and $K \in \mathbf{S}$, $P \otimes K$ = the homotopy colimit of the simplicial object $n \mapsto \underset{P_n}{\oplus} K$ in $\mathbf{S}$. Observe that, as $\mathbf{S}$ is closed under all small colimits, this homotopy colimit exists.

We assume again that the site $\mathcal{S}$ has a conservative family of points. A map in $C(Mod(\mathcal{S}, \underline{R}))$ is a *quasi-isomorphism if it induces an isomorphism on all cohomology sheaves.*

(6.2.5) In each of the above situations one has the notion of *distinguished triangles*. In the situation of (6.2.1) or (6.2.2) what corresponds to distinguished triangles are fibration sequences: it is shown in [B-K] XI, (7.2) that the functor $\underset{\Delta}{\text{holim}}$ preserves these. In the situation of (6.2.3) what corresponds to distinguished triangles are diagrams of presheaves $\Omega P'' \to P' \to P \to P''$ which are fibration sequences at each stalk. It is shown in [T] (5.12) that $\underset{\Delta}{\text{holim}}$ preserves these. One may define distinguished triangles in the following uniform manner in all of the above situations. First one has the notion of a *Path object* associated to any object Y. One defines this as $\mathcal{H}om(F(\Delta[1]_+), Y)$ along with the obvious maps $d^i = \mathcal{H}om(d_i, Y) : Y \cong \mathcal{H}om(F(\Delta[0]_+), Y) \to \mathcal{H}om(F(\Delta[1]_+), Y)$, $i = 0, 1$. We will denote this object as PY. Observe that since Y is a fibrant object in (6.2.1) or (6.2.2), the maps d^i are fibrations; in the case of (6.2.3), since Y is stalk-wise-fibrant, the induced maps d^i are fibrations stalk-wise. Finally in the case of (6.2.4) the maps d^i are surjective homomorphisms stalk-wise. If $f : X \to Y$ is a map of objects, one defines $P(f) = X \underset{f,Y,d^0}{\times} PY$ and $\Omega(f)$ = the kernel of the map $P(f) \to Y$ induced by the map d^1. We call $\Omega(f)$ the *canonical homotopy fiber* of f. A diagram $Z \to X \xrightarrow{f} Y$ is called a *distinguished triangle* if there exists a map $Z \to \Omega(f)$ which is a quasi-isomorphism. Observe that if the composition $Z \to X \xrightarrow{f} Y$ is strictly trivial, there exists an induced map $Z \to \Omega(f)$ which is a quasi-isomorphism if $Z \to X \to Y$ is a distinguished triangle. We define $\Omega Y = \Omega(Y \to *)$.

(6.2.6) One may also define the *derived categories* associated to the categories of presheaves in (6.2.1) through (6.2.4) by inverting all maps that are quasi-isomorphisms. This may be verified to be a triangulated category with the triangles given by the distinguished triangles and $Y[-1] = \Omega Y$ for any object Y.

(6.3.1) Let $\mathbf{S}$ denote any one of the complete pointed simplicial categories in (6.2.1) through (6.2.4). Let $C^\bullet$, $D^\bullet \,\epsilon\, Cosimp(\mathbf{S})$. A map of cosimplicial objects $C^\bullet = \{C^m|m\} \to D^\bullet = \{D^m|m\}$ in any one of the situations (6.2.1) or (6.2.2) is called a *quasi-isomorphism* if it is a weak-equivalence for every $m \geq 0$. In the situation of (6.2.3) such a map will be a quasi-isomorphism if it induces a weak-equivalence at every stalk and for every m. Finally in the situation of (6.2.4), such a map will be a quasi-isomorphism if it is one at each stalk and for every m.

(6.3.2) Observe that, in any one of the above contexts, a map $f^\bullet : M^\bullet \to N^\bullet$ in $(Cosimp(\mathbf{S}))$ is a quasi-isomorphism if each f^n is a quasi-isomorphism. We proceed to show that the functor $\underset{\Delta}{\text{holim}}$ preserves quasi-isomorphisms so that it will induce a functor at the level of the associated derived categories. (These derived categories are defined by inverting maps that are quasi-isomorphisms.) Since the natural map $Const(M) \to L(Const(M))$ is a quasi-isomorphism (in any of the above contexts), it will follow that the functor induced by $L(Const)$ at the level of the derived categories will be the functor $Const$. Since $R\underset{\Delta}{\lim}$ is right adjoint to the functor $Const$ at the level of derived categories, it will follow that $\underset{\Delta}{\text{holim}}$ induces $R\underset{\Delta}{\lim}$ at the level of derived categories.

(6.3.3) If S is a simplicial category, we will define a bi-functor $\underline{Hom} : Cosimpl(\mathbf{S}) \times Cosimpl(\mathbf{S}) \to \mathbf{S}$ by

$$\underline{Hom}(M^\bullet, N^\bullet) = Equalizer(\underset{n}{\Pi}\mathcal{H}om(M^n, N^n) \underset{b}{\overset{a}{\rightrightarrows}} \underset{\gamma:n\to m}{\Pi} \mathcal{H}om(M^n, N^m))$$

where a (b) is the map sending $\mathcal{H}om(M^n, N^n)$ to $\mathcal{H}om(M^n, N^m)$ by the map $N(\gamma) : N^n \to N^m$ (to $\mathcal{H}om(M^m, N^n)$ by the map $\mathcal{H}om(\gamma, id)$, respectively). Now observe that in the situation of (6.2.4) one obtains the isomorphism:

$\underset{\Delta}{\text{holim}}\{M^m|m\} \cong \underline{Hom}(L(Const(\underline{R})), M^\bullet)$, $M^\bullet \,\epsilon\, Cosimpl(C(Mod(\mathcal{S}, \underline{R})))$.

(6.3.3') *Remark.* In the setting of (6.2.4) observe the existence of a pairing $\otimes : \mathbf{S} \times Cosimpl(\mathbf{S}) \to Cosimpl(\mathbf{S})$ that is left-adjoint to the bi-functor $\underline{Hom}$ defined above. This is defined by $(K \otimes M)^n = K \underset{\underline{R}}{\otimes} M^n$ where $K \,\epsilon\, \mathbf{S}$ and $M \,\epsilon\, Cosimpl(\mathbf{S})$.

(6.3.4)**Proposition**. The functor $\underset{\Delta}{\text{holim}}$ preserves quasi-isomorphisms in the situations of (6.2.1) through (6.2.4).

Proof. This is well-known in the situations in (6.2.1) through (6.2.3): see [B-K] XI, (7.2) and [T] (5.12). Therefore, we will provide a proof only for the situation in (6.2.4).

For each integer $n \geq 0$, let $Cosimp_{\leq n}(C(Mod(\mathcal{S}, \underline{R}))$ denote the category of all truncated cosimplicial objects in $C(Mod(\mathcal{S}, \underline{R}))$ truncated up to level n. Let $tr_n : Cosimp(C(Mod(\mathcal{S}, \underline{R})) \to Cosimp_{\leq n}(C(Mod(\mathcal{S}, \underline{R}))$ denote the n-truncation functor that truncates a cosimplicial object at level n. This functor has a left adjoint which will be denoted $cosk_n$. The composition $cosk_n \circ tr_n$ will be denoted $Cosk_n$. We will now show using ascending induction on n that the functor $\underline{Hom}(Cosk_n(L(Const(\underline{R}))), -) : Cosimp_{\leq n}(C(Mod(\mathcal{S}, \underline{R})) \rightarrow C(Mod(\mathcal{S}, \underline{R}))$ preserves quasi-isomorphisms in the second argument. (The above functor is defined as in (6.3.3).)

(6.3.4.0) *n=0.* Observe that $tr_0(L(Const(\underline{R}))) \cong \underline{R}$ and therefore the functor $\underline{Hom}(tr_0(L(Const(\underline{R}))), -) : Cosimpl_{\leq 0}(C(Mod(\mathcal{S}, \underline{R})) \cong C(Mod(\mathcal{S}, \underline{R})) \rightarrow C(Mod(\mathcal{S}, \underline{R}))$ preserves quasi-isomorphisms in the second argument. Now the adjunction between $cosk_0$ and tr_0 proves $\underline{Hom}(Cosk_0(L(Const(\underline{R}))), -) : Cosimp(C(Mod(\mathcal{S}, \underline{R})) \rightarrow C(Mod(\mathcal{S}, \underline{R}))$ preserves quasi-isomorphisms.

(6.3.4.1) Let $n > 0$. Now one may check readily that the map $Cosk_{n-1}(L(Const(\underline{R}))) \rightarrow Cosk_n(L(Const(\underline{R})))$ is injective. (Observe that both the above cosimplicial objects are isomorphic in degrees 0 through $n-1$. In degree n, $Cosk_{n-1}(L(Const(\underline{R})))$ is given by $\overset{o}{\Delta}[n]_+ \otimes \underline{R}$. This proves the above assertion.) Moreover it follows that the cokernel of the map $Cosk_{n-1}(L(Const(\underline{R}))) \rightarrow Cosk_n(L(Const(\underline{R})))$ is the cosimplicial object $cosk_n(\Delta[n]_+/\overset{o}{\Delta}[n]_+[n]) \otimes \underline{R}$. (Here $\Delta[n]_+/\overset{o}{\Delta}[n]_+[n]$ is the n-truncated cosimplicial object defined by $\Delta[n]_+/\overset{o}{\Delta}[n]_+$ in degree n and $*$ everywhere else.) One may readily see from the above description that $\underline{Hom}(Cokernel(Cosk_{n-1}(L(Const(\underline{R}))) \rightarrow Cosk_n(L(Const(\underline{R})))), -) : Cosimp(C(Mod(\mathcal{S}, \underline{R})) \rightarrow C(Mod(\mathcal{S}, \underline{R}))$ preserves quasi-isomorphisms in the second argument. This follows because one may identify $H^i(\underline{Hom}(\Delta[n]_+/\overset{o}{\Delta}[n]_+[n] \otimes R, M)) = \underset{0 \leq i \leq n}{\cap} ker(s^i : \mathcal{H}^{-n}(M^n) \rightarrow \mathcal{H}^{-n}(M^{n-1}))$, and $= 0$ if $i \neq 0$ for any $M \,\epsilon\, Cosimp(C(Mod(\mathcal{S}, \underline{R}))$. (See [B-K] Chapter X, Proposition (6.3)(ii) for example.) Therefore, in order to prove that

$$\underline{Hom}(Cosk_n(L(Const(\underline{R}))), -) : Cosimp(C(Mod(\mathcal{S}, \underline{R})) \rightarrow C(Mod(\mathcal{S}, \underline{R})),$$

preserves quasi-isomorphisms, it suffices to observe the following:

Let $K^\bullet \,\epsilon\, Cosimp(C(Mod(\mathcal{S}, \underline{R}))$. Now (see (6.3.5) below) one obtains a degree-wise split short exact sequence:

(6.3.4.2) $0 \rightarrow \underline{Hom}(Cokernel(Cosk_{n-1}(L(Const(\underline{R}))) \rightarrow Cosk_n(L(Const(\underline{R})))), K^\bullet)$

$\rightarrow \underline{Hom}(Cosk_n(L(Const(\underline{R}))), K^\bullet) \rightarrow \underline{Hom}(Cosk_{n-1}(L(Const(\underline{R}))), K^\bullet) \rightarrow 0$

Since the functor $\underline{Hom}$ is left-exact it suffices to show that the last map is split surjective degree-wise. This follows from (6.3.5) below.

Clearly $L(Const(\underline{R})) = \lim_{n \rightarrow \infty} Cosk_n(L(Const(\underline{R})))$. Now $\underline{Hom}(L(Const(\underline{R})), K^\bullet)$ $= \underline{Hom}(\lim_{n \rightarrow \infty} Cosk_n(L(Const(\underline{R}))), K^\bullet) \cong \lim_{\infty \leftarrow n} \underline{Hom}(Cosk_n(L(Const(\underline{R}))), K^\bullet)$. Moreover in view of the surjectivity of the last map in (6.3.4.2), it follows that the higher derived functors of the above inverse limit are trivial, thereby identifying it with its right derived functor. It follows that $\underline{Hom}(L(Const(\underline{R})), -)$ preserves quasi-isomorphisms in the second argument. □

(6.3.5) **Lemma** Assume the situation of (6.3.4.2). Now the last map in (6.3.4.2) is degree-wise split surjective.

Proof. Use the adjunction between $cosk_n$ and tr_n to identify:

$\underline{Hom}(Cosk_n(L(Const(\underline{R}))), K^\bullet) \cong \underline{Hom}(tr_n(L(Const(\underline{R}))), tr_n K^\bullet)$ and

$\underline{Hom}(Cosk_{n-1}(L(Const(\underline{R}))), K^\bullet) \cong \underline{Hom}(Cosk_n(Cosk_{n-1}(L(Const(\underline{R})))), K^\bullet)$

$\cong \underline{Hom}(tr_n(Cosk_{n-1}L(Const(\underline{R}))), tr_n K^\bullet)$

(One may first observe similar isomorphisms when the internal hom, $\underline{Hom}$, is replaced by the external Hom Hom in the category $C(Mod(\mathcal{S}, \underline{R}))$. This follows readily from the adjunction between the functors $cosk_n$ and tr_n. Now apply $Hom(M, \quad)$, where M is a fixed object in $C(Mod(\mathcal{S}, \underline{R}))$ to all of the above terms, use the adjunction between $\underline{Hom}$ and $\underset{\underline{R}}{\otimes}$ along with the observation that $Cosk_n(M \underset{\underline{R}}{\otimes} P^\bullet) \cong M \underset{\underline{R}}{\otimes} Cosk_n(P^\bullet)$ for any cosimplicial object $P^\bullet \,\epsilon\, C(Mod(\mathcal{S}, \underline{R})$. This will establish the above isomorphisms.)

Let $P = tr_n(Cosk_{n-1}(L(Const(\underline{R}))))$ and $Q = tr_n(L(Const(\underline{R})))$. Observe that P and Q are objects in $Cosimp_{\leq n}(C(Mod(\mathfrak{S}; \underline{R})))$. Let $P = \{P^{k,l} | k, l\}$ and $Q = \{Q^{k,l} | k, l\}$ where the index k is cosimplicial index while the index l denotes the index of the complexes in $Mod(\mathfrak{S}; \underline{R})$. Now it clearly suffices to show that the map obvious map $i^{\cdot,l} : P^{\cdot,l} \rightarrow Q^{\cdot,l}$ of objects in $Cosimp_{\leq n}(Simpl(Mod(\mathcal{S}; \underline{R}))$ is split injective for each fixed l.

Now observe that the map $i^{k,l}$ is an isomorphism for all $k \leq n-1$ and all l and also for $k = n$ and all $l \leq n-1$. For each $l \geq n$ we define a splitting $\epsilon^l : Q^{\cdot,l} \rightarrow P^{\cdot,l}$ so that $\epsilon^l \circ i^{\cdot,l} = id_{P^{\cdot,l}}$ as follows. One may readily verify that, it suffices to define $\epsilon^l : Q^{n,l} \rightarrow P^{n,l}$ so that

(6.3.5.1) (i) $\epsilon^l \circ i^{n,l} = id_{P^{n,l}}$ and (ii) the diagram

$$\begin{array}{ccc} P^{n,l} & \xleftarrow{\epsilon^l} & Q^{n,l} \\ {\scriptstyle s^i}\downarrow & & \downarrow{\scriptstyle s^i} \\ P^{n-1,l} & \xleftarrow{id} & Q^{n-1,l} \end{array}$$

commutes for all $0 \leq i \leq n-1$. (One may readily verify that the diagram

$$\begin{array}{ccc} P^{n,l} & \xleftarrow{\epsilon^l} & Q^{n,l} \\ {\scriptstyle d^i}\uparrow & & \uparrow{\scriptstyle s^i} \\ P^{n-1,l} & \xleftarrow{id} & Q^{n-1,l} \end{array}$$

commutes for all $0 \leq i \leq n$ and for any splitting ϵ^l in the top row.)

Moreover, observe that $P = tr_n(Cosk_{n-1}(L(Const(\mathbb{Z}))))\underset{\mathbb{Z}}{\otimes}\underline{R}$ and $Q = tr_n(L(Const(\mathbb{Z})))\underset{\mathbb{Z}}{\otimes}\underline{R}$ where by $\mathbb{Z}$ we denote the constant sheaf on the site $\mathcal{S}$ whose stalks are all isomorphic to the ring of integers. One may also observe that the map $i^{\cdot,l}$ is itself induced from a corresponding map $tr_n(Cosk_{n-1}(L(Const(\mathbb{Z})))) \rightarrow tr_n(L(Const(\mathbb{Z})))$. *In other words, one may assume for the rest of the proof that the site $\mathcal{S}$ is trivial and that $\underline{R}$ is a constant sheaf.*

Let $M^{n-1}(P^\bullet) = \{(p^0, ..., p^{n-1}) \ \epsilon \ P^{n-1,\cdot} \times ... \times P^{n-1,\cdot} | s^i(p^j) = s^{j-1}(p^i),\ 0 \leq i < j \leq n\}$ and

$M^{n-1}(Q^\bullet) = \{(q^0, ..., q^{n-1}) \ \epsilon \ Q^{n-1,\cdot} \times ... \times Q^{n-1,\cdot} | s^i(q^j) = s^{j-1}(q^i),\ 0 \leq i < j \leq n\}$.

These are the so-called *matching spaces* as in [B-K] p.274. Clearly the map $x \rightarrow (s^0(x), ..., s^{n-1}(x))$, $x \ \epsilon \ P^{n,\cdot}$ defines a map $s_P : P^{n,\cdot} \rightarrow M^{n-1}(P^\bullet)$. One defines a map $s_Q : Q^{n,\cdot} \rightarrow M^{n-1}(Q^\bullet)$ similarly. Now it is a basic result of [B-K] pp.275-276 that the maps s_P and s_Q are surjective since $Q^\bullet$ and $P^\bullet$ are $n-$truncated cosimplicial *groups*. Now consider the commutative diagram:

$$\begin{array}{ccc} ker(s_P)^{n,l} & \longrightarrow & ker(s_Q)^{n,l} \\ {\scriptstyle s_P^{n,l}}\downarrow & & \downarrow{\scriptstyle s_Q^{n,l}} \\ P^{n,l} & \longrightarrow & Q^{n,l} \\ \downarrow{\scriptstyle s_P^{n,l}} & & \downarrow{\scriptstyle s_Q^{n,l}} \\ M^{n-1}(P^\bullet) & \xrightarrow{id} & M^{n-1}(Q^\bullet) \end{array}$$

Observe that the two vertical rows are short-exact sequences: in fact since all the objects in the above diagram are projective modules over R, one may assume the two vertical rows are split short exact sequences. Therefore one first chooses a splitting to the map $ker(s_P^{n,l}) \rightarrow ker(s_Q^{n,l})$ which will define a splitting to the map $P^{n,l} \rightarrow Q^{n,l}$. One can see readily that if ϵ^l denotes this splitting, it satisfies the conditions in (6.3.5.1). □

(6.3.6)**Corollary**. Assume the situation of (6.2.4).

(i) Now the functor $\underset{\Delta}{\text{holim}}$ induces $R\underset{\Delta}{\lim}$ at the level of the corresponding derived category.

(ii) The functor $\underset{\Delta}{\text{holim}}$ preserves distinguished triangles in $C(Mod(\mathcal{S}, \underline{R}))$.

(iii) There exists a spectral sequence:

$E_2^{s,t} = \mathcal{H}^s(\mathcal{H}^t(M^n)|n\}) \Rightarrow \mathcal{H}^{s+t}(\underset{\Delta}{\text{holim}}\{M^n|n\})$, $\{M^n|n\} \ \epsilon \ Cosimp(C(Mod(\mathcal{S}, \underline{R})))$.

Proof. Recall that $\underset{\Delta}{\text{holim}}$ preserves quasi-isomorphisms and that therefore it induces a functor at the level of the corresponding derived categories. Recall also that $\underset{\Delta}{\text{holim}}$ is right adjoint to the functor $L(Const)$ which is naturally quasi-isomorphic to the functor $Const$. The functor $Const$ clearly induces a functor at the level of the corresponding derived categories and its right adjoint is the derived functor of the inverse limit, $R\underset{\Delta}{\lim}$. Therefore $\underset{\Delta}{\mathrm{holim}}$ represents $R\underset{\Delta}{\lim}$. This proves (i) and now (ii) is immediate. Moreover we now obtain a spectral sequence:

$$E_2^{s,t} = R^s\underset{\Delta}{\lim}\{\mathcal{H}^t(M^n)|n\} \Rightarrow \mathcal{H}^{s+t}(\underset{\Delta}{\mathrm{holim}}\{M^n|n\}).$$

Now $\{\mathcal{H}^t(M^n)|n\}$ is a cosimplicial object in $Mod(\mathcal{S}, \underline{R})$. Therefore, one may identify $R^s\underset{\Delta}{\lim}\{\mathcal{H}^t(M^n)|n\}$ with the s-th cohomology of the associated co-chain complex in $Mod(\mathcal{S}, \underline{R})$. This provides the identification of the E_2-terms as in (iii). □

(6.3.7) **Remark**. Spectral sequences similar to that in (6.3.6) (iii) are shown to hold in the situations in (6.2.1) through (6.2.3). These take the following form:

$$E_2^{s,t} = \mathcal{H}^s(\{\pi_t(M^n)|n\}) \Rightarrow \pi_{-s+t}(\underset{\Delta}{\mathrm{holim}}\{M^n|n\})$$

(See [T] (5.13) for details.)

(6.3.8) **Lemma**. The functor $\underset{\Delta}{\mathrm{holim}}$ commutes with taking stalks upto quasi-isomorphism in all of the above situations.

Proof. First we need to recall certain truncation functors. In the context of (6.2.1) through (6.2.4) these are variants of the functorial Postnikov truncation. If $S \in \mathbf{S}$, and n is an integer, $\tau_{\leq n}S$ will denote an object in $\mathbf{S}$ so that $\pi_i(\tau_{\leq n}S) \cong \pi_i(S)$, $i \leq n$ and $\cong 0$ if $i > n$. (π_i denotes the homotopy groups in (6.2.1) and (6.2.2) while it denotes a sheaf of homotopy groups in (6.2.3) and the cohomology sheaf $\mathcal{H}^{-i}$ in (6.2.4).)

Now let $C^\bullet = \{C^n|n\} \in Cosimpl(\mathbf{S})$. We first replace $C^\bullet$ by $\tau_{\leq n}C^\bullet$. In this case we have a strongly convergent spectral sequence:

$$E_2^{s,t} = H^s(\pi_t(\tau_{\leq n}C^\bullet)) \Rightarrow \pi_{-s+t}(\underset{\Delta}{\mathrm{holim}}\tau_{\leq n}C^\bullet)$$

Let p denote a point of the site $\mathcal{C}$. Now one obtains a map $p^*(\underset{\Delta}{\mathrm{holim}}C^\bullet) \to \underset{\Delta}{\mathrm{holim}}(p^*C^\bullet)$ which is natural in $C^\bullet$. Now the corresponding map for $\tau_{\leq n}C^\bullet$ may be verified to be a quasi-isomorphism using the above spectral sequence.

Next the definition of the $\underset{\Delta}{\mathrm{holim}}$ as an end shows that it commutes with the inverse limit as $n \to \infty$. Now observe the existence of a short exact sequence:

$$* \to \underset{\infty \leftarrow n}{\lim}{}^1\pi_{i+1}(\underset{\Delta}{\mathrm{holim}}\tau_{\leq n}C^\bullet) \to \pi_i(\underset{\infty \leftarrow n}{\lim}\underset{\Delta}{\mathrm{holim}}\tau_{\leq n}C^\bullet) \to \underset{\infty \leftarrow n}{\lim}\pi_i(\underset{\Delta}{\mathrm{holim}}\tau_{\leq n}C^\bullet) \to *$$

This along with the observation that $\underset{\infty \leftarrow n}{\lim}\underset{\Delta}{\mathrm{holim}}\tau_{\leq n}C^\bullet \simeq \underset{\Delta}{\mathrm{holim}}\underset{\infty \leftarrow n}{\lim}\tau_{\leq n}C^\bullet \simeq \underset{\Delta}{\mathrm{holim}}C^\bullet$ completes the proof of the lemma. □

References.

[Ar-1] M. Artin, *Grothendieck topologies*, Harvard Notes, (1962)

[Ar-2] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math., **27**, (1974), 169-189

[Ar-3] M. Artin, *On the join of Hensel rings*, Adv. Math., **7**, (1971), 282-296

[B-K] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*, Lect. Notes in Mathematics, **304**, (1972)

[B-M] K. Behrend and Y. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math Journal, **85**, 1, (1996), 1-60

[BL] A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Physics, **164**, 385-419, (1994)

[Bl-1] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math., **61**, (1986), 267-304

[Bl-2] S. Bloch, *The moving lemma for higher Chow groups*, Jour. Alg. Geometry, **3**, (1994), 537-568

[Bl-3] S. Bloch, *Notes on Algebraic cycles*, (unpublished yet)

[Bl-4] S. Bloch, *Algebraic cycles and the Beilinson conjectures*, Contemp. Math., **58**, (1986), 65-79

[Board] J. M. Boardman, *Conditionally convergent spectral sequences*, Contemp. Math., **239**, (1999), 49-84

[BG] K.S. Brown and S. Gersten, *Algebraic K theory as generalized sheaf cohomology*, Springer lecture notes, **341**, 266-292, (1973)

[De] P. Deligne, *Theorie de Hodge-III*, Publ. Math. IHES, **44**, (1974), 5-77

[D-M] P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, Publ. Math. IHES, **36**, (1969), 75-109

[EGA] *Elements de geometrie algebrique*, Publ. Math. IHES **20** (1964), **24** (1965), **28** (1968), **32**(1967)

[EG] D. Edidin and W. Graham, *Equivariant intersection Theory*, Invent. Math, **131**, (1998), 595-634

[Fr] E. Friedlander, *Étale homotopy of simplicial schemes*, Ann. Math study, **104**, (1983)

[Fr-S] E. Friedlander and A. Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology*, preprint, May, (1999)

[F-K] E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjecture*, Ergb. Math, **3, 13**, (1987), Springer Verlag

[F] W. Fulton, *Intersection Theory*, Erg. Math., Folge **3**, Band **2**, (1984), Springer

[F-P] W. Fulton and R. Pandharipande, *Notes on quantum cohomology*, University of Chicago, (1997)

[Gi] H. Gillet, *Intersection theory on algebraic stacks and* $\mathbb{Q}$*-varieties*, Jour. Pure and Appl. algebra, **34**, (1984), 193-240

[H] R. Hartshorne, *Algebraic Geometry*, Springer, GTM **52**, (1978)

[Hil] H. Hiller, λ*-rings and algebraic K-theory*, JPAA, **20**, (1981), 241-266

[J-1] R. Joshua, *Higher intersection theory for algebraic stacks:II*, Preprint, June, 2002

[J-2] R. Joshua, *Riemann-Roch on algebraic stacks:I*, to appear in Compositio Math

[KM] S. Keel and S. Mori, *Quotients by groupoids*, Ann. Math, **145**, (1997), 193-213

[Kn] D. Knutson, *Algebraic spaces*, Springer Lecture Notes, **203**, (1971), Springer

[Krat] C. Kratzer, *λ-structure en K-théorie algebrique*, Comm. Math. Hel., **55**, (1970), 233-254

[Kr] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math., **138**, (1999), 495-536

[L-MB1] G. Laumon and L. Moret-Bailly, *Champs Algébriques*, Prépublication d'Orsay, (1995)

[L-MB2] G. Laumon and L. Moret-Bailly, *Champs Algébriques*, Ergebnisse der Math., **39**, 3. Folge, Springer-Verlag, (2000)

[Lev] M. Levine, *Techniques of localization in the theory of algebraic cycles*, Preprint, (1999)

[Licht] S. Lichtenbaum, *values of zeta-functions at non-negative integers*, Number Theory, Lect. Notes in Math., **1068**, Springer, (1983), 127-138

[Mac] S. Maclane, *Categories for the working mathematician*, GTM **5**, Springer, (1971)

[Mum] D. Mumford, *Picard groups of moduli functors*, in *Arithmetic algebraic geometry*, (1965) 33-81

[Qu] D. Quillen, *Higher algebraic K-theory-I*, Lecture Notes in Math., **341**, (1973), 85-147

[SGA4] M. Artin, A. Grothendieck, et J. L. Verdier, *Théorie des Topos et Cohomologie Étale des schemas*, Seminaire de geometrie algebrique, Springer Lect. Notes, **269**, **270**, **305**, (1971)

[SGA6] P. Berthelot, A. Grothendieck et L. Illusie, *Théorie des intersections et Théoremes de Riemann-Roch*, Springer Lect. Notes, **225**, (1971)

[Su] H. Sumihiro, *Equivariant Completion II*, J. Math. Kyoto, **15**, (1975), 573-605

[T] R. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Scient. Ecole Norm Sup, **18**, (1985), 437-552

[Toe] B. Toen, *Théoreme de Riemann-Roch pour les champs de Deligne-Mumford*, K-theory, **18**, (1999), 33-76

[Tot] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), 249-281, Proc. Symposia in Pure Math, **67**, AMS, Providence, (1999)

[Vis] A. Vistoli, *Intersection Theory on algebraic stacks and on their moduli spaces*, Invent. Math., **97**, (1989), 613-670

[Voev-1] V. Voevodsky, *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not., **2002**, no. 7, 351-355

[Voev-2] V. Voevodsky, *On 2-torsion in motivic cohomology*, Preprint, (2001)

Department of Mathematics,
The Ohio State University,
Columbus, Ohio, 43210, USA and

Department of Mathematics,
Miami University, Oxford, Ohio, 45056, USA