Algebraic cycles and motives: a bird's eye-view

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Outline

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Top spaces vs. Alg varieties

	Top Spaces	Alg varieties
Objects	Spaces, CW complexes,	Schemes, alg varieties

Basic idea: $H^*, \pi_* : (spaces) \rightarrow (groups)$

$H^*, \pi_* : (alg.vars) \rightarrow (groups)$

H^* = a suitable cohomology theory, π_*= suitable homotopy groups

The picture on the left - far too simple: (i) all coh theories characterized by E-S axioms (ii) the spaces have *nice* topologies (for example, most are Hausdorff) and (iii) the spaces usually have finite coh dimension.

Top spaces vs. Alg varieties (contd)

As a result the coh theories are singly graded by integers and can be easily defined with $\mathbb{Z}$-coeffs.

For schemes and alg. vars, the situation much more complicated.

- Schemes and alg. vars more rigid - need more sophisticated techniques. For example, the topology not so nice.
- For e.g. $H^i(X, A) = 0$ for $i > 0$ for any (irreducible) variety and constant sheaf A.
- Usually no well-behaved $\mathbb{Z}$-valued cohomology unless defined over $\mathbb{C}$.

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Top spaces vs. Alg varieties (contd)

Consequence: cohomology theories for schemes and vars-usually graded by degree and a weight and also defined for various primes ℓ, often different from the characteristic of the base field and often equal to it. No $\mathbb{Z}$ or $\mathbb{Q}$-valued coh theories with any good properties.

$H^*_{et}(X, \mathbb{Q}_\ell)$ ($\ell \neq char(k)$), $H^*_{cris}(X)$ ($\ell = char(k)$).

Main observation: these families of cohomology theories share many key important properties - a deep mystery.

Quote from A. GROTHENDIECK, Récoltes et semailles:

Contrary to what occurs in ordinary topology, one finds oneself confronting a disconcerting abundance of different cohomological theories. One has the distinct impression (but in a sense that remains vague) that each of these theories "amount to the same thing", that they "give the same results". In order to express this intuition, of the kinship of these different cohomological theories, I formulated the notion of "motive" associated to an algebraic variety. By this term, I want to suggest that it is the "common motive" (or "common reason")

Quotes from Grothendieck (contd)

behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible.

Again in Grothendieck's letter to Serre, 16.8.1964:

I call a "motif" over k something like an ℓ-adic cohomology group of an algebraic scheme over k, but considered as independent of ℓ, and with its "integral" structure, or, let us say for the moment "Q" structure, deduced from the theory of algebraic cycles. The sad truth is that for the moment I do not know how to define the abelian category of motives, even though I am beginning to have a rather precise yoga on this category.

Grothendieck's standard conjectures

Then Grothendieck restricted to projective smooth varieties over a field and formulated a series of conjectures now known as standard conjectures, several of them still unproven even now, though several of them are known in special cases.

We will discuss one of these soon.

Key role of K-theory

- $K_{top}(X)$ = Isom. classes of top vector bundles on X. Related to usual cohomology theories by Chern-character: $H^{even}(X, \mathbb{Q}) \cong K_{top}(X) \otimes \mathbb{Q}$ and there is an Atiyah-Hirzebruch spectral sequence:

$$E_2^{s,t} = H^s(X, \pi_{-t}(K)) \Rightarrow K_{top}^{s-t}(X)$$

- $K_{alg}(X)$ = Isom. classes of alg. vector bundles on X. Any candidate for "motivic cohomology" should then be similarly related to alg. K-theory as singular cohomology of a top. space is related to its top K-theory.

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Algebraic cycles

Similar to the definition of (singular) (co-)homology of a cell-complex: take the free abelian group on n-dimensional cells and mod-out by certain relations to define the n-th homology group: $H_n(X; \mathbb{Z})$.

For an alg variety X: take the free abelian group on dimension n integral subvarieties and mod-out by a relation, which will be a suitable form of homotopy: $CH_n(X)$.

The relation used is called *rational equivalence*.

In case X is a smooth variety, one can define a ring structure

Algebraic cycles (contd)

on $CH_*(X)$ by *moving* cycles so that they intersect in *general position*. This theory is classical: dates from the mid 40s or 50s.

But prior to the mid 1980s, only known in a weak-form. For example, $CH^*(X) \otimes Q \cong K^0_{alg}(X) \otimes \mathbb{Q}$, but no such relation known for higher K-theory. Moreover, no long-exact sequences for computing these Chow-groups. These provided by Spencer Bloch in the mid 1980s with the introduction of the *Higher Chow groups* which are bigraded groups $CH^*(X, \bullet)$.

Three equivalence relations on algebraic cycles

First define two other relations on cycles:

(i) *homological equivalence* $\gamma \ \sim_{hom} \gamma'$ if the associated cycles in a chosen cohomology theory are equal.

(ii) *numerical equivalence* $\gamma \ \sim_{num} \gamma'$ if $\gamma \bullet \alpha = \gamma' \bullet \alpha$ for all cycles α of complimentary dim.

Of these three *rational equivalence* the strongest and *numerical equivalence* the weakest. Motives can be defined using any of these three notions.

Correspondences

Typically there are only few maps between alg vars, so the coh. theories need to be functorial w.r.t to *multivalued functions*, which are in fact *correspondences*.

- $C^{cor,r}_{\sim}(X, Y) = C^{dim(X)+r}_{\sim}(X \times Y) \otimes \mathbb{Q}$
- $f : X \to Y \mapsto \Gamma_f$, Compose correspondences
- Naive def of $\mathcal{M}(k)$: objects smooth projective vars over k, morphisms $X \to Y$ given by $\Gamma \varepsilon C^{cor,r}_{\sim}(X, Y)$.

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Correspondences (cont)

- Improve to add images of idempotents or projectors. Now objects are pairs (X, e), X as before and $e \epsilon Cor_{\sim}(X, X)$ an idempotent. This suffices for the most part and the resulting category: effective motives w.r.t $\sim$.

Category of motives

- Objects: triples (X, e, m), X, e as before and $m \varepsilon Z$.
- $Hom((X, e, m), (Y, f, n)) = f \circ C^{n-m}_{\sim}(X, Y) \circ e$.
- Effective notives contained here by taking the integer $= 0$: amounts to inverting the *Tate motive*.
- $\mathcal{M}_{rat}(k)$: the resulting category for $\sim = rat$: Chow motives.
- $\mathcal{M}_{num}(k)$: the resulting category for $\sim = num$: Grothendieck (numerical) motives

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Chow motives

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- Objects (X, p), X pseudo-smooth projective equidimensional scheme, $p \varepsilon CH^*(X \times X) \otimes Q = C_{rat}(X, X)$, $p^2 = p$.

  Morphisms $(X, p) \rightarrow (Y, q) = q \circ Hom(X, Y) \circ p$, $Hom(X, Y)$ = Hom in the category of correspondences.
- Conjectured Bloch-Beilinson filtration on $CH^*(X)$: a descending filtration F^i so that $F^0 = CH^*_{\mathbb{Q}}(X)$, $F^1 = ker(cycl), \cdots$. with certain nice properties.

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The Chow-Kunneth projectors of Murre: an equivalent formulation

- $d = dim(X)$. Then X has a Chow Kunneth decomposition if there exist $\pi_i \varepsilon CH^d_{\mathbb{Q}}(X \underset{k}{\times} X)$ so that $[\Delta_X] = \Sigma_{i=0}^{2d} \pi_i$, $\pi_i \circ \pi_j = 0$, $i \neq j$ and $\pi_i \circ \pi_i = \pi_i$. If *cl* denotes the cycle map into any Weil cohomology, $cl(\pi_i)$ = a Kunneth component of $cl([\Delta_X])$.
- *Conjectures:I.* π_i acts trivially on $CH^j_{\mathbb{Q}}(X)$, $i < j$ and also for $i > 2j$. This equivalent to the existence of the Bloch-Beilinson filtration on $CH^*_{\mathbb{Q}}(X)$ with all the expected properties. For example: define $F^1_{|CH^j_{\mathbb{Q}}(X)} = ker(\pi_{2j})$, $F^2 = ker(\pi_{2j-1}|F^1)$, etc.

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Known for curves, surfaces, certain 3-folds and all Abelian varieties, Abelian schemes etc.

Further conjectures: Hard-Lefschetz

If $\mathcal{L}$ is an ample line bundle, its first Chern class defines a class $[L]$ in $CH^*(X \underset{k}{\times} X)$ so that each projector π_i admits a further decomposition and moreover, the iterated compositions induce isomorphisms $[L]^i : \pi_{d-i} \xrightarrow{\cong} \pi_{d+i}$.

Known till recently only for Abelian varieties.

In recent work, with Reza Akhtar, we extended some of these to quotients of Abelian varieties by finite group actions and in very recent work to some other related varieties.

Key properties of the categories of motives

- $\mathcal{M}_{\sim}(k)$: additive and not abelian in general. Only $\mathcal{M}_{num}(k)$ is known to be abelian and also semi-simple.
- Products exist: $(X, e, m) \times (Y, f, n) = (X \times Y, e \times f, m + n)$.
- Duals exist: $(X, e, m)^{\vee} = (X, e^t, dim(X) - m)$.
- Motive of X: $h(X) = (X, id_X, dim(X))$.

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Mixed motivic complexes

Some-what reminiscent of the ℓ-adic derived categories that one can associate to schemes, one has the following conjectural formalism for a motivic derived category.

$S \mapsto D_M(S)$, $(schemes/S) \to (derived\ categories)$ is supposed to exist with several key properties:

- Realization functors, $real : D_M(S) \to D(S_{et}, \mathbb{Q}_\ell)$ etc. exist which preserve many of the key properties
- For example, direct and inverse images for maps $f : X \to Y$ of schemes are compatible with the realization functors
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Mixed motivic complexes: Voevodsky's construction

He considers a derived category made up of complexes of abelian sheaves on the (big) Nisnevich topology on smooth schemes that have certain additional properties as a candidate.

Main missing property: existence of a *t*-structure whose heart is the catgeory of (mixed) motives of smooth schemes.

Other advances

A key advance is the existence of an analogue of the Atiyah-Hirzebruch spectral sequence whose E_2-terms are motivic cohomology and which converges to algebraic K-theory. (This was first worked out by Bloch and Lichtenbaum (still unpublished) and then by several others: Suslin, Friedlander, Levine, Grayson...)

Another key missing piece: the Beilinson-Soulé vanishing conjecture

For a topological space X, it is trivial that $H^i(X, Z) = 0$ for all $i < 0$.

The corresponding statement in the motivic context is:

$H^i_M(X, \mathbb{Z}(n)) = 0, i < 0$

which denotes motivic cohomology with weight n and degree i.

In fact this is needed to prove the existence of a t-structure on the derived category of motives.

Deep connections with arithmetic and number theoretic questions

First no surprise here, since algebraic geometry over the integers is arithmetic geometry.

Already these show up in étale cohomology: the number of rational point of a smooth projective varieties defined over a finite field can be computed by the Lefschtez-fixed-point-formula applied to the Frobenius.

Many open questions remain in this area.

Key points of motivic homotopy theory:I

To conclude this survey: Voedvodsky and Morel also introduced a formalism of homotopy groups in the motivic setting.

- A key underlying idea is to use the affine line $\mathbb{A}^1$ as the analogue of the unit interval I and build a homotopy theory for simplicial presheaves and sheaves on the category of all smooth schemes provided with certain Grothendieck topology.
- A big problem in carrying out this program is that the category of schemes, unlike the category of topological spaces, usually do not have colimits (i.e. direct limits) or quotients. This rectified formally by not working with schemes alone, but by working with all sheaves of sets defined on schemes with some topology.

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Key points of motivic homotopy theory:II

- In this framework, often called *motivic homotopy theory*, one uses techniques from classical homotopy theory to study algebraic varieties and schemes.
- The appropriate framework for all of this is Quillen's homotopy theory of model categories. Quillen, in the late 1960s had axiomatized homotopy theory and made it more like a non-abelian version of homological algebra. Motivic homotopy theory fits exactly into this general framework.
- Most generalized cohomology theories in algebraic topology have motivic analogues: motivic cohomology (analogue of singular cohomology), algebraic K-theory (analogue of topological K-theory), algebraic cobordism (analogue of complex cobordism), etc.

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The Nisnevich topology

As one finds out early on, the Zariski topology: too coarse to tell anything useful about the local structure of the variety. One solution: refine this topology by trying to use the classical inverse function theorem: a map $f : X \to Y$ is a local isomorphism if it induces an isomorphim on tangent spaces.

This gives the étale topology. But this topology does not have finite cohomological dimension, unlike the Zariski topology.

Intermediate between these two is the Nisnevich topology which seems to capture enough good properties of the étale topology and the Zariski topology. Much of the work of Morel and Voevodsky takes place with this topology.

Emerging Picture: Grothendieck was absolutely right!

He was absolutely right on pursuing algebraic cycles and motives as a unifying technique for unifying diverse cohomology theories for algebraic varieties.

Moreover, using algebraic cycles, the differences in technique for handling top.spaces and algebraic alg. varieties seem to disappear and many similarities emerge.

Expository References

J. Milne: *What is a motive?*: see www.jmilne.org/xnotes/MOT102.pdf

B. Mazur: *What is the motivation behind the theory of motives?*, see: www.math.harvard.edu/ mazur/preprints