# ALGEBRAIC K-THEORY AND HIGHER CHOW GROUPS OF LINEAR VARIETIES

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Abstract. The main focus in this paper is the algebraic K-theory and higher Chow groups of linear varieties and schemes. We provide Kunneth spectral sequences for the higher algebraic K-theory of linear schemes flat over a base scheme and for the motivic cohomology of linear varieties defined over a field. The latter provides a Kunneth formula for the usual Chow groups of linear varieties originally obtained by different means by Totaro. We also obtain a general condition under which the higher cycle maps of Bloch from mod-$l^\nu$ higher Chow groups to mod-$l^\nu$ étale cohomology are isomorphisms for projective non-singular varieties defined over an algebraically closed field of arbitrary characteristic $p \geq 0$ with $l \neq p$. It is observed that the Kunneth formula for the Chow groups implies this condition for linear varieties and we compute the mod-$l^\nu$motivic cohomology and mod-$l^\nu$ algebraic K-theory of projective nonsingular linear varieties to be free $\mathbb{Z}/l^\nu$-modules.

## 1. Introduction.

In this paper we study the algebraic K-theory and higher Chow groups of linear varieties and schemes. The class of linear varieties contains all toric and spherical varieties; it also contains all varieties on which a connected solvable group acts with finitely many orbits and also varieties that are stratified by strata which are products of tori and affine spaces. (See [Tot-1] for a general discussion on linear varieties.)

In the second section of this paper we begin with a quick review of linear varieties. We review the derived tensor product functors in section 3: these are used in the next section to establish Kunneth spectral sequences in the algebraic K-theory of linear schemes flat over a given smooth base scheme and for the motivic cohomology of linear varieties defined over a field $k$. These appear in Theorems 4.2 and 4.5 and may summarized as follows:

**Theorem 1.1.** *(i) Let $X$ and $Y$ denote quasi-projective schemes flat over a smooth base-scheme $S$ and assume further that one of them is linear. Now there exists a Kunneth spectral sequence*

$$E^2_{s,t} = Tor^{\pi_*(K(S))}_{s,t}(\pi_*G(X), \quad \pi_*G(Y)) \Rightarrow \pi_{s+t}(G(X \underset{S}{\times} Y)) \tag{1.1}$$

*(ii) Let $X$ and $Y$ denote two quasi-projective varieties over a field $k$ and assume further that one of them is linear. Now there exists a Kunneth spectral sequence*

$$E^2_{s,t} = Tor^{\pi_*(z^*(Speck;.))}_{s,t}(\pi_*(z^*(X;.), \quad \pi_*(z^*(Y;.)))) \Rightarrow \pi_{s+t}(z^*(X \times Y;.)) \tag{1.2}$$

It is shown that these Kunneth spectral sequences reduce to Kunneth formulae for the Grothendieck groups of coherent sheaves and also for the usual Chow groups. The spectral sequence in algebraic K-theory only holds integrally, whereas the spectral sequence in motivic cohomology is shown to hold also with mod-$l^\nu$ coefficients. Burt Totaro also obtained a Kunneth formula for the usual Chow groups of linear varieties (see [Tot-1]) using entirely different techniques and our interest on linear varieties owes a great deal to his work and also to conversations with Michel Brion.

In the rest of the paper, we consider only projective nonsingular varieties defined over an algebraically closed field $k$ of arbitrary characteristic $p \geq 0$, $l$ a prime different from the characteristic and $\nu > 0$ an integer. The main result in the fifth section is the following theorem.

**Theorem 1.2.** *(See Theorem 5.2) Let $X$ denote a projective nonsingular variety over $k$ so that the class of the diagonal $\Delta \,\epsilon\, CH^*(X \times X; \mathbb{Z}_l)$ can be written as $\Sigma_i \alpha_i \times \beta_i$, $\alpha_i$, $\beta_i \,\epsilon\, CH^*(X; \quad \mathbb{Z}_l)$. (i) Now the higher cycle-map*

$$cl_l : CH^r(X, \quad n; \quad \mathbb{Z}_l) \rightarrow H^{2r-n}_{et}(X; \mathbb{Z}_l(r))$$

*is an isomorphism for each fixed $n \geq 0$. In particular, both terms are trivial if $n$ is odd.*

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*(ii) For each fixed integer $n \geq 0$, $\underset{r}{\oplus} CH^r(X, \quad n; \quad \mathbb{Z}_l) \cong \underset{r}{\oplus} H_M^{2r-n}(X, \quad \mathbb{Z}_l(r))$ is generated over $\underset{r}{\oplus} H_M^*(Spec$ $k, \quad \mathbb{Z}_l(r)) \cong \underset{r}{\oplus}\mathbb{Z}_l(r)$ by classes $\{\alpha_i\} \subseteq CH^*(X; \mathbb{Z}_l)$. Moreover for each fixed pair of integers $u$ and $v$, the motivic cohomology $H_M^u(X, \quad \mathbb{Z}_l(v))$ is a free $\mathbb{Z}_l$-module which is trivial if $u$ is odd.*

*(iii) All the above assertions hold with $\mathbb{Z}_l$ replaced $\mathbb{Z}/l^\nu$.* □

Recall that many deep conjectures in the theory of motives are related to properties of the Kunneth components of the diagonal. The above theorem should be interpreted as saying that a sufficiently strong Chow Kunneth decomposition for the class of the diagonal implies the mod-$l^\nu$ motivic cohomology of projective non-singular varieties is isomorphic to mod-$l^\nu$ étale cohomology. As a corollary to the above theorem and the Kunneth formula from section 4 (or from [Tot-1]) we obtain the following result in section 6.

**Corollary 1.3.** *Let $X$ be a projective smooth linear variety defined over $k$ and $\nu >> 0$. Then $\pi_n(K/l^\nu(X)) \cong H^*_{et}(X; \mathbb{Z}/l^\nu)$ if $n \geq 0$ is even and trivial otherwise. Similarly $\pi_n(K/l^\nu_{top}(X)) \cong H^*_{et}(X; \mathbb{Z}/l^\nu)$ for all $n$ even and trivial otherwise. (Here $K/l^\nu_{top}$ denotes the mod-$l^\nu$ étale K-theory in [Fr-1] or equivalently the mod-$l^\nu$ topological K-theory in [T].) In particular these groups are fee $\mathbb{Z}/l^\nu$-modules for all $n$.*

The last section is devoted to examples. There are two appendices: Appendix A where we summarize the properties of the higher Chow groups and Appendix B where we provide a quick review of $A^\infty$ and $E^\infty$-differential graded algebras, their modules as well as ring and module spectra.

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## 2. Linear schemes and varieties

We will briefly recall the definition from [Tot-1].

**Definition 2.1.** Let $S$ denote a base-scheme. (i) A scheme *flat* over $S$ is 0-linear if it is either empty or isomorphic to any affine space $\mathbb{A}^n_S$.

(ii) Let $n > 0$ be an integer. An $S$-scheme $Z$, flat over $S$, is $n$-linear, if there exists a triple $(U, X, Y)$ of $S$-schemes so that $Y \subseteq X$ is an $S$-closed immersion with $U$ its complement, $Y$ and one of the schemes $U$ or $X$ is $(n-1)$-linear and $Z$ is the other member in $\{U, X\}$. We say $Z$ is linear if it is $n$-linear for some $n \geq 0$.

(iii) If $S$ is the spectrum of a field $k$, any reduced scheme $X$ of finite type over $k$ will be called a variety. Linear varieties over $k$ are varieties over $k$ that are linear schemes.

**Example 2.2.** The following are common examples of linear varieties. In these examples we fix a base field $k$ and consider only varieties over $k$.

- All toric varieties
- All spherical varieties (A variety $X$ is spherical if there exists a reductive group $G$ acting on $X$ so that there exists a Borel subgroup having a dense orbit.)
- Any variety on which a connected solvable group acts with finitely many orbits. (For example projective spaces and flag varieties.)
- Any variety that has a stratification into strata each of which is the product of a torus with an affine space.

## 3. Derived tensor products

We will first review derived tensor products in the category of spectra. The homotopy category of spectra is well known to be an additive category. In fact one can do *homotopical algebra* in this category in a manner entirely similar to doing homological algebra in an abelian category. See [Qu-1] section 5. However to be able to consider the analogue of tensor products and Hom for modules over a ring, one first needs a symmetric monoidal structure on the category of spectra. Till recently this was not quite available (but see the some-what preliminary work by Alan Robinson in [Rob]). Now one has two distinct approaches available: (i) in the context of topological spectra as in [EKMM] and (ii) in the simplicial context as in [HSS]. (See also [Lyd].) For the purposes of this paper, either of the above approaches is quite satisfactory.

To define topological spectra one first fixes a universe, namely a real inner product space $U \cong \mathbb{R}^\infty$. Now a topological spectrum is a sequence of pointed topological spaces $\{X_V | \text{V} = \text{a finite dimensional (vector) sub-space of } U\}$ so that if $V \subseteq W$ are both finite dimensional vector sub-spaces of $U$, there is given a continuous map

$\sigma_{V,W}: X_V \to \Omega^{W-V}X_W$. (Here $W-V$ is the orthogonal complement of $V$ in $W$ and $\Omega^{W-V}X_W =$ the space of based maps $S^{W-V} \to X_W$, $S^{W-V}$ being the one-point compactification of $W-V$. The above function space is provided with the function space topology in the category of weak Hausdorff spaces.) These maps are also required to satisfy an associativity condition. A map $f: \{X_V|V\} \to \{Y_V|V\}$ of spectra is a collection of maps $f_V: X_V \to Y_V$ of pointed spaces commuting with the above structure. It is known that the above category is complete and co-complete. Let (top spectra) denote this category. (See [EKMM] for the basic theory. Observe that what we call spectra are called pre-spectra there.)

A simplicial spectrum is a sequence of pointed simplicial sets $\{X(n)|n \geq 0\}$ provided with suspension maps $\Sigma X(n) = S^1 \wedge X(n) \to X(n+1)$ for all $n$. Maps between simplicial spectra are defined in the obvious manner. This category is also complete and co-complete. Let (simpl spectra) denote this category.

In either set-up one may define homotopies between two maps in the customary manner and consider the homotopy category where the objects are the same, but morphisms are homotopy classes of maps. One may also define the (stable) homotopy groups associated to spectra - these are all abelian. A map of spectra is a *weak-equivalence* if it induces an isomorphism on all the homotopy groups. A *fibrant* simplicial spectrum $X = \{X_n|n\}$ is a simplicial spectrum where each $X_n$ is a fibrant pointed simplicial set and the obvious maps $X_n \to \Omega X_{n+1}$ adjoint to the given maps $\Sigma X_n \to X_{n+1}$ are weak-equivalences. A fibrant topological spectrum is one where the maps $\sigma_{V,W}$ are all isomorphisms. In either setting, there exists a functor $Q$ : ( spectra) $\to$ (fibrant spectra) that produces a weakly-equivalent *fibrant* spectrum.

Now we will axiomatize our basic frame-work so that either of the above approaches to spectra will suffice for our work.

- **(Sp.0):** Let $Sp$ denote a category of either topological spectra or simplicial spectra, provided with a coherently associative and commutative operation $\otimes: Sp \times Sp \to Sp$.
- **(Sp.1):** There exists a special spectrum called *the sphere spectrum*, which is a unit for the operation $\otimes$. This will be denoted $S$.
- **(Sp.2):** An associative *ring spectrum* $R$ will be a coherently associative monoid in the category $Sp$; given such a ring spectrum $R$, a left module spectrum $M$ (a right module spectrum $N$) over $R$ will denote a spectrum provided with a coherently associative pairing: $R \otimes M \to M$ ($N \otimes R \to N$, respectively ). Let $Mod_l(R)$ ($Mod_r(R)$) denote the category of all left $R$-module-spectra (right $R$-module spectra, respectively ). An associative and commutative ring spectrum is an associative and commutative monoid in the category $Sp$. Now we will summarize the main results in [EKMM] and [HSS] that we require:
- **(Sp.3):** There exists a pairing: $\underset{R}{\otimes}: Mod_r(R) \times Mod_l(R) \to Sp$.
- **(Sp.4):** The above functor has a left-derived functor $\overset{L}{\underset{R}{\otimes}}$ that preserves weak-equivalences and (stable-) fibration sequences in either argument. Strictly speaking, $\overset{L}{\underset{R}{\otimes}}$ is a functor only at the level of a suitable *derived category*. For our purposes, we may simply state this as follows. For each $M \in Mod_l(R)$ ($N \in Mod_r(R)$, respectively ), one may find a weak-equivalence $M' \overset{\simeq}{\to} M$ in $Mod_l(R)$ ($N' \overset{\simeq}{\to} N$ in $Mod_r(R)$, respectively ) having the following property: $- \overset{L}{\underset{R}{\otimes}} M'$ ($N' \overset{L}{\underset{R}{\otimes}} -$) preserves stable fibration sequences and weak-equivalences in the first (second, respectively ) argument. In addition, there exists a spectral sequence:

$$E^2_{s,t} = Tor^{\pi_*(R)}_{s,t}(\pi_*(N), \pi_*(M)) \Rightarrow \pi_{s+t}(N \overset{L}{\underset{R}{\otimes}} M)$$

  Here $E^2_{s,t} = (Tor^{\pi_*(R)}_s(\pi_*(M), \pi_*(N)))_t =$ the $t$-th graded piece of $Tor^{\pi_*(R)}_s$. It follows readily that $R \overset{L}{\underset{R}{\otimes}} M \simeq M$ and $N \simeq N \overset{L}{\underset{R}{\otimes}} R$ where $\simeq$ denotes weak-equivalences.
- **(Sp.5):** Let $R$ denote an associative and commutative ring spectrum. An $R$ algebra is another associative and commutative ring spectrum $S$ provided with a map $f: R \to S$ of ring spectra. Given $R$ algebras $S$, $T$ and $U$ along with maps $S \to U$ and $T \to U$ which are compatible with the given map $R \to U$, there exists an induced map $S \overset{L}{\underset{R}{\otimes}} T \to U$ that is natural in all the arguments.

**Example 3.1.** Let $S$ denote a base-scheme. For each scheme $X$ over $S$, let $K(X)$ ($G(X)$) denote the spectrum associated in the usual manner to the symmetric monoidal category of locally free coherent sheaves on $X$

(coherent sheaves on $X$, respectively ). The tensor product pairing on the category of locally free coherent sheaves makes the spectra $K(S)$ and $K(X)$ $E^{\infty}$-ring spectra as in (B.2.4). It is shown in [EKMM] chapter II that one may associate to any $E^{\infty}$ ring spectrum, a weakly-equivalent ring spectrum in the sense of $(Sp.2)$. We will denote the ring spectrum associated to $K(S)$ $(K(X))$ by $K(S)$ ($K(X)$, respectively ) itself. The tensor product pairing between a locally free coherent sheaf and a coherent sheaf on $X$ is coherently associative and commutative. These higher order coherences provide the higher order homotopies required to make $G(X)$ an $E^{\infty}$-module spectrum as in (B.2.4). Now the machinery in [EKMM] shows that any such $E^{\infty}$-module spectrum provides a weakly-equivalent module spectrum (over the ring spectrum $K(X)$) in the sense of $(Sp.2)$. Let $p : X \to S$ denote the obvious structure map of $X$. Now the induced map $p^* : K(S) \to K(X)$ will be a map of ring spectra. It follows that if $X$ is any scheme over $S$, $G(X)$ will have the structure of a module spectrum over $K(S)$. Moreover observe that if $i : Y \to X$ is a closed immersion, the induced map $i_* : G(Y) \to G(X)$ is a map of module spectra over $K(S)$ in the same sense. (This follows readily from the above discussion.)

Next we consider similar derived tensor product functors in the category of differential graded modules over a differential graded algebra. Let $R$ denote a fixed commutative ring and let $\mathbf{A}$ denote the abelian tensor category of modules over $R$. Now $C(\mathbf{A})$ will denote the category of complexes in $\mathbf{A}$. Let $\mathcal{A}$ denote a differential graded algebra or an $A^{\infty}$ differential graded algebra in $C(\mathbf{A})$. The 'differential graded algebra' we have in mind is the one constructed from the (integral) higher *Chow complex* of Bloch which is not quite a differential graded algebra, but only one upto coherent associativity and commutativity. Therefore, we require the more general notion of $A^{\infty}$ differential graded algebras in the sense of [K-M]. This is discussed in appendix B. We require the following axioms hold in our setting: these have all been established in [K-M].

**DG.0:** Now assume that $\mathcal{A}$ is an $A^{\infty}$ differential graded algebra in $C(\mathbf{A})$ and let $Mod_l(\mathcal{A})$ $(Mod_r(\mathcal{A}))$ denote the category of left-modules (right-modules, respectively ) over $\mathcal{A}$. We require that there exist a bi-functor:$\underset{\mathcal{A}}{\otimes} : Mod_r(\mathcal{A}) \times Mod_l(\mathcal{A}) \to C(\mathbf{A})$.

**DG.1:** Moreover we require that there exist a left-derived functor associated to the above functor. Strictly speaking the derived functor is a functor only at the level of the appropriate derived categories. For our purposes, we may restate this as follows. For each $M \in Mod_l(\mathcal{A})$ $(N \in Mod_r(\mathcal{A})$, respectively ), one may find a quasi-isomorphism $M' \overset{\simeq}{\to} M$ in $Mod_l(\mathcal{A})$ ($N' \overset{\simeq}{\to} N$ in $Mod_r(\mathcal{A})$, respectively ) having the following property:

$- \overset{L}{\underset{\mathcal{A}}{\otimes}} M'$ $(N' \overset{L}{\underset{\mathcal{A}}{\otimes}} -)$ preserves distinguished triangles and quasi-isomorphisms in the first (second, respectively ) argument.

**DG.2:** There exists a spectral sequence:

$E_2^{s,t} = Tor_{s,t}^{H^*(\mathcal{A})}(H^*(M), H^*(N)) \Rightarrow H^{-s+t}(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N)$. The $E_2$-term has an interpretation as in (Sp.4).

**DG.3:** Let $\mathcal{A}$ denote an $E^{\infty}$ differential graded algebra and let $s : \mathcal{A} \to S$, $t : \mathcal{A} \to T$ and $u : \mathcal{A} \to U$ denote maps of $E^{\infty}$ differential graded algebras. Given maps $S \to U$ and $T \to U$ of $E^{\infty}$ differential graded algebras compatible with the given map $\mathcal{A} \to U$, there exists an induced map $S \overset{L}{\underset{\mathcal{A}}{\otimes}} T \to U$ of $E^{\infty}$ differential graded algebras which is natural in all the arguments.

**Example 3.2.** The main example of the above situation that we consider will be the $E^{\infty}$ differential graded algebras and modules associated to the *higher Chow complexes* of Bloch. (See [Bl-1].) Let $k$ denote a fixed field and let $X$, $Y$ be quasi-projective varieties over $k$. Now consider the external product $\times : CH^r(X; n) \otimes CH^s(Y; m) \to CH^{r+s}(X \times Y; n+m)$. This involves choosing a triangulation of $\Delta[n] \times \Delta[m]$ identifying it with $\Delta[n+m]$. Nevertheless this product is only a *partially defined product* at the level of cycles - see [Bl-1] section 5. If $X$ and $Y$ are varieties over $k$, there is also a *partially defined* external product $\times$ on $z^*(X; .) \times z^*(Y; .)$ taking values in $z^*(X \times Y; .)$. As shown in [K-M] example (6.3) one may view $z^*(\text{Spec } k; .)$ as a partially defined algebra over an $E^{\infty}$-operad. Similarly if $X$ is a variety over $k$, one may view $z^*(X; .)$ as a partially defined $E^{\infty}$-module over the above partially defined algebra. (This partially defined $E^{\infty}$-module structure may be obtained by taking $Y = \text{Spec } k$ and by using the partially defined external product considered above.) Now the conversion theorem (Theorem (1.1), Part II of [K-M]) applies to produce a quasi-isomorphic $E^{\infty}$-differential graded algebra associated to $z^*(\text{Spec } k; .)$. We will continue to denote this $E^{\infty}$ differential graded algebra by $z^*(\text{Spec } k; .)$. Similarly if $X$ is a variety over Spec $k$, we obtain an $E^{\infty}$ differential graded module over the above $E^{\infty}$ differential graded algebra: this will be also denoted $z^*(X;.)$. The module structure is induced by the external product. If $X$ is, in addition smooth, (it is claimed in [K-M] without proofs that) one also obtains an $E^{\infty}$ differential graded algebra over $z^*(\text{Spec }\ k;.)$ in a functorial manner from the higher Chow complex $z^*(X;.)$. Once again the resulting $E^{\infty}$ algebra will be denoted $z^*(X;.)$. (Observe that unless we tensor the above $E^{\infty}$ differential graded algebra (module) with $\mathbb{Q}$, it may not be possible to obtain a (*strict*) differential graded algebra (module, respectively ) - see the conversion theorems in [K-M] Part II.)

## 4. The Kunneth formula for higher K-theory and motivic cohomology

Throughout this section $S$ will denote a *smooth* base-scheme. We will restrict to schemes that are *flat* over $S$. Let $X$, $Y$ denote two quasi-projective schemes over $S$. Let $p_1 : X \underset{S}{\times} Y \to X$ and $p_2 : X \underset{S}{\times} Y \to Y$ denote the two projections. Since both $X$ and $Y$ are flat over $S$, these maps are also flat; therefore they induce maps $p_1^* : G(X) \to G(X \underset{S}{\times} Y)$ and $p_2^* : G(Y) \to G(X \underset{S}{\times} Y)$ of module-spectra over the ring spectrum $K(S)$.

**Proposition 4.1.** *Assume the above situation. Now we obtain an induced map of spectra* $G(X) \overset{L}{\underset{K(S)}{\otimes}} G(Y) \to G(X \underset{S}{\times} Y)$.

*Moreover, if* $U \to X$ *is an open immersion, one obtains a homotopy commutative diagram*

$$\begin{array}{ccc} G(X) \overset{L}{\underset{K(S)}{\otimes}} G(Y) & \longrightarrow & G(U) \overset{L}{\underset{K(S)}{\otimes}} G(Y) \\ \downarrow & & \downarrow \\ G(X \underset{S}{\times} Y) & \longrightarrow & G(U \underset{S}{\times} Y) \end{array}$$

*Proof.* We will first show that the required map exists when both the schemes $X$ and $Y$ are smooth. In this case we may replace $G$-theory everywhere by $K$-theory. Now $K(X)$, $K(Y)$ and $K(X \underset{S}{\times} Y)$ are $K(S)$- algebras and the maps $p_1^* : K(X) \to K(X \underset{S}{\times} Y)$, $p_2^* : K(Y) \to K(X \underset{S}{\times} Y)$ are maps of $K(S)$-module spectra. Therefore the axiom (Sp.5) shows that we obtain an induced map $K(X) \overset{L}{\underset{K(S)}{\otimes}} K(Y) \to K(X \underset{S}{\times} Y)$ of spectra.

Now we consider the case when $X$ is smooth and $Y$ is an arbitrary quasi-projective scheme. Clearly we can find a quasi-projective *smooth* scheme $\tilde{Y}$ that contains $Y$ as a closed subscheme. Now $\tilde{Y} - Y$ is open in $\tilde{Y}$ and therefore also smooth. Therefore we obtain the diagram:

$$\begin{array}{ccccc} G(X) \overset{L}{\underset{K(S)}{\otimes}} G(Y) & \longrightarrow & G(X) \overset{L}{\underset{K(S)}{\otimes}} G(\tilde{Y}) & \longrightarrow & G(X) \overset{L}{\underset{K(S)}{\otimes}} G(\tilde{Y} - Y) \\ & & \downarrow & & \downarrow \\ G(X \underset{S}{\times} Y) & \longrightarrow & G(X \underset{S}{\times} \tilde{Y}) & \longrightarrow & G(X \underset{S}{\times} \tilde{Y} - Y) \end{array}$$

Since both rows are fibration sequences, one obtains an induced map $G(X) \overset{L}{\underset{K(S)}{\otimes}} G(Y) \to G(X \underset{S}{\times} Y)$. One may readily prove the second assertion in this case (i.e. when $X$ is smooth and $U$ is open in $X$) by comparing the commutative diagrams above for $X$ and $X$ replaced everywhere by $U$. This in turn implies the first assertion in the general case when both $X$ and $Y$ are not necessarily smooth. To complete the proof of the second statement in this case, observe that if $\tilde{X}$ is a smooth scheme containing $X$ as a closed subscheme and $U$ is open in $X$, one may find an open subscheme $\tilde{U}$ of $\tilde{X}$ that contains $U$ as a closed subscheme. □

**Theorem 4.2. Kunneth formula for higher K-theory**. *Assume in addition to the above situation that at least one of the schemes* $X$ *or* $Y$ *is linear.*

*(i) Now one obtains a weak-equivalence of spectra:*

$$G(X \underset{S}{\times} Y) \simeq G(X) \overset{L}{\underset{K(S)}{\otimes}} G(Y)$$

*The derived functor of the tensor-product on the right is the derived tensor product over the ring spectrum $K(S)$ considered in (Sp.4).*

*(ii) There exists a spectral sequence:*

$$E^2_{s,t} = Tor^{\pi_*(K(S))}_{s,t}(\pi_* G(X),\ \pi_* G(Y)) \Rightarrow \pi_{s+t}(G(X \underset{S}{\times} Y))$$

*Proof.* The spectral sequence is the one in [EKMM] chapter IV, Theorem (6.2). (A similar approach works in the setting of symmetric spectra, though no explicit derivation of such a spectral sequence appears in [HSS].) We may assume $X$ is an $n$-linear scheme and $Y$ arbitrary. We will use ascending induction on $n$ to establish the theorem. One may clearly start the induction with $n = 0$, since the homotopy property of $G$-theory establishes (i) in this case. Now assume the theorem is true for all $n-1$ linear schemes , where $n > 0$ is an integer. Let $X$ denote an $n$-linear scheme. According to the definition in (2.1)(ii) there are two cases to consider : (i) when there exist an open $n-1$ linear sub-scheme $U$ of $X$ so that its complement $X - U$ is also $(n-1)$ linear and (ii) when there exists a larger $(n-1)$ linear scheme $Z$ so that $X$ is an open subscheme of $Z$ and $Z - X$ is also an $(n-1)$ linear scheme. In the first case consider the commutative diagram:

$$\begin{array}{ccccc} G(X-U) \overset{L}{\underset{K(S)}{\otimes}} G(Y) & \longrightarrow & G(X) \overset{L}{\underset{K(S)}{\otimes}} G(Y) & \longrightarrow & G(U) \overset{L}{\underset{K(S)}{\otimes}} G(Y) \\ \downarrow & & \downarrow & & \downarrow \\ G((X-U) \underset{S}{\times} Y) & \longrightarrow & G(X \underset{S}{\times} Y) & \longrightarrow & G(U \underset{S}{\times} Y) \end{array}$$

The vertical maps are defined above. Both of the above rows are localization sequences (i.e. fibration sequences of spectra). To see the top row is a fibration sequence, see (Sp.4) in section 1. Now the first and last vertical maps are weak-equivalences by the inductive assumption. Therefore so is the middle map proving (i) in this case.

In case (ii) one may consider instead the commutative diagram:

$$\begin{array}{ccccc} G(Z-X) \overset{L}{\underset{K(S)}{\otimes}} G(Y) & \longrightarrow & G(Z) \overset{L}{\underset{K(S)}{\otimes}} G(Y) & \longrightarrow & G(X) \overset{L}{\underset{K(S)}{\otimes}} G(Y) \\ \downarrow & & \downarrow & & \downarrow \\ G((Z-X) \underset{S}{\times} Y) & \longrightarrow & G(Z \underset{S}{\times} Y) & \longrightarrow & G(X \underset{S}{\times} Y) \end{array}$$

Now the first two vertical maps are weak-equivalences; both rows are fibration sequences for the same reasons as above. It follows that the last vertical map must also be a weak-equivalence. □

**Corollary 4.3.** *Assume $S$ is the spectrum of a field $k$. If $X$ is a linear scheme and $Y$ is any scheme over $S$, one obtains the Kunneth-formula:*

$$G_0(X \times Y) \cong G_0(X) \underset{\mathbb{Z}}{\otimes} G_0(Y)$$

*Proof.* Observe that the $G$-theory spectra are connected i.e. trivial in negative degrees. Therefore $E^2_{s,t} = 0$ for all $s$ or $t$ negative. It follows that the only $E^2_{s,t}$-term that contributes to $\pi_0(G(X \times Y))$ is $E^2_{0,0} \cong G_0(X) \underset{\mathbb{Z}}{\otimes} G_0(Y)$. □

Now we consider motivic cohomology: this will be identified with the higher Chow groups. Henceforth we will assume the base scheme $S$ is the spectrum of a field and that all schemes we consider are reduced and of finite type over $k$. (Such schemes will be called varieties.) As explained in example (2.2) the complex $z^*(Spec\ \ k; .)$ of Bloch defines an $E^\infty$ differential graded algebra and if $X$ is any variety over $k$, $z^*(X; .)$ is an $E^\infty$ differential graded module over $z^*(Spec\ \ \mathrm{k};\ .)$.

**Proposition 4.4.** *If $X$ and $Y$ are two quasi-projective varieties over $k$, one obtains an induced map $\times$ : $z^*(X;.) \overset{L}{\underset{z^*(Spec\ \ k;.)}{\otimes}} z^*(Y;.) \rightarrow z^*(X \underset{Spec\ \ k}{\times} Y;.)$. Moreover this is compatible with restriction to open subschemes of $X$ and $Y$.*

*Proof.* This is established in a manner entirely similar to the proof of (4.1). The proof will be exactly the same assuming that for any smooth quasi-projective variety, the higher cycle complex may be replaced by a quasi-isomorphic $E^{\infty}$ differential graded algebra as is claimed in [K-M]. We sketch an alternate approach here. Recall that the higher cycle complex associated to $Spec\ \ k$ may in fact be replaced by a quasi-isomorphic $E^{\infty}$-differential graded algebra and that the higher cycle complex associated to any quasi-projective variety over $k$ may replaced by a quasi-isomorphic $E^{\infty}$-differential graded module over this algebra. Now one may use the explicit bar-resolution to define $z^*(X;.) \overset{L}{\underset{z^*(Spec\ \ k;.)}{\otimes}} z^*(Y;.)$. Moreover, in this case, the external product defines a natural map from this to $z^*(X \underset{Spec\ \ k}{\times} Y;.)$. One may prove this is a quasi-isomorphism by induction on the dimension of $X$ as in the case of $G$-theory. □

**Theorem 4.5. Kunneth formula in motivic cohomology**. *Assume in addition to the above situation that at least one of the varieties $X$ or $Y$ is linear and both are quasi-projective.*

*(i) Now the external product $\times : z^*(X;.) \overset{L}{\underset{z^*(Spec\ \ k;.)}{\otimes}} z^*(Y;.) \rightarrow z^*(X \times Y;.)$ is a quasi-isomorphism.*

*(ii) There exists a spectral sequence:*

$$E^2_{s,t} = Tor^{\pi_*(z^*(Speck;.))}_{s,t}(\pi_*(z^*(X;.),\ \pi_*(z^*(Y;.)))) \Rightarrow \pi_{s+t}(z^*(X \times Y;.))$$

*Proof.* This is entirely similar to that of (4.1). One shows that under the hypotheses of the theorem the external product induces an isomorphism on the homotopy groups:

$$\times : z^*(X;.) \overset{L}{\underset{z^*(Speck;.)}{\otimes}} z^*(Y;.) \rightarrow z^*(X \times Y;.)$$

The only major difference is that for the localization sequence to exist, one needs to assume the varieties are also quasi-projective. (See [Bl-1] Theorem (3.1).). Now the spectral sequence in **DG.2** provides the required spectral sequence by viewing simplicial abelian groups as chain-complexes trivial in positive degrees. □

**Corollary 4.6.** *Assume the hypotheses of Theorem 4.5. Now one obtains the isomorphism $CH^*(X) \underset{\mathbb{Z}}{\otimes} CH^*(Y) \cong CH^*(X \times Y)$.*

*Proof.* Observe that the higher Chow complexes of Bloch are infact simplicial abelian groups. Therefore their homotopy groups are trivial in negative degrees. Therefore $E^2_{s,t} = 0$ for all $t$ negative or $s$ negative. It follows that for $s + t = 0$, one must have $s = 0$ and $t = 0$; therefore the only $E^2_{s,t}$-term that contributes to $CH^*(X \times Y) = \pi_0(z^*(X \times Y;.))$ is $E^2_{0,0} \cong CH^*(X) \underset{\mathbb{Z}}{\otimes} CH^*(Y)$. □

*Remarks* 4.7. See [Tot-1] Proposition 1 which has a different derivation of the above Kunneth formula at the level of the usual Chow groups. Also observe that the spectral sequence in algebraic K-theory only holds integrally since smashing with Moore-spectra does not in general preserve the $E^{\infty}$ or $A^{\infty}$ structure. On the other hand, the spectral sequence in motivic cohomology clearly holds also with mod-$l^{\nu}$ coefficients. *Observe also that the above Kunneth formulae with one of $X$ or $Y$ linear is an extension of the familar formulae for the G-theory and Chow groups of the product of a variety with an affine space, a projective space or a torus.*

## 5. The Strong Chow Kunneth decomposition implies mod-$l^{\nu}$ motivic cohomology is isomorphic to mod-$l^{\nu}$ étale cohomology

We will adopt the following conventions and notations throughout the rest of the paper. Let $k$ denote an algebraically closed field of characteristic $p \geq 0$, let $l$ denote a fixed prime different from $p$ and let $\nu > 0$ be an integer. A variety will mean a reduced scheme of finite type over $k$. The higher cycle map $CH^m(Z,\ \ 2m-n) \rightarrow H^n_{et}(Z; \mathbb{Z}/l^{\nu}(m))$ (see [Bl-2]) will be denoted $cl$. (The latter denotes $mod-l^{\nu}$ étale cohomology.) If $l$ and $\nu$ are as above, we will let $z^r(Z, \quad s; \quad \mathbb{Z}/l^\nu) = z^r(Z, \quad s) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/l^\nu$. The $s$-th homology group of the above chain complex will be denoted $CH^r(Z, \quad s; \quad \mathbb{Z}/l^\nu)$; we will identify this with the corresponding mod-$l^\nu$ *motivic cohomology group*. The induced cycle map $CH^m(Z, \quad 2m-n; \quad \mathbb{Z}/l^\nu) \rightarrow H^n_{et}(Z; \quad \mathbb{Z}/l^\nu(m))$ will be denoted $cl_{l^\nu}$. We will let $CH^r(Z, \quad s; \mathbb{Z}_l) = \lim_{\infty \leftarrow \nu} CH^r(Z, \quad s; \mathbb{Z}/l^\nu)$. We will identify this with the motivic cohomology group denoted $H^{2r-s}_M(X, \mathbb{Z}_l(r))$. (If $s = 0$, we will let $CH^r(Z; \mathbb{Z}_l)$ denote the above group.) The corresponding cycle map into $H^n_{et}(Z; \quad \mathbb{Z}_l(m)) = \lim_{\infty \leftarrow \nu} H^n_{et}(Z; \quad \mathbb{Z}/l^\nu(m))$ will be denoted $cl_l$.

**Definition 5.1.** Let $X$ denote a variety over $k$. We say that a cycle $\gamma \,\epsilon\, CH^*(X \times X; \mathbb{Z}_l)$ has the *strong Chow Kunneth decomposition* if there exist classes $\alpha_i$ , $\beta_i \,\epsilon\, CH^*(X; \mathbb{Z}_l)$ so that $\gamma = \Sigma_i \alpha_i \times \beta_i$.

The goal of this section is to prove the following theorem.

**Theorem 5.2.** *Let $X$ denote a projective nonsingular variety over $k$ so that the class of the diagonal $\Delta \,\epsilon\, CH^*(X \times X; \mathbb{Z}_l)$ has a strong Chow Kunneth decomposition. (i) Now the higher cycle-map*

$$cl_l : CH^r(X, \quad n; \quad \mathbb{Z}_l) \rightarrow H^{2r-n}_{et}(X; \mathbb{Z}_l(r))$$

*is an isomorphism for each fixed $n$, $r \geq 0$. In particular, both terms are trivial if $n$ is odd.*

*(ii) For each fixed integer $n \geq 0$, $\underset{r}{\oplus} CH^r(X, \quad n; \quad \mathbb{Z}_l) \cong \underset{r}{\oplus} H^{2r-n}_M(X, \quad \mathbb{Z}_l(r))$ is generated over $\underset{r}{\oplus} H^*_M$(Spec $k, \quad \mathbb{Z}_l(r)) \cong \underset{r}{\oplus} \mathbb{Z}_l(r)$ by classes $\{\alpha_i\} \subseteq CH^*(X; \mathbb{Z}_l)$. Moreover for each fixed pair of integers $u$ and $v$, the motivic cohomology $H^u_M(X, \quad \mathbb{Z}_l(v))$ is a free $\mathbb{Z}_l$-module which is trivial if $u$ is odd.*

*(iii) All the above assertions hold with $\mathbb{Z}_l$ replaced $\mathbb{Z}/l^\nu$.* □

*Proof.* We begin with the key hypothesis that if $[\Delta]$ denotes the class of the diagonal of $X \times X$ in $CH^*(X \times X; \mathbb{Z}_l)$, it decomposes in the form

$$[\Delta] = \Sigma_i(\alpha_i \times \beta_i) = \Sigma_i p_1^*(\alpha_i) \circ p_2^*(\beta_i), \quad \alpha_i, \quad \beta_i \,\epsilon\, CH^*(X; \mathbb{Z}_l) \tag{5.1}$$

(Here $p_i : X \times X \rightarrow X$, $i = 1, 2$ denotes the obvious projection to the $i$-th factor and $\circ$ denotes the intersection product on $CH^*(X \times X; \mathbb{Z}_l$. We will continue to let $\circ$ denote the intersection product on the higher Chow groups.) We may assume $\beta_i \,\epsilon\, CH^i(X; \mathbb{Z}_l)$ and that $\alpha_i \,\epsilon\, CH^{d-i}(X; \mathbb{Z}_l)$. Next let $n \geq 0$ denote a fixed integer. Now we obtain:

$$x = p_{1*}([\Delta] \circ p_2^*(x)), \quad x \,\epsilon\, CH^*(X, \quad n; \quad \mathbb{Z}_l) \tag{5.2}$$

where $\circ$ denotes the intersection product. By the projection formula and the observation that the class $[\Delta] = \Delta_*(1)$, $1 = [X] \,\epsilon\, CH^*(X; \mathbb{Z}_l)$ (see (A.1.3)), we obtain equality of the classes $[\Delta] \circ p_2^*(x) = \Delta_*(\Delta^*(p_2^*(x))$. Therefore $p_{1*}([\Delta] \circ p_2^*(x)) = p_{1*}(\Delta_*(\Delta^*(p_2^*(x)))) = (p_1 \circ \Delta)_*((p_2 \circ \Delta)^*(x)) = x$, for any class $x \,\epsilon\, CH^*(X, \quad n; \quad \mathbb{Z}_l)$. Now substitute the formula for $[\Delta]$ from (5.1) in (5.2) and use the projection formula to obtain:

$$x = \Sigma_i \alpha_i \circ p_{1*}(p_2^*(\beta_i \circ x)) \tag{5.3}$$

The cartesian square

$$\begin{array}{ccc} X \times X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1'} \\ X & \xrightarrow{p_2'} & Spec \quad k \end{array}$$

and flat base-change (see (A.1.4)) provide the identification $p_{1*}(p_2^*(\beta_i \circ x)) = {p_2'}^* p'_{1*}(\beta_i \circ x)$. Now the class $p'_{1*}(\beta_i \circ x) \,\epsilon\, CH^*(\text{Spec} \quad k, \quad n; \quad \mathbb{Z}_l)$.

(5.4) It follows that the classes $\{\alpha_i | i \,\epsilon\, I\}$ generate $CH^*(X, \quad n; \quad \mathbb{Z}_l)$ over $\underset{r,s}{\oplus} CH^r(\text{Spec} \quad k, \quad s; \quad \mathbb{Z}_l)$.

This proves the first assertion in (ii).

Observe from Suslin's computation of the mod-$l^\nu$ higher Chow groups of Spec $k$ and the observation that it readily extends to positive characteristic using de Jong's theory of alterations by Geisser (see [Sus-2], [Geis] Theorem (3.6) and [Oort]) that:

$$(5.5) \qquad CH^r(\text{Spec } k, \quad s; \quad \mathbb{Z}_l) \cong H^{2r-s}_{et}(\text{Spec } k; \mathbb{Z}_l(r)) \cong \mathbb{Z}_l(r), \qquad s = 2r$$
$$\cong 0, \qquad s \neq 2r$$

(For the sake of completeness, we add a couple of remarks on extending Suslin's computations to varieties over algebraically closed fields of positive characteristics. The main result we need is the following comparison theorem:

(5.6) **Theorem**(Suslin-Voevodsky) (See [S-V].) Let $k$ denote an algebraically closed field and let $\mathcal{F}$ be a homotopy invariant presheaf on the big site of schemes of finite type over $k$. If $\tilde{\mathcal{F}}_h$ is the sheafification of $\mathcal{F}$ for the $h$-topology and $l$ is prime to the characteristic of $k$, there exists a canonical isomorphism:

$$Ext^*_h(\tilde{\mathcal{F}}_h, \mathbb{Z}/l^\nu) \overset{\cong}{\to} Ext^*_{Ab}(\mathcal{F}(\text{Spec } k), \mathbb{Z}/l^\nu) \qquad \square$$

The $Ext$ on the left (on the right) is the one computed in the category of sheaves on the $h$-topology (is the one computed in the category of abelian groups, respectively ). In the original proof, the characteristic is restricted to be 0 so that one may apply resolution of singularities and thereby show that each scheme of finite type over $k$ admits an $h$-cover which is smooth over $k$. The theory of alterations due to de Jong (see [Oort]) enables one to find such a smooth $h$-cover making the original proof still valid in positive characteristics.)

Moreover, it follows from Proposition (5.3) (see below) that we may assume the above isomorphism is in fact given by Bloch's higher cycle maps. Observe also that now (5.4) and (5.5) show that $CH^*(X, \quad n; \quad \mathbb{Z}_l) \cong 0$ if $n$ is *odd*.

Next we show that *for each fixed integer* $n$, the $l$-adic cycle map

$$cl_l : CH^*(X, \quad n; \quad \mathbb{Z}_l) \to \underset{r}{\oplus} H^{2r-n}_{et}(X; \mathbb{Z}_l(r))$$

is *surjective*. Since $X$ is projective and smooth the projections $p_i$ induce maps $p_{i*} : H^*_{et}(X \times X; \mathbb{Z}_l(r)) \to H^{*-2d}_{et}(X; \mathbb{Z}_l(r-d))$ and $p_i^* : H^*_{et}(X; \mathbb{Z}_l(r)) \to H^*_{et}(X \times X; \mathbb{Z}_l(r))$ where $d$ is the dimension of $X$. Moreover the cycle-class of the diagonal $cl_l([\Delta])$ belongs to $H^{2d}_{et}(X \times X; \mathbb{Z}_l(d))$. Now $cl_l(\beta_i) \in H^{2i}_{et}(X; \mathbb{Z}_l(i))$. Let $x' \in H^{2r-n}_{et}(X; \mathbb{Z}_l(r))$. Now the formula

$$(5.7) \qquad x' = p_{1*}(cl_l([\Delta]) \cup p_2^*(x'))$$

holds by an argument as in the proof of (5.3). Moreover $cl_l([\Delta]) = \Sigma_i p_1^*(cl_l(\alpha_i)) \cup p_2^*(cl_l(\beta_i))$. Substituting this into (5.7) and using the projection formula, we obtain $x' = \Sigma_i cl_l(\alpha_i) \cup p_{1*}p_2^*(cl_l(\beta_i) \cup x')$.

Now $p_{1*}p_2^*(cl_l(\beta_i) \cup x') = {p'_2}^*(p'_{1*}(cl_l(\beta_i) \cup x'))$ and $p'_{1*}(cl_l(\beta_i) \cup x') \in H^{2i+2r-n-2d}_{et}(\text{Spec } k; \mathbb{Z}_l(i+r-d))$ since $cl_l(\beta_i) \cup x' \in H^{2i+2r-n}_{et}(X; \mathbb{Z}_l(i+r))$. By (5.5) and proposition (5.3) (see below), one observes the isomorphism $cl_l : CH^{i+r-d}(\text{Spec } k, \quad n; \quad \mathbb{Z}_l) \overset{\cong}{\to} H^{2i+2r-2d-n}_{et}(\text{Spec } k; \mathbb{Z}_l(i+r-d))$. Therefore one may write $p'_{1*}(cl_l(\beta_i) \cup x) = cl_l(\gamma_i)$ for some $\gamma_i \in CH^{i+r-d}(\text{Spec } k, \quad n; \quad \mathbb{Z}_l)$. It follows

$$(5.8) \qquad x' = \Sigma_i cl_l(\alpha_i) \cup cl_l({p'_2}^*(\gamma_i)) = \Sigma_i cl_l(\alpha_i \circ {p'_2}^*(\gamma_i))$$

This proves the surjectivity of the $l$-adic cycle map for each fixed integer $n$. Taking $n = 1$, it follows in particular that $H^i_{et}(X; \mathbb{Z}_l(r)) \cong H^i_{et}(X; \mathbb{Z}_l) = 0$ if $i$ is *odd*.

Next we will show the above cycle map is *injective* as well proving (i). To see this suppose $x \in CH^*(X, \quad n; \quad \mathbb{Z}_l)$ so that $cl_l(x) = 0$. Now $cl_l(p_{1*}p_2^*(\beta_i \circ x)) = p_{1*}p_2^*(cl_l(\beta_i) \cup cl_l(x)) = 0$ for all $i$. (See (A.1.7).) Since $p_{1*}p_2^*(\beta_i \circ x) = {p'_2}^* p'_{1*}(\beta_i \circ x)$, $cl_l(p_{1*}p_2^*(\beta_i \circ x)) = cl_l({p'_2}^* p'_{1*}(\beta_i \circ x)) = {p'_2}^*(cl_l(p'_{1*}(\beta_i \circ x))) = 0$. However ${p'_2}^* : \underset{r}{\oplus} H^*_{et}(\text{Spec } k; \mathbb{Z}_l(r)) \to \underset{r}{\oplus} H^*_{et}(X; \mathbb{Z}_l(r))$ is injective, it follows that $cl_l(p'_{1*}(\beta_i \circ x)) = 0$. Since the cycle class map $cl_l : CH^r(\text{Spec } k, \quad n; \quad \mathbb{Z}_l) \to H^{2r-n}_{et}(\text{Spec } k; \mathbb{Z}_l(r))$ is an isomorphism (see Proposition (5.3) below) it follows that $p'_{1*}(\beta_i \circ x) = 0$. Therefore ${p'_2}^*(p'_{1*}(\beta_i \circ x) = p_{1*}(p_2^*(\beta_i \circ x) = 0$ for all $i$. Now (5.3) shows that $x$ itself is trivial. This completes the proof of the assertion in (i).

Now we recall the short exact sequence (from [Lau] Proposition (2.2)(ii)):

$$0 \to Hom(Tors(H^{i+1}_{et}(X;\mathbb{Z}_l)), \mathbb{Q}_l/\mathbb{Z}_l) \to H^{et}_i(X;\mathbb{Z}_l) \to Hom(H^i_{et}(X;\mathbb{Z}_l), \mathbb{Z}_l) \to 0 \tag{5.9}$$

Here $Hom$ is in the category of $\mathbb{Z}_l$-modules and $Tors(H^{i+1}_{et}(X;\mathbb{Z}_l))$ denotes the torsion part of $H^{i+1}_{et}(X;\mathbb{Z}_l)$. (This is nothing other than the universal coefficient sequence and the first term is $Ext^1(H^{i+1}_{et}(X;\mathbb{Z}_l);\mathbb{Z}_l)$ computed using the injective resolution $\mathbb{Q}_l \to \mathbb{Q}_l/\mathbb{Z}_l$ of $\mathbb{Z}_l$.) Now let $i = 2j$ be an even integer. By the above arguments, $H^{i+1}_{et}(X;\mathbb{Z}_l) \cong 0$. Therefore $H^{et}_i(X;\mathbb{Z}_l) \cong Hom(H^i_{et}(X;\mathbb{Z}_l);\mathbb{Z}_l)$ is a free $\mathbb{Z}_l$ -module. Now Poincaré-duality shows that $H^i_{et}(X;\mathbb{Z}_l(r)) \cong H^{et}_{2d-i}(X;\mathbb{Z}_l(r-d)) \cong H^{et}_{2d-i}(X;\mathbb{Z}_l)$. These show that each $H^i_{et}(X;\mathbb{Z}_l(r))$ is a free $\mathbb{Z}_l$-module if $i$ is *even* and trivial if $i$ is *odd*. Now the observation that the $l$-adic cycle map is an isomorphism shows $CH^r(X, \quad n;\mathbb{Z}_l)$ is also a free $\mathbb{Z}_l$ module for all $r$ and $n$. Since all the modules we are considering are free over $\mathbb{Z}_l$, tensoring with $\mathbb{Z}/l^\nu$ now completes the proof of Theorem (5.2). □

**Proposition 5.3.** *The higher cycle map*

$$cl_{l^\nu} : CH^q(Spec \quad k, \quad p;\mathbb{Z}/l^\nu) \to H^{2q-p}_{et}(Spec \quad k;\mathbb{Z}/l^\nu(q)) \tag{5.10}$$

*is an isomorphism for each fixed integer $p \geq 0$.*

*Proof.* By the homotopy property one obtains the isomorphism:

$$CH^q(\text{Spec} \quad k, \quad p;\mathbb{Z}/l^\nu) \cong CH^q(\mathbb{A}^q, \quad p;\mathbb{Z}/l^\nu) \tag{5.11}$$

Now by Suslin's computation and the observation that it extends to positive characteristic by Geisser (see [Sus-2] and [Geis] Theorem (3.6))), $CH^q(\mathbb{A}^q, \quad p;\mathbb{Z}/l^\nu) \cong 0$ unless $p = 2q$ and $\cong \mathbb{Z}/l^\nu$ if $p = 2q$. Therefore the left-hand-side in (5.10) is trivial unless $p = 2q \geq 0$. i.e. we may assume that $p = 2q$. Moreover, since $CH^q(\text{Spec} \quad k, \quad 2q; \quad \mathbb{Z}/l^\nu) \cong CH^q(\mathbb{A}^q, \quad 2q; \quad \mathbb{Z}/l^\nu) \cong \mathbb{Z}/l^\nu$, it suffices to show that higher cycle map in (5.10) with $p = 2q$ is *surjective*. By the multiplicative property of the cycle map we now reduce to the case where $q = 1$ and $p = 2$. In this case, this follows immediately from [Bl-2] p. 73 which identifies $CH^1(X, \quad 2; \quad \mathbb{Z}/l^\nu)$ with $\Gamma(X, \mu_{l^\nu})$ for any scheme $X$. (Here $\mu_{l^\nu}$ is the sheaf of $l^\nu$-th roots of unity on $X$.) □

**Corollary 5.4.** *Let $X$ denote a projective nonsingular linear variety over an algebraically closed field $k$ of arbitrary characteristic. Now the conclusions of theorem (5.2) hold for $X$.*

*Proof.* This is clear. Observe that the strong Chow-Kunneth decomposition of the class of $\Delta \in CH^*(X \times X;\mathbb{Z}_l)$ follows from Corollary 4.6. □

**Corollary 5.5. Kunneth formula for the mod-$l^\nu$ higher Chow groups**. *Let $X$ denote a projective nonsingular linear variety and let $Y$ denote any quasi-projective variety over an algebraically closed field $k$ of arbitrary characteristic. Now the Kunneth spectral sequence*

$$E^2_{s,t} = Tor^{\pi_*(z^*(Spec \quad k),.;\mathbb{Z}/l^\nu)}_{s,t}(\pi_*(z^*(X,.;\mathbb{Z}/l^\nu)), \quad \pi_*(z^*(Y,.;\mathbb{Z}/l^\nu))) \Rightarrow \pi_{s+t}(z^*(X \times Y,.;\mathbb{Z}/l^\nu))$$

*in Theorem (4.5) degenerates and therefore one obtains the isomorphism:*

$$CH^*(X \times Y,.;\mathbb{Z}/l^\nu) \cong CH^*(X,.;\mathbb{Z}/l^\nu) \underset{CH^*(Spec \quad k,.;\mathbb{Z}/l^\nu)}{\otimes} CH^*(Y,.;\mathbb{Z}/l^\nu)$$

*Proof.* This is again clear from corollary (5.4), Theorem (4.5) and remark (4.7). □

## 6. Mod-$l^\nu$ Higher K-groups of linear varieties

In this section we make use several fundamental results in the literature on the mod-$l^\nu$ algebraic K-theory of algebraic varieties along with the results of the previous section to compute the mod-$l^\nu$ algebraic K-theory of coherent sheaves on linear varieties, where $l$ i is prime to the residue characteristics. If $l$ is as above, $K/l^\nu(Z)$ ($G/l^\nu(Z)$) will denote the smash product of $K(Z)$ ($G(Z)$, respectively ) with an appropriate Moore-spectrum. $K/l^\nu_{top}(Z)$ ( $G/l^\nu_{top}(Z)$) will denote the corresponding mod-$l^\nu$ topological K-theory (G-theory, respectively ) which may be obtained from $K/l^\nu(Z)$ ($G/l^\nu(Z)$, respectively ) by inverting the Bott element. (See [T]. The mod-$l^\nu$ topological K-theory may be identified with the mod-$l^\nu$ étale K-theory in [Fr-2].) It is shown in [T] (A.6) that, if $\nu$ is sufficiently large, (i.e. if $l > 3$, all $\nu \geq 1$ are allowed, while if $l = 3$, $\nu \geq 2$ and if $l = 2$, $\nu \geq 4$), the above spectra are homotopy associative and commutative. Therefore we will assume $\nu$ is sufficiently large throughout.

We will first recall the definition of the hypercohomology spectrum for presheaves of spectra essentially from [T]. Let $X$ denote a scheme of finite type over $k$ and let $F$ denote a presheaf of spectra on the big étale site of $X$. *Throughout we will restrict to presheaves of spectra $F$ so that for each integer $n$ and each $U$ in the site, $\pi_n(\Gamma(U, F))$ is finite with torsion prime to the characteristic.* Now we let $\mathcal{G}^{\cdot}F$ denote the cosimplicial object provided by the Godement resolution $\{\mathcal{G}F...\mathcal{G}^nF|n\}$ *on the small étale site* of $X$. (See for example [T](1.31).) We let

(6.1) $$\mathbb{H}(X; F) = \underset{\Delta}{\text{holim}}\, \Gamma(X, \mathcal{G}^n F)$$

which is the homotopy limit of the cosimplicial object $\{\Gamma(X, \mathcal{G}^nF)|n\}$. This is clearly functorial in $F$; one may also observe (using the properties of the homotopy inverse limit) that it preserves fibration sequences in $F$. It also preserves weak-equivalences in $F$ (provided $F$ is a presheaf of fibrant spectra). Now the usual spectral sequence for the homotopy inverse limit of a spectrum provides a spectral sequence:

(6.2) $$E_2^{s,t} = H^s_{et}(X;\quad \pi_t(F)\tilde{\ }) \Rightarrow \pi_{-s+t}(\mathbb{H}(X;\quad F))$$

This spectral sequence converges strongly since there is a uniform bound on the $l$-cohomological dimension of the schemes considered here. Finally observe that there is a natural augmentation $\Gamma(X; F) \to \mathbb{H}(X; F)$.

(6.3) Let $F = G$ be the presheaf $X \mapsto G(X)$. Quillen (see [Qu-2]) shows that now we obtain localization sequences in the following sense. Let $i : Z \to X$ denote a closed immersion with $j : U \to X$ the open immersion of its complement. In this situation we obtain cofibration sequence $G(Z) \to G(X) \to G(U)$ of spectra. Moreover he establishes the continuity property (see [Qu-2] I, section 2); this and the properties of homotopy inverse limits show that now we obtain the commutative diagram:

$$\begin{array}{ccccccc}
\Gamma(Z;F) & \longrightarrow & \Gamma(X;F) & \longrightarrow & \Gamma(U;F) & \longrightarrow & \Sigma\Gamma(Z;F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{H}(Z;F) & \longrightarrow & \mathbb{H}(X;F) & \longrightarrow & \mathbb{H}(U;F) & \longrightarrow & \mathbb{H}(Z;\Sigma F)
\end{array}$$

where each row is a cofibration sequence of spectra and the vertical maps are the obvious augmentation maps. (Here $F = G$.) Since smashing with a Moore-spectrum or inverting the Bott element preserves cofibration sequences of spectra, we obtain a similar commutative diagram with the presheaves $F = G/l^\nu$ and $F = G/l^\nu_{top}$.

**Proposition 6.1.** *Let $X$ denote a quasi-projective variety that has a decomposition into a finite number of strata each of which is isomorphic to an affine space. Now the natural map $G/l^\nu(X) \to \mathbb{H}(X; G/l^\nu(\quad))$ induces an isomorphism:*

$\pi_n(G/l^\nu(X)) \to \pi_n(\mathbb{H}(X; G/l^\nu(\quad)))$, $n \geq 0$

*Proof.* Let $d$ denote the dimension of $X$. Now let $U_0 =$ the union of the strata of dimension $d$; let $U_k - U_{k-1}$ denote the union of strata of dimension $d-k$. Now we obtain a commutative diagram of localization sequences:

$$\begin{array}{ccccccc}
\pi_{n+1}(G/l^\nu(U_0)) & \longrightarrow & \pi_n(G/l^\nu(X-U_0)) & \longrightarrow & \pi_n(G/l^\nu(X)) & \longrightarrow & \pi_n(G/l^\nu(U_0)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{n+1}(\mathbb{H}(U_0;G/l^\nu())) & \longrightarrow & \pi_n(\mathbb{H}(X-U_0;G/l^\nu())) & \longrightarrow & \pi_n(\mathbb{H}(X;G/l^\nu())) & \longrightarrow & \pi_n(\mathbb{H}(U_0;G/l^\nu()))
\end{array}$$

Since $U_0$ is a disjoint union of affine spaces, the first and last vertical maps are isomorphisms. To see this we need to compute the sheaves $U \to \pi_n(G/l^\nu(U))\tilde{\ }$, $U$ on the étale site of $X$. The computation of the K-theory of strictly Hensel local rings (and the proof of the Lichtenbaum-Quillen conjecture) by Suslin (see [Sus-1] and also [G-T]) shows that the sheaf $U \to \pi_n(G/l^\nu(U))\tilde{\ }$, $\quad U$ in the étale topology of $X$ is the constant sheaf $\mathbb{Z}/l^\nu$ if $n \geq 0$ is *even* and is trivial otherwise. Therefore the spectral sequence in (6.1), the homotopy property of $G/l^\nu$-theory and the homotopy property of étale cohomology with $\mathbb{Z}/l^\nu$-coefficients provide the required isomorphism. Observe that the dimension of $X - U_0$ is strictly less than $d$. Therefore, by ascending induction on the dimension of $X - U_0$ one may assume the second vertical map above is also an isomorphism. Now consider the case $n = 0$; observe that now the last map in the top row is surjective. Therefore a five-lemma argument shows that the map $\pi_0(G/l^\nu(X)) \to \pi_0(\mathbb{H}(X; G/l^\nu(\ \ )))$ is an isomorphism. If $n > 0$, the term following the last term on the top row (bottom row) is $\pi_{n-1}(G/l^\nu(X - U_0))$ ($\pi_{n-1}(\mathbb{H}(X - U_0; G/l^\nu(\ \ )))$, respectively ). The map $\pi_{n-1}(G/l^\nu(X - U_0)) \to \pi_{n-1}(\mathbb{H}(X - U_0; G/l^\nu()))$ is also an isomorphism by the inductive hypothesis on the dimension of $X$. Now a five lemma argument once again shows that the map $\pi_n(G/l^\nu(X)) \to \pi_n(\mathbb{H}(X; G/l^\nu(\ \ )))$ is also an isomorphism. □

**Corollary 6.2.** *(i) If $X$ is a variety as in (6.1) there exists a spectral sequence:*

$$E_2^{s,t} = H^s_{et}(X; \pi_t(G/l^\nu(\ \ ))\tilde{}\ ) \Rightarrow \pi_{-s+t}(G/l^\nu(X))$$

*(ii) The hypothesis of (6.1) is satisfied by all projective smooth toric and spherical varieties and by all projective smooth varieties provided with the action of a torus so that the fixed point scheme is discrete.*

*Proof.* (i) follows readily from (6.1) and the spectral sequence in (6.1). All varieties in (ii) come provided with the action of a torus that has only finitely many fixed points. Therefore they all come provided with a decomposition into strata that are affine spaces as shown by Bialynicki-Birula. (See [B-B].) □

Now we consider the following theorem.

**Theorem 6.3.** *(i) Let $X$ denote a projective non-singular toric variety or a projective nonsingular variety on which a torus acts with finitely many fixed points. Now there exists a spectral sequence*

$$E_2^{s,t} = H^s_{et}(X;\ \pi_t(K/l^\nu(\ \ ))\tilde{}\ \ ) \Rightarrow \pi_{-s+t}(K/l^\nu(X))$$

*Here $\pi_t(K/l^\nu(\ \ ))\tilde{}$ denotes the sheaf of $t$-th homotopy groups associated to the presheaf $U \to K/l^\nu(U)$, $U$ in the étale topology of $X$. Moreover the above spectral sequence degenerates at the $E_2$-level and $\pi_n(K/l^\nu(X))$ is trivial for all odd integers $n$ and negative integers while it is a free $\mathbb{Z}/l^\nu$ -module for all non-negative even integers $n$. Moreover one obtains an isomorphism:*

$$\pi_n(K/l^\nu(X)) \cong \underset{k}{\oplus} H^{2k}_{et}(X; \mathbb{Z}/l^\nu),\ \textit{for all } n \textit{ even, } n \geq 0.$$

*This isomorphism is not natural in $X$.*

*(ii) Assuming the existence of a mod $- l^\nu$ Bloch-Lichtenbaum spectral sequence (see [Bl-L]) for all projective non-singular linear varieties, the results of (i) extend to all projective non-singular linear varieties.*

*Proof.* The existence of the spectral sequence in (i) is clear from Corollary (6.2). Again we need to compute the sheaves $U \to \pi_n(G/l^\nu(U))\tilde{}$, $U \to \pi_n(G/l^\nu_{top}(U))\tilde{}$, $U$ on the étale site of $X$. The computation of the K-theory of strictly Hensel local rings (and the proof of the Lichtenbaum-Quillen conjecture) by Suslin (see [Sus-1] and also [G-T]) shows that the sheaf $U \to \pi_n(G/l^\nu(U))\tilde{}$, $U$ in the étale topology of $X$ is the constant sheaf $\mathbb{Z}/l^\nu$ if $n \geq 0$ is even and is trivial otherwise. It follows from Theorem (5.2) that the $E_2^{s,t}$ terms are all projective and hence free (since $\mathbb{Z}/l^\nu$ is a local ring) modules over $\mathbb{Z}/l^\nu$. Now the abutment has a filtration whose successive terms are these $E_2$-terms; it follows the abutment is a split extension of the $E_2$-terms. This proves (i) in Theorem (6.3).

Now we consider (ii). Recall the Bloch-Lichtenbaum spectral sequence is of the form:

$$E_2^{s,t} = H^{s-t}_M(X; \mathbb{Z}/l^\nu(-t)) \cong H^{s-t}_{et}(X; \mathbb{Z}/l^\nu(-t)) \Rightarrow \pi_{-s-t}(G/l^\nu(X))$$

Observe also that the differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. Clearly either $s-t$ or $s+r-(t-r+1) = s-t+(2r-1)$ is *odd*. Therefore Theorem (5.2)and ascending induction on $r$ shows that these differentials are all zero. It follows that the above spectral sequence degenerates and $E_2^{s,t} = E_\infty^{s,t}$ for all $s$, $t$. Since these are free modules over $\mathbb{Z}/l^\nu$ it follows that the abutment is in fact a split extension of these $E_2^{s,t}$-terms. This completes the proof of theorem (6.3). □

Recall from [T] the existence of a spectral sequence

$$E_2^{s,t} = H^s_{et}(X;\ \pi_t(K/l^\nu_{top}(\ \ ))\tilde{}\ \ ) \Rightarrow \pi_{-s+t}(K/l^\nu_{top}(X))$$

Here $\pi_t(K/l^\nu_{top}(\ \ ))\tilde{}$ denotes the sheaf of $t$-th homotopy groups associated to the presheaf $U \to K/l^\nu_{top}(U)$, $U$ in the étale topology of $X$. (This holds for all varieties over $k$.)

**Theorem 6.4.** *(i) Let $X$ denote a projective non-singular linear variety. Now the above spectral sequence degenerates at the $E_2$-level and $\pi_n(K/l^{\nu}_{top}(X))$ is trivial for all odd integers $n$ while it is a free $\mathbb{Z}/l^{\nu}$ -module for all even integers $n$. One also obtains an isomorphism:*

$$\pi_n(K/l^{\nu}_{top}(X)) \cong \underset{k}{\oplus} H^{2k}_{et}(X;\mathbb{Z}/l^{\nu}), \textit{ for all } n \textit{ even.}$$

*This isomorphism is, once again, not natural in $X$.*

*(ii) Let $X$ denote a variety provided with a stratification by finite strata each of which is isomorphic to the product of a torus and an affine space. The natural map $\pi_n(G/l^{\nu}(X)) \to \pi_n(G/l^{\nu}_{top}(X))$ is an isomorphism for all $n \geq d-1$ if $d$ is the dimension of $X$.*

*Proof.* Take the presheaf $F = G/l^{\nu}_{top}$ in (6.1) and use [T] Theorem (2.47), to obtain the spectral sequence in Theorem (6.4). (Observe that [T] (2.47) provides the weak-equivalence $\Gamma(X; G/l^{\nu}_{top})@ > \simeq >> \mathbb{H}(X; G/l^{\nu}_{top})$. The proof of the first assertion now follows along the same lines as the proof of Theorem (6.3). Next we consider the last assertion in Theorem (6.4). For this one considers the commutative diagram

$$\begin{array}{ccccccc}
\pi_n(G/l^{\nu}(X-U)) & \longrightarrow & \pi_n(G/l^{\nu}(X)) & \longrightarrow & \pi_n(G/l^{\nu}(U)) & \longrightarrow & \pi_{n-1}(G/l^{\nu}(X-U)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(G/l^{\nu}_{top}(X-U)) & \longrightarrow & \pi_n(G/l^{\nu}_{top}(X)) & \longrightarrow & \pi_n(G/l^{\nu}_{top}(U)) & \longrightarrow & \pi_{n-1}(G/l^{\nu}_{top}(X-U))
\end{array}$$

Here $U$ is the open dense stratum. The homotopy property shows that one may assume $U$ is a torus of dimension $\leq d$. In this case the computation in [Fr-2] Proposition (3.4) shows that the third vertical map is an isomorphism for all $n \geq d-1$. We may assume the same holds for the first and last vertical map using ascending induction. (Observe that the dimension of $X - U$ is strictly less than that of $X$.) Therefore a five lemma argument shows the second vertical map is also an isomorphism if $n \geq d-1$. □

*Remarks* 6.5. Theorem (6.3) shows that one may compute the higher mod-$l^{\nu}$ K-theory of the above varieties knowing their mod-$l^{\nu}$ étale cohomology. However the mod-$l^{\nu}$ topological K-theory of a non-singular toric or spherical variety is *not*, in general, isomorphic (in non-negative degrees) to the mod-$l^{\nu}$ K-theory as the simple example of the torus $\mathbb{G}^2_m$ will show. (See [Fr-2] Proposition (3.4).) Consequently the descent spectral sequence as in (6.3) does not exist in general for all non-singular linear varieties (i.e. unless one restricts to projective non-singular linear varieties). It is a recent result of Friedlander and Suslin (see [Fr-S]) that the conjectured Bloch-Lichtenbaum spectral sequence as in (6.3) (ii) indeed exists. Therefore the descent spectral sequence in (6.3)(i) in fact holds for all projective non-singular linear varieties.

Theorem (5.2) plays a key role in the computations in (6.3) and (6.4). The hypothesis that the variety be projective seems also essential in (5.2).

## 7. Examples

In this section we verify Theorems (5.2) and (6.3) by computing the mod$-l^{\nu}$-K-theory and mod-$l^{\nu}$ motivic cohomology of several toric surfaces and three-folds using *elementary methods* i.e. we do not use any of the sophisticated computations in [Sus-2] , but instead rely totally on the localization sequence and the resulting blow-up formula. We also give an example of a singular projective toric variety for which the strong Chow Kunneth decomposition for the class of the diagonal fails.

**Proposition 7.1.** *Let $i: Y \to X$ denote a regular closed immersion of smooth schemes over an algebraically closed field k. Let $X'$ denote the blow-up of $X$ along $Y$ and assume that $Y$ is of pure co-dimension $c$ in $X$. (i) Now there exists a weak-equivalence:*

$$G/l^{\nu}(X') \simeq G/l^{\nu}(X) \times \overset{(c-1)}{\Pi} G/l^{\nu}(Y)$$

*Taking the homotopy groups, one obtains:*

$$\pi_n(G/l^{\nu}(X')) \cong \pi_n(G/l^{\nu}(X)) \oplus \overset{(c-1)}{\oplus} \pi_n(G/l^{\nu}(Y)), \textit{ all } n.$$

*(ii) Similarly there exists an isomorphism:*

$$CH^*(X',n) \simeq CH^*(X,n) \oplus \overset{(c-1)}{\oplus} CH^*(Y,n)$$

*for each fixed integer $n$.*

*Proof.* The proof of (i) being very similar, we will consider (ii) only. We begin with the commutative diagram:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & CH^*(Y',n) & \longrightarrow & CH^*(X',n) & \longrightarrow & CH^*(X'-Y',n) & \longrightarrow & CH^*(Y',n-1) & \longrightarrow \dots \\
& \uparrow{\scriptstyle \pi_{Y*}} & & \uparrow{\scriptstyle \pi_X^*} & & \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \pi_{Y*}} & \\
\dots \longrightarrow & CH^*(Y,n) & \longrightarrow & CH^*(X,n) & \longrightarrow & CH^*(X-Y,n) & \longrightarrow & CH^*(Y,n-1) & \longrightarrow \dots
\end{array}$$

where $Y'$ is the exceptional divisor and $\pi_X : X' \to X$, $\pi_Y : Y' \to Y$ are the obvious maps. The two rows are long exact sequences by the localization theorem. This readily shows that the diagram

$$\dots \to CH^*(Y,n) \to CH^*(Y',n) \oplus CH^*(X,n) \to CH^*(X',n) \to CH^*(Y,n-1) \to \dots.$$

is a long exact sequence. Now the observation that $Y'$ is a projective space bundle over $Y$ (associated to the normal bundle of $Y$ in $X$), the computation in [Bl-1] Theorem (7.1), and the observation that $\pi_{Y*} \circ \pi_Y^* = id$ shows the above long exact sequences breaks up into split short exact sequences and also provides the isomorphism in (ii). □

Observe that since all the schemes are smooth, one may replace the mod$-l^\nu$ $G$-theory by the corresponding mod$-l^\nu$ K-theory.

One has a similar formula at the level of étale cohomology with $\mathbb{Z}/l^\nu$-coefficients. i.e.

$$H^*_{et}(X';\mathbb{Z}/l^\nu) \cong H^*_{et}(X;\mathbb{Z}/l^\nu) \oplus \overset{(c-1)}{\oplus} H^*_{et}(Y;\mathbb{Z}/l^\nu) \tag{7.1}$$

Now we begin by considering non-singular complete toric surfaces. It is shown in [Ful-1] p.43 that all such varieties are obtained by a finite sequence of blow-ups centered at the fixed points of the given torus starting with either the projective space $\mathbb{P}^2$ or the *Hirzebruch surface* $F_a$, which is a ruled surface.

**Proposition 7.2.** *Let $X$ denote a non-singular complete toric variety of dimension $\leq 2$ over an algebraically closed field $k$. For each even integer $n \geq 0$, one obtains an (abstract) isomorphism (of $\mathbb{Z}/l^\nu$-modules):*

$$\pi_n(K/l^\nu(X)) \cong H^*_{et}(X;\mathbb{Z}/l^\nu) = \underset{k}{\oplus} H^{2k}_{et}(X;\mathbb{Z}/l^\nu) \text{ and}$$

$$CH^*(X, \quad n; \quad \mathbb{Z}/l^\nu) \cong \underset{k}{\oplus} H^{2k}_{et}(X;\mathbb{Z}/l^\nu)$$

*Proof.* We will observe that the above result holds for points and nonsingular complete toric varieties of dimension 1 which are all isomorphic to $\mathbb{P}^1$. This follows from the computation of the mod-$l^\nu$ K-theory and motivic cohomology of projective spaces:

if $\mathcal{E}$ is a vector bundle of dimension $d > 1$ over a variety $Y$,

(7.2) $\pi_n(K/l^\nu(Proj(\mathcal{E}))) \cong \overset{d}{\oplus} \pi_n(K/l^\nu(Y))$, $CH^*(Proj(\mathcal{E});n) \cong \overset{d}{\oplus} CH^*(Y;n)$ and

$H^*_{et}(Proj(\mathcal{E});\mathbb{Z}/l^\nu) \cong \overset{d}{\oplus} H^*_{et}(Y;\mathbb{Z}/l^\nu)$

The above formula therefore proves the proposition for $\mathbb{P}^2$ and $F_a$. Now assume we are considering a toric surface $X$ obtained by $n$ successive blow-up of points starting with either $\mathbb{P}^2$ or $F_a$. Let $X_k$ denote the variety at the $k$-th stage. If $k = 1$, clearly the proposition is true for $X_1$, since now $X_k$ is either $\mathbb{P}^2$ or $F_a$. Therefore we may use ascending induction to prove the proposition. Assume $k$ is a fixed integer $> 1$ and the proposition is true for all $X_n$, $n < k$. Now $X_k$ is obtained from $X_{k-1}$ by blowing up a finite number of points. Therefore one may take $Y =$ a finite number of points in the formulae in (7.1) to prove the proposition for $X_k$. This completes the inductive step and hence the proof. □

We will conclude by computing the mod$-l^\nu$ K-theory of toric *Fano* three-folds. (See [Oda] p.90 for their classification.)

**Proposition 7.3.** *The same conclusion as in (7.3) holds for all toric Fano three-folds.*

*Proof.* Let $X$ denote such a three-fold. Observe that this is a non-singular projective toric variety of dimension 3. According to the classification in [Oda] p. 90, $X$ is obtained by successive blow-ups at a finite number of points or at a finite number of closed 1-dimensional non-singular toric sub-varieties by starting with either (i) $\mathbb{P}^3$, (ii) $\mathbb{P}^2 \times \mathbb{P}^1$, (iii) certain $\mathbb{P}^1$-bundles over $\mathbb{P}^2$, (iv) certain $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, (v) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, (vi) certain $\mathbb{P}^1$-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, (vii) $\mathbb{P}^1 \times F_1$, where $F_1$ is the Hirzebruch surface (as above), (viii) certain $\mathbb{P}^1$-bundles over $F_1$ and (ix) $\mathbb{P}^1 \times Y_2$, where $Y_2$ is the del Pezzo surface obtained from $\mathbb{P}^2$ by blowing up certain fixed points for the torus.

Now assume $X$ is one of the nine cases above. The conclusion of the proposition holds in case (i) by (7.3.1). They also hold for cases (iii) and (iv) by the same argument. Now consider $Y \times \mathbb{P}^n$; one may view this as a projective space over $Y$ and apply the computation in (7.3.1) and the corresponding ones in mod-$l^\nu$ cohomology. Therefore, one may see that the proposition holds in the remaining cases as well. Finally assume $X$ is obtained from one of the above varieties by a finite succession of equivariant blow-ups. Now one may apply an inductive argument as in the proof of (7.3) ( making use of (7.1)) to complete the proof. □

Next consider the projective, but singular toric threefold defined by the fan with edges through the vertices $(\pm 1, \pm 1, \pm 1)$ of a cube, in the sublattice of $\mathbb{Z}^3$ generated by these vertices (see [Ful] p. 105). We may assume for simplicity that the ground field is the field of complex numbers. Now it is an exercise in [Ful] p. 105 to show that if $X$ denotes this toric threefold,

$$CH^3(X) = H^6(X;\mathbb{Z}) = \mathbb{Z}, CH^0(X) = H^0(X;\mathbb{Z}) = Z \quad \text{and} \tag{7.3}$$
$$CH^2(X) = H^4(X;\mathbb{Z}) = \mathbb{Z}, H^3(X;\mathbb{Z}) = \mathbb{Z}^2, CH^1(X) = H^2(X;\mathbb{Z}) = \mathbb{Z}^5$$

Now consider the variety $Y = X \times X$. One may readily compute the mod-$l^\nu$ étale cohomology of this variety in degree 6 to be $(\mathbb{Z}/l^\nu)^{16}$. If $T$ denotes the dense torus in $X$, $X \times X$ is also a toric variety for the action of $T \times T$. Therefore its Chow groups are generated by the closures of the $T \times T$-orbits. (See [Ful] p. 96.) Therefore one may compute $CH^3(X \times X) = \mathbb{Z}^{12}$ and therefore $CH^3(X \times X; \mathbb{Z}/l^\nu) = (\mathbb{Z}/l^\nu)^{12}$. Thus the hypothesis of theorem 5.2 cannot be true for the variety $Y$. Observe that $Y$ is a toric variety that is projective, but singular.

Finally consider the example of the nonsingular, but affine toric variety $\mathbb{G}_m^2$. Now $CH^*(\mathbb{G}_m^2; \mathbb{Z}/l^\nu) = \mathbb{Z}/l^\nu$ while $H^*_{et}(\mathbb{G}_m^2; \mathbb{Z}/l^\nu) = \mathbb{Z}/l^\nu \oplus \mathbb{Z}/l^\nu$. Thus the hypothesis that the variety be projective is necessary in Theorem 5.2.

## 8. Appendix A: Properties of the higher Chow groups

In this appendix we collect together all the relevant properties of the higher Chow groups we use in this paper.

(A.0) For any scheme $X$ over $k$, let $z^i(X, j) = z^i(X \times \Delta_k[j])$ denote the free abelian group on the codimension $i$ cycles on $X \times \Delta_k[j]$. Now each face of $\Delta_k[j]$ defines a principal divisor on $X \times \Delta_k[j]$. Therefore one may define pull-backs by the face maps. One readily shows that $z^i(X, .)$ defines a chain complex with the differential $d : z^i(X, *) \to z^i(X, * - 1)$ defined by $d(\alpha) = \Sigma_i(-1)^i d_i^*(\alpha)$. We refer to this as the *cycle complex of Bloch.* The homology groups of this complex will be denoted $CH^i(X, .)$ or alternatively as $CH_{d-i}(X, .)$ where $d$ is the dimension of $X$ over $k$.

Next we recall some of the properties established by Bloch in ([Bl-1] ) for the higher cycle complexes and the higher Chow groups. We state these results integrally, but the same properties carry over to the mod-$l^\nu$ higher Chow groups.

(A.1.1) The higher cycle complex is *contravariantly functorial* for *flat* maps and *covariantly functorial* for *proper* maps with an appropriate shift. Moreover, the higher Chow groups are contravariantly functorial for arbitrary maps between smooth varieties.

(A.1.2) If $X$ and $Y$ are two schemes over $k$, there exists an external product $\times : CH^i(X, n) \otimes CH^j(Y, m) \to CH^{i+j}(X \times Y; n + m)$. If $X$ is a *smooth* variety over $k$, one obtains an internal product on $CH^*(X, .)$ by pulling back the external product using the diagonal. This will be denoted $\circ$.

If $f : X \to Y$ is a *proper* map between *smooth* varieties over $k$, one obtains the projection formula:

(A.1.3) $f_*(\alpha \circ f^*(\beta)) = f_*(\alpha) \circ \beta$, $\alpha \, \epsilon \, CH^*(X, p)$ and $\beta \, \epsilon \, CH^*(Y, q)$ for integers $p$ and $q$.

(A.1.4) Moreover, given a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with $f$, $f'$ *proper* and $g$, $g'$ *flat* of relative dimension $d$, we obtain:

$(f')_*(g'^*(\alpha) = g^* f_*(\alpha)$, $\alpha \, \epsilon \, z^i(X, j)$.

(A.1.5) *Homotopy property.* If $X$ is any scheme over $k$, and $\pi : X \times \mathbb{A}^1 \to X$ is the obvious projection, $\pi^* : z^*(X, .) \to z^*(X \times \mathbb{A}^1, .)$ is a quasi-isomorphism. (See [Bl-1] Theorem (2.1).)

(A.1.6) *Localization sequence.* Given a quasi-projective variety $X$ over $k$, with $Y$ a closed subvariety of pure codimension $c$ and $U = X - Y$ its complement, the obvious (restriction) homomorphism $z^*(X, .)/z^{*-c}(Y, .) \to z^*(U, .)$ is a quasi-isomorphism. (See [Bl-1] (3.1) )

(A.1.7) *Higher cycle maps.* There exist cycle maps

$cl^i(j) : CH^i(X, j) \to H^{2i-j}_{et}(X; \mathbb{Z}/l^\nu(i))$

which commute with push-forward by proper maps and pull-backs by smooth maps. (The cycle maps are established in [Bl-2].) Recall that Bloch's higher cycle maps are defined in terms of the usual cycle maps

$cl : z^r(X \times \Delta[s]) \to \lim_{\substack{\to \\ |Z|}} H^{2r}_{Z,et}(X \times \Delta[s]; \mathbb{Z}/l^\nu(r))$

where the colimit $\lim_{\substack{\to \\ |Z|}}$ denotes the colimit over all $|Z|$ when $Z$ runs through cycles in $z^r(X \times \Delta[s])$. Therefore, the commutativity with respect to proper push-forward and smooth pull-back follows from the fact that these properties hold for the usual cycle maps.

## 9. Appendix B: $A^{\infty}$ and $E^{\infty}$ differential graded algebras and their modules

We will adopt the approach from [K-M]. We first recall the definition of operads. Let $R$ denote a fixed commutative ring with unit. Let $\mathbf{A}$ denote the abelian tensor category of modules over $R$ and let $C(\mathbf{A})$ denote the category of all chain complexes in $\mathbf{A}$ with differentials of degree $+1$. We call the objects in $C(\mathbf{A})$ *differential graded objects* in $\mathbf{A}$. Now the tensor structure on $\mathbf{A}$ induces a tensor structure on $C(\mathbf{A})$ which we will denote by $\otimes$. An (algebraic) operad $\mathcal{O}$ in $C(\mathbf{A})$ is given by a sequence $\{\mathcal{O}(k)|k \geq 0\}$ of differential graded objects in $\mathbf{A}$ along with the following data:

(B.1.1) for every integer $k \geq 1$ and every sequence $(j_1, \ldots, j_k)$ of non-negative integers so that $\Sigma_s j_s = j$ there is given a map $\gamma_k : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ...\mathcal{O}(j_k) \rightarrow \mathcal{O}(j)$.

(B.1.2) The following associativity diagrams commute, where $\Sigma j_s = j$ and $\Sigma i_t = i$; we set $g_s = j_1 + ... + j_s$ and $h_s = i_{g_{s-1}+1} + \ldots + i_{g_s}$ for $1 \leq s \leq k$.

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) & \xrightarrow[\gamma \otimes id]{} & \mathcal{O}(j) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) \\
\downarrow{\scriptstyle shuffle} & & \downarrow{\scriptstyle \gamma} \\
 & & \mathcal{O}(i) \\
 & & \uparrow{\scriptstyle \gamma} \\
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j_s}{\underset{q=1}{\otimes}} \mathcal{O}(i_{g_{s-1}+q})) & \xrightarrow{id \otimes (\otimes_s \gamma)} & \mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(h_s))
\end{array}$$

In addition one is provided with a unit map $\eta : R \rightarrow \mathcal{O}(1)$ so that the diagrams

(B.1.3)
$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (R^{\overset{k}{\otimes}}) & \xrightarrow[\cong]{} & \mathcal{O}(k) \\
\downarrow{\scriptstyle id \otimes \eta^k} & \nearrow{\scriptstyle \gamma} & \\
\mathcal{O}(k) \otimes \mathcal{O}(1)^{\overset{k}{\otimes}} & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
R \otimes \mathcal{O}(j) & \xrightarrow[\cong]{} & \mathcal{O}(j) \\
\downarrow{\scriptstyle \eta \otimes id} & \nearrow{\scriptstyle \gamma} & \\
\mathcal{O}(1) \otimes \mathcal{O}(j) & &
\end{array}$$

*commute.*

(B.1.4) An $A^{\infty}$-operad is an algebraic operad so that each $\mathcal{O}(k)$ is acyclic and concentrated in non-positive degrees. Such an operad is an $E^{\infty}$-operad if each $\mathcal{O}(k)$ is provided with an action by the symmetric group $\Sigma_k$ so that the above diagrams are equivariant with respect to the actions by the appropriate symmetric groups. Moreover it is required that $\mathcal{O}(k)$ is, in each degree, a projective $R[\Sigma_k]$ module. (See [K-M] p. 14.) See ([K-M] pp. 134-138) for an explicit construction of algebraic operads in some of the typical situations we consider.

An $A^{\infty}$-differential graded algebra $\mathcal{A}$ over an $A^{\infty}$-operad $\mathcal{O}$ is an object in $C(\mathbf{A})$ provided with maps $\theta$ : $\mathcal{O}(j) \otimes \mathcal{A}^j \rightarrow \mathcal{A}$ for all $j \geq 0$ that are associative and unital in the sense that the following diagrams commute:

(B.2.1)
$$\begin{array}{ccc}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) \otimes \mathcal{A}^j & \xrightarrow[\gamma \otimes id]{} & \mathcal{O}(j) \otimes \mathcal{A}^j \\
\downarrow{\scriptstyle shuffle} & & \downarrow{\scriptstyle \theta} \\
 & & \mathcal{A} \\
 & & \uparrow{\scriptstyle \theta} \\
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \mathcal{A}^{j_1} \otimes ... \otimes \mathcal{O}(j_k) \otimes \mathcal{A}^{j_k} & \xrightarrow[id \otimes \theta^k]{} & \mathcal{O}(k) \otimes \mathcal{A}^k
\end{array}
\quad \text{and}$$

$$\begin{array}{ccc}
R \otimes \mathcal{A} & \xrightarrow[\simeq]{} & \mathcal{A} \\
\downarrow{\scriptstyle \eta \otimes id} & \nearrow{\scriptstyle \theta} & \\
\mathcal{O}(1) \otimes \mathcal{A} & &
\end{array}$$

(B.2.2) If $\mathcal{A}$ is an $A^{\infty}$ algebra over an operad $\mathcal{O}$ as above one defines a left $\mathcal{A}$-module $M$ to be an object in $C(\mathbf{A})$ provided with maps $\lambda : \mathcal{O}(j) \otimes \mathcal{A}^{j-1} \otimes M \rightarrow M$ satisfying similar associativity and unital conditions. Right-modules are defined similarly.

(B.2.1)' An $E^{\infty}$ algebra over an $E^{\infty}$-operad $\mathcal{O}$ is an $A^{\infty}$ algebra over the corresponding $A^{\infty}$-operad $\mathcal{O}$ so that the following diagrams commute (where $\sigma \quad in \quad \Sigma_j$):

$$\begin{array}{ccc} \mathcal{O}(j) \otimes \mathcal{A}^{\otimes^j} & \xrightarrow[\sigma \otimes \sigma^{-1}]{} & \mathcal{O}(j) \otimes \mathcal{A}^{\otimes^j} \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta} & \\ & \mathcal{A} & \end{array}$$

Given an $E^\infty$-algebra $\mathcal{A}$ over an $E^\infty$ operad $\mathcal{O}$, an $E^\infty$ left-module $M$ over $\mathcal{A}$ is an $A^\infty$ left-module $M$ so that the following diagrams commute:

(B.2.2)'

$$\begin{array}{ccc} \mathcal{O}(j) \otimes \mathcal{A}^{j-1} \otimes M & \xrightarrow[\sigma \otimes \sigma^{-1} \otimes id]{} & \mathcal{O}(j) \otimes \mathcal{A}^{j-1} \otimes M \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle\lambda} & \\ & M & \end{array}$$

If $\mathcal{A}$ denotes either an $A^\infty$ or $E^\infty$-algebra in $C(\mathbf{A})$, the category of all left modules (right modules) over $\mathcal{A}$ will be denoted $Mod_l(C(\mathbf{A}))$ ($Mod_r(C(\mathbf{A}))$, respectively ).

(B.2.3) As an example of operads one may apply the singular functor to the linear isometries operad discussed in detail in [K-M]. This will provide an operad in the category of all chain complexes of abelian groups.

(B.2.4) We conclude by defining $E^\infty$ *ring and module spectra* in an entirely parallel manner using the key observation that currently there is a strictly associative *smash product* at the level of spectra. A *topological operad* is given by a sequence $\{\mathcal{O}(k)|k \geq 0\}$ of spectra along with the data as in (B.1.1) -(B.1.3) where $\otimes$ denotes the above mentioned smash product of spectra. We may define an $A^\infty$-*topological operad* to be a topological operad where each $\mathcal{O}(k)$ is contractible; an $E^\infty$-*topological operad* will be an $A^\infty$-topological operad so that each $\mathcal{O}(k)$ is provided with the action of the symmetric group $\Sigma_k$ so that the corresponding diagrams are equivariant with respect to the action of the appropriate symmetric groups. Now one may define $E^\infty$ ring and module spectra exactly as in (B.2.1) and (B.2.2) where $\otimes$ denotes the smash product of spectra.

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