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ON THE CONVERGENCE OF THE LAGGED DIFFUSIVITY FIXED POINT METHOD IN TOTAL VARIATION IMAGE RESTORATION*

TONY F. CHAN[†] AND PEP MULET[‡]

Abstract. In this paper we show that the *lagged diffusivity fixed point* algorithm introduced by Vogel and Oman in [*SIAM J. Sci. Comput.*, 17 (1996), pp. 227–238] to solve the problem of total variation denoising, proposed by Rudin, Osher, and Fatemi in [*Phys. D*, 60 (1992), pp. 259–268], is a particular instance of a class of algorithms introduced by Voss and Eckhardt in [*Computing*, 25 (1980), pp. 243–251], whose origins can be traced back to Weiszfeld’s original work for minimizing a sum of Euclidean lengths [*Tôhoku Math. J.*, 43 (1937), pp. 355–386]. There have recently appeared several proofs for the convergence of this algorithm [G. Aubert et al., Technical report 94-01, Informatique, Signaux et Systèmes de Sophia Antipolis, 1994], [A. Chambolle and P.-L. Lions, Technical report 9509, CEREMADE, 1995], and [D. C. Dobson and C. R. Vogel, *SIAM J. Numer. Anal.*, 34 (1997), pp. 1779–1791]. Here we present a proof of the global and linear convergence using the framework introduced in [H. Voss and U. Eckhardt, *Computing*, 25 (1980), pp. 243–251] and give a bound for the convergence rate of the fixed point iteration that agrees with our experimental results. These results are also valid for suitable generalizations of the fixed point algorithm.

Key words. image restoration, total variation, Weiszfeld’s method, fixed point

AMS subject classifications.

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1. Introduction. Recently, a new class of nonlinear PDE based techniques has emerged for image restoration problems, primarily because they preserve sharp edges better. A particularly popular technique is the *total variation* (TV) restoration method, introduced by Rudin, Osher, and Fatemi in [10]. This can be posed as a variational problem, which gives a highly nonlinear Euler–Lagrange equation. In [10], the solution of this equation was originally obtained as the steady state solution of an auxiliary time dependent equation which was numerically integrated by an explicit method. Because of stability constraints, this algorithm can be slowly convergent. To improve the convergence behavior, Vogel and Oman proposed in [11] the use of a linearization technique for this nonlinear equation, resulting in the *lagged diffusivity fixed point algorithm* (FP). This idea is quite commonly used in other PDE applications, e.g., CFD, and our theory may have applications there as well. The implementation of this iterative method amounts to solving a linear elliptic equation at each iteration. Experiments show that this algorithm is very robust and globally and linearly convergent.

Although faster convergent methods have been developed since the appearance of FP (e.g., [6]), it remains one of the most efficient and robust methods for total variation image restoration, and thus its convergence theory has attracted some attention

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in the literature. In particular, there have recently appeared several convergence proofs [2], [4], [7]. The main idea in these proofs is based on the introduction of an auxiliary variable v and alternately minimizing an associated *half-quadratic functional* (see [9] for details) in the original variable u and the new variable v .

In this paper, we present a different proof of the global and linear convergence for the discrete version of this algorithm. We view our contributions as follows:

1. We point out that the fixed point algorithm is a generalized version of an algorithm proposed by Weiszfeld in 1937 for minimizing a sum of Euclidean lengths. Although his paper appeared 60 years ago, it does not seem to be known in the image processing literature.
2. Our proof is based on concepts appearing in a 1980 paper by Voss and Eckhardt [12] and relies only on elementary facts of linear algebra and nonlinear equations, and not on the differential nature of the operators involved. This, we think, gives a new insight to the subject.
3. Our proof applies to a generalized version of the FP algorithm in which the regularization functional can be more general than TV. In particular it contains as a special case the regularization functional considered by Aubert et al. in [2].
4. The linear convergence of the fixed point algorithm is proven under no stringent assumptions on the data and parameters. Some experiments aiming to study the dependence of the convergence rate with respect to various parameters are shown as well. As far as we know, these constitute novel results not found in [2], [4], [7].

The outline of this paper is as follows. We introduce the image denoising problem using the total variation as regularization functional in section 2; in section 3 we define the denoising problem for a more general functional. The concept of the generalized Weiszfeld's method is introduced in section 4. In section 5 we give a proof of the global convergence of these methods and in section 6 we obtain their linear convergence, as well as an estimate for the rate of convergence in terms of the spectral structure of a preconditioned Hessian. Finally, section 7 contains some experiments that validate the bounds obtained in the previous section, as well as a numerical study of this rate for varying parameters and data.

2. Total variation denoising. The total variation denoising method that we are going to briefly survey in this section can be formulated for data of any dimensionality, but since it is normally utilized for two-dimensional data (images), we will use its two-dimensional formulation for the sake of clarity.

An image u can be interpreted as a two-dimensional array $(u_{i,j})$, $i = 1, \dots, n$, $j = 1, \dots, m$. For notational simplicity let us assume $m = n$, so that $N = n^2$. We use $\|y\|$ or $|y|$ to denote the 2-norm of a vector y .

Our interest is in restoring an image which is contaminated with noise. The restoration process should recover the edges of the image. Let us denote by z the observed image and by u the real image. The model of degradation we assume is $u + \tilde{n} = z$, where \tilde{n} is a Gaussian white noise, with $\|\tilde{n}\|$ known.

Of course, the problem $u + \tilde{n} = z$, $\|\tilde{n}\| = \sigma$, has many solutions, but if we impose a certain *regularity* condition on the solution u , then it may become well posed. In [10], it is proposed to use as a regularization functional the so-called TV functional. If we denote $h = \frac{1}{n}$, then a standard discretization of this functional is

$$(2.1) \quad TV(u) = \sum_{i,j=1}^n h^2 \sqrt{\left(\frac{u_{i+1,j} - u_{i,j}}{h}\right)^2 + \left(\frac{u_{i,j+1} - u_{i,j}}{h}\right)^2} = \frac{1}{N} \sum_{i,j=1}^n |\nabla_{i,j}u|,$$

where we have taken into account that $N = n^2$ and used the notation

$$\nabla_{i,j}u = \left(\frac{u_{i+1,j} - u_{i,j}}{h}, \frac{u_{i,j+1} - u_{i,j}}{h}\right)$$

for the linear operator $\nabla_{i,j}$ from \mathbb{R}^N to \mathbb{R}^2 (which is a discrete gradient operator).

The TV functional does not penalize discontinuities in u and thus allows us to recover the edges of the original image. The restoration problem can now be written as

$$(2.2) \quad \min_u \frac{1}{N} \alpha \sum_{i,j=1}^n |\nabla_{i,j}u| + \frac{1}{N} \sum_{i,j=1}^n \frac{1}{2} (u_{ij} - z_{ij})^2,$$

or simply as

$$(2.3) \quad \min_u \sum_{i,j=1}^n \left[\alpha |\nabla_{i,j}u| + \frac{1}{2} (u_{ij} - z_{ij})^2 \right],$$

where $\alpha > 0$. Since this functional is not differentiable, a commonly used technique to overcome this difficulty is to slightly perturb the TV norm to

$$\frac{1}{N} \sum_{i,j=1}^n \sqrt{|\nabla_{i,j}u|^2 + \beta},$$

where β is a small positive parameter. The problem is stable with respect to this perturbation, in the sense that we can obtain the solution as the limit of the solutions of the perturbed problems when $\beta \rightarrow 0$; see [1]. We will denote $\sqrt{|\nabla_{i,j}u|^2 + \beta}$ by $|\nabla_{i,j}u|_\beta$.

The first order condition for problem (2.3) is

$$(2.4) \quad \alpha \sum_{i,j=1}^n \nabla_{i,j}^T \left(\frac{\nabla_{i,j}u}{|\nabla_{i,j}u|_\beta} \right) + u - z = 0,$$

where $\nabla_{i,j}^T$ is the transpose of the linear operator $\nabla_{i,j}$.

Note that (2.4) can be viewed as a nonlinear algebraic system of discretized partial differential equations. The presence of the diffusivity coefficient $\frac{1}{|\nabla_{i,j}u|_\beta}$ in (2.4) makes it highly nonlinear and Newton's method does not work satisfactorily on it (see [11], [5], [6]).

Vogel and Oman [11] proposed the following fixed point iteration to linearize equation (2.4) by solving

$$(2.5) \quad \alpha \sum_{i,j=1}^n \nabla_{i,j}^T \left(\frac{\nabla_{i,j}u^{k+1}}{|\nabla_{i,j}u^k|_\beta} \right) + u^{k+1} - z = 0$$

for u^{k+1} , given u^k . The rest of the paper discusses a convergence proof for the iteration given in (2.5).

Let us briefly review the *half-quadratic regularization* approach of [9] that has been used in [2] to obtain a proof of the global convergence of the lagged diffusivity fixed point iteration.

Using the fact that $t = \min_v vt^2 + \frac{1}{4v}$, attained for $v = \frac{1}{2t}$, the minimization problem (2.3) can be written as

$$\begin{aligned} & \min_u \sum_{i,j=1}^n \left[\alpha |\nabla_{i,j} u|_\beta + \frac{1}{2} (u_{ij} - z_{ij})^2 \right] \\ &= \min_u \sum_{i,j=1}^n \left[\alpha \min_{v_{ij}} \left(v_{ij} |\nabla_{i,j} u|_\beta^2 + \frac{1}{4v_{ij}} \right) + \frac{1}{2} (u_{ij} - z_{ij})^2 \right] \\ &= \min_u \min_v \sum_{i,j=1}^n \left[\alpha \left(v_{ij} |\nabla_{i,j} u|_\beta^2 + \frac{1}{4v_{ij}} \right) + \frac{1}{2} (u_{ij} - z_{ij})^2 \right] \\ &= \min_u \min_v \Theta(u, v), \end{aligned}$$

where $\frac{1}{2|\nabla_{i,j} u|_\beta} = \operatorname{argmin}_{v_{ij}} (v_{ij} |\nabla_{i,j} u|_\beta^2 + \frac{1}{4v_{ij}}) \forall i, j$. Note that $\Theta(u, v)$ is quadratic in u , hence the name *half-quadratic*.

The algorithm ARTUR used in [2] consists of the following alternate minimization procedure:

$$\begin{aligned} & v^{k+1} = \operatorname{argmin}_v \Theta(u^k, v), \\ (2.6) \quad & u^{k+1} = \operatorname{argmin}_u \Theta(u, v^{k+1}). \end{aligned}$$

The first equation is trivially solved by $v_{ij}^{k+1} = \frac{1}{2|\nabla_{i,j} u^k|_\beta} \forall i, j$. Since $\Theta(u, v^{k+1})$ is a convex quadratic function of u , it then follows that u^{k+1} is given as the solution of the equation

$$\begin{aligned} (2.7) \quad 0 &= \nabla_u \Theta(u^{k+1}, v^{k+1}) = \sum_{i,j=1}^n \left[\alpha 2v_{ij}^{k+1} \nabla_{ij}^T \nabla_{ij} u^{k+1} + u_{ij}^{k+1} - z_{ij} \right] \\ &= \sum_{i,j=1}^n \left[\alpha \frac{\nabla_{ij}^T \nabla_{ij} u^{k+1}}{|\nabla_{i,j} u^k|_\beta} + u_{ij}^{k+1} - z_{ij} \right], \end{aligned}$$

which is precisely the fixed point iteration (2.5).

3. A generalized problem. In this section we introduce a twofold generalization of problem (2.3). Let us consider the problem

$$(3.1) \quad \min_u F(u) \equiv \min_u \alpha \sum_{i=1}^m \phi(|A_i^T u|_\beta) + \frac{1}{2} \|u - z\|^2,$$

where $A_i, i = 1, \dots, m$, is a $n \times 2$ matrix, and ϕ is a real function, which, respectively, generalize $\nabla_{i,j}^T$ and $\phi(x) = x$. We assume that ϕ verifies the following.

HYPOTHESIS 3.1.

1. ϕ is C^2 .
2. $\phi(0) = 0$.
3. ϕ is increasing for $x \geq 0$, i.e., $\phi'(x) > 0 \ x \geq 0$.
4. ϕ is convex, i.e., $\phi''(x) \geq 0$.

5. $\psi(x) = \phi(\sqrt{x})$ is concave for $x \geq 0$.

Note that by 2 and 3 in Hypothesis 3.1 $\phi(x) \geq 0 \forall x \geq 0$. We can cite as examples of such functions ϕ the functions $x^p, 1 \leq p \leq 2$. Note that problem (2.3) corresponds to the case $A_i^T = \nabla_{i,j}$ and $p = 1$ and that $p = 2$ corresponds to the H^1 -norm restoration. The exponents p such that $1 < p < 2$ might be potentially useful for image restoration applications (cf. [3]).

It is easy to show that

$$(3.2) \quad F'(u) = \alpha \sum_i A_i \left(\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} A_i^T u \right) + u - z,$$

$$(3.3) \quad F''(u) = \alpha \sum_i A_i \left(\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} \left(I_2 - \frac{A_i^T u \otimes A_i^T u}{|A_i^T u|_\beta^2} \right) + \phi''(|A_i^T u|_\beta) \frac{A_i^T u \otimes A_i^T u}{|A_i^T u|_\beta^2} \right) A_i^T + I_n,$$

where \otimes denotes the Kronecker product.

LEMMA 3.2. Under Hypothesis 3.1 on ϕ , we have the following:

1. F is twice continuously differentiable and strictly convex.
2. F is coercive, i.e., $\lim_{\|x\| \rightarrow \infty} F(x) = \infty$.
3. F is bounded below.

Therefore problem (3.1) has a unique solution.

Proof. Since $F(u) \geq \frac{1}{2} \|u - z\|^2$, F is coercive and bounded below by 0. By (3.2) and the fact that $|A_i^T u|_\beta \neq 0$, F is twice continuously differentiable. By Cauchy-Schwarz, we have $|A_i^T v|^2 - \frac{(A_i^T u, A_i^T v)^2}{|A_i^T u|_\beta^2} \geq 0$. Moreover, since $\phi'(x) \geq 0$ if $x > 0$ and $\phi''(x) \geq 0$, we have for $v \neq 0$

$$F''(u)(v, v) = \alpha \sum_i \left[\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} \left(|A_i^T v|^2 - \frac{(A_i^T u, A_i^T v)^2}{|A_i^T u|_\beta^2} \right) + \phi''(|A_i^T u|_\beta) \frac{(A_i^T u, A_i^T v)^2}{|A_i^T u|_\beta^2} \right] + \|v\|^2 > 0;$$

hence F is a strictly convex functional. By standard arguments in convex analysis (cf. [8], for example), problem (3.1) has a unique solution. \square

We consider the more general fixed point algorithm given by solving for u^{k+1} , given u^k , in

$$(3.4) \quad \alpha \sum_i A_i \left(\frac{\phi'(|A_i^T u^k|_\beta)}{|A_i^T u^k|_\beta} A_i^T u^{k+1} \right) + u^{k+1} - z = 0.$$

Note the connection with (3.2); also, if $\phi(x) = x$ and $A_i^T \equiv \nabla_{i,j}$, i.e., the original functional is the total variation functional, then (3.4) is just (2.5).

4. Generalized Weiszfeld's method. Let us consider now a general minimization problem

$$(4.1) \quad \min_u F(u)$$

for a functional F defined on \mathbb{R}^n , and let us assume the usual hypothesis that ensures that problem (4.1) has a unique solution.

HYPOTHESIS 4.1.

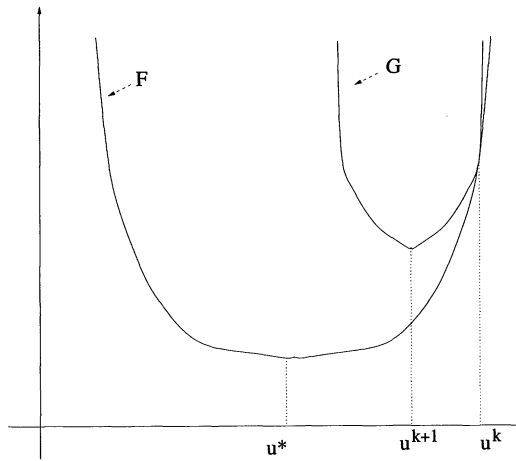


FIG. 4.1. Weiszfeld's algorithm. G is a quadratic osculating super-approximation to F .

1. F is twice continuously differentiable and strictly convex.
2. F is coercive, i.e., $\lim_{\|x\| \rightarrow \infty} F(x) = \infty$.
3. F is bounded below.

Note that (3.1) verifies these conditions by Lemma 3.2.

Weiszfeld proposed in [13] a linearization technique for a class of problems similar to (3.1) that differs from Newton's method in that the original functional is approximated from above. This *generalized Weiszfeld's algorithm* for (4.1) consists, first of all, of the choice of a *uniformly strictly convex* quadratic functional $G(v, u)$ that is an approximation from above for $F(u)$, i.e., the following hypothesis.

HYPOTHESIS 4.2.

1. $G(v, u) = F(u) + (v - u, F'(u)) + \frac{1}{2}(v - u, C(u)(v - u))$.
2. C is continuous.
3. $\lambda_{\min}(C(u)) \geq \mu > 0 \forall u$.
4. $F(v) \leq G(v, u) \forall v$.

Then one defines the iteration for this generalized Weiszfeld's method by

$$(4.2) \quad u^{k+1} = \underset{v}{\operatorname{argmin}} G(v, u^k);$$

see Figure 4.1. Since, for fixed u , G is C^2 , coercive, bounded below, and strictly convex, this minimum exists and it is given by the following equation:

$$(4.3) \quad 0 = G_v(u^{k+1}, u^k) = F'(u^k) + C(u^k)(u^{k+1} - u^k).$$

We show next that the fixed point algorithm (2.5) is an instance of a generalized Weiszfeld's method.

LEMMA 4.3. *If $C(u)$ is given by*

$$(4.4) \quad C(u) = \alpha A \operatorname{diag} \left(\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} I_2 \right) A^T + I_n, \quad A = [A_1 \dots A_m],$$

then the functional

$$G(v, u) = F(u) + (v - u, F'(u)) + \frac{1}{2}(v - u, C(u)(v - u))$$

defines a generalized Weiszfeld's method for problem (3.1), and (3.4) corresponds to its iteration.

Proof. The smallest eigenvalue of $C(u)$ is characterized as

$$\lambda_{\min} = \min_{\|v\|_2=1} (v, C(u)v).$$

Since $(v, C(u)v) \geq \|v\|^2$, we deduce that $\lambda_{\min}(C(u)) \geq 1$, so we can take $\mu = 1$ in 3 of Hypothesis 4.2.

The continuity of $C(u)$ can be deduced easily from the fact that $|A_i^T u|_\beta \neq 0$.

Let us now see that $F(v) \leq G(v, u) \forall v$. By using 1 in Hypothesis 4.2 and the definition of F in (3.1) we have

$$\begin{aligned} G(v, u) - F(v) &= F(u) - F(v) + (v - u, F'(u)) + \frac{1}{2}(v - u, C(u)(v - u)) \\ &= \alpha \sum_i \left[\phi(|A_i^T u|_\beta) - \phi(|A_i^T v|_\beta) \right. \\ &\quad \left. + \frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} \left((A_i^T u, A_i^T v - A_i^T u) + \frac{1}{2}|A_i^T v - A_i^T u|^2 \right) \right] \\ &= \alpha \sum_i \left[\phi(|A_i^T u|_\beta) - \phi(|A_i^T v|_\beta) + \frac{\phi'(|A_i^T u|_\beta)}{2|A_i^T u|_\beta} (|A_i^T v|^2 - |A_i^T u|^2) \right] \\ (4.5) \quad &= \alpha \sum_i \left[\phi(|A_i^T u|_\beta) - \phi(|A_i^T v|_\beta) + \frac{\phi'(|A_i^T u|_\beta)}{2|A_i^T u|_\beta} (|A_i^T v|_\beta^2 - |A_i^T u|_\beta^2) \right]. \end{aligned}$$

In the equations above we have used the fact that

$$|A_i^T v|_\beta^2 - |A_i^T u|_\beta^2 = |A_i^T v|^2 + \beta - (|A_i^T u|^2 + \beta) = |A_i^T v|^2 - |A_i^T u|^2.$$

Each of the terms in this sum is of the form

$$\phi(a) - \phi(b) + \frac{\phi'(a)}{2a}(b^2 - a^2),$$

and, by the definition of ψ in Hypothesis 3.1, it can be written as

$$(4.6) \quad \psi(a^2) - \psi(b^2) + \psi'(a^2)(b^2 - a^2).$$

Since ψ is concave by hypothesis,

$$(4.7) \quad \psi(c) + \psi'(c)(d - c) \geq \psi(d) \quad \forall c, d \geq 0.$$

From (4.5), (4.6), and (4.7), we deduce $F(v) \leq G(v, u)$.

Let us now finally see that (3.4) corresponds to the iteration of the generalized Weiszfeld's method. From (3.2) and (4.4) we deduce that (4.3) can be written as

$$\begin{aligned} 0 &= \alpha \sum_i \left[A_i \left(\frac{\phi'(|A_i^T u^k|_\beta)}{|A_i^T u^k|_\beta} A_i^T u^k \right) \right] + u^k - z \\ &\quad + \alpha \sum_i \left[A_i \left(\frac{\phi'(|A_i^T u^k|_\beta)}{|A_i^T u^k|_\beta} A_i^T (u^{k+1} - u^k) \right) \right] + u^{k+1} - u^k \\ (4.8) \quad &= \alpha \sum_i \left[A_i \left(\frac{\phi'(|A_i^T u^k|_\beta)}{|A_i^T u^k|_\beta} A_i^T u^{k+1} \right) \right] + u^{k+1} - z. \end{aligned}$$

Therefore, (4.3) is equivalent to the generalized fixed point iteration given in (3.4). \square

5. Global convergence theory. In this section, we establish the global convergence of the generalized Weiszfeld’s algorithm. Voss and Eckhardt have shown in [12] that the generalized Weiszfeld’s method has linear and monotonic convergence. In our proof, we follow essentially the same approach.

LEMMA 5.1. *If (u^k) is obtained from (4.2), then the following hold:*

1. $F(u^{k+1}) \leq F(u^k) \forall k$.
2. $\lim_{k \rightarrow \infty} \|u^{k+1} - u^k\| = 0$.
3. *The sequence (u^k) converges to the unique global minimum of (3.1).*

Proof. 1. By 4 in Hypothesis 4.2, $F(u^{k+1}) \leq G(u^{k+1}, u^k)$. Now, $u^{k+1} = \operatorname{argmin}_v G(v, u^k)$ implies $G(u^{k+1}, u^k) \leq G(u^k, u^k) = F(u^k)$.

2. By 4 in Hypothesis 4.2 and (4.3),

$$\begin{aligned} F(u^{k+1}) &\leq G(u^{k+1}, u^k) \\ &= F(u^k) + (u^{k+1} - u^k, F'(u^k)) + \frac{1}{2}(u^{k+1} - u^k, C(u^k)(u^{k+1} - u^k)) \\ &= F(u^k) - \frac{1}{2}(u^{k+1} - u^k, C(u^k)(u^{k+1} - u^k)). \end{aligned}$$

From this, taking into account that $\mu \leq \lambda_{\min}(C(u)) \forall u$, by 3 in Hypothesis 4.2 we have

$$\frac{1}{2}\mu \|u^{k+1} - u^k\|^2 \leq \frac{1}{2}(u^{k+1} - u^k, C(u^k)(u^{k+1} - u^k)) \leq F(u^k) - F(u^{k+1}),$$

which implies

$$(5.1) \quad 0 \leq \|u^{k+1} - u^k\| \leq \sqrt{\frac{2}{\mu}(F(u^k) - F(u^{k+1}))}.$$

On the other hand, since F is bounded below and $(F(u^k))$ is monotonically decreasing, then $F(u^k)$ converges and, in particular, $\lim_{k \rightarrow \infty} F(u^k) - F(u^{k+1}) = 0$. This and (5.1) give us that $\lim_{k \rightarrow \infty} \|u^{k+1} - u^k\| = 0$.

Let us finally see that (u^k) converges to the unique minimum of (3.1). Since $(F(u^k))$ is monotonically decreasing, it is bounded above. The fact that F is coercive then implies that (u^k) is bounded. By compactness, it suffices to show that the limit of every convergent subsequence of (u^k) is the global minimum of (3.1). By the strict convexity of F , it will then suffice to show that those limits are stationary points of F .

Let (u^{k_i}) be a subsequence of (u^k) that converges to \bar{u} . Since $G_v(v, u) = F'(u) + C(u)(v - u)$ and by the hypothesis on F and C , we deduce that G_v is continuous: Since $\lim_{i \rightarrow \infty} \|u^{k_i+1} - u^{k_i}\| = 0$, as we have just seen, we have that $\lim_{i \rightarrow \infty} u^{k_i+1} = \lim_{i \rightarrow \infty} u^{k_i} = \bar{u}$. By (4.3),

$$0 = G_v(u^{k_i+1}, u^{k_i}) = F'(u^{k_i}) + C(u^{k_i})(u^{k_i+1} - u^{k_i}).$$

Taking limits, we deduce from the facts stated above that

$$\begin{aligned} 0 &= \lim_{i \rightarrow \infty} G_v(u^{k_i+1}, u^{k_i}) = G_v\left(\lim_{i \rightarrow \infty} u^{k_i+1}, \lim_{i \rightarrow \infty} u^{k_i}\right) \\ &= G_v(\bar{u}, \bar{u}) = F'(\bar{u}) + C(\bar{u})(\bar{u} - \bar{u}) = F'(\bar{u}). \quad \square \end{aligned}$$

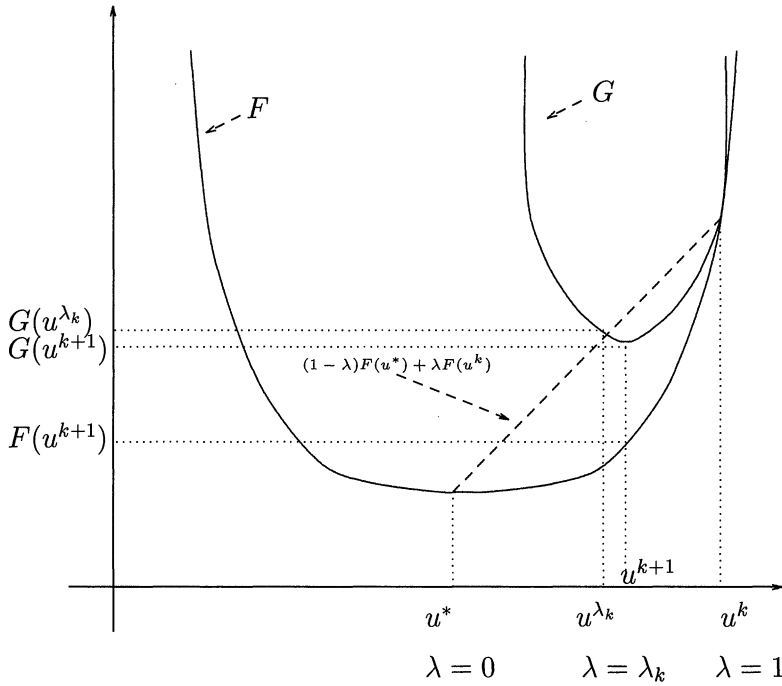


FIG. 6.1. Convergence of Weiszfeld's algorithm. $u^{\lambda_k} = \lambda_k u^k + (1 - \lambda_k)u^*$.

6. Linear convergence analysis. We derive in this section the linear convergence of the generalized Weiszfeld's method and give a bound of its convergence rate in terms of spectral structure of a preconditioned Hessian.

Let us denote by u^* the unique minimum of (3.1) and

$$(6.1) \quad \lambda_k = \frac{G(u^*, u^k) - F(u^*)}{\frac{1}{2}(u^* - u^k, C(u^k)(u^* - u^k))},$$

$$\Lambda = 1 - \lambda_{\min}(C(u^*)^{-1}F''(u^*)).$$

See Figure 6.1 and (6.3) below for a geometrical interpretation of λ_k .

THEOREM 6.1.

1. $F(u^{k+1}) - F(u^*) \leq \lambda_k(F(u^k) - F(u^*))$.
2. $\Lambda < 1$ and $0 \leq \lambda_k \leq \Lambda$, for k sufficiently large. In particular, $F(u^k)$ has a linear convergence rate of at most Λ .
3. u^k is r -linearly convergent with a convergence rate of at most $\sqrt{\Lambda}$.

Proof. By 4 in Hypothesis 4.2 and (4.2),

$$(6.2) \quad F(u^{k+1}) \leq G(u^{k+1}, u^k) \leq G(\lambda_k u^k + (1 - \lambda_k)u^*, u^k).$$

A direct computation using (6.1) shows that

$$\begin{aligned}
 & G(\lambda_k u^k + (1 - \lambda_k)u^*, u^k) \\
 &= F(u^k) + (1 - \lambda_k)(u^* - u^k, F'(u^k)) + \frac{(1 - \lambda_k)^2}{2}(u^* - u^k, C(u^k)(u^* - u^k)) \\
 &= \lambda_k F(u^k) + (1 - \lambda_k)\left(G(u^*, u^k) - \frac{\lambda_k}{2}(u^* - u^k, C(u^k)(u^* - u^k))\right) \\
 (6.3) \quad &= \lambda_k F(u^k) + (1 - \lambda_k)F(u^*).
 \end{aligned}$$

Substituting (6.3) into (6.2), subtracting $F(u^*)$ from both sides, and dividing by λ_k (which can be assumed to be nonzero; otherwise $u_i = u^*$, $i > k$, and the result is trivial) gives the first statement (see Figure 6.1 for an illustration).

Recall that $F''(u)$ is the Hessian of F at u . By using the definition of G and Taylor expansions at u^k ,

$$(6.4) \quad G(u^*, u^k) = F(u^k) + (u^* - u^k, F'(u^k)) + \frac{1}{2}(u^* - u^k, C(u^k)(u^* - u^k)),$$

$$(6.5) \quad F(u^*) = F(u^k) + (u^* - u^k, F'(u^k)) + \frac{1}{2}(u^* - u^k, F''(\hat{u}^k)(u^* - u^k)),$$

with $\|\hat{u}^k - u^k\| \leq \|u^* - u^k\|$. If we subtract (6.5) from (6.4), divide by $\frac{1}{2}(u^* - u^k, C(u^k)(u^* - u^k))$, and take into account that $(x, Ax)/(x, Bx) \geq \lambda_{\min}(B^{-1}A)$ we get

$$\begin{aligned}
 \lambda_k &= \frac{G(u^*, u^k) - F(u^*)}{\frac{1}{2}(u^* - u^k, C(u^k)(u^* - u^k))} \\
 &= \frac{\frac{1}{2}(u^* - u^k, C(u^k)(u^* - u^k)) - \frac{1}{2}(u^* - u^k, F''(\hat{u}^k)(u^* - u^k))}{\frac{1}{2}(u^* - u^k, C(u^k)(u^* - u^k))} \\
 &= 1 - \frac{(u^* - u^k, F''(\hat{u}^k)(u^* - u^k))}{(u^* - u^k, C(u^k)(u^* - u^k))} \\
 &= 1 - \frac{(u^* - u^k, F''(\hat{u}^k)(u^* - u^k))}{(u^* - u^k, C(u^k)(u^* - u^k))} \\
 (6.6) \quad &\leq 1 - \lambda_{\min}(C(u^k)^{-1}F''(\hat{u}^k)).
 \end{aligned}$$

Since F'' and C are continuous,

$$(6.7) \quad 1 - \lambda_{\min}(C(u^k)^{-1}F''(\hat{u}^k)) \rightarrow 1 - \lambda_{\min}(C(u^*)^{-1}F''(u^*)).$$

Since F is strictly convex, its Hessian at u^* , $F''(u^*)$, is symmetric positive definite. By assumption, $C(u^*)$ is also symmetric positive definite. Since

$$C(u^*)^{-1}F''(u^*) = C(u^*)^{-\frac{1}{2}}C(u^*)^{-\frac{1}{2}}F''(u^*)C(u^*)^{-\frac{1}{2}}C(u^*)^{\frac{1}{2}},$$

$C(u^*)^{-1}F''(u^*)$ is similar to $C(u^*)^{-\frac{1}{2}}F''(u^*)C(u^*)^{-\frac{1}{2}}$, which is positive definite; thus

$$(6.8) \quad \lambda_{\min}(C(u^*)^{-1}F''(u^*)) = \lambda_{\min}(C(u^*)^{-\frac{1}{2}}F''(u^*)C(u^*)^{-\frac{1}{2}}) > 0.$$

By (6.7) and (6.8),

$$(6.9) \quad \Lambda = 1 - \lambda_{\min}(C(u^*)^{-1}F''(u^*)) < 1,$$

and together with (6.6) we get that $\lambda_k \leq \Lambda < 1$, for k sufficiently big. To conclude the second statement, note that by 4 in Hypothesis 4.2 we have $F(u^*) \leq G(u^*, u^k)$, which, by the definition of λ_k , in turn implies that $\lambda_k \geq 0$.

Since $F'(u^*) = 0$, by using the Taylor expansion at u^* we get

$$F(u^k) - F(u^*) = \frac{1}{2}(u^k - u^*, F''(\tilde{u}^k)(u^k - u^*)) \geq \frac{1}{2}\lambda_{\min}(F''(\tilde{u}^k)) \|u^k - u^*\|^2$$

for \tilde{u}^k with $\|\tilde{u}^k - u^*\| \leq \|u^k - u^*\|$. Therefore, $\tilde{u}^k \rightarrow u^*$ and, due to the continuity of F'' , we deduce the existence of $K > 2$ such that

$$(6.10) \quad \|u^k - u^*\| \leq \sqrt{\frac{K}{\lambda_{\min}(F''(u^*))}(F(u^k) - F(u^*))} \stackrel{\text{def}}{=} y_k$$

for k sufficiently big. Now, by 1 in Theorem 6.1,

$$y_{k+1}^2 = \frac{K}{\lambda_{\min}(F''(u^*))}(F(u^{k+1}) - F(u^*)) \leq \frac{K}{\lambda_{\min}(F''(u^*))}\Lambda(F(u^k) - F(u^*)) = \Lambda y_k^2,$$

that is, $y_{k+1} \leq \sqrt{\Lambda}y_k$, hence (u^k) is r -linearly convergent with at most $\sqrt{\Lambda}$ as the r -convergence rate by (6.10). \square

7. Numerical results. For typical image restoration problems it has been observed that the rate of convergence of the fixed point algorithm dramatically deteriorates when the regularizing parameter β takes the small values that are necessary for obtaining solutions with sharp edges. The purpose of this section is to present a numerical study of the dependence of the linear convergence rate of the FP algorithm on the parameter β . The result that we obtain in this section is that, although the convergence of the fixed point method can be slow, the convergence rate does not get asymptotically to 1 when $\beta \rightarrow 0$.

We have used one-dimensional signals u , to which we have added Gaussian white noise with some variance σ^2 to obtain z . In the following experiments, the parameter α is selected such that the solution of the problem with that parameter and $\beta = 10^{-4}$ matches the noise constraints $\|u - z\|^2 = \sigma^2$.

We estimate the linear convergence rate of the sequence $z_k(\beta) = F_\beta(u^k) - F_\beta(u^*)$ appearing in Theorem 6.1 for the functional of problem (2.3) and for several values of the parameter β forming a geometric sequence between 1 and 10^{-16} . The convergence rate for $z_k(\beta)$, $r(\beta) = \lim_k z_{k+1}/z_k$, is related to that for $u^k(\beta)$ by (6.10); thus it is enough to consider the convergence rate for $z_k(\beta)$.

The computations have been carried out in high-precision (33 digits) arithmetic to ensure the accuracy of the estimates. The solution of each problem, which is needed to estimate its convergence rate, is computed to full precision by the primal-dual method introduced in [6].

The signal in the first experiment has a size of 64 and it has been generated by adding Gaussian white noise to a piecewise constant signal with nine jump discontinuities, resulting in a *signal to noise ratio* ($\text{SNR} = \frac{\|u - \text{mean}(u)\|_{\mathcal{L}^\infty}}{\|u - z\|_{\mathcal{L}^\infty}}$) of approximately 1.

We show in the second column of Table 7.1 $r(\beta)$ for the different β 's displayed in the first column. In the third column we display $\bar{r}(\beta) = \lim_{k \rightarrow \infty} \lambda_k(\beta)$, for $\lambda_k = \lambda_k(\beta)$ defined in (6.1), since this limit is precisely what we have used in (6.6), (6.7) and have used in (6.6), (6.7), and (6.9) to estimate $\Lambda = 1 - \lambda_{\min}(C(u^*)^{-1}F''(u^*))$, which is shown in the fourth column of this table. We plot in Figure 7.1 columns 2, 3, and 4 versus column 1 of this table. The conclusions we can draw from this experiment are the following:

TABLE 7.1

Estimates of the convergence rate $r(\beta)$ and predicted convergence rate $\bar{r}(\beta)$; see section 7 for an explanation.

β	$r(\beta)$	$\bar{r}(\beta)$	Λ
1.0e + 00	8.52e - 01	9.232202e - 01	9.241033e - 01
7.8e - 02	8.85e - 01	9.412004e - 01	9.414071e - 01
6.2e - 03	9.09e - 01	9.536306e - 01	9.536308e - 01
4.8e - 04	9.08e - 01	9.533806e - 01	9.533810e - 01
3.8e - 05	9.06e - 01	9.518686e - 01	9.518715e - 01
3.0e - 06	9.05e - 01	9.513539e - 01	9.513572e - 01
2.3e - 07	9.04e - 01	9.512019e - 01	9.512063e - 01
1.8e - 08	9.04e - 01	9.511583e - 01	9.511635e - 01
1.4e - 09	9.04e - 01	9.511459e - 01	9.511514e - 01
1.1e - 10	9.04e - 01	9.511433e - 01	9.511480e - 01
9.2e - 12	9.04e - 01	9.511406e - 01	9.511471e - 01
7.2e - 13	9.04e - 01	9.511422e - 01	9.511468e - 01
5.7e - 14	9.04e - 01	9.511388e - 01	9.511467e - 01
4.5e - 15	9.04e - 01	9.511407e - 01	9.511467e - 01
3.5e - 16	9.04e - 01	9.511417e - 01	9.511467e - 01
1.0e - 16	9.04e - 01	9.511420e - 01	9.511467e - 01

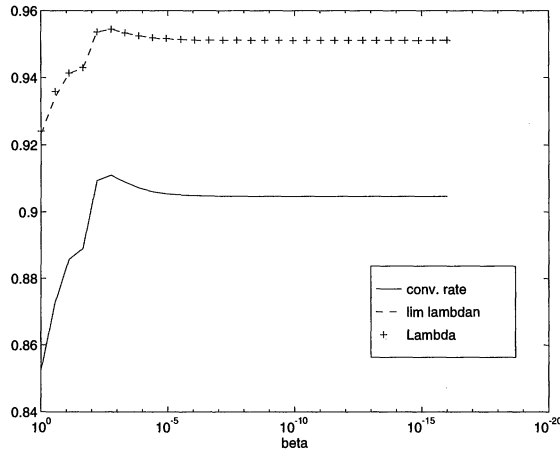


FIG. 7.1. Comparison of convergence rate and estimates for signal of size 64 and SNR ≈ 1.

- The convergence rate of the fixed point algorithm seems to approach asymptotically a value strictly smaller than 1, when $\beta \rightarrow 0$, i.e., $\lim_{\beta \rightarrow 0} r(\beta) < 1$. To the best of the authors' knowledge, this result seems not to have been noticed before.
- The estimate of convergence rate of the fixed point algorithm $\bar{r}(\beta) = \lim_k \lambda_k$, obtained in Theorem 6.1 (1), seems to approach asymptotically a value strictly smaller than 1, when $\beta \rightarrow 0$, i.e., $\lim_{\beta \rightarrow 0} \bar{r}(\beta) < 1$.
- $\lim_{\beta \rightarrow 0} r(\beta) < \lim_{\beta \rightarrow 0} \bar{r}(\beta) < 1$.
- The estimate of (6.6) seems to be asymptotically sharp.

The significance of the first point is that the convergence rate of the fixed point iteration does not get significantly worse once the parameter β gets below some threshold. The other points imply that a study of the convergence rate of the fixed point iteration can be based upon an analysis of the spectrum of the preconditioned matrix

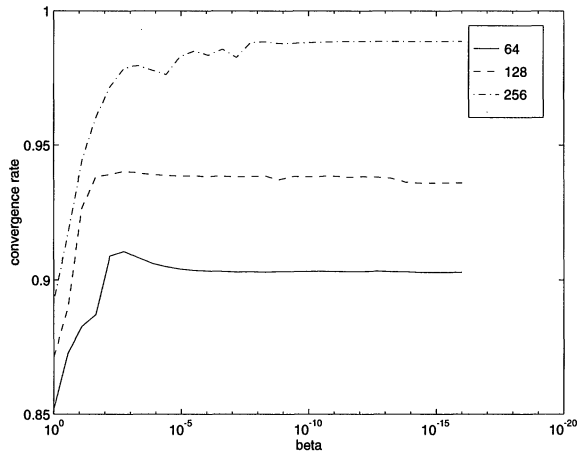


FIG. 7.2. Comparison of convergence rate for signals of sizes 64, 128, 256 and $\text{SNR} \approx 1$.

$C(u^*)^{-1}F''(u^*)$ for the typical piecewise constant solutions u^* of the total variation image restoration problem. Finally, we have not found any correlation between the latter asymptotic convergence rates and some characteristics of the data, such as noise level or number of jump discontinuities.

In the second experiment we obtain the convergence rate of the fixed point algorithm applied to signals of sizes 64, 128, and 256. These signals are obtained by adding Gaussian white noise to some original signals, resulting in noisy signals with a $\text{SNR} \approx 1$. The signals of sizes 64 and 128 have been obtained by repeated subsampling of the one of size 256. The values of β are the same as in the previous experiment and the results are shown in Figure 7.2. This experiment seems to suggest that the gap between 1 and the asymptotic convergence rate of the fixed point algorithm tends to shrink, i.e., the convergence gets slower, when the computational grid size grows, but the dependence of this gap on several parameters, such as the original image, α , and noise level, is still an open research question.

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