# Higher Intersection Theory on Algebraic Stacks

*Abstract.* In this talk we establish a theory of Chow groups and higher Chow groups on algebraic stacks locally of finite presentation over a field and establish their basic properties. This includes algebraic stacks in the sense of Deligne-Mumford as well as Artin.

*Example*: We begin with the quotient stack $[X/G]$

Borel style cohomology theories of $[X/G]$ defined in terms of its simplicial resolution, or the classifying space $EG \underset{G}{\times} X$. One approach to Borel-style cohomology theories for $[X/G]$: via *approximations* to $EG \underset{G}{\times} X$ *by a scheme through a certain finite range.* An alternative is to consider the cohomology of the simplicial scheme $EG \underset{G}{\times} X$ itself. The approach to intersection theory on stacks we take, via their motivic cohomology, is closely related to this.

## Notation

$k$: a fixed field of arbitrary characteristic

Objects (schemes, algebraic spaces and algebraic stacks): locally of finite presentation (often convenient to include locally Noetherian objects also) over $k$ and often finite dimensional

Unless mentioned to the contrary all stacks are Artin stacks.

## Table of Contents

## An overview of (some of) the main results

• Higher Chow groups for all Artin stacks are defined and come with long-exact localization sequences. Covariant for strongly projective morphisms and contravariant for flat representable morphisms.

• Motivic cohomology for all smooth Artin stacks is defined. This agrees with the higher Chow groups for all smooth Artin stacks. In particular an intersection pairing is defined for all smooth Artin stacks at the level of higher Chow groups. Contravariant functoriality in general for maps between smooth stacks.

• Our theory comes equipped with a theory of (higher) Chern classes with values in the higher Chow groups for all smooth stacks.

• The higher Chow groups are intrinsic to the stack for all smooth stacks and also for stacks of finite type.

• Agrees with other definitions modulo torsion for all Deligne-Mumford stacks with quasi-projective coarse moduli space and for all smooth Deligne-Mumford stacks.

Note: our theory has reasonable properties only modulo torsion

## 1. The naive Higher Chow groups of algebraic stacks

(1.1) **Definition** *The naive higher Chow groups.*

$\mathcal{S}$: an algebraic stack, $d$: an integer. A *dimension* $d$ cycle on $\mathcal{S} \times \Delta_k[n]$ is an element of the free abelian group on dimension $d$ integral closed sub-stacks of $\mathcal{S}\times\Delta_k[n]$. (Recall: the dimension of an algebraic stack could be a $\leq 0$.)

We restrict to those cycles that intersect all the faces of $\mathcal{S} \times \Delta_k[n]$ properly. This leads to a simplicial abelian group denoted $z_d(\mathcal{S}, .)$. (Note: if $\mathcal{S}$ of finite dimension $(= N)$, we may define for each integer $c \geq 0$, $z^c(\mathcal{S}, n)$ similarly and obtain the isomorphism $z^c(\mathcal{S}, n) \cong z_{N+n-c}(\mathcal{S}, n)$.) Define $CH_d^{naive}(\mathcal{S}, n) = \pi_n(z_d(\mathcal{S}, .))$.

(1.2) *Functoriality.* Let $f : \mathcal{S}' \to \mathcal{S}$ be a flat map of algebraic stacks of relative dimension $m$. Now $f$ induces a map $f^* : z_d(\mathcal{S}, n) \to z_{d+m}(\mathcal{S}', n)$. Therefore, $\mathcal{S} \to z_*(\mathcal{S}, n)$, for each fixed integer $n$, a *contravariant* functor for flat maps of algebraic stacks and defines an *additive presheaf* on $(alg.stacks/k)^{l.Noeth}_{Res}$. We will denote this *presheaf* by

$\mathcal{Z}_*(\quad, n)$. (We may define $\mathcal{Z}^*(\quad, n)$ similarly.)

Next assume that $\mathcal{S}$ is an algebraic stack. The restriction of the presheaf $\mathcal{Z}_*(\quad, n)$ to $\mathcal{S}_{res.smt}$ defines an *additive* presheaf denoted $\mathcal{Z}_*^{\mathcal{S}_{res.smt}}(\quad, .)$. The restriction of $\mathcal{Z}_*^{\mathcal{S}_{res.smt}}(\quad, .)$ to the étale site of $\mathcal{S}$ will be denoted $\mathcal{Z}_*^{\mathcal{S}_{et}}(\quad, .)$.

Next let $p : \mathcal{S}' \to \mathcal{S}$ denote a *proper representable* map of algebraic stacks. Define the direct image $p_* : \mathcal{Z}_*(\mathcal{S}', n) \to \mathcal{Z}_*(\mathcal{S}, n)$ for each fixed $n \geq 0$ by $p_*([T']) = degree(p_{|T'}).[p(T')]$ for any integral sub-stack $T'$ of $\mathcal{S}' \times \Delta_k[n]$ that belongs to $\mathcal{Z}_*(\mathcal{S}', n)$.

(1.3) Next consider Deligne-Mumford stacks (after Gillet and Vistoli).

If $Z$ is an integral sub-stack of $\mathcal{S}$, let $k(Z)^*$ denote the multiplicative group of rational functions on $Z$. Let $W_*(\mathcal{S})$ = the *rational equivalences* on $\mathcal{S}$, namely $\underset{j}{\oplus} W_j(\mathcal{S})$ where $W_j(\mathcal{S})$ is the direct sum of $k(Z)^*$ over integral closed sub-stacks $Z$ of dimension $j+1$. It is rather well-known that these are sheaves on $\mathcal{S}_{\underline{et}}$ and that one obtains a homomorphism $\delta : W_*(\mathcal{S}) \to \mathcal{Z}_*(\mathcal{S}, 0)$. The *naive Chow group* $CH_q^{naive}(\mathcal{S})$ is defined to be the cokernel of $\delta : W_q(\mathcal{S}) \to \mathcal{Z}_q(\mathcal{S}, 0)$. We conclude this section with the following result.

(1.4) **Proposition**. If $\mathcal{S}$ is a Deligne-Mumford stack, one obtains the isomorphism:

$$CH_q^{naive}(\mathcal{S})\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \cong CH_q^{naive}(\mathcal{S}, 0; \mathbb{Q}) = \pi_0(\mathcal{Z}_q(\mathcal{S}, .)\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$$

**Remark**. The proof will use the fact $\mathcal{Z}_q(\mathcal{S}, .)$ restricted to $\mathcal{S}_{et}$ is a complex of sheaves. We have shown the *naive higher Chow groups* $\pi_n(\mathcal{Z}_q(\mathcal{S}, .) \otimes \mathbb{Q})$ have reasonable functorial properties with respect to flat-pull backs and proper push-forwards. The main drawback: lack of long exact localization sequences associated to closed immersions of algebraic stacks. This will be rectified in the next section.

## 2. The Higher Chow groups with respect to a presentation

We begin by showing that even Zariski hypercohomology with respect to the cycle complex extends Bloch's localization sequences from quasi-projective schemes to all schemes.

(2.1) **Definition**. $X$ :a scheme locally of finite presentation over $k$. For each integer $d$, $\mathcal{Z}_d^{X_{Zar}}(\quad,.)$ = the restriction of the presheaf $\mathcal{Z}_d^{X_{res.smt}}(\quad,.)$ to $X_{Zar}$. Let $\mathbf{CH}_d(X,.) = \mathbb{H}_{Zar}(X, \mathcal{Z}_d^{X_{Zar}}(\quad,.))$ and $CH_d(X,n) = \pi n_{Zar}(X, \mathcal{Z}_d^{X_{Zar}}(\quad,.))$.

*Remark.* If $X$ is a quasi-projective scheme over $k$, the above group is isomorphic to the naive Chow group $CH_*^{naive}(X,.) = \pi_0(z_*(X,.))$. (This follows from the localization theorem of Bloch.)

(2.2) **Proposition**. $X$: a scheme locally of finite presentation over $k$. $Z \to X$ :the closed immersion of a closed subscheme with $U = X - Z$. Then one obtains a distinguished triangle:

$$\mathbf{CH}_m(Z,.) \to \mathbf{CH}_m(X,.) \to \mathbf{CH}_m(U,.) \to \mathbf{CH}_m(Z,.)[1]$$

and therefore a long-exact sequence:

$$... \to CH_m(Z,n) \to CH_m(X,n) \to CH_m(X-Z,n)$$

$$\to CH_m(Z,n-1) \to ...$$

The proof will be clear from a more general result discussed below. In the above situation, it is however, not clear that $CH_*(X,n) = 0$ for $n < 0$. However we can readily show this is the case modulo torsion provided $X$ is of finite type over $k$ by using Riemann-Roch to identify the complex $\mathcal{Z}_*(\quad,.) \otimes \mathbb{Q}$ with $G(\quad)_{\mathbb{Q}}$.

(2.3)**Definition**. Let $\mathcal{S}$ = an algebraic stack, $x : X \to \mathcal{S}$ a fixed atlas, $B_x\mathcal{S}$ the corresponding classifying simplicial space and $d$ a fixed integer. (i) Define $\mathbf{CH}_d(\mathcal{S}, x, .) = \mathbb{H}_{et}(B_x\mathcal{S}, \mathcal{Z}_d^{\mathcal{S}_{res.smt}}(\quad,.))$.

(ii) If $R = \mathbb{Q}$, or $\mathbb{Z}/l^\nu$, $\nu > 0$, define $\mathbf{CH}_d(\mathcal{S}, x, .; R)$

$= \mathbb{H}_{et}(\mathcal{S}, \mathcal{Z}_d^{\mathcal{S}_{res.smt}}(\quad,.)\underset{\mathbb{Z}}{\otimes} R)$.

(iii) If $n$ is an integer, let $CH_d(\mathcal{S}, x, n) = \pi_n(\mathbf{CH}_d(\mathcal{S}, x, .))$ while $CH_d(\mathcal{S}, x, n; R) = \pi_n(\mathbf{CH}_d(\mathcal{S}, x, .; R))$

(2.4) **Theorem** (Localization sequence). $i : \mathcal{S}' \to \mathcal{S}$ : a closed immersion of algebraic stacks with $\mathcal{S}''$ = the complement of $\mathcal{S}'$ in $\mathcal{S}$. The stack $\mathcal{S}$ is locally of finite presentation over $k$. $x : X \to \mathcal{S}$

denotes a fixed atlas, $x' = x \underset{\mathcal{S}}{\times} \mathcal{S}'$ and $x'' = x \underset{\mathcal{S}}{\times} \mathcal{S}''$. Then one obtains a long exact sequence:

$$... \to CH_*(\mathcal{S}', x', n; \mathbb{Q}) \to CH_*(\mathcal{S}, x, n; \mathbb{Q}) \to CH_*(\mathcal{S}'', x'', n; \mathbb{Q}) \to ...$$

*Proof.* This follows immediately from the following more general result.

Let $P$ denote a complex of *additive* presheaves on a site $\mathcal{C}$. $P$ has *cohomological descent* if the obvious augmentation $\Gamma(U, P) \to \mathbb{H}_{\mathcal{C}}(U, P)$ is a quasi-isomorphism for all $U \,\epsilon\, \mathcal{C}$. In particular the presheaf $\mathcal{Z}_*(\quad, .) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ has cohomological descent on the étale site of any quasi-projective scheme. $P$ is *covariant with respect to closed immersions*, if for any $Y$ and $i : Y' \to Y$ a closed immersion there exists a map $\Gamma(Y', P) \to \Gamma(Y, P)$ natural in $i$.

(2.5) $P$ has the *localization property*, if $P$ is covariant with respect to closed immersions and if for any $Y$, $i : Y' \to Y$ a closed immersion with $j : U \to Y$ its open complement, one obtains a distinguished triangle

$$\Gamma(Y', P) \to \Gamma(Y, P) \to \Gamma(U, P) \to \Gamma(Y', P)[1]$$

which is natural in $i$.

(2.6) **Theorem**. $P:$ a complex of additive presheaves on $(alg.stacks/k)_{Res}^{l.Noeth}$ having the localization property on the Zariski site of any quasi-projective scheme over $k$.

(i) Assume further $P$ is covariant with respect to closed immersions for all closed immersions of schemes and that if $Z \to X$ is a closed immersion of schemes with $U$ the open complement of $Z$, then the composition $\Gamma(Z,P) \to \Gamma(X,P) \to \Gamma(U,P)$ is strictly trivial. Then the presheaf of hypercohomology spectra $U \to \mathbb{H}_{Zar}(U,P)$ has the localization property on any scheme locally of finite type over $k$.

(ii) Assume that $P$ has the continuity property on the Zariski site of any quasi-projective scheme, the weak-transfer property on restriction to the étale site of any Artin local ring whose residue field is a finite extension of $k$ and the presheaves $U \to \pi_i(\Gamma(U,P))$, are presheaves of $\mathbb{Q}$-vector spaces. We further assume that $P$ is covariant with respect to closed immersions of all algebraic spaces and that if $i : \mathcal{S}' \to \mathcal{S}$ is a closed immersion of algebraic spaces with $\mathcal{S}'' = \mathcal{S} - \mathcal{S}'$, the composition $\Gamma(\mathcal{S}',P) \to \Gamma(\mathcal{S},P) \to \Gamma(\mathcal{S}'',P)$ is strictly trivial.

Let $\mathcal{S}$ denote any algebraic stack locally of finite presentation over $k$, $i : \mathcal{S}' \to \mathcal{S}$ a closed immersion of algebraic stacks and $j : \mathcal{S}'' =$

$\mathcal{S}-\mathcal{S}' \to \mathcal{S}$ the open immersion of its complement. Let $x : X \to \mathcal{S}$ denote an atlas, $x' = x \underset{\mathcal{S}}{\times} \mathcal{S}'$ and $x'' = x \underset{\mathcal{S}}{\times} \mathcal{S}''$. Then one obtains a distinguished triangle:

$$... \to \mathbb{H}_{et}(B_{x'}\mathcal{S}'; P) \to \mathbb{H}_{et}(B_x\mathcal{S}, P) \to \mathbb{H}_{et}(B_{x''}\mathcal{S}'', P) \to ...$$

(2.7) **Corollary** (Mayer-Vietoris). Let $\mathcal{S}$ denote an algebraic stack with $\mathcal{S}_0$ and $\mathcal{S}_1$ two *open* algebraic sub-stacks so that $\mathcal{S}$ is isomorphic to $\mathcal{S}_0 \cup \mathcal{S}_1$. Let $x : X \to \mathcal{S}$ denote a fixed atlas for $\mathcal{S}$. Let $x_0$, $x_1$ and $x_{01}$ denote the induced atlases for $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_0 \cap \mathcal{S}_1$. Then one obtains a long-exact sequence:

$$... \to CH_*(\mathcal{S}, x, n; \mathbb{Q}) \to CH_*(\mathcal{S}_0, x_0, n; \mathbb{Q}) \oplus CH_*(\mathcal{S}_1, x_1, n; \mathbb{Q}) \to CH_*(\mathcal{S}_0 \cap \mathcal{S}_1, x_{01}, n; \mathbb{Q}) \to ...$$

*Proof.* This follows in the usual manner from (2.4).

(2.8)**Examples**:

(i) Let $G$ denote a smooth affine group scheme acting on a scheme locally of finite presentation over $k$ and let $i : Y \to X$ denote the closed immersion of a $G$-stable closed subscheme with $j : U = X - Y \to X$ the open immersion of its complement. Then one obtains a long-exact-sequence:

$$... \to CH_*([Y/G], Y, n; \mathbb{Q}) \to CH_*([X/G], X, n; \mathbb{Q}) \to CH_*([U/G], U, n; \mathbb{Q})$$

$$\to CH_*([Y/G], Y, n-1; \mathbb{Q}) \to ....$$

where $X$ ($Y$, $U$) denotes the obvious atlas for the stack $[X/G]$ ($[Y/G]$, $[U/G]$, respectively ). (Observe in view of what we prove below, that if $X$ is smooth or of finite type over $k$, the groups $CH_*([X/G], X, .; \mathbb{Q})$ are in fact independent of the atlas $X$.)

(ii) Let $X$ denote a smooth projective curve over an algebraically closed field $k$ and let $r > 0$ denote an integer. Let $\mathcal{SL}_X(r)$ denote the moduli stack of rank $r$ vector bundles on $X$ with trivial determinant and let $\mathcal{SL}_X(r)^{ss}$ denote the open sub-stack of semi-stable bundles. Let $z : Z \to \mathcal{SL}_X(r)$ denote an atlas for the first stack and let $z''$, $z'$ denote the induced atlases for $\mathcal{SL}_X(r)^{ss}$, $\mathcal{SL}_X(r) - \mathcal{SL}_X(r)^{ss}$, respectively . Then one obtains a long-exact sequence:

$$... \to CH_*(\mathcal{SL}_X(r) - \mathcal{SL}_X(r)^{ss}, z', n; \mathbb{Q}) \to CH_*(\mathcal{SL}_X(r), z, n; \mathbb{Q}) \to CH_*(\mathcal{SL}_X(r)^{ss}, z'', n; \mathbb{Q}) \to ...$$

(2.9) **Theorem** (Functoriality). Let $f : \mathcal{S}' \to \mathcal{S}$ denote a representable map of algebraic stacks. Let $x : X \to \mathcal{S}$ denote a fixed atlas and let $x' : x \underset{\mathcal{S}}{\times} \mathcal{S}'$. (i) If $f$ is *flat* of relative dimension $m$, it induces a map $f^* : CH_*(\mathcal{S}, x, n; \mathbb{Q}) \to CH_{*+m}(\mathcal{S}', x', n; \mathbb{Q})$ for every $n \geq 0$.

(ii) If $f$ is a *representable* finite map or if $f$ is *strongly projective*, $f$ induces a map $f_* : CH_*(\mathcal{S}', x', n; \mathbb{Q}) \to CH_*(\mathcal{S}, x, n; \mathbb{Q})$.

(iii) Let $f : \mathcal{S}' \to \mathcal{S}$ denote any finite map of Deligne-Mumford stacks. Now $f$ induces a map $f_* : CH_*(\mathcal{S}', .; \mathbb{Q}) \to CH_*(\mathcal{S}, .; \mathbb{Q})$.

(2.10) **Proposition** (i) Let $\mathcal{S}$ denote a Deligne-Mumford stack with $\mathfrak{M}_{\mathcal{S}}$ its coarse moduli space. If $\mathfrak{M}_{\mathcal{S}}$ is an algebraic space over $k$, then $CH_*(\mathcal{S}, .; \mathbb{Q}) \cong CH_*(\mathfrak{M}_{\mathcal{S}}, .; \mathbb{Q})$ where the right hand side is defined as the hypercohomology on the étale site with respect to the complex $\mathcal{Z}(\quad, .) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

(ii) If, in addition, $\mathfrak{M}_{\mathcal{S}}$ is quasi-projective, one obtains an isomorphism of $CH_*(\mathcal{S}, 0; \mathbb{Q})$ with the naive Chow group $CH_*^{naive}(\mathcal{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

**Examples**: (i) Let $G$ denote a finite group acting on a scheme $X$ of finite over $k$ so that $\mathfrak{M}_{[X/G]}$ is a coarse moduli space. Then $CH_*([X/G], .; \mathbb{Q}) \cong CH_*(\mathfrak{M}_{[X/G]}, .; \mathbb{Q})$. (For example $CH_*(BG, .; \mathbb{Q}) \cong CH_*(Spec \quad k, .; \mathbb{Q})$.)

(ii)Let $G$ denote an affine smooth group scheme acting locally properly on a scheme $X$ (of finite type over $k$) so that a coarse moduli space $\mathfrak{M}_{[X/G]}$ exists as a quasi-projective scheme over $k$. Assume further that the stack $[X/G]$ is Deligne-Mumford (for example the stabilizers are all reduced and finite). Then one obtains the isomorphisms: $CH_*(\mathfrak{M}_{[X/G]}, 0; \mathbb{Q}) \cong CH_*([X/G], 0; \mathbb{Q}) \cong CH_*^G(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ where the right hand side is the equivariant intersection theory.

(iii) Let $X$ denote a projective variety. Let $\bar{M}_{g,n}(X, \beta)$ denote the stack of stable families of maps of $n$-pointed genus $g$-curves to $X$ and let $\bar{\mathfrak{M}}_{g,n}(X, \beta)$ denote the corresponding coarse-moduli space. Here $\beta$ denotes a class in $CH^1(X)$. Then one obtains an isomorphism: $CH_*(\bar{\mathfrak{M}}_{g,n}(X, \beta), .; \mathbb{Q}) \cong CH_*(\bar{M}_{g,n}(X, \beta), .; \mathbb{Q})$.

## 3. Comparison with motivic cohomology and intersection theory for all smooth stacks

(3.1) **Definitions** Let $\mathcal{S}$ denote a smooth algebraic stack and let $\mathcal{S}_{\underline{smt}}$ denote the smooth site whose objects are smooth maps $u : U \to \mathcal{S}$ with $U$ an algebraic space. Assume that for each integer $i \geq 0$, $\mathbb{Q}(i)[2i]$ = the shifted (rational) motivic complex of weight $i$ defined on $\mathcal{S}_{\underline{smt}}$. Similarly $\mathcal{Z}^i(\quad, .; \mathbb{Q})$ = the presheafification of the codimension $i$ higher cycle complex on $\mathcal{S}_{res.\underline{smt}}$. We will let

(3.1.1) $H^{2i-n}_{\mathcal{M}}(\mathcal{S}, i; \mathbb{Q}) = \pi_n(\mathbb{H}_{smt}(\mathcal{S}, \mathbb{Q}(i)[2i]))$

and call it *the rational motivic cohomology of the stack* $\mathcal{S}$. We define

(3.1.2) $\mathbf{CH}^i(\mathcal{S}, x, .; \mathbb{Q}) = \mathbb{H}_{et}(B_x\mathcal{S}, \mathcal{Z}^i(\quad, .; \mathbb{Q}))$,

$CH^i(\mathcal{S}, x, n) = \pi_n(\mathbf{CH}^i(\mathcal{S}, x, .; \mathbb{Q}))$

for an atlas $x : X \to \mathcal{S}$.

(3.2) **Theorem**. Let $\mathcal{S}$ denote a smooth (equi-dimensional) algebraic stack and let $x : X \to \mathcal{S}$ denote a given atlas. Then there exists a quasi-isomorphism:

$\mathbb{H}_{smt}(\mathcal{S}, \mathbb{Q}(i)[2i]) \simeq \mathbf{CH}^i(\mathcal{S}, x, .; \mathbb{Q})$.

Therefore one obtains the isomorphism $H^{2i-n}_{\mathcal{M}}(\mathcal{S}, i; \mathbb{Q}) \cong CH^i(\mathcal{S}, x, n; \mathbb{Q})$ for all $i \geq 0$ and all $n$.

(3.3) **Corollary**.

(i) The higher Chow-groups $CH^i(\mathcal{S}, x, n; \mathbb{Q})$ are in fact independent of the choice of the atlas and therefore intrinsic to the stack for all *smooth* algebraic stacks. The higher Chow groups $CH^i(\mathcal{S}, x, n; \mathbb{Q})$ are independent of the choice of the atlas $x$ for all *stacks of finite type* over the field $k$.

(ii) If $f : \mathcal{S}' \to \mathcal{S}$ is a map of smooth algebraic stacks, one obtains an induced map $f^* : CH^*(\mathcal{S}, .; \mathbb{Q}) \to CH^*(\mathcal{S}', .; \mathbb{Q})$.

(iii) If $\mathcal{S}$ is a *smooth* algebraic stack, one obtains an *intersection-pairing*:

$\cup : CH^i(\mathcal{S}, n; \mathbb{Q}) \otimes CH^j(\mathcal{S}, m; \mathbb{Q}) \to CH^{i+j}(\mathcal{S}, n+m; \mathbb{Q})$.

In addition, if $f : \mathcal{S}' \to \mathcal{S}$ is a map of smooth algebraic stacks, the induced map $f^* : CH^*(\mathcal{S}, .; \mathbb{Q})) \to CH^*(\mathcal{S}', .; \mathbb{Q})$ is compatible with the above intersection pairing.

(3.4) **Examples**: (i) Let $X$ be a smooth scheme of finite type, $G$ an affine smooth group scheme acting on $X$. Then there exists an intersection pairing

$\cup : CH^i([X/G], n; \mathbb{Q}) \otimes CH^j([X/G], m; \mathbb{Q}) \to CH^{i+j}([X/G], n + m; \mathbb{Q})$

(ii) Next assume the action is locally proper and the stabilizers are all finite and reduced and $\mathfrak{M}_{[X/G]}$ is a coarse moduli space. Then one obtains an isomorphism $CH^*([X/G], .; \mathbb{Q}) \cong CH^*(\mathfrak{M}_{[X/G]}, .; \mathbb{Q})$ and hence an induced intersection pairing on the latter provided $X$ is smooth.

(iii) Let $k$ be algebraically closed, $X$ a smooth projective curve of genus $g$ over $k$ and $\mathcal{M}_G$ the stack of principal $G$-bundles over $X$. (See [LS] for example.) (This is a smooth stack of pure dimension $(g-1).dim(G)$.) Then there exists an intersection pairing

$\cup : CH^*(\mathcal{M}_G, .; \mathbb{Q}) \otimes CH^*(\mathcal{M}_G, .; \mathbb{Q}) \to CH^*(\mathcal{M}_G, , .; \mathbb{Q})$

(iv) Let $X$ denote a smooth projective variety which is *convex* in the sense of [F-P] p.6. Let $\bar{M}_{g,n}(X, \beta)$ denote the stack of stable families of maps of $n$-pointed genus $g$-curves to $X$ and let $\bar{\mathfrak{M}}_{g,n}(X, \beta)$ denote the corresponding coarse-moduli space. Here $\beta$ denotes a class in $CH^1(X)$. If $3g-3+n \geq 0$, one obtains an intersection pairing

$$\cup : CH^*(\bar{M}_{g,n}(X,\beta), .; \mathbb{Q}) \otimes CH^*(\bar{M}_{g,n}(X,\beta), .; \mathbb{Q})$$

$$\to CH^*(\bar{M}_{g,n}(X,\beta), .; \mathbb{Q}).$$

Moreover, one also obtains an isomorphism $CH^*(\bar{M}_{g,n}(X,\beta), .; \mathbb{Q}) \cong CH^*(\bar{\mathfrak{M}}_{g,n}(X,\beta), .; \mathbb{Q})$ and therefore an induced pairing

$$\cup : CH^*(\bar{\mathfrak{M}}_{g,n}(X,\beta), .; \mathbb{Q}) \otimes CH^*(\bar{\mathfrak{M}}_{g,n}(X,\beta), .; \mathbb{Q})$$

$$\to CH^*(\bar{\mathfrak{M}}_{g,n}(X,\beta), .; \mathbb{Q}).$$

A *theory of Chern classes and higher Chern classes* (in the sense of Gillet) comes for free by our techniques as in the following:

(3.5) **Theorem** (Chern classes) Let $\mathcal{S}$ denote an algebraic stack of dimension $d$ and let $x : X \rightarrow \mathcal{S}$ denote a fixed atlas for $\mathcal{S}$. Let $K^0(\mathcal{S})$ denote the Grothendieck group of vector bundles on $\mathcal{S}$.

(i) If $\mathcal{S}$ is *smooth* and equi-dimensional, one obtains Chern-classes

$c_i : K^0(\mathcal{S}) \rightarrow CH_{d-i}(\mathcal{S}, x, 0; \mathbb{Q}),\ i \geq 0$

which pull-back under any representable flat map $\mathcal{S}' \rightarrow \mathcal{S}$.

(ii) *Projective space bundle theorem.* Let $\mathcal{E}$ denote a vector bundle of rank $r$ on the algebraic stack $\mathcal{S}$ and let $\pi : \mathbb{P}(\mathcal{E}) = Proj(\mathcal{E}) \rightarrow \mathcal{S}$ denote the associated projective space bundle. Let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ denote the tautological bundle on $\mathbb{P}(\mathcal{E})$. This defines a class $\psi_{\mathcal{E}} \in CH_{d+r-2}(\mathbb{P}(\mathcal{E}), 0; \mathbb{Q})$. Then the map $\oplus_{i=0}^{r-1} CH_*(\mathcal{S}, .; \mathbb{Q}) \rightarrow CH_*(\mathbb{P}(\mathcal{E}), .; \mathbb{Q})$ sending $(a_0, ..., a_{r-1})$ to $\Sigma_i \pi^*(a_i).\psi_{\mathcal{E}}^i$ is an isomorphism.

(iii) For each integer $i$, let $\Gamma(i)$ denote a complex of Abelian sheaves on the big smooth site of all algebraic stacks locally of finite presentation over $k$ so that there exist universal Chern classes $C_i^p \in \mathbb{H}_{et}^{di}(B.GL_p, \Gamma(i))$. Let $\mathbf{K}(\mathcal{S})$ denote the Waldhausen K-theory space of the category of perfect complexes on $\mathcal{S}$. Then one obtains *higher Chern classes*

$$C_i(n) : \pi_n(\mathbf{K}(\mathcal{S})) \to \mathbb{H}^{di-n}_{smt}(\mathcal{S}, \Gamma(i))$$

where $d$ is an integer depending on the complex $\Gamma(i)$ and the right hand side denotes hypercohomology on the smooth site of the stack $\mathcal{S}$. These pull-back under any representable map. In case the stack $\mathcal{S}$ is smooth, there exist higher Chern classes $C_i(n) : \pi_n K(\mathcal{S}) \to CH^i(\mathcal{S}, n; \mathbb{Q})$.

*Remark* The Chern classes in (iii) are also obtained in the thesis of Toen for the K-theory of the exact category of vector bundles on $\mathcal{S}$ by similar methods for certain other complexes.

(3.6) **Proposition**

(i) If $\mathcal{S}$ is any smooth Deligne-Mumford stack of finite type over $k$, there exists an isomorphism $CH^*(\mathcal{S}, 0; \mathbb{Q}) \cong CH^*_{naive}(\mathcal{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

(ii) The intersection pairing on $CH^*(\mathcal{S}, 0; \mathbb{Q})$ agrees with the known intersection pairing on the naive Chow group $CH^*_{naive}(\mathcal{S}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

*Remark.* The proof involves showing the existence of $\lambda$-operations on the (higher) étale K-theory of any smooth Deligne-Mumford stack with rational coefficients and comparing with the older definitions of intersection product on Deligne-Mumford stacks using K-theoretic techniques (as was done by Gillet).

## 4. Some concluding remarks.

• *Comparison with the Totaro-Edidin-Graham theory for quotient stacks*

Should be the same with rational coefficients for all quotient stacks.

Example: $B\mathbb{G}_m$ vs $\mathbb{P}^\infty$

Fortunately they are identified in the motivic homotopy category of schemes ( of Morel and Voevodsky)!

• All Borel style cohomology theories are rather coarse invariants of stacks: in particular they are not suited as the target of a RR transformation (if one considers non-representable morphisms). The work of Toen (based on prior work by Vistoli) shows how to get finer invariants for Deligne-Mumford stacks. In an ongoing project, we show how to define *Bredon* style cohomology and homology theories for algebraic stacks that have a coarse moduli space with respect to any of the standard cohomology-homology theories for the moduli space. In particular, using the motivic cohomology complex on the moduli space, this construction provides a Bredon style cohmology theory for the stack that would be finer than the motivic cohomology of the stack considered above.

See http://www.math.ohio-state.edu/~ joshua/pub.html