THE STRONG KUNNETH DECOMPOSITION AND APPLICATIONS

ROY JOSHUA

Abstract. In this talk we will explore in detail the strong Kunneth decomposition for the class of the diagonal for the product of two schemes. In the first part of the talk I will discuss the strong Kunneth decomposition and its consequences in a general setting. The second part of the talk will focus on motivic cohomology with finite coefficients for schemes especially over algebraically closed fields.

1. Introduction

In the first part of the talk we extend and explore in detail some of the techniques and results obtained in earlier works of the author (see [J-1] and [J-2]) and several others (for example: [Pan], [LSW]) in connection with a strong Kunneth decomposition of the diagonal, especially for linear schemes over sufficiently general base schemes. (See the references for a detailed bibliography.)

Our results hold for sufficiently general cohomology and homology theories which range from algebraic K-theory to motivic and Deligne cohomology (the first two also with suitable coefficients). To state our results in as much generality as possible, we will first axiomatize the theories we consider. While motivic, absolute and Deligne cohomology are bigraded, algebraic K-theory is singly graded. This makes it necessary to consider both singly and doubly graded theories: we show how to pass from a doubly graded theory to a singly graded one. (As seen in the proof of Theorem 3.1, this simplifies even the statements of our results).

Here is a brief outline of the talk. To motivate our discussions, we consider, in section 2, an elementary example: namely a Kunneth decomposition for the class of the diagonal in the product of two projective spaces. Section 3 is devoted to the statement and proof of Theorem 3.1 which shows that a strong Kunneth decomposition of the diagonal implies what we call the relative Kunneth decomposition of cohomology. We end this section by interpreting Theorem 3.1 for various cohomology theories. The next section is devoted to the statement and proof of Theorem 4.3, which establishes a strong Kunneth decomposition for the diagonal for all linear schemes over smooth base schemes. We conclude the first part of the talk by explicitly considering various examples and counter examples. In the second part of the talk we will consider in detail the relation of the strong Kunneth decomposition with the higher cycle map; see section 6 for more details.

Conventions and terminology. Throughout the talk we will let S denote a smooth base scheme (often of finite type over a field: this restriction comes into play only when considering motivic cohomology.) We will let $(schemes/S)_{Zar}$ denote the category of all equidimensional schemes of finite type over S provided with the Zariski topology. We will also let $(schemes/S)_{Res.Zar}$ denote the restricted big Zariski site associated to $(schemes/S)$, i.e. where the morphisms are restricted to be flat maps and where coverings are Zariski open coverings as before. The only schemes we consider will be those belonging to these sites and often they will be assumed to be smooth. It may be often necessary to make further restrictions on the category of schemes we consider and also on the sites to be able to define some of the cohomology theories. Rather than discuss these issues here, we will consider them on a case by case basis.

If $\mathcal{C}$ denotes any of these sites and P is a presheaf of spectra (or a complex of abelian sheaves) on the same site and $X \varepsilon \mathcal{C}$, $\mathbb{H}(X; P)$ will denote the hypercohomology defined as in section 8 using the Godement resolution. We let $Presh(\mathcal{C})$ denote either a category of presheaves of spectra that is both complete and co-complete or a category of complexes of abelian sheaves that is both complete and co-complete. Moreover, we let $D(Presh(\mathcal{C}))$ denote the corresponding derived category which may be defined as follows. We will first consider the obvious homotopy category which will be an additive category. Then we will invert quasi-isomorphisms: a map between presheaves of spectra will be a quasi-isomorphism if it induces and isomorphism on the sheaves of homotopy groups whereas a map between co-chain complexes of abelian sheaves will be a quasi-isomorphism if it induces an isomorphism on the cohomology sheaves. To keep things uniform we will let $\pi_i(P)$ denote the sheaf of i-th homotopy groups if P is

These are private notes, of a preliminary nature and some of it based on unpublished work: not meant for further circulation.

a presheaf of spectra and $\mathcal{H}^{-i}(P)$ = the i-th cohomology sheaf of the complex P in case P is a complex of abelian sheaves. Moreover one may define distinguished triangles in $Presh(\mathcal{C})$ in the obvious manner: for presheaves of spectra these will denote fibration sequences whereas for complexes of abelian sheaves these will mean usual distinguished triangles.)

It is often convenient and even necessary that our *cohomology theories* be defined either by a presheaf of spectra P on one of the above sites or by a complex of abelian sheaves Γ. To obtain bigraded cohomology theories we will assume that the complex $\Gamma = \underset{r}{\oplus}\Gamma(r)$ where each $\Gamma(r)$ is a complex of abelian sheaves on the same site. In this context, $\Gamma(r)$ will be called the *weight* r-part of the complex Γ. We let $H^s(X, \Gamma(r)) = H^s(\mathbb{H}(X, \Gamma(r))$ with s called the *degree*. (To understand the role of weight, one may consider absolute cohomology defined below.) We assume the cohomology theories we consider are contravariant for arbitrary maps between smooth schemes and for flat maps between equidimensional schemes. If f is a flat map or if $f : X \to Y$ is an arbitrary map between smooth schemes, we assume that the induced map f^* is of the form $f^* : H^r(X, \Gamma(s)) \to H^r(X, \Gamma(s))$, i.e. f^* preserves both the degree and the weight.

The homology theories we consider will also be defined by either a presheaf of spectra P' or a complex of abelian sheaves Γ' on the restricted big Zariski site. To obtain bigraded homology theories we will assume that the complex $\Gamma' = \underset{r}{\oplus}\Gamma'(r)$ where each $\Gamma'(r)$ is a complex of abelian sheaves on the same site. We let $H_s(X, \Gamma'(r)) = H^{-s}(\mathbb{H}(X, \Gamma'(r))$ with s the *the degree* and r *the weight*. We will further assume that if $f : X \to Y$ is a proper map, there is a natural induced map $f_* : \mathbb{H}(X; P') \to \mathbb{H}(Y; P)$ ($f_* : \mathbb{H}(X; \Gamma'(r)) \to \mathbb{H}(Y; \Gamma'(r))$, respectively) and that it is natural in the derived category considered above. (It follows that the homology theories we consider are all Borel-Moore homology theories.) We will further require that if $f : X \to Y$ is a flat map of relative dimension d, there is an induced map of homology and that f^* maps $H_s(Y, \Gamma'(r)) \to H_{s+2d}(X, \Gamma'(r + d))$.

Examples 1.1. • To see that algebraic K-theory fits in this frame-work, we will let $P = \mathbf{K}$ the presheaf of spectra defining algebraic K-theory. The homology theory associated to algebraic K-theory is G-theory and clearly this is covariant for proper maps as required while being contravariant for flat maps. (One may also consider suitable completions of these presheaves in the sense of [B-K].) Observe that the K-theory and G-theory spectra obtained this way have Zariski cohomological descent so that both K-theory and G-theory may be realized as hypercohomology.

• To obtain motivic cohomology, we let $\Gamma(r) = \mathbb{Z}^{mot}(r)$ = the motivic complex of weight r. (To obtain motivic cohomology with R-coefficients, where R is a commutative ring with 1, one may consider $\Gamma(r) = \mathbb{Z}^{mot}(r) \otimes R$.) The motivic Borel-Moore homology theory will be the one defined by the complex as in [FSV] chapter 4, section 9. When restricting to quasi-projective schemes over a field k, one may identify the motivic Borel-Moore homology with the higher cycle complex of Bloch graded by the dimension of the cycles.

• Recall that absolute cohomology is defined as $H^i_A(X, \mathbb{Q}(j)) = K^{(j)}_{2j-i}(X) \otimes \mathbb{Q} = \{x \varepsilon K_{2j-i}(X) \otimes \mathbb{Q} | \psi^k(x) = k^j(x), k \geq 1\}$, where ψ is the Adams' operator. This is covariant for proper maps between smooth schemes and therefore one may define absolute homology for smooth schemes as $H^A_i(X, \mathbb{Q}(j)) = H^{2d-i}_A(X, \mathbb{Q}(d - j))$ if X is a smooth scheme of pure dimension d.

• Deligne cohomology and homology are defined for schemes of finite type over the field of complex or real numbers. (See [EV] and [Jan-2] for more details.)

• Lawson cohomology (often called morphic cohomology) and Lawson homology appropriately indexed. I leave it as an exercise to show these fit into the description given above.

• One may add *singular cohomology and homology* to this list: we define $H^*_{sing}(X, (r))$ by $H^s_{sing}(X, \mathbb{Z}(r))$ where $\mathbb{Z}(r) = \mathbb{Z}^{\overset{r}{\otimes}}$. The corresponding homology theory is Borel-Moore homology with $\mathbb{Z}(r)$-coefficients. Observe that since $\mathbb{Z}(r) \cong \mathbb{Z}$ for all r, weights are less important for this theory.

• If l is a prime different from the residue characteristics and $\nu ge 0$, one may consider $H^s_{et}(X, \mathbb{Z}/l^\nu(r))$ and the corresponding *étale homology* similarly.

Acknowledgments. We would like to thank, V. Srinivas and Stefan Müller-Stach for asking some questions at the Morelia conference on algebraic geometry that lead to the first few sections and which is based on [J-2]. The author also thanks Eric Friedlander, Burt Totaro, Zig Fiedorowicz and Rainer Vogt for discussions on the material in [J-1] and Spencer Bloch, Peter May, Patrick Brosnan and Bertrand Toen for discussions on the material in [J-3].

2. An elementary example: the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$

In any reasonable cohomology theory (as in section 3, for example), one has the following computation of the cohomology of a projective space:

$$H^s(\mathbb{P}^n_S, r) \cong \oplus_{i=0}^{i=n+1} H^{s-2i}(S, r-i)$$

Moreover as a graded algebra over $H^*(S,.)$, the cohomology of P^n_S is a truncated polynomial ring generated by an element of degree 2 and weight 1. Therefore one may readily obtain an isomorphism

$$H^*(\mathbb{P}^n_S \underset{S}{\times} \mathbb{P}^n_S, .) \cong H^*(\mathbb{P}^n_S, .) \underset{H^*(S,.)}{\otimes} H^*(\mathbb{P}^n, .).$$

It follows that the class of the diagonal may be written as a sum $\Sigma_i a_i \times b_{n-i}$ with a_i, $b_{n-i} \varepsilon H^*(\mathbb{P}^n, .)$. *We call such a decomposition a strong Kunneth decomposition of the class of the diagonal.*

Much less obvious is the fact that the structure sheaf of the diagonal has a bounded resolution by vector bundles on $\mathbb{P}^n \times \mathbb{P}^n$ each of the form $p_1^*(\mathcal{E}_i) \otimes p_2^*(\mathcal{F}_i)$ where $p_i : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ is the projection to the i-th factor and $\mathcal{E}_i$, $\mathcal{F}_i$ are vector bundles on $\mathbb{P}^n$. This is due to Beilinson: see [Be]. This has been generalized to Grassmanians and twisted Grassmanians. (See [Pan].)

Remarks 2.1. 1. Often showing the class of the diagonal has a strong Kunneth decomposition is easier than obtaining an explicit formula for the Kunneth decomposition.

2. When the base scheme S is not a field, obtaining a strong Kunneth decomposition for the diagonal in $X \underset{S}{\times} X$ is difficult. This will become clear in section 4.

3. In the next section we show that the strong Kunneth decomposition of the diagonal implies a relative Kunneth decomposition under fairly general hypotheses.

3. The strong relative Kunneth decomposition of cohomology

It should be remarked that in this section, it suffices to assume the cohomology theory is, at least in principle, part of a twisted duality theory in the sense of Bloch-Ogus. (See [B-O].) Accordingly we will denote $H^s(X, \Gamma(r))$ by $H^s(X,(r))$. Throughout this section we will make the following additional hypotheses on our cohomology theories. (Observe these hypotheses are not identical to the ones in [B-O], but are implied by them.)

(H.1): for every flat map $f : X \to Y$, there is an induced map $f^* : H^s(Y,(r)) \to H^s(X,(r))$ and this is natural in f.

(H.2): for every proper smooth map $f : X \to Y$ of relative dimension d, there is a push-forward $f_* : H^i(X;j) \to H^{i-2d}(Y; j-d)$ so that if $g : Y \to Z$ is another proper smooth map of relative dimension d', one obtains $g_* \circ f_* = (g \circ f)_*$. In this case the obvious projection formula $f_*(x \circ f^*(y)) = f_*(x) \circ y$, $x \varepsilon H^s(X,(r))$, $y \varepsilon H^s(Y,(r))$ holds.

(H.3) : for each smooth scheme X and closed smooth subscheme Y of pure codimension c, there exists a canonical class $[Y] \varepsilon H^{2c}(X;c)$. Moreover the last class lifts to a canonical class $[Y] \varepsilon H^{2c}_Y(X;c)$. (The latter has the obvious meaning in the setting Bloch-Ogus twisted duality theories. In case the cohomology theory is defined as hypercohomology with respect to a complex, we let $\mathbb{H}_Y(X;c) =$ the canonical homotopy fiber of the obvious map $\mathbb{H}(X;c) \to \mathbb{H}(X-Y;c)$; now $H^{2c}_Y(X,c) = H^{2c}(\mathbb{H}_Y(X;\Gamma(c)))$.) The cycle classes are required to pull-back under flat pull-back and push-forward under proper push-forwards.

(H.4): if X is a smooth scheme, there exists the structure of a *graded commutative ring* on $H^*(X;.) = \underset{r,s}{\oplus} H^r(X;s)$. i.e. $\circ : H^r(X;s) \otimes H^{r'}(X;s') \to H^{r+r'}(X;s+s')$. In addition to this, there exists an external product $H^r(X;s) \otimes H^{r'}(X;s') \to H^{r+r'}(X \times X; s+s')$ so that the internal product is obtained from the latter by pull-back with the diagonal.

For the purposes of this section it is also convenient to consider only cohomology theories that are singly graded or non-weighted. Given a bigraded cohomology theory $H^s(X;(r))$, we will re-index it as follows: we let

$$h^r(X; 2r-s) = H^s(X;(r)) \quad and \quad h^*(X;n) = \underset{r}{\oplus} h^r(X;n) \tag{3.0.1}$$

We view $\{h^*(X;n)|n\}$ as a singly graded cohomology theory. Observe that if $f: X \to Y$ is a proper smooth map of relative dimension d, the induced map f_* sends $h^*(X;n)$ to $h^*(Y;n)$. Similarly if $f: X \to Y$ is a flat map, the induced map f^* sends $h^*(Y;n)$ to $h^*(X;n)$.

Theorem 3.1. *Let $f: X \to Y$ denote a proper smooth map of smooth schemes of relative dimension d and let $[\Delta]\varepsilon H^{2d}(X \underset{Y}{\times} X; d)$ denote the class of the diagonal. Assume that $[\Delta] = \Sigma_{i,j} a_{i,j} \times b_{d-i,j}$, with each $a_{i,j}\varepsilon H^{2i}(X;i)$, $b_{d-i,j}\varepsilon H^{2d-2i}(X;d-i)$. Then for every fixed integer n one obtains the isomorphism:*

$$h^*(X;n) \cong h^*(X;0) \underset{h^*(Y;0)}{\otimes} h^*(Y;n) \tag{3.0.2}$$

Proof. We will first prove that the classes $\{a_{i,j}|i\}$ generate $h^*(X;n)$ as a module over $h^*(Y;.)$ i.e. the obvious map from the right hand side to the left hand side of 3.0.2 (which we will denote by ρ) is *surjective.*

Let $p_i : X \underset{Y}{\times} X \to X$ denote the projection to the i-th factor. For each $x\varepsilon h^*(X;n)$ we will first observe the equality :

$$x = p_{1*}(\Delta \circ p_{2*}(x)) \tag{3.0.3}$$

(To see this observe that $[\Delta] = \Delta_*(1)$, $1\varepsilon H^*(X;\Gamma(.))$. Therefore, $\Delta \circ p_{2*}(x) = \Delta_*(\Delta^* p_2^*(x))$ and hence $p_{1*}(\Delta \circ p_2^*(x)) = p_{1*}\Delta_*(\Delta^* p_2^*(x)) = (p_1 \circ \Delta)_*((p_2 \circ \Delta)^*(x)) = x$.)

Now we substitute $[\Delta] = \Sigma_{i,j} p_1^*(a_{i,j}) \circ p_2^*(b_{d-i,j})$ into the above formula to obtain:

$$\begin{aligned} x &= p_{1*}(\Sigma_{i,j} p_1^*(a_{i,j}) \circ p_2^*(b_{d-i,j} \circ p_2^*(x))) \\ &= p_{1*}(\Sigma_{i,j} p_1^*(a_{i,j}) \circ p_2^*(b_{d-i,j} \circ x)) \\ &= \Sigma_{i,j} a_{i,j} \circ p_{1*}p_2^*(b_{d-i,j} \circ x) \\ &= \Sigma_{i,j} a_{i,j} \circ f^*(f_*(b_{d-i,j} \circ x)) \end{aligned} \tag{3.0.1}$$

This proves the assertion that the classes $\{a_{i,j}|i\}$ generate $h^*(X;.)$ i.e. the map ρ is surjective.

The rest of the proof is to show that the map ρ is *injective.* The key is the following diagram:

$$\begin{array}{ccc} h^*(X;n) & \xleftarrow{\rho} & h^*(X,0) \underset{h^*(Y,0)}{\otimes} h^*(Y;n) \\ & \searrow^{\mu} & \downarrow \alpha \\ & & Hom_{h^*(Y;0)}(h^*(X,0), h^*(Y;n)) \end{array}$$

where the map α ($\mu(x)$, $x\varepsilon h^*(X,0)$) is defined by $\alpha(x \otimes y) =$ the map $x' \mapsto f_*(x' \circ x) \circ y$ (the map $x' \mapsto f_*(x' \circ x)$, respectively). The commutativity of the above diagram is an immediate consequence of the projection formula: observe $\rho(x \otimes y) = x \circ f^*(y)$. Therefore, to show the map ρ is injective, it suffices to show the map α is injective. For this we define a map β to be a splitting for α as follows: if $\phi\varepsilon Hom_{h^*(Y;0)}(h^*(X,0), h^*(Y;n))$, we let $\beta(\phi) = \Sigma_{i,j} a_{i,j} \otimes (\phi(b_{d-i,j}))$. Observe that $\beta(\alpha(x \otimes y)) = \beta(the \quad map \quad x' \to f_*(x' \circ x) \otimes y) = (\Sigma_{i,j} a_{i,j} \otimes f_*(b_{d-i,j} \circ x)) \otimes y$. Now observe that $f_*(b_{d-i,j} \circ x)\varepsilon h^*(Y;0)$ so that we may write the last term as $= (\Sigma_{i,j} a_{i,j} \circ f^* f_*(b_{d-i,j} \circ x)) \otimes y$. By (3.0.1), the last term $= x \otimes y$. This proves that α is injective and hence that so is ρ. □

We end this section by considering some explicit examples of what the last theorem implies for various cohomology theories.

Examples 3.2. 1. *K-theory.* In this case the theorem takes on the form:

$$K^i(X) \cong K^0(X) \underset{K^0(Y)}{\otimes} K^i(Y) \tag{3.0.2}$$

See [Pan] for a proof of this. The last part of the proof of the above theorem is clearly an adaptation of this argument.

2. The theorem takes on the following simple form in the case of any of the following bigraded cohomology theories: absolute, motivic or Deligne cohomology.

$$H^n(X;t) = (\underset{a}{\oplus} H^{2a}(X;a)) \underset{(\underset{a}{\oplus} H^{2a}(Y;a))}{\otimes} (\underset{a}{\oplus} H^{n-2a}(Y;t-a)) \tag{3.0.3}$$

Remarks 3.3. 1. Taking $n = 0$, the argument in the last part of the proof of the theorem shows that one obtains a non-degenerate pairing $< \quad , \quad >: h^*(X,0) \underset{h^*(Y,0)}{\otimes} h^*(X,0) \rightarrow h^*(Y,0)$ by $\alpha \otimes \beta \mapsto f_*(\alpha \otimes \beta)$.

2. Next assume that $Y = Spec \quad k$ and that $h^*(\quad ,.) = CH^*(\quad ,0;\mathbb{Q})$. Now we leave it an easy exercise to verify that a strong Kunneth decomposition of the diagonal implies a Kunneth decomposition of the diagonal in the sense of Murre. (See [Murre] or [Jan-2].) Extra credit: show that if $k = \mathbb{C}$, show that the cycle map into singular cohomology with rational coefficients is an isomorphism and maps the Kunneth components of the diagonal to the Kunneth components of the class of the diagonal in singular cohomology.

3. As an example of the usefulness of the last theorem or the formula in 2, we consider the following result. Let H denote motivic cohomology. Recall that the Beilinson-Soulé vanishing conjecture for the motivic cohomology of a scheme is the following statement: $H^s(X,r;\mathbb{Q}) = 0$ if $s < 0$ or if ($s = 0$ and $r \neq 0$) while $H^0(X,0;\mathbb{Q}) = \mathbb{Q}$. Now we leave it as an easy exercise to prove from 2 the following proposition.

Proposition 3.4. *If $X \rightarrow Y$ satisfies the hypotheses of 3.1 and are both connected, then the Beilinson-Soulé vanishing conjecture holds for X if it holds for Y.*

Exercise 3.5. Show, using the above proposition and Remarks 4.7, 4 of the next section that the Beilinson-Soulé vanishing conjecture will hold for all schemes in the list in example 4.2 with S a field k provided it holds for the field k.

4. The strong Kunneth decomposition for the diagonal for linear schemes

Throughout this section we will mostly consider homology theories, which when the schemes we consider are smooth will be assumed to coincide with cohomology theories via Poincaré duality. More specifically we will make the following assumptions. We will assume that the homology theories are defined as hypercohomology with respect to a complex of sheaves as in section 1.

(H1): *Localization sequences.* If $Y \rightarrow X$ is a closed immersion with open complement U, one obtains the distinguished triangle: $\mathbb{H}(Y;\Gamma') \rightarrow \mathbb{H}(X;\Gamma') \rightarrow \mathbb{H}(U;\Gamma')$ which, in the case of weighted homology theories, preserves the weights. i.e. For each integer r, one obtains a distinguished triangle

$$\mathbb{H}(Y;\Gamma'(r)) \rightarrow \mathbb{H}(X;\Gamma'(r)) \rightarrow \mathbb{H}(U;\Gamma'(r))$$

(H2): *Homotopy property* If $\pi : X \times \mathbb{A}^n \rightarrow X$ denotes the obvious projection, $\pi^* : \mathbb{H}(X;\Gamma') \rightarrow \mathbb{H}(X \times \mathbb{A}^n;\Gamma')$ is a quasi-isomorphism.

(H3): If X is a smooth scheme of pure dimension, one obtains a quasi-isomorphism $\mathbb{H}(X,\Gamma') \simeq \mathbb{H}(X,\Gamma)$. In the case of weighted homology theories this quasi-isomorphism is induced by quasi-isomorphisms: $\mathbb{H}(X,\Gamma'(r)) \simeq \mathbb{H}(X,\Gamma(d-r))[2d]$ where d is the dimension of X. (i.e. $H_s(X,r) \cong H^{2d-s}(X,d-r)$.

(H4): *E^∞-DGA structure on cohomology.* We assume that there is given an E^∞ algebra $\mathcal{A}$ in *Presh* quasi-isomorphic to Γ that defines the cohomology. (In case Γ is a graded object we will assume that $\mathcal{A}$ is also compatibly graded.) If X is a smooth scheme, let $\mathcal{A}_{|X}$ denote the restriction of $\mathcal{A}$ to the Zariski site of X. Now we require, in addition that, if $f : Z \rightarrow X$ is a scheme of finite type over X, the complex $\Gamma'_{|Y}$ (= the restriction of Γ' to the Zariski site of Y) has the structure of an E^∞-module over the presheaf of E^∞-algebras $f^{-1}\mathcal{A}_{|X}$.

Next we recall the definition of linear schemes over a given base-scheme from [Jan-1] or [J-1].

Definition 4.1. Let S denote a base-scheme. (i) A scheme *flat* over S is 0-linear if it is either empty or isomorphic to any affine space $\mathbb{A}^n_S$.

(ii) Let $n > 0$ be an integer. An S-scheme Z, flat over S, is n-linear, if there exists a triple (U, X, Y) of S-schemes so that $Y \subseteq X$ is an S-closed immersion with U its complement, Y and one of the schemes U or X is $(n-1)$-linear and Z is the other member in $\{U, X\}$. We say Z is linear if it is n-linear for some $n \geq 0$.

(iii) If S is the spectrum of a field k, any reduced scheme X of finite type over k will be called a variety. Linear varieties over k are varieties over k that are linear schemes.

Example 4.2. The following are common examples of linear schemes over a given base scheme S. (All the tori appearing in the strata will be assumed to be defined and split over S.)

- All toric schemes over S

- All spherical schemes over S. (A scheme X over S is spherical if there exists a reductive group G scheme defined over S acting on X so that there exists a Borel subgroup scheme S having a dense orbit. The orbits are products of tori and affine spaces (over S).)
- Any scheme over S on which a connected solvable group scheme defined over S acts with finitely many orbits. (For example projective spaces and flag schemes over S.)
- Any scheme over S that has a stratification into strata each of which is the product of a torus with an affine space.

Theorem 4.3. *Let $f : X \to Y$, $g : Z \to Y$ denote proper flat maps with Y smooth. We will further assume that one may find closed immersions $X \to \bar{X}$ and $Z \to \bar{Z}$ with $\bar{X}$ and $\bar{Z}$ proper smooth over Y.*

(i) In this case there exists a natural map $\mathbb{H}(X, \Gamma') \overset{L}{\underset{\mathbb{H}(Y,\Gamma)}{\otimes}} \mathbb{H}(Z, \Gamma') \to \mathbb{H}(X \underset{Y}{\times} Z, \Gamma')$

(ii) This map is a quasi-isomorphism if X or Z is a linear scheme over Y.

Proof. We will first consider the case where X and Z are smooth. In this case we may may use the properties (H3) and (H4) to replace Γ' by $\mathcal{A}$. We will denote the restriction of $\mathcal{A}$ to the Zariski site of Y (X, Z) by $\mathcal{A}_{|Y}$ ($\mathcal{A}_{|X}$, $\mathcal{A}_{|Z}$, respectively). Therefore the existence of a natural map $\mathbb{H}(X, \mathcal{A}_{|X}) \overset{L}{\underset{\mathbb{H}(Y,\mathcal{A}_{|Y})}{\otimes}} \mathbb{H}(Z, \mathcal{A}_{|Z}) \to \mathbb{H}(X \underset{Y}{\times} Z, \mathcal{A})$ is a formal consequence of the E^{∞}-DGA structure. The hypotheses show that, in general we may imbed X and Z into schemes proper and smooth over Y. Making use of the localization sequences as in (H1) and the observation that the derived functor of the tensor product functor preserves distinguished triangles in either argument, one may now complete the proof of (i) readily.

To show this map is a quasi-isomorphism, we use the inductive definition of linear schemes. Observe that if X or Z is an affine space over Y, then the homotopy property as in (H2) applies to show the above map is a quasi-isomorphism. (See [J-1] section 4 for more details.) □

Remark 4.4. Observe that the second hypothesis in the theorem is satisfied if X and Z are quasi-projective over Y.

To consider the Kunneth formula in the next corollary it is again convenient to re-grade bigraded (weighted) homology theories by a single degree. This is done exactly as for cohomology. i.e. We let $h_r(X, s-2r) = H_s(X; \Gamma'(r))$ and $h_*(X, n) = \underset{r}{\oplus} h_r(X, n)$.

Corollary 4.5. *(Kunneth formula) Assume that $h_*(V, n) = 0$ for all $n < 0$ and all schemes V. Then one obtains a natural isomorphism:*

$$h_*(X, 0) \underset{h_*(Y,0)}{\otimes} h_*(Z, 0) \overset{\cong}{\to} h_*(X \underset{Y}{\times} Z, 0) \tag{4.0.1}$$

Proof. One considers the Kunneth spectral sequence:

$$E^2_{s,t} = Tor^{h_*(Y,.)}_{s,t}(h_*(X, .), h_*(Z, .)) \Rightarrow h_*(X \underset{Y}{\times} Z; s+t).$$

(The above $E^2_{s,t}$-term is the s-th Tor and the t-graded piece.) In view of the hypotheses, $E^2_{s,t} = 0$ for all $s < 0$ or $t < 0$. Therefore the only $E^2_{s,t}$-term that contributes to $h_*(X \underset{Y}{\times} Z; 0)$ is $E^2_{0,0}$. □

Corollary 4.6. *Under the hypotheses of the previous corollary, the hypotheses and therefore the conclusions of Theorem 3.1 hold for any linear scheme smooth and proper over a smooth scheme Y.*

Proof. This is clear in view of the last corollary. □

Remarks 4.7. 1. The hypotheses (H4) seems necessary to prove Theorem 4.3 for Y any general smooth scheme. In case Y is the spectrum of a field, the schemes X and Z are quasi-projective over k and the homology theory is motivic homology (higher Chow groups), one can prove the last corollary as in [Tot-1]; however this uses the special property that $CH^*(Speck \quad k) = \mathbb{Z}$ and therefore does not apply in any more generality.

2. The list of linear schemes above show that the conclusions of theorems 3.1 and 4.3 apply to a large collection of schemes over a base scheme.

3. It is shown in [J-1] that the hypotheses (H1) through (H4) are satisfied by G-theory. The hypotheses (H1) through (H3) are satisfied by most homology theories, for example motivic homology. However to show (H4) is

satisfied by motivic homology, one needs to invoke the results of [J-3] which establishes the existence of an E^∞-DGA in this case.

4. If $h_*(\ \ ;0) = G_0$, in the last corollary, we obtain a relative Kunneth formula in the Grothendieck group of coherent sheaves. (This uses the results of [J-1].) If we take $h_*(\ \ ;0) = CH_*(\ \ ;0)$ and restrict to quasi-projective schemes over a field k, we obtain a relative Kunneth formula for the Chow groups assuming the results of [J-3].

5. Examples and Counter Examples

In this section we will consider some examples and counter examples for our main results.

1. Beilinson's resolution of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$. The terms of the resolution are given by $p_1^*(\Lambda^i \mathcal{E}^*) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^n}(-i))$. (See [Be].) Here $\mathcal{E}$ is the universal rank-n quotient bundle on $\mathbb{P}^n$.

2. This has been generalized to flag manifolds in characteristic 0 by Kapranov (see [Kap]) and to twisted flag manifolds by Panin (see [Pan]).

3. Another explicit formula for the class of the diagonal for G/B (and G/P) has been obtained by Brion (see [Br-P]).

4. Theorem 3.1 is false unless the map $f : X \to Y$ is proper. To see this take $Y = Spec\ \ k$, $X = \mathbb{G}_m$. Now one knows $K^i(X) = K^i(Y) \oplus K^{i-1}(Y)$ (see [Qu-2]), whereas $K^0(X) = K^0(Y)$. Therefore, $K^1(X)$ is not isomorphic to $K^0(X) \underset{K^0(Y)}{\otimes} K^1(Y)$.

6. The higher cycle complex and the higher cycle maps of Bloch

Next we will restrict to varieties defined over an algebraically closed field and explore in detail some further consequences of the strong Kunneth decomposition for the diagonal. We will begin by recalling the definition of the higher cycle complex and the higher cycle maps: these are due to Bloch. In the next section we consider one the main theorem of this talk which shows that the strong Kunneth decomposition of the diagonal is a necessary and sufficient condition for the higher cycle maps from mod-l^ν motivic cohomology to mod-l^ν étale cohomology is an isomorphism. In the following section we explore some consequences of this theorem for the mod-l^ν algebraic K-theory of linear varieties. We conclude by discussing various examples and counter examples.

For any scheme X over k, let $z^i(X, j) = z^i(X \times \Delta_k[j])$ denote the free abelian group on the codimension i cycles on $X \times \Delta_k[j]$. Now each face of $\Delta_k[j]$ defines a principal divisor on $X \times \Delta_k[j]$. Therefore one may define pull-backs by the face maps. One readily shows that $z^i(X, .)$ defines a chain complex with the differential $d : z^i(X, *) \to z^i(X, * - 1)$ defined by $d(\alpha) = \Sigma_i(-1)^i d_i{}^*(\alpha)$. We refer to this as the *cycle complex of Bloch* or the *higher cycle complex*. The homology groups of this complex will be denoted $CH^i(X, .)$ or alternatively as $CH_{d-i}(X, .)$ where d is the dimension of X over k. If l is a prime different from the characteristic of k and $\nu > 0$ is an integer, we may consider the complex $z^i(X, *) \otimes \mathbb{Z}/l^\nu$. The homology groups of this complex will be denoted $CH^i(X, .; \mathbb{Z}/l^\nu)$. (If R is any commutative ring with 1, one may define $CH^*(X, .; R)$ in an entirely similar manner.) One may take the inverse limit over $\nu \to \infty$ to obtain the corresponding l-adic higher Chow-groups. These will be denoted $CH^i(X, .; \mathbb{Z}_l)$.

Next we recall some of the properties established by Bloch in ([Bl-1]) for the higher cycle complexes and the higher Chow groups. We state these results integrally, but the same properties carry over to the mod-l^ν higher Chow groups.

(P.1) The higher cycle complex is *contravariantly functorial* for *flat* maps and *covariantly functorial* for *proper* maps with an appropriate shift. Moreover, the higher Chow groups are contravariantly functorial for arbitrary maps between smooth varieties.

(P.2) If X and Y are two schemes over k, there exists an external product $\times : CH^i(X, n) \otimes CH^j(Y, m) \to CH^{i+j}(X \times Y; n + m)$. If X is a *smooth* variety over k, one obtains an internal product on $CH^*(X, .\ )$ by pulling back the external product using the diagonal. This will be denoted $\circ$.

(P.3) If $f : X \to Y$ is a *proper* map between *smooth* varieties over k, one obtains the projection formula:

$f_*(\alpha \circ f^*(\beta)) = f_*(\alpha) \circ \beta$, $\alpha \varepsilon CH^*(X, p\ )$ and $\beta \varepsilon CH^*(Y, q)$ for integers p and q.

(P.4) Moreover, given a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with f, f' *proper* and g, g' *flat* of relative dimension d, we obtain:

$(f')_*(g'^*(\alpha) = g^* f_*(\alpha)$, $\alpha \varepsilon z^i(X, j)$.

(P.5) *Homotopy property.* If X is any scheme over k, and $\pi : X \times A^1 \to X$ is the obvious projection, $\pi^* : z^*(X,.) \to z^*(X \times \mathbb{A}^1,.)$ is a quasi-isomorphism. (See [Bl-1] Theorem (2.1).)

(P.6) *Localization sequence.* Given a quasi-projective variety X over k, with Y a closed subvariety of pure codimension c and $U = X - Y$ its complement, the obvious (restriction) homomorphism $z^*(X,.)/z^{*-c}(Y,.) \to z^*(U,.)$ is a quasi-isomorphism. (See [Bl-1] (3.1))

The next remaining property is the existence of higher cycle maps. Since this is a key property for us, we will recall Bloch's definition of these maps in detail.

6.1. **The Higher Cycle maps.** Though we will be interested only in cycle maps into l-adic étale cohomology, we will give Bloch's general construction valid also for other cohomology theories. (See (Bl-2) for more details.) Let $\{H^s(X,t)|s,t\}$ denote a bigraded cohomology theory as in the first part of the talk. In addition to the hypotheses there, we will require *weak purity* of cohomology: i.e. if $Y \subseteq X$ is a closed subscheme of pure codimension b, then $H^a_Y(X, b) = 0$ for all $a < 2r$. We will also require that this cohomology theory has the *homotopy property*, i.e. $H^a(X \times \mathbb{A}^n, b) \cong H^a(X, b)$ for all a and b.

To each scheme X we associate the diagram

(6.1.1) $$X \rightrightarrows X \times \Delta[1] \quad \cdots X \times \Delta[2] \quad \cdots$$

On applying hypercohomology with respect to $\Gamma(b)$ to this diagram, one obtains a spectral sequence as the spectral sequence of the total complex associated to the resulting double complex. Since the resulting complex will be unbounded (in general), there is a convergence problem. However, using the homotopy property and the observation that each $\Delta[n] \cong \mathbb{A}^n$, one may make the following observations: $E_2^{p,q} = H^q(X, b)$ if $p = 0$ and $= 0$ if $p \neq 0$. Moreover the boundary maps on the E_1 terms are either 0 or isomorphisms. Therefore we may write:

$E_1^{p,q} \Rightarrow H^*(X, b)$.

In order to make use of the weak-purity, we will next consider the following cohomology: $\bar{H}^a(X \times \Delta[p], b) = \varinjlim H^a_{|Z|}(X \times \Delta[p], b)$ where Z runs through $z^b(X, p)$ and $|Z| =$ *support* (Z). To avoid convergence problems we truncate the diagram in 6.1.1 at some finite N even and apply hypercohomology with respect to $\Gamma(b)$) to the resulting finite diagram; now we obtain the spectral sequence

$\bar{E}_1^{p,q} = \bar{H}^q(X \times \Delta[-p], b)$, $-p \leq N$

Moreover, one obtains a natural map of spectral sequences $\bar{E}_1^{p,q} \to E_1^{p,q}$. The cycle map and its contravariant functoriality shows that it induces a map of complexes:

$t_N z^b(X,.) \to (\bar{E}_1^{.,2b}, d_1)$

where t_N denotes *naive* truncation at level N. By weak-purity, it follows that $d_r : E_r^{p,2b} \to E_r^{p+r,2b-r+1} = 0$, for $r > 1$, so that we obtain a map $CH^b(X, n) \to \bar{E}_2^{-n,2b} \to \bar{E}_\infty^{-n,2b}$. Moreover, $\bar{E}_\infty^{x,y} = 0$ for $y < 2b$ so that we obtain

$CH^b(X, n) \to \bar{E}_\infty^{-n,2b} \to$ the limit of $\bar{E}_1$-spectral sequence in $deg(2b - n) \to H^{2b-n}(X, b)$.

This is the definition of the higher cycle map of Bloch. In view of our hypothesis that the usual cycle maps commute with flat pull-backs and proper push-forwards, the same properties carry over to the higher cycle maps. Further conditions on the cohomology theory (satisfied in cases of interest) will ensure compatibility with the product structure.

We may consider étale cohomology with mod-l^ν coefficients. We obtain the higher cycle map

$$cl : CH^r(X, .; \mathbb{Z}/l^\nu) \to H^{2r-n}_{et}(X, \mathbb{Z}/l^\nu).$$

In this case one may verify the cycle map is compatible with products.

7. The strong Kunneth decomposition and higher cycle maps

Theorem 7.1. *Let X denote a projective nonsingular variety over k so that the class of the diagonal $\Delta \varepsilon CH^*(X \times X; \mathbb{Z}_l)$ has a strong Chow Kunneth decomposition. (i) Now the higher cycle-map*

$$cl_l : CH^r(X, \quad n; \quad \mathbb{Z}_l) \to H^{2r-n}_{et}(X; \mathbb{Z}_l(r))$$

is an isomorphism for each fixed n, $r \geq 0$. In particular, both terms are trivial if n is odd.

(ii) For each fixed integer $n \geq 0$, $\underset{r}{\oplus} CH^r(X, \quad n; \quad \mathbb{Z}_l) \cong \underset{r}{\oplus} H^{2r-n}_M(X, \quad \mathbb{Z}_l(r))$ is generated over $\underset{r}{\oplus} H^*_M(Spec \quad k, \mathbb{Z}_l(r)) \cong \underset{r}{\oplus} \mathbb{Z}_l(r)$ by classes $\{\alpha_{i,j}\} \subseteq CH^*(X; \mathbb{Z}_l)$. Moreover for each fixed pair of integers u and v, the motivic cohomology $H^u_M(X, \quad \mathbb{Z}_l(v))$ is a free $\mathbb{Z}_l$-module which is trivial if u is odd.

(iii) All the above assertions hold with $\mathbb{Z}_l$ replaced $\mathbb{Z}/l^\nu$. □

Proof. We begin with the key hypothesis that if $[\Delta]$ denotes the class of the diagonal of $X \times X$ in $CH^*(X \times X; \mathbb{Z}_l)$, it decomposes in the form

(7.0.1) $$[\Delta] = \Sigma_{i,j}(\alpha_{i,j} \times \beta_{d-i,j}) = \Sigma_{i,j} p_1^*(\alpha_{i,j}) \circ p_2^*(\beta_{d-i,j}), \quad \alpha_{i,j}, \quad \beta_{d-i,j} \varepsilon CH^*(X; \mathbb{Z}_l)$$

(Here $p_i : X \times X \to X$, $i = 1, 2$ denotes the obvious projection to the i-th factor and $\circ$ denotes the intersection product on $CH^*(X \times X; \mathbb{Z}_l$. We will continue to let $\circ$ denote the intersection product on the higher Chow groups.) We may assume $\beta_{d-i,j} \varepsilon CH^{d-i}(X; \mathbb{Z}_l)$ and that $\alpha_i \varepsilon CH^i(X; \mathbb{Z}_l)$. Next let $n \geq 0$ denote a fixed integer. Now we obtain:

(7.0.2) $$x = p_{1*}([\Delta] \circ p_2^*(x)), \quad x \varepsilon CH^*(X, \quad n; \quad \mathbb{Z}_l)$$

where $\circ$ denotes the intersection product. By the projection formula and the observation that the class $[\Delta] = \Delta_*(1)$, $1 = [X] \varepsilon CH^*(X; \mathbb{Z}_l)$, we obtain equality of the classes $[\Delta] \circ p_2^*(x) = \Delta_*(\Delta^*(p_2^*(x))$. Therefore $p_{1*}([\Delta] \circ p_2^*(x)) = p_{1*}(\Delta_*(\Delta^*(p_2^*(x)))) = (p_1 \circ \Delta)_*((p_2 \circ \Delta)^*(x)) = x$, for any class $x \varepsilon CH^*(X, \quad n; \quad \mathbb{Z}_l)$. Now substitute the formula for $[\Delta]$ from (5.1) in (5.2) and use the projection formula to obtain:

(7.0.3) $$x = \Sigma_{i,j} \alpha_{i,j} \circ p_{1*}(p_2^*(\beta_{d-i,j} \circ x))$$

The cartesian square

$$\begin{array}{ccc} X \times X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1'} \\ X & \xrightarrow{p_2'} & Spec \quad k \end{array}$$

and flat base-change provide the identification $p_{1*}(p_2^*(\beta_{d-i,j} \circ x)) = {p'_2}^* p'_{1*}(\beta_{d-i,j} \circ x)$. Now the class $p'_{1*}(\beta_{d-i,j} \circ x) \varepsilon CH^*(\text{Spec} \quad k, \quad n; \quad \mathbb{Z}_l)$.

(7.0.4) It follows that the classes $\{\alpha_{i,j} | i \varepsilon I\}$ generate $CH^*(X, \quad n; \quad \mathbb{Z}_l)$ over $\underset{r,s}{\oplus} CH^r(\text{Spec} \quad k, \quad s; \quad \mathbb{Z}_l)$.

This proves the first assertion in (ii).

Observe from Suslin's computation of the mod-l^ν higher Chow groups of Spec k and the observation that it readily extends to positive characteristic using de Jong's theory of alterations by Geisser (see [Sus-2], [Geis] Theorem (3.6) and [Oort]) that:

(7.0.5) $$\begin{aligned} CH^r(\text{Spec} \quad k, \quad s; \quad \mathbb{Z}_l) &\cong H^{2r-s}_{et}(\text{Spec} \quad k; \mathbb{Z}_l(r)) \cong \mathbb{Z}_l(r), \qquad s = 2r \\ &\cong 0, \qquad s \neq 2r \end{aligned}$$

(For the sake of completeness, we add a couple of remarks on extending Suslin's computations to varieties over algebraically closed fields of positive characteristics. The main result we need is the following comparison theorem:

Theorem(Suslin-Voevodsky) (See [S-V].) Let k denote an algebraically closed field and let $\mathcal{F}$ be a homotopy invariant presheaf on the big site of schemes of finite type over k. If $\tilde{\mathcal{F}}_h$ is the sheafification of $\mathcal{F}$ for the h-topology and l is prime to the characteristic of k, there exists a canonical isomorphism:

$$Ext^*_h(\tilde{\mathcal{F}}_h, \mathbb{Z}/l^\nu) \overset{\cong}{\rightarrow} Ext^*_{Ab}(\mathcal{F}(\text{Spec}\quad k), \mathbb{Z}/l^\nu)$$ □

The Ext on the left (on the right) is the one computed in the category of sheaves on the h-topology (is the one computed in the category of abelian groups, respectively). In the original proof, the characteristic is restricted to be 0 so that one may apply resolution of singularities and thereby show that each scheme of finite type over k admits an h-cover which is smooth over k. The theory of alterations due to de Jong (see [Oort]) enables one to find such a smooth h-cover making the original proof still valid in positive characteristics.)

Moreover, it follows from the Proposition below that we may assume the above isomorphism is in fact given by Bloch's higher cycle maps. Observe also that now (7.0.4) and (7.0.5) show that $CH^*(X, \quad n; \quad \mathbb{Z}_l) \cong 0$ if n is *odd*.

Next we show that *for each fixed integer* n, the l-adic cycle map

$$cl_l : CH^*(X, \quad n; \quad \mathbb{Z}_l) \rightarrow \underset{r}{\oplus} H^{2r-n}_{et}(X; \mathbb{Z}_l(r))$$

is *surjective*. Since X is projective and smooth the projections p_i induce maps $p_{i*} : H^*_{et}(X \times X; \mathbb{Z}_l(r)) \rightarrow H^{*-2d}_{et}(X; \mathbb{Z}_l(r-d))$ and $p^*_i : H^*_{et}(X; \mathbb{Z}_l(r)) \rightarrow H^*_{et}(X \times X; \mathbb{Z}_l(r))$ where d is the dimension of X. Moreover the cycle-class of the diagonal $cl_l([\Delta])$ belongs to $H^{2d}_{et}(X \times X; \mathbb{Z}_l(d))$. Now $cl_l(\beta_{d-i,j}) \varepsilon H^{2i}_{et}(X; \mathbb{Z}_l(i))$. Let $x' \varepsilon H^{2r-n}_{et}(X; \mathbb{Z}_l(r))$. Now the formula

$$x' = p_{1*}(cl_l([\Delta]) \cup p^*_2(x')) \tag{7.0.7}$$

holds by an argument as in the proof of 7.0.3. Moreover $cl_l([\Delta]) = \Sigma_{i,j} p^*_1(cl_l(\alpha_{i,j})) \cup p^*_2(cl_l(\beta_{d-i,j}))$. Substituting this into 7.0.7 and using the projection formula, we obtain $x' = \Sigma_{i,j} cl_l(\alpha_{i,j}) \cup p_{1*}p^*_2(cl_l(\beta_{d-i,j}) \cup x')$.

Now $p_{1*}p^*_2(cl_l(\beta_{d-i,j}) \cup x') = {p'_2}^*(p'_{1*}(cl_l(\beta_{d-i,j}) \cup x'))$ and $p'_{1*}(cl_l(\beta_{d-i,j}) \cup x') \varepsilon H^{2i+2r-n-2d}_{et}(\text{Spec}\quad k; \mathbb{Z}_l(i + r - d))$ since $cl_l(\beta_{d-i,j}) \cup x' \varepsilon H^{2i+2r-n}_{et}(X; \mathbb{Z}_l(i + r))$. By (5.5) and proposition (5.3) (see below), one observes the isomorphism $cl_l : CH^{i+r-d}(\text{Spec}\quad k, \quad n; \quad \mathbb{Z}_l) \overset{\cong}{\rightarrow} H^{2i+2r-2d-n}_{et}(\text{Spec}\quad k; \mathbb{Z}_l(i + r - d))$. Therefore one may write $p'_{1*}(cl_l(\beta_{d-i,j}) \cup x) = cl_l(\gamma_{i,j})$ for some $\gamma_{i,j} \varepsilon CH^{i+r-d}(\text{Spec}\quad k, \quad n; \quad \mathbb{Z}_l)$. It follows

$$x' = \Sigma_{i,j} cl_l(\alpha_{i,j}) \cup cl_l({p'_2}^*(\gamma_{i,j})) = \Sigma_{i,j} cl_l(\alpha_{i,j} \circ {p'_2}^*(\gamma_{i,j})) \tag{7.0.8}$$

This proves the surjectivity of the l-adic cycle map for each fixed integer n. Taking $n = 1$, it follows in particular that $H^i_{et}(X; \mathbb{Z}_l(r)) \cong H^i_{et}(X; \mathbb{Z}_l) = 0$ if i is *odd*.

Next we will show the above cycle map is *injective* as well proving (i). To see this suppose $x \varepsilon CH^*(X, \quad n; \quad \mathbb{Z}_l)$ so that $cl_l(x) = 0$. Now $cl_l(p_{1*}p^*_2(\beta_{d-i,j} \circ x)) = p_{1*}p^*_2(cl_l(\beta_{d-i,j}) \cup cl_l(x)) = 0$ for all i. (See (A.1.7).) Since $p_{1*}p^*_2(\beta_{d-i,j} \circ x) = {p'_2}^* p'_{1*}(\beta_{d-i,j} \circ x)$, $cl_l(p_{1*}p^*_2(\beta_{d-i,j} \circ x)) = cl_l({p'_2}^* p'_{1*}(\beta_{d-i,j} \circ x)) = {p'_2}^*(cl_l(p'_{1*}(\beta_{d-i,j} \circ x))) = 0$. However ${p'_2}^* : \underset{r}{\oplus} H^*_{et}(\text{Spec}\quad k; \mathbb{Z}_l(r)) \rightarrow \underset{r}{\oplus} H^*_{et}(X; \mathbb{Z}_l(r))$ is injective, it follows that $cl_l(p'_{1*}(\beta_{d-i,j} \circ x)) = 0$. Since the cycle class map $cl_l : CH^r(\text{Spec}\quad k, \quad n; \quad \mathbb{Z}_l) \rightarrow H^{2r-n}_{et}(\text{Spec}\quad k; \mathbb{Z}_l(r))$ is an isomorphism (see Proposition (5.3) below) it follows that $p'_{1*}(\beta_{d-i,j} \circ x) = 0$. Therefore ${p'_2}^*(p'_{1*}(\beta_{d-i,j} \circ x) = p_{1*}(p^*_2(\beta_{d-i,j} \circ x) = 0$ for all i. Now (5.3) shows that x itself is trivial. This completes the proof of the assertion in (i).

Now we recall the short exact sequence (from [Lau] Proposition (2.2)(ii)):

$$0 \rightarrow Hom(Tors(H^{i+1}_{et}(X; \mathbb{Z}_l)), \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow H^{et}_i(X; \mathbb{Z}_l) \rightarrow Hom(H^i_{et}(X; \mathbb{Z}_l), \mathbb{Z}_l) \rightarrow 0 \tag{7.0.9}$$

Here Hom is in the category of $\mathbb{Z}_l$-modules and $Tors(H^{i+1}_{et}(X; \mathbb{Z}_l))$ denotes the torsion part of $H^{i+1}_{et}(X; \mathbb{Z}_l)$. (This is nothing other than the universal coefficient sequence and the first term is $Ext^1(H^{i+1}_{et}(X; \mathbb{Z}_l); \mathbb{Z}_l)$ computed using the injective resolution $\mathbb{Q}_l \rightarrow \mathbb{Q}_l/\mathbb{Z}_l$ of $\mathbb{Z}_l$.) Now let $i = 2j$ be an even integer. By the above arguments, $H^{i+1}_{et}(X; \mathbb{Z}_l) \cong 0$. Therefore $H^{et}_i(X; \mathbb{Z}_l) \cong Hom(H^i_{et}(X; \mathbb{Z}_l); \mathbb{Z}_l)$ is a free $\mathbb{Z}_l$ -module. Now Poincaré-duality shows that $H^i_{et}(X; \mathbb{Z}_l(r)) \cong H^{et}_{2d-i}(X; \mathbb{Z}_l(r - d)) \cong H^{et}_{2d-i}(X; \mathbb{Z}_l)$. These show that each $H^i_{et}(X; \mathbb{Z}_l(r))$ is a free $\mathbb{Z}_l$-module if i is *even* and trivial if i is *odd*. Now the observation that the l-adic cycle map is an isomorphism shows $CH^r(X, \quad n; \mathbb{Z}_l)$ is also a free $\mathbb{Z}_l$ module for all r and n. Since all the modules we are considering are free over $\mathbb{Z}_l$, tensoring with $\mathbb{Z}/l^\nu$ now completes the proof of Theorem 7.1. □

Proposition 7.2. *The higher cycle map*

$$cl_{l^\nu} : CH^q(Spec\quad k, \quad p; \mathbb{Z}/l^\nu) \rightarrow H^{2q-p}_{et}(Spec\quad k; \mathbb{Z}/l^\nu(q)) \tag{7.0.10}$$

is an isomorphism for each fixed integer $p \geq 0$.

Proof. By the homotopy property one obtains the isomorphism:

$$CH^q(\text{Spec}\ \ k,\quad p;\mathbb{Z}/l^\nu) \cong CH^q(\mathbb{A}^q,\quad p;\mathbb{Z}/l^\nu) \tag{7.0.11}$$

Now by Suslin's computation and the observation that it extends to positive characteristic by Geisser (see [Sus-2] and [Geis] Theorem (3.6))), $CH^q(\mathbb{A}^q,\ \ p;\mathbb{Z}/l^\nu) \cong 0$ unless $p = 2q$ and $\cong \mathbb{Z}/l^\nu$ if $p = 2q$. Therefore the left-hand-side in (5.10) is trivial unless $p = 2q \geq 0$. i.e. we may assume that $p = 2q$. Moreover, since $CH^q(\text{Spec}\ \ k,\ \ 2q;\ \ \mathbb{Z}/l^\nu) \cong CH^q(\mathbb{A}^q,\ \ 2q;\ \ \mathbb{Z}/l^\nu) \cong \mathbb{Z}/l^\nu$, it suffices to show that higher cycle map in (5.10) with $p = 2q$ is *surjective*. By the multiplicative property of the cycle map we now reduce to the case where $q = 1$ and $p = 2$. In this case, this follows immediately from [Bl-2] p. 73 which identifies $CH^1(X,\ \ 2;\ \ \mathbb{Z}/l^\nu)$ with $\Gamma(X, \mu_{l^\nu})$ for any scheme X. (Here μ_{l^ν} is the sheaf of l^ν-th roots of unity on X.) □

Corollary 7.3. *Let X denote a projective nonsingular linear variety over an algebraically closed field k of arbitrary characteristic. Now the conclusions of theorem 7.1 hold for X.*

Proof. This is clear. Observe that the strong Chow-Kunneth decomposition of the class of $\Delta \varepsilon CH^*(X \times X;\mathbb{Z}_l)$ follows from Corollary 4.6 with $h^*(X,.) = z^*(X,.)$ assuming the result of [J-3]. □

Corollary 7.4. Kunneth formula for the mod-l^ν higher Chow groups. *Let X denote a projective nonsingular linear variety and let Y denote any quasi-projective variety over an algebraically closed field k of arbitrary characteristic. Now the Kunneth spectral sequence*

$$E^2_{s,t} = Tor^{\pi_*(z^*(Spec\ \ k),.;\mathbb{Z}/l^\nu)}_{s,t}(\pi_*(z^*(X,.;\mathbb{Z}/l^\nu)),\quad \pi_*(z^*(Y,.;\mathbb{Z}/l^\nu))) \Rightarrow \pi_{s+t}(z^*(X \times Y,.;\mathbb{Z}/l^\nu))$$

in Theorem 4.5 degenerates and therefore one obtains the isomorphism:

$$CH^*(X \times Y,.;\mathbb{Z}/l^\nu) \cong CH^*(X,.;\mathbb{Z}/l^\nu) \underset{CH^*(Spec\ \ k,.;\mathbb{Z}/l^\nu)}{\otimes} CH^*(Y,.;\mathbb{Z}/l^\nu)$$

Proof. This is again clear from the above corollary , Theorem 4.3 and Corollary (4.6). □

Exercise 7.5. Show that the strong Kunneth decomposition for the diagonal of a projective smooth scheme X as well as $X \times X$ is equivalent to the higher cycle maps as in Theorem 7.1 being isomorphisms for both X and $X \times X$.

We end this section by stating a generalization of theorem 7.1 to any cohomology theory in the sense of the first part of the talk. Let $H^*(\quad,.)$ denote a cohomology theory in the sense of section 2 for which a higher cycle map $cl : CH^r(X,n;R) \to H^{2r-n}(X;r)$ (where R is a suitable ring) is defined and having the properties as in section 2 for all schemes we consider.

Theorem 7.6. *Let Y denote a scheme and let $f : X \to Y$ denote a proper smooth map so that the class of the diagonal $\Delta \varepsilon CH^*(X \underset{Y}{\times} X;R)$ has a strong Kunneth decomposition. If the cycle maps*

$$cl : CH^r(Y,n;R) \to H^{2r-n}(Y;r)$$

are surjective for all r and n, then the same holds for the cycle map $cl : CH^r(X,n;R) \to H^{2r-n}(X;r)$ for all r and n. If, in addition, the map $f^* : H^{2r-n}(Y;r) \to H^{2r-n}(X;r)$ is injective and the cycle map $cl : CH^r(Y,n;R) \to H^{2r-n}(Y;n)$ is injective for all r and n, then so is the cycle map $cl : CH^r(X,n;R) \to H^{2r-n}(X;n)$ for all r and n.

Exercise 7.7. (For fans of Lawson cohomology only!). Compute the Lawson cohomology of *Spec* $\mathbb{C}$ with $\mathbb{Z}$-coefficients. Now compute the Lawson cohomology of any non-singular projective linear variety over $\mathbb{C}$. (Hint: exactly the same arguments as in the proof of Theorem 7.1 should carry over.)

8. Mod-l^ν Higher K-groups of linear varieties

In this section we make use several fundamental results in the literature on the mod-l^ν algebraic K-theory of algebraic varieties along with the results of the previous section to compute the mod-l^ν algebraic K-theory of coherent sheaves on linear varieties, where l i is prime to the residue characteristics. If l is as above, $K/l^\nu(Z)$ ($G/l^\nu(Z)$) will denote the smash product of $K(Z)$ ($G(Z)$, respectively) with an appropriate Moore-spectrum. $K/l^\nu_{top}(Z)$ ($G/l^\nu_{top}(Z)$) will denote the corresponding mod-l^ν topological K-theory (G-theory, respectively) which may be obtained from $K/l^\nu(Z)$ ($G/l^\nu(Z)$, respectively) by inverting the Bott element. (See [T]. The mod-l^ν

topological K-theory may be identified with the mod-l^ν étale K-theory in [Fr-2].) It is shown in [T] (A.6) that, if ν is sufficiently large, (i.e. if $l > 3$, all $\nu \geq 1$ are allowed, while if $l = 3$, $\nu \geq 2$ and if $l = 2$, $\nu \geq 4$), the above spectra are homotopy associative and commutative. Therefore we will assume ν is sufficiently large throughout.

We will first recall the definition of the hypercohomology spectrum for presheaves of spectra essentially from [T]. Let X denote a scheme of finite type over k and let F denote a presheaf of spectra on the big étale site of X. *Throughout we will restrict to presheaves of spectra F so that for each integer n and each U in the site, $\pi_n(\Gamma(U, F))$ is finite with torsion prime to the characteristic.* Now we let $\mathcal{G}^{\cdot}F$ denote the cosimplicial object provided by the Godement resolution $\{\mathcal{G}F ... \mathcal{G}^n F|n\}$ *on the small étale site* of X. (See for example [T](1.31).) We let

(8.0.1) $$\mathbb{H}(X; F) = \underset{\Delta}{\text{holim}}\, \Gamma(X, \mathcal{G}^n F)$$

which is the homotopy limit of the cosimplicial object $\{\Gamma(X, \mathcal{G}^n F)|n\}$. This is clearly functorial in F; one may also observe (using the properties of the homotopy inverse limit) that it preserves fibration sequences in F. It also preserves weak-equivalences in F (provided F is a presheaf of fibrant spectra). Now the usual spectral sequence for the homotopy inverse limit of a spectrum provides a spectral sequence:

(8.0.2) $$E_2^{s,t} = H^s_{et}(X; \quad \pi_t(F)\tilde{\ }) \Rightarrow \pi_{-s+t}(\mathbb{H}(X; \quad F))$$

This spectral sequence converges strongly since there is a uniform bound on the l-cohomological dimension of the schemes considered here. Finally observe that there is a natural augmentation $\Gamma(X; F) \to \mathbb{H}(X; F)$.

8.1. Let $F = G$ be the presheaf $X \mapsto G(X)$. Quillen (see [Qu-2]) shows that now we obtain localization sequences in the following sense. Let $i : Z \to X$ denote a closed immersion with $j : U \to X$ the open immersion of its complement. In this situation we obtain cofibration sequence $G(Z) \to G(X) \to G(U)$ of spectra. Moreover he establishes the continuity property (see [Qu-2] I, section 2); this and the properties of homotopy inverse limits show that now we obtain the commutative diagram:

$$\begin{array}{ccccccc} \Gamma(Z;F) & \longrightarrow & \Gamma(X;F) & \longrightarrow & \Gamma(U;F) & \longrightarrow & \Sigma\Gamma(Z;F) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}(Z;F) & \longrightarrow & \mathbb{H}(X;F) & \longrightarrow & \mathbb{H}(U;F) & \longrightarrow & \mathbb{H}(Z;\Sigma F) \end{array}$$

where each row is a cofibration sequence of spectra and the vertical maps are the obvious augmentation maps. (Here $F = G$.) Since smashing with a Moore-spectrum or inverting the Bott element preserves cofibration sequences of spectra, we obtain a similar commutative diagram with the presheaves $F = G/l^\nu$ and $F = G/l^\nu_{top}$.

Proposition 8.1. *Let X denote a quasi-projective variety that has a decomposition into a finite number of strata each of which is isomorphic to an affine space. Now the natural map $G/l^\nu(X) \to \mathbb{H}(X; G/l^\nu(\quad))$ induces an isomorphism:*

$$\pi_n(G/l^\nu(X)) \to \pi_n(\mathbb{H}(X; G/l^\nu(\quad))),\ n \geq 0$$

Proof. Let d denote the dimension of X. Now let U_0 = the union of the strata of dimension d; let $U_k - U_{k-1}$ denote the union of strata of dimension $d - k$. Now we obtain a commutative diagram of localization sequences:

$$\begin{array}{ccccccc} \pi_{n+1}(G/l^\nu(U_0)) & \longrightarrow & \pi_n(G/l^\nu(X - U_0)) & \longrightarrow & \pi_n(G/l^\nu(X)) & \longrightarrow & \pi_n(G/l^\nu(U_0)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_{n+1}(\mathbb{H}(U_0; G/l^\nu())) & \longrightarrow & \pi_n(\mathbb{H}(X - U_0; G/l^\nu())) & \longrightarrow & \pi_n(\mathbb{H}(X; G/l^\nu())) & \longrightarrow & \pi_n(\mathbb{H}(U_0; G/l^\nu())) \end{array}$$

Since U_0 is a disjoint union of affine spaces, the first and last vertical maps are isomorphisms. To see this we need to compute the sheaves $U \to \pi_n(G/l^\nu(U))\tilde{\ }$, U on the étale site of X. The computation of the K-theory of strictly Hensel local rings (and the proof of the Lichtenbaum-Quillen conjecture) by Suslin (see [Sus-1] and also [G-T]) shows that the sheaf $U \to \pi_n(G/l^\nu(U))\tilde{\ }$, U in the étale topology of X is the constant sheaf $\mathbb{Z}/l^\nu$ if $n \geq 0$ is *even* and is trivial otherwise. Therefore the spectral sequence in (8.0.2), the homotopy property of G/l^ν-theory and the homotopy property of étale cohomology with $\mathbb{Z}/l^\nu$-coefficients provide the required isomorphism. Observe that the dimension of $X - U_0$ is strictly less than d. Therefore, by ascending induction on the dimension of $X - U_0$ one may assume the second vertical map above is also an isomorphism. Now consider the case $n = 0$; observe that now the last map in the top row is surjective. Therefore a five-lemma argument shows that the map $\pi_0(G/l^\nu(X)) \to \pi_0(\mathbb{H}(X; G/l^\nu(\quad)))$ is an isomorphism. If $n > 0$, the term following the last term on the top row (bottom row) is $\pi_{n-1}(G/l^\nu(X - U_0))$ ($\pi_{n-1}(\mathbb{H}(X - U_0; G/l^\nu(\quad)))$, respectively). The map $\pi_{n-1}(G/l^\nu(X - U_0)) \to$

$\pi_{n-1}(\mathbb{H}(X - U_0; G/l^\nu()))$ is also an isomorphism by the inductive hypothesis on the dimension of X. Now a five lemma argument once again shows that the map $\pi_n(G/l^\nu(X)) \to \pi_n(\mathbb{H}(X; G/l^\nu(\ \ )))$ is also an isomorphism. □

Corollary 8.2. *(i) If X is a variety as in (8.0.2) there exists a spectral sequence:*

$$E_2^{s,t} = H^s_{et}(X; \pi_t(G/l^\nu(\ \ ))\tilde{\ }) \Rightarrow \pi_{-s+t}(G/l^\nu(X))$$

(ii) The hypothesis of the previous Proposition is satisfied by all projective smooth toric and spherical varieties and by all projective smooth varieties provided with the action of a torus so that the fixed point scheme is discrete.

Proof. (i) follows readily from (8.0.1) and the spectral sequence in (8.0.2). All varieties in (ii) come provided with the action of a torus that has only finitely many fixed points. Therefore they all come provided with a decomposition into strata that are affine spaces as shown by Bialynicki-Birula. (See [B-B].) □

Now we consider the following theorem.

Theorem 8.3. *(i) Let X denote a projective non-singular toric variety or a projective nonsingular variety on which a torus acts with finitely many fixed points. Now there exists a spectral sequence*

$$E_2^{s,t} = H^s_{et}(X;\, \pi_t(K/l^\nu(\ \ ))\tilde{\ }\ ) \Rightarrow \pi_{-s+t}(K/l^\nu(X))$$

Here $\pi_t(K/l^\nu(\ \ ))\tilde{\ }$ denotes the sheaf of t-th homotopy groups associated to the presheaf $U \to K/l^\nu(U)$, U in the étale topology of X. Moreover the above spectral sequence degenerates at the E_2-level and $\pi_n(K/l^\nu(X))$ is trivial for all odd integers n and negative integers while it is a free $\mathbb{Z}/l^\nu$ -module for all non-negative even integers n. Moreover one obtains an isomorphism:

$$\pi_n(K/l^\nu(X)) \cong \bigoplus_k H^{2k}_{et}(X; \mathbb{Z}/l^\nu), \text{ for all } n \text{ even, } n \geq 0.$$

This isomorphism is not natural in X.

(ii) Assuming the existence of a mod $- l^\nu$ Bloch-Lichtenbaum spectral sequence (see [Bl-L]) for all projective non-singular linear varieties, the results of (i) extend to all projective non-singular linear varieties.

Proof. The existence of the spectral sequence in (i) is clear from the last corollary. Again we need to compute the sheaves $U \to \pi_n(G/l^\nu(U))\tilde{\ }$, $U \to \pi_n(G/l^\nu_{top}(U))\tilde{\ }$, U on the étale site of X. The computation of the K-theory of strictly Hensel local rings (and the proof of the Lichtenbaum-Quillen conjecture) by Suslin (see [Sus-1] and also [G-T]) shows that the sheaf $U \to \pi_n(G/l^\nu(U))\tilde{\ }$, U in the étale topology of X is the constant sheaf $\mathbb{Z}/l^\nu$ if $n \geq 0$ is even and is trivial otherwise. It follows from Theorem 7.1 that the $E_2^{s,t}$ terms are all projective and hence free (since $\mathbb{Z}/l^\nu$ is a local ring) modules over $\mathbb{Z}/l^\nu$. Now the abutment has a filtration whose successive terms are these E_2-terms; it follows the abutment is a split extension of the E_2-terms. This proves (i) in the Theorem.

Now we consider (ii). Recall the Bloch-Lichtenbaum spectral sequence is of the form:

$$E_2^{s,t} = H^{s-t}_M(X; \mathbb{Z}/l^\nu(-t)) \cong H^{s-t}_{et}(X; \mathbb{Z}/l^\nu(-t)) \Rightarrow \pi_{-s-t}(G/l^\nu(X))$$

Observe also that the differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. Clearly either $s-t$ or $s+r-(t-r+1) = s-t+(2r-1)$ is *odd*. Therefore Theorem 7.1 and ascending induction on r shows that these differentials are all zero. It follows that the above spectral sequence degenerates and $E_2^{s,t} = E_\infty^{s,t}$ for all s, t. Since these are free modules over $\mathbb{Z}/l^\nu$ it follows that the abutment is in fact a split extension of these $E_2^{s,t}$-terms. This completes the proof of the theorem. □

Recall from [T] the existence of a spectral sequence

$$E_2^{s,t} = H^s_{et}(X;\, \pi_t(K/l^\nu_{top}(\ \ ))\tilde{\ }\ ) \Rightarrow \pi_{-s+t}(K/l^\nu_{top}(X))$$

Here $\pi_t(K/l^\nu_{top}(\ \ ))\tilde{\ }$ denotes the sheaf of t-th homotopy groups associated to the presheaf $U \to K/l^\nu_{top}(U)$, U in the étale topology of X. (This holds for all varieties over k.)

Theorem 8.4. *(i) Let X denote a projective non-singular linear variety. Now the above spectral sequence degenerates at the E_2-level and $\pi_n(K/l^\nu_{top}(X))$ is trivial for all odd integers n while it is a free $\mathbb{Z}/l^\nu$ -module for all even integers n. One also obtains an isomorphism:*

$$\pi_n(K/l^\nu_{top}(X)) \cong \bigoplus_k H^{2k}_{et}(X; \mathbb{Z}/l^\nu), \text{ for all } n \text{ even.}$$

This isomorphism is, once again, not natural in X.

(ii) Let X denote a variety provided with a stratification by finite strata each of which is isomorphic to the product of a torus and an affine space. The natural map $\pi_n(G/l^\nu(X)) \to \pi_n(G/l^\nu_{top}(X))$ is an isomorphism for all $n \geq d-1$ if d is the dimension of X.

Proof. Take the presheaf $F = G/l^\nu_{top}$ in (8.0.1) and use [T] Theorem (2.47), to obtain the spectral sequence in the first statement of the Theorem. (Observe that [T] (2.47) provides the weak-equivalence $\Gamma(X; G/l^\nu_{top})@>\simeq>> \mathbb{H}(X; G/l^\nu_{top})$. The proof of the first assertion now follows along the same lines as the proof of the previous Theorem. Next we consider the last assertion in the Theorem. For this one considers the commutative diagram

$$\begin{array}{ccccccc}
\pi_n(G/l^\nu(X-U)) & \longrightarrow & \pi_n(G/l^\nu(X)) & \longrightarrow & \pi_n(G/l^\nu(U)) & \longrightarrow & \pi_{n-1}(G/l^\nu(X-U)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(G/l^\nu_{top}(X-U)) & \longrightarrow & \pi_n(G/l^\nu_{top}(X)) & \longrightarrow & \pi_n(G/l^\nu_{top}(U)) & \longrightarrow & \pi_{n-1}(G/l^\nu_{top}(X-U))
\end{array}$$

Here U is the open dense stratum. The homotopy property shows that one may assume U is a torus of dimension $\leq d$. In this case the computation in [Fr-2] Proposition 3.4 shows that the third vertical map is an isomorphism for all $n \geq d-1$. We may assume the same holds for the first and last vertical map using ascending induction. (Observe that the dimension of $X-U$ is strictly less than that of X.) Therefore a five lemma argument shows the second vertical map is also an isomorphism if $n \geq d-1$. □

Remarks 8.5. Theorem 8.3 shows that one may compute the higher mod-l^ν K-theory of the above varieties knowing their mod-l^ν étale cohomology. However the mod-l^ν topological K-theory of a non-singular toric or spherical variety is *not*, in general, isomorphic (in non-negative degrees) to the mod-l^ν K-theory as the simple example of the torus $\mathbb{G}_m^2$ will show. (See [Fr-2] Proposition 3.4.) Consequently the descent spectral sequence as in Theorem 8.3 does not exist in general for all non-singular linear varieties (i.e. unless one restricts to projective non-singular linear varieties). It is a recent result of Friedlander and Suslin (see [Fr-S]) that the conjectured Bloch-Lichtenbaum spectral sequence as in Theorem 8.3(ii) indeed exists. Therefore the descent spectral sequence in Theorem 8.3(i) in fact holds for all projective non-singular linear varieties.

Theorem 7.1 plays a key role in the computations in Theorems 8.3 and 8.4. The hypothesis that the variety be projective seems also essential in theorem (7.1).

9. **Examples and Counter-examples**

In this section we verify Theorems (7.1) and (8.3) by computing the mod$-l^\nu$-K-theory and mod-l^ν motivic cohomology of several toric surfaces and three-folds using *elementary methods* i.e. we do not use any of the sophisticated computations in [Sus-2] , but instead rely totally on the localization sequence and the resulting blow-up formula. We also give an example of a singular projective toric variety for which the strong Chow Kunneth decomposition for the class of the diagonal fails.

Proposition 9.1. *Let $i : Y \to X$ denote a regular closed immersion of smooth schemes over an algebraically closed field k. Let X' denote the blow-up of X along Y and assume that Y is of pure co-dimension c in X. (i) Now there exists a weak-equivalence:*

$$G/l^\nu(X') \simeq G/l^\nu(X) \times \overset{(c-1)}{\Pi} G/l^\nu(Y)$$

Taking the homotopy groups, one obtains:

$$\pi_n(G/l^\nu(X')) \cong \pi_n(G/l^\nu(X)) \oplus \overset{(c-1)}{\oplus} \pi_n(G/l^\nu(Y)), \text{ all } n.$$

(ii) Similarly there exists an isomorphism:

$$CH^*(X', n) \simeq CH^*(X, n) \oplus \overset{(c-1)}{\oplus} CH^*(Y, n)$$

for each fixed integer n.

Proof. The proof of (i) being very similar, we will consider (ii) only. We begin with the commutative diagram:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & CH^*(Y',n) & \longrightarrow & CH^*(X',n) & \longrightarrow & CH^*(X'-Y',n) & \longrightarrow & CH^*(Y',n-1) & \longrightarrow \ldots \\ & \pi_{Y*}\uparrow & & \pi_X^*\uparrow & & \uparrow\simeq & & \uparrow\pi_{Y*} & \\ \ldots \longrightarrow & CH^*(Y,n) & \longrightarrow & CH^*(X,n) & \longrightarrow & CH^*(X-Y,n) & \longrightarrow & CH^*(Y,n-1) & \longrightarrow \ldots \end{array}$$

where Y' is the exceptional divisor and $\pi_X : X' \to X$, $\pi_Y : Y' \to Y$ are the obvious maps. The two rows are long exact sequences by the localization theorem. This readily shows that the diagram

$$\ldots \to CH^*(Y,n) \to CH^*(Y',n) \oplus CH^*(X,n) \to CH^*(X',n) \to CH^*(Y,n-1) \to \ldots.$$

is a long exact sequence. Now the observation that Y' is a projective space bundle over Y (associated to the normal bundle of Y in X), the computation in [Bl-1] Theorem 7.1, and the observation that $\pi_{Y*} \circ \pi_Y^* = id$ shows the above long exact sequences breaks up into split short exact sequences and also provides the isomorphism in (ii). □

Observe that since all the schemes are smooth, one may replace the mod$-l^\nu$ G-theory by the corresponding mod$-l^\nu$ K-theory.

One has a similar formula at the level of étale cohomology with $\mathbb{Z}/l^\nu$-coefficients. i.e.

$$H^*_{et}(X';\mathbb{Z}/l^\nu) \cong H^*_{et}(X;\mathbb{Z}/l^\nu) \oplus \overset{(c-1)}{\oplus} H^*_{et}(Y;\mathbb{Z}/l^\nu) \tag{9.0.1}$$

Now we begin by considering non-singular complete toric surfaces. It is shown in [Ful-1] p.43 that all such varieties are obtained by a finite sequence of blow-ups centered at the fixed points of the given torus starting with either the projective space $\mathbb{P}^2$ or the *Hirzebruch surface* F_a, which is a ruled surface.

Proposition 9.2. *Let X denote a non-singular complete toric variety of dimension ≤ 2 over an algebraically closed field k. For each even integer $n \geq 0$, one obtains an (abstract) isomorphism (of $\mathbb{Z}/l^\nu$-modules):*

$$\pi_n(K/l^\nu(X)) \cong H^*_{et}(X;\mathbb{Z}/l^\nu) = \underset{k}{\oplus} H^{2k}_{et}(X;\mathbb{Z}/l^\nu) \text{ and}$$

$$CH^*(X,\quad n;\quad \mathbb{Z}/l^\nu) \cong \underset{k}{\oplus} H^{2k}_{et}(X;\mathbb{Z}/l^\nu)$$

Proof. We will observe that the above result holds for points and nonsingular complete toric varieties of dimension 1 which are all isomorphic to $\mathbb{P}^1$. This follows from the computation of the mod-l^ν K-theory and motivic cohomology of projective spaces:

if $\mathcal{E}$ is a vector bundle of dimension $d > 1$ over a variety Y,

(9.2) $\pi_n(K/l^\nu(Proj(\mathcal{E}))) \cong \overset{d}{\oplus} \pi_n(K/l^\nu(Y))$, $CH^*(Proj(\mathcal{E});n) \cong \overset{d}{\oplus} CH^*(Y;n)$ and

$H^*_{et}(Proj(\mathcal{E});\mathbb{Z}/l^\nu) \cong \overset{d}{\oplus} H^*_{et}(Y;\mathbb{Z}/l^\nu)$

The above formula therefore proves the proposition for $\mathbb{P}^2$ and F_a. Now assume we are considering a toric surface X obtained by n successive blow-up of points starting with either $\mathbb{P}^2$ or F_a. Let X_k denote the variety at the k-th stage. If $k = 1$, clearly the proposition is true for X_1, since now X_k is either $\mathbb{P}^2$ or F_a. Therefore we may use ascending induction to prove the proposition. Assume k is a fixed integer > 1 and the proposition is true for all X_n, $n < k$. Now X_k is obtained from X_{k-1} by blowing up a finite number of points. Therefore one may take $Y =$ a finite number of points in the formulae in Proposition 9.1 to prove the proposition for X_k. This completes the inductive step and hence the proof. □

We will conclude by computing the mod$-l^\nu$ K-theory of toric *Fano* three-folds. (See [Oda] p.90 for their classification.)

Proposition 9.3. *The same conclusion as in the previous theorem holds for all toric Fano three-folds.*

Proof. Let X denote such a three-fold. Observe that this is a non-singular projective toric variety of dimension 3. According to the classification in [Oda] p. 90, X is obtained by successive blow-ups at a finite number of points or at a finite number of closed 1-dimensional non-singular toric sub-varieties by starting with either (i) $\mathbb{P}^3$, (ii) $\mathbb{P}^2 \times \mathbb{P}^1$, (iii) certain $\mathbb{P}^1$-bundles over $\mathbb{P}^2$, (iv) certain $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, (v) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, (vi) certain $\mathbb{P}^1$-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, (vii) $\mathbb{P}^1 \times F_1$, where F_1 is the Hirzebruch surface (as above), (viii) certain $\mathbb{P}^1$-bundles over F_1 and (ix) $\mathbb{P}^1 \times Y_2$, where Y_2 is the del Pezzo surface obtained from $\mathbb{P}^2$ by blowing up certain fixed points for the torus.

Now assume X is one of the nine cases above. The conclusion of the proposition holds in case (i) by (9.2) and Proposition 9.1. They also hold for cases (iii) and (iv) by the same argument. Now consider $Y \times \mathbb{P}^n$; one may view this as a projective space over Y and apply the computation in (9.2) and the corresponding ones in mod-l^ν cohomology. Therefore, one may see that the proposition holds in the remaining cases as well. Finally assume X is obtained from one of the above varieties by a finite succession of equivariant blow-ups. Now one may apply an inductive argument as in the proof of the last proposition to complete the proof. □

Next consider the projective, but singular toric threefold defined by the fan with edges through the vertices $(\pm 1, \pm 1, \pm 1)$ of a cube, in the sublattice of $\mathbb{Z}^3$ generated by these vertices (see [Ful] p. 105). We may assume for simplicity that the ground field is the field of complex numbers. Now it is an exercise in [Ful] p. 105 to show that if X denotes this toric threefold,

(9.0.3)
$$CH^3(X) = H^6(X;\mathbb{Z}) = \mathbb{Z}, CH^0(X) = H^0(X;\mathbb{Z}) = Z \quad \text{and}$$
$$CH^2(X) = H^4(X;\mathbb{Z}) = \mathbb{Z}, H^3(X;\mathbb{Z}) = \mathbb{Z}^2, CH^1(X) = H^2(X;\mathbb{Z}) = \mathbb{Z}^5$$

Now consider the variety $Y = X \times X$. One may readily compute the mod-l^ν étale cohomology of this variety in degree 6 to be $(\mathbb{Z}/l^\nu)^{16}$. If T denotes the dense torus in X, $X \times X$ is also a toric variety for the action of $T \times T$. Therefore its Chow groups are generated by the closures of the $T \times T$-orbits. (See [Ful] p. 96.) Therefore one may compute $CH^3(X \times X) = \mathbb{Z}^{12}$ and therefore $CH^3(X \times X; \mathbb{Z}/l^\nu) = (\mathbb{Z}/l^\nu)^{12}$. Thus the hypothesis of theorem 7.1 cannot be true for the variety Y. Observe that Y is a toric variety that is projective, but singular.

Finally consider the example of the nonsingular, but affine toric variety $\mathbb{G}_m^2$. Now $CH^*(\mathbb{G}_m^2; \mathbb{Z}/l^\nu) = \mathbb{Z}/l^\nu$ while $H^*_{et}(\mathbb{G}_m^2; \mathbb{Z}/l^\nu) = \mathbb{Z}/l^\nu \oplus \mathbb{Z}/l^\nu$. Thus the hypothesis that the variety be projective is necessary in Theorem 7.1 .

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Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA.
E-mail address: joshua@math.ohio-state.edu