

REMARKS ON DEFINITION OF KHOVANOV HOMOLOGY

OLEG VIRO

Uppsala University, Uppsala, Sweden
POMI, St. Petersburg, Russia

ABSTRACT. Mikhail Khovanov defined, for a diagram of an oriented classical link, a collection of groups numerated by pairs of integers. These groups were constructed as homology groups of certain chain complexes. The Euler characteristics of these complexes are coefficients of the Jones polynomial of the link. The goal of this note is to rewrite this construction in terms more friendly to topologists. A version of Khovanov homology for framed links is introduced. For framed links whose Kauffman brackets are involved in a skein relation, these homology groups are related by an exact sequence.

1. Introduction

For a diagram D of an oriented link L , Mikhail Khovanov [4] constructed a collection of groups $\mathcal{H}^{i,j}(D)$ such that

$$K(L)(q) = \sum_{i,j} q^j (-1)^i \dim_{\mathbb{Q}}(\mathcal{H}^{i,j} \otimes \mathbb{Q}),$$

where $K(L)$ is a version of the Jones polynomial of L . These groups are constructed as homology groups of chain complexes. The primary goal of this note is to rewrite this construction in terms more pleasant for topologists.

To some extent this has been done recently by Dror Bar-Natan in his preprint [1]. This came together with progress in calculation and understanding of Khovanov homology, see [6] and [7]. The outbreak of activity that happened two years after the release of initial paper has proved that stripping Khovanov's construction of its fancy formal decorations was useful.

I hope that a further chewing¹ of Khovanov's construction that is presented below occur to be useful. I am grateful to Khovanov, who made these my

¹From Bar-Natan's paper: "Not being able to really digest it (construction of Khovanov's invariant – O.V.) we decided to just chew some, and then provide our output as a note containing a description of his construction, complete and consistent and accompanied by computer code and examples but stripped of all philosophy and of all the linguistic gymnastics that is necessary for the philosophy but isn't necessary for the mere purpose of having a working construction. Such a note may be more accessible than the original papers. It may lead more people to read Khovanov at the source, and maybe somebody reading such a note will figure out what the Khovanov invariants really are. Congratulations! You are reading this note right now."

considerations possible by not only the very initiating of the subject, but also by keeping (at least during the Fall of 1998) his text secret, while giving talks about his work at various seminars. The rumors reached me, and I tried to figure out how homology with the Euler characteristic equal to the Kauffman bracket could be defined. When the preprint became available, I found that Khovanov's construction can be reformulated in the way that I had guessed. I could upload then most of the text presented below, but hesitated, since I had no real new results based on my chewing. I still have no them, but feel that a text showing how a transition from Kauffman bracket to Khovanov homology could be motivated for a topologist may be appreciated. Some irresponsible speculations about possible generalizations are included at the end of Section 2.

The Khovanov homology is closer to Kauffman bracket, which is an invariant of non-oriented, but framed links, rather than to the Jones polynomial, which is an invariant of oriented, but non-framed links. The corresponding modification of Khovanov homology is presented in Section 3. This allows us to write down a categorification of the Kauffman skein relation for Kauffman bracket. The skein relation gives rise to a homology sequence.

2. From Kauffman bracket to Khovanov homology

Since the Euler characteristic does not change under passing from a chain complex to its homology, it is natural to expect that $\mathcal{H}^{i,j}$ appear as the homology groups of a chain complex $\mathcal{C}^{i,j}$ such that its polynomial Euler characteristic $\sum_{i,j} q^j (-1)^i \text{rk } \mathcal{C}^{i,j}$ is the Jones polynomial $K(L)(q)$.

Khovanov constructs such a complex starting with the Kauffman state sum presentation of the Jones polynomial. However, the construction proceeds as a chain of algebraic constructions of auxiliary objects. This hides a simple geometric meaning. A natural generators of Khovanov's complex can be represented as enhancements of the states from the Kauffman state sum presentation of the Jones polynomial.

2.1. Kauffman's states of a link diagram. Recall that for a state sum representation of the Jones polynomial, Kauffman [3] introduced the following states of a link diagram. Each state is a collection of markers. At each crossing of the diagram there should be a marker specifying a pair of vertical angles. See Figure 1.

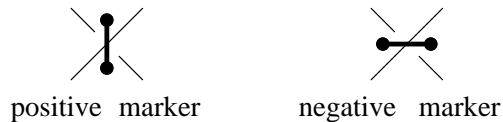


FIGURE 1. Markers comprising a Kauffman state.

2.2. Numerical characteristics of states and diagrams. Each marker of a Kauffman state has a *sign*, which is $+$ (or, rather, $+1$) if the direction of rotation of the upper string towards the lower one through the specified pair of vertical angles is counter-clock-wise, and $-$ (or -1) otherwise. For a state s of a diagram D denote by $\sigma(s)$ the difference between the numbers of positive and negative markers. A state of a diagram defines a *smoothing* of the diagram: at each of its double points the marked angles are united in a connected area, see Figure 2.

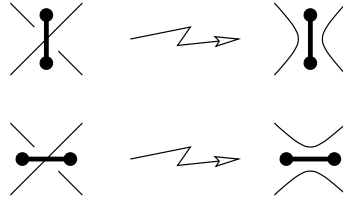


FIGURE 2. Smoothing of a diagram according to markers.

The result of the smoothing is a collection of circles embedded into the plane. Denote the union of these circles by D_s . The number of the circles is denoted by $|s|$.

With a crossing point of a diagram of an oriented link we associate *local writhe number* equal to

- $+1$, if at the point the diagram looks like $\nearrow \nwarrow$, and
- -1 , if it looks like $\nwarrow \nearrow$.

The sum of local writhe numbers over all crossing points of a link diagram D is called the *writhe number* of D and denoted by $w(D)$.

To a Kauffman state s of an oriented link diagram D we assign a polynomial

$$V_s(A) = A^{\sigma(s)}(-A)^{-3w(D)}(-A^2 - A^{-2})^{|s|},$$

where $w(D)$ is the writhe number of D .

2.3. Jones Polynomial. The sum of $V_s(A)$ over all states s of the diagram is a version the Jones polynomial of L . It is denoted by $V_L(A)$.

Khovanov [4] uses another variable q instead of A . It is related to A by $q = -A^{-2}$. Following Khovanov, we denote the corresponding version of the Jones polynomial by $K(L)$. It is defined by $K(L)(-A^{-2}) = V_L(A)$.

Certainly, $K(L)$ can be presented as the sum of the corresponding versions of $V_s(A)$ over all Kauffman states s . The summand of $K(L)$ corresponding to s can be defined via $K(s)(-A^2) = V_s(A)$ or, more directly, by

$$K(s)(q) = (-1)^{\frac{w(D)-\sigma(s)}{2}} q^{\frac{3w(D)-\sigma(s)}{2}} (q + q^{-1})^{|s|}.$$

2.4. Enhanced Kauffman states. By an *enhanced Kauffman state* S of an oriented link diagram D we shall mean a collection of markers comprising a usual Kauffman state s of D enhanced by an assignment of a plus or minus sign to each of the circles obtained by the smoothing of D according to the markers.

Denote by $\tau(S)$ the difference between the numbers of pluses and minuses assigned to the circles of D_s . Put

$$j(S) = -\frac{\sigma(s) + 2\tau(S) - 3w(D)}{2}.$$

Observe that both $\sigma(s)$ and $w(D)$ are congruent modulo 2 to the number of crossing points. Therefore $j(S)$ is an integer.

2.5. Monomial state sum representation of Jones polynomial. Recall that $V_s(A) = A^{\sigma(s)}(-A)^{-3w(D)}(-A^2 - A^{-2})^{|s|}$. We can associate to each component of D_s one of $|s|$ factors $(-A^2 - A^{-2})$ of the right hand side of this formula.

Passing to the enhanced Kauffman states, we associate the summand $-A^2$ of the sum $-A^2 - A^{-2}$ to a component of $D_{|s|}$ equipped with $+$. To the same component, but equipped with $-$, we associate the summand $-A^{-2}$.

Thus an enhancement of a Kauffman state gives rise to a choice of a summand from each of the binomial factors in the formula defining $V_s(A)$. Now by opening the brackets we obtain the following presentation for $V_s(A)$:

$$(1) \quad V_s(A) = \sum_{S \in s} A^{\sigma(s)} (-A^2)^{\tau(S)} (-A)^{-3w(D)} = \sum_{S \in s} (-1)^{\tau(S)+w(D)} A^{-2j(S)}.$$

Denote by $V_S(A)$ the summand $(-1)^{\tau(S)+w(D)} A^{-2j(S)}$ which corresponds to S . Hence $V_s(A) = \sum_{S \in s} V_S(A)$ and

$$(2) \quad V_L(A) = \sum_{\substack{\text{Kauffman} \\ \text{states } s \text{ of } D}} V_s(A) = \sum_{\substack{\text{enhanced Kauffman} \\ \text{states } S \text{ of } D}} V_S(A) = \sum_{\substack{\text{enhanced Kauffman} \\ \text{states } S \text{ of } D}} (-1)^{\tau(S)+w(D)} A^{-2j(S)}.$$

2.6. Change of variable. Let us switch to variable q used by Khovanov.

$$\begin{aligned}
 (3) \quad V_S(A) &= (-1)^{\tau(S)+w(D)} A^{-2j(S)} = \\
 &= (-1)^{\tau(S)+w(D)} (A^{-2})^{j(S)} = \\
 &= (-1)^{\tau(S)+w(D)-j(S)} (-A^{-2})^{j(S)} = \\
 &= (-1)^{\tau(S)+w(D)+\frac{1}{2}\sigma(s)+\tau(S)-\frac{3}{2}w(D)} q^{j(S)} = \\
 &= (-1)^{\frac{\sigma(s)-w(D)}{2}} q^{j(S)} = \\
 &= (-1)^{\frac{w(D)-\sigma(s)}{2}} q^{j(S)}.
 \end{aligned}$$

Denote $(-1)^{\frac{w(D)-\sigma(s)}{2}} q^{j(S)}$ by $K(S)(q)$. Clearly,

$$(4) \quad K(L)(q) = \sum_{\substack{\text{enhanced Kauffman} \\ \text{states } S \text{ of } D}} (-1)^{\frac{w(D)-\sigma(s)}{2}} q^{j(S)} = \sum_{\substack{\text{enhanced Kauffman} \\ \text{states } S \text{ of } D}} K(S)(q).$$

2.7. Khovanov chain groups. Denote the free abelian group generated by enhanced Kauffman states of a link diagram D by $\mathcal{C}(D)$. Denote by $\mathcal{C}^j(D)$ the subgroup of $\mathcal{C}(D)$ generated by enhanced Kauffman states S of D with $j(S) = j$. Thus $\mathcal{C}(D)$ is a \mathbb{Z} -graded free abelian group:

$$\mathcal{C}(D) = \bigoplus_{j \in \mathbb{Z}} \mathcal{C}^j(D).$$

For an enhanced Kauffman state S belonging to a Kauffman state s of a link diagram D , put

$$i(S) = \frac{w(D) - \sigma(s)}{2}.$$

Denote by $\mathcal{C}^{i,j}(D)$ the subgroup of $\mathcal{C}^j(D)$ generated by enhanced Kauffman states S with $i(S) = i$.

Notice that as follows from (4),

$$(5) \quad K(L)(q) = \sum_{j=-\infty}^{\infty} q^j \sum_{i=-\infty}^{\infty} (-1)^i \text{rk } \mathcal{C}^{i,j}(D).$$

Remark. The Kauffman choice of variable (A instead of q) in the Jones polynomial discussed above suggests another choice of the second grading of $\mathcal{C}(D)$. In fact, the choices incorporate already two different \mathbb{Z}_2 -grading: $\tau(S) + w(D) \pmod{2}$ and $\frac{w(D)-\sigma(s)}{2} \pmod{2}$. However these \mathbb{Z}_2 -grading differ just by the reduction modulo 2 of the first one,

$$\tau(S) + w(D) - \frac{w(D) - \sigma(s)}{2} = j(S) \pmod{2}.$$

One can prove that the Kauffman choice of variable gives rise to the same homology groups.

2.8. Differential. Let D be a diagram of an oriented link L . In [4] Khovanov defined in $\mathcal{C}^{i,j}(D)$ a differential of bidegree $(1, 0)$ and proved that its homology groups $\mathcal{H}^{i,j}(D)$ do not depend on D . The construction of the differential depends on ordering of crossing points of D , although the homology groups do not. Suppose that the crossing points of D are numerated by natural numbers $1, \dots, n$.

Khovanov's description of the differential is somewhat complicated by several auxiliary algebraic constructions. I give here its simplified version. I just describe the matrix elements. In the context of chain complex the matrix elements are traditionally called *incidence numbers*. For enhanced Kauffman states S_1 and S_2 , denote their incidence number by $(S_1 : S_2)$.

Recall that $\mathcal{C}^{i,j}(D)$ is generated by enhanced Kauffman states. Thus an incidence number is a function of two enhanced Kauffman states, say S_1 and S_2 (which are generators of $\mathcal{C}^{i,j}(D)$ and $\mathcal{C}^{i+1,j}(D)$, respectively). Enhanced Kauffman states with a nonzero incidence number are said to be *incident* to each other. Pairs of incident states satisfy natural restrictions. Surprisingly, these restrictions give exact description of the set of incident states: each pair of enhanced Kauffman states which is not thrown away by the restrictions consists of incident states.

The first restriction emerges from our desire to have a differential of bidegree $(1, 0)$. Since the differential preserves j and increases i by one, $j(S_1) = j(S_2)$ and $i(S_2) = i(S_1) + 1$. Recall that $i(S) = \frac{w(D) - \sigma(s)}{2}$. Therefore $i(S_2) = i(S_1) + 1$ implies that $\sigma(s_2) = \sigma(s_1) - 2$, in other words the number of negative markers of S_2 is greater by one than the number of negative markers of S_1 .

It is natural to enforce this numerological restriction in the following way:

The incidence number is zero, unless only at one crossing point of D the markers of S_1 and S_2 differ and at this crossing the marker of S_1 is positive, while the marker of S_2 is negative.

In our description of the incidence number of S_1 and S_2 we will assume that this is the case. The crossing point where the markers differ is called the *difference point* of S_1 and S_2 . Let it have number k .

Since exactly at one crossing the markers differ, D_{S_2} is obtained from D_{S_1} by a single oriented Morse modification of index 1. Hence $|S_1| - |S_2| = \pm 1$. In other words either D_{S_2} is obtained from D_{S_1} by joining two circles or by splitting a circle of D_{S_1} into two circles.

Here is the next natural restriction on incident states:

The incidence number of S_1 and S_2 vanishes, unless the common circles of D_{S_1} and D_{S_2} have the same signs.

This, together with equality $j(S_2) = j(S_1)$, gives a strong restriction also on the signs of the circles of D_{S_i} adjacent to the k th crossing point. Indeed,

$$\begin{aligned} j(S_2) &= \frac{3w(D) - \sigma(s_2) - 2\tau(S_2)}{2} \\ &= j(S_1) = \frac{3w(D) - \sigma(s_1) - 2\tau(S_1)}{2} = \frac{3w(D) - \sigma(s_2) - 2 - 2\tau(S_1)}{2}, \end{aligned}$$

thus $\tau(S_2) = \tau(S_1) + 1$.

Now we can easily list all the situations which satisfy these restrictions (see Figure 3):

1. If $|S_2| = |S_1| - 1$ and both joining circles of D_{S_1} are negative then the resulting circle of S_2 should be negative.
2. If $|S_2| = |S_1| - 1$ and the joining circles of D_{S_1} have different signs then the resulting circle of S_2 should be positive.
3. If the joining circles of D_{S_1} are positive, none S_2 can be incident.
4. If $|S_2| = |S_1| + 1$ and the splitting circle of D_{S_1} is positive then both of the circles of D_{S_2} obtained from it should be positive.
5. If $|S_2| = |S_1| + 1$ and the splitting circle of D_{S_1} is negative then the circles of D_{S_2} obtained from it should be of different signs.

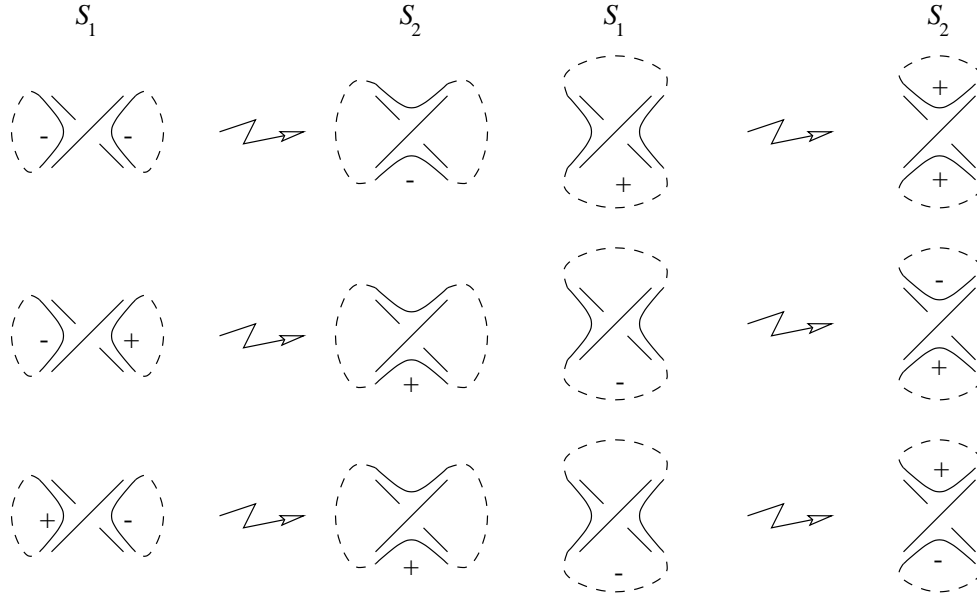


FIGURE 3. Pairs of incident enhanced Kauffman states. Dotted arcs show how the fragments of D_s at a crossing point are connected in the whole D_s .

In each of the cases listed above the incidence number $(S_1 : S_2)$ is ± 1 . The sign depends on the ordering of the crossing points: it is equal to $(-1)^t$ where t is the number of negative markers in S_1 numerated with numbers greater than k .

This sign is needed to make the differential satisfying identity $d^2 = 0$. However, the proof of this identity requires some routine check. At first glance, Khovanov [4] escaped it. In fact, it is hidden in the checking (which was left to the reader) that F is a functor, see [4], Section 2.2.

2.9. Enhanced Kauffman states with polynomial coefficients. Khovanov constructed not only groups $\mathcal{H}^{i,j}$, but also graded modules H^i over the ring $\mathbb{Z}[c]$ of polynomials with integer coefficients in variable c of degree 2. The grading is a representation of H^i as a direct sum of abelian subgroups $H^{i,j}$ such that multiplication by c in H^i gives rise to a homomorphism $H^{i,j} \rightarrow H^{i,j+2}$.

To construct homology groups $H^{i,j}$, let us define the corresponding complex of graded $\mathbb{Z}[c]$ -modules C^i . The module C^i is the sum of its subgroups $C^{i,j}$. The group $C^{i,j}$ is generated by formal products $c^k S$, where $k \geq 0$ and S is an enhanced Kauffman states S with $i(S) = i$ and $j(S) = j - 2k$.

The differential is defined almost in the same way. The states which were adjacent above are adjacent here, as well as their products by the same power of c . Products of enhanced Kauffman states by different powers of c are not adjacent, besides the following situations: $c^{k+1}S_2$ is adjacent to $c^k S_1$, if the markers of S_1 and S_2 differ exactly at one point, where the marker of S_1 is positive and the marker of S_2 negative, the signs of S_1 and S_2 on the common circles of D_{S_1} and D_{S_2} are the same, $|S_2| = |S_1| + 1$, the splitting circle of D_{S_1} is negative, and the circles of D_{S_2} obtained from it are positive, see Figure 4, where this situation is shown symbolically in the style of Figure 3.

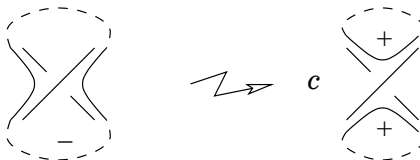


FIGURE 4.

Each group $C^{i,j}$ is finitely generated, but there are infinitely many non-trivial groups. Groups $H^{i,j}$ with fixed i and sufficiently large j are isomorphic to each other.

2.10. Speculations. How can one understand construction of Khovanov homology? By understanding I mean a formulation which would give, true or false, indication to possible generalizations. The general program of categorification pictured by Khovanov in [4] is an attractive approach to the problem of constructing combinatorial counterparts for Donaldson and Seiberg-Witten invariants, although it does not promise a fast and technically easy track.

In the construction above the specific of Kauffman bracket is used so much that it is difficult to imagine a generalization to other polynomial link invariants. A state sum over smoothings along Kauffman markers does not appear in formulas representing any other quantum invariant. Of course, there are many

state sum formulas. One can try to decompose any of them to a sum of monomials and consider decompositions as chains. Attach to them appropriate dimensions and define differentials such that the homology groups would be invariant under Reidemeister moves. Success of the whole project depends on quality of several guesses. The most difficult of the guesses seems to be the choice of differentials. It may involve and rely on a delicate analysis of the effects of Reidemeister moves. However a wrong choice at the previous step can make any efforts at this last step unfruitful.

An additional hint comes from Khovanov's mention of Lusztig's canonical bases. Indeed, signs which are attached to circles in a smoothened link diagram can be replaced with orientations and an orientation was interpreted by Frenkel and Khovanov [2] as an element of Lusztig's canonical basis. Smoothings at crossing points correspond then to monomials of the entries of the R -matrix. The decomposition of the entries of R -matrix to monomials not as clear as it seems: in the only known case considered above a zero entry of R -matrix is decomposed to two non-zero monomials canceling each other. However, the main guess is still to be done after the entries of the R -matrix in crystal bases are decomposed to monomials: one should find a differential that would give homology invariant under Reidemeister moves.

If the guesses made above about the rôles of crystal bases and monomials of R -matrices are correct, 1 + 1-dimensional TQFTs would not appear in the future more involved categorifications. Instead of cobordisms of 1-manifolds, there appear probably cobordisms of 4-valent graphs with some additional structure.

How can one estimate chances for further categorification of quantum topology? Still, they do not seem to be clear, despite of high optimism of Khovanov. As for chain groups, the observations made above are encouraging. However, we are not even able to discuss the choice of differentials on a comparable level of generality.

It would be very interesting, especially for prospective 4-dimensional applications, to find a categorification of face state models.

3. Frame version

3.1. Kauffman bracket versus Jones polynomial. Involvement of an orientation of link in the definition of the Jones polynomial in Section 2.3 is easy to localize. It is limited to the writhe number $w(D)$ of D . If we remove $w(D)$, we get the *Kauffman bracket* of diagram D . To eliminate fractional powers, we stay with Kauffman's variable A . Thus Kauffman bracket of a link diagram D is

$$\langle D \rangle = \sum_{\substack{\text{Kauffman} \\ \text{states } s \text{ of } D}} A^{\sigma(s)} (-A^2 - A^{-2})^{|s|}.$$

It is not invariant under first Reidemeister moves. Hence it is not an ambient isotopy invariant of link. However it is invariant with respect to second and third Reidemeister moves.

A link diagram defines the blackboard framing on the link. The isotopies which accompany second and third Reidemeister moves can be extended to the isotopy of the corresponding framings. Moreover, embedding of the diagrams to the 2-sphere makes this relation two-sided: link diagrams on S^2 can be converted to each other by a sequence of second and third Reidemeister moves iff the corresponding links with blackboard framings are isotopic in the class of framed links. Since the Kauffman bracket can be defined for a link diagram on 2-sphere, this makes the Kauffman bracket invariant for non-oriented framed links.

The Kauffman bracket is categorified below.

3.2. Framed Khovanov homology. For an enhanced Kauffman state of a link diagram D , put

$$J(S) = \sigma(s) + 2\tau(S) \quad \text{and} \quad I(S) = \sigma(s).$$

If this is an oriented link, and hence $w(D)$ is defined, then

$$J(S) = 3w(D) - 2j(S) \quad \text{and} \quad I(S) = w(D) - 2i(S).$$

In terms of $J(S)$ and $I(S)$, the Kauffman bracket is expressed as follows:

$$\langle D \rangle = \sum_{\substack{\text{enhanced Kauffman} \\ \text{states } S \text{ of } D}} (-1)^{\frac{I(S)}{2}} A^{J(S)}.$$

Denote the free abelian group generated by enhanced Kauffman states S of D with $I(S) = i$ and $J(S) = j$ by $\mathcal{C}_{i,j}(D)$. If one orients the link, the Khovanov chain groups $\mathcal{C}^{i,j}(D)$ appear, and

$$\mathcal{C}_{i,j}(D) = \mathcal{C}^{\frac{w(D)-i}{2}, \frac{3w(D)-j}{2}}(D).$$

Under this identification, the differentials of the Khovanov complex turn into differentials

$$\partial : \mathcal{C}_{i,j}(D) \rightarrow \mathcal{C}_{i-2,j}(D)$$

(the construction of the differentials does not involve orientation of the link, hence it does not matter what orientation is used). Denote the homology group of the complex obtained by $H_{i,j}(D)$.

3.3. Skein homology sequence. Let D be a link diagram, c its crossing and D_+ , D_- be link diagrams obtained from D by smoothing at c along positive and negative markers, respectively. As is well-known, the Kauffman brackets of D , D_+ and D_- are related as follows

$$\langle D \rangle = A \langle D_+ \rangle + A^{-1} \langle D_- \rangle.$$

This equality is called the *Kauffman skein relation*. It allows one to define the Kauffman bracket up to a normalization (which can be done by fixing the Kauffman bracket of the unknot).

Let us categorify this skein relation. Consider map

$$\alpha : \mathcal{C}_{i,j}(D_-) \rightarrow \mathcal{C}_{i-1,j-1}(D)$$

which sends an enhanced Kauffman state S of D_- to the enhanced Kauffman state of D smoothing along which coincides with the smoothing of D_- along S and signs of the ovals are the same, too. The collection of these maps is a homomorphism of complex, that is they commute with ∂ , provided c is the last crossing of D in the ordering of crossings which is used in the construction of ∂ . Indeed, the incidence coefficients are the same for an enhanced Kauffman states of D_- and its images in D , and the latter cannot contain in the boundary a state with positive marker at c .

Now consider map

$$\beta : \mathcal{C}_{i,j}(D) \rightarrow \mathcal{C}_{i-1,j-1}(D_+)$$

which sends each enhanced Kauffman state with negative marker at c to 0 and each enhanced Kauffman state with positive marker at c to the enhanced Kauffman state of D_+ with the same smoothing and signs of the ovals. This is again a complex homomorphism.

Homomorphisms α and β form a short exact sequence of complexes:

$$0 \longrightarrow \mathcal{C}_{*,*}(D_-) \xrightarrow{\alpha} \mathcal{C}_{*-1,*-1}(D) \xrightarrow{\beta} \mathcal{C}_{*-2,*-2}(D_+) \longrightarrow 0$$

It induces collection of long homology sequences:

$$\begin{array}{ccccccc} \xrightarrow{\partial} & H_{i,j}(D_-) & \xrightarrow{\alpha_*} & H_{i-1,j-1}(D) & \xrightarrow{\beta_*} & H_{i-2,j-2}(D_+) & \\ \xrightarrow{\partial} & H_{i-2,j}(D_-) & \xrightarrow{\alpha_*} & H_{i-3,j-1}(D) & \xrightarrow{\beta_*} & H_{i-4,j-2}(D_+) & \xrightarrow{\partial} \end{array}$$

A special case of this sequence, which relates the groups of connected sum and disjoint sum of knots, can be found in Section 7.4 of Khovanov's paper [4]. This special case is the only one which can be formulated for the original version of homology depending on orientations of links.

I was not able to categorify the other skein relation, the one which involves the Jones polynomial of *oriented* links. It follows from a couple of skein relations of the type considered above.

References

- [1] Dror Bar-Natan, Khovanov's categorification of the Jones polynomial, arXiv: math.QA /0201043.
- [2] I. B. Frenkel, M. Khovanov, Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$, *Duke Math. Journal*, v. 87, (1997), 409–480.
- [3] L.H.Kauffman, State models and the Jones polynomial, *Topology*, **26:3** (1987), 395–407.
- [4] Mikhail Khovanov, A categorification of the Jones polynomial, *Duke Math. J.*, 101 (3), 359–426, 1999, arXiv: math.QA /9908171.

- [5] Mikhail Khovanov, A functor-valued invariant of tangles, arXiv: math.QA /0103190
- [6] Mikhail Khovanov, Patterns in knot cohomology I, arXiv: math.QA /0201306.
- [7] E. S. Lee, The support of the Khovanov's invariants for alternating knots, arXiv: math.GT /0201105.

DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, BOX 480, S-751 06 UPPSALA, SWEDEN

E-mail address: oleg@math.uu.se