Conservation Laws

Past and Future

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Introduction

Cecilia Krieger and Evelyn Nelson

Outline

Systems of quasilinear hyperbolic PDE (conservation laws)

- Where they come from; why they are studied
- Some of the challenges (well-posedness)
- How symmetry is broken in a system that is formally time-reversible

Analysis of conservation laws

- Results on Riemann problems (geometric)
- BV spaces and well-posedness in one space dimension
- Open questions

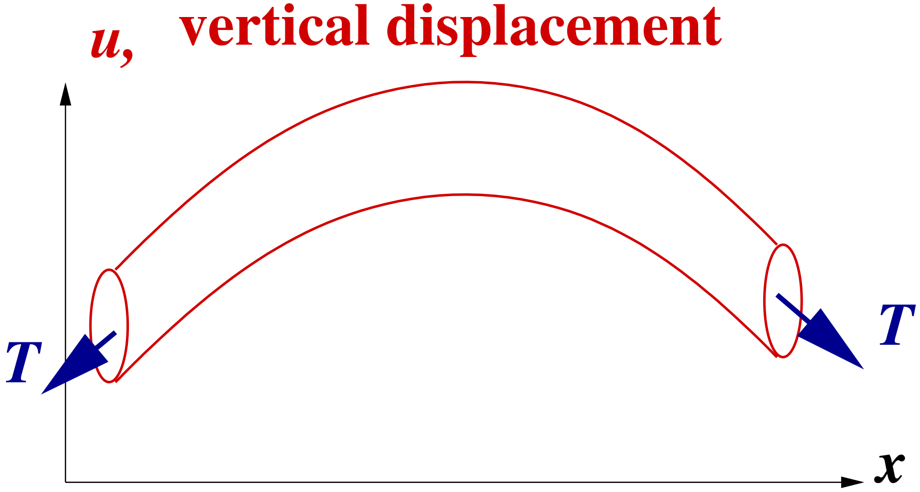
Partial Differential Equations

How PDE arise: local information (u, Du at a point)
Why solve them: obtain global conclusions about function

Example
Wave Eqn (1-D string)

$$Pu \equiv u_{tt} - c^2 u_{xx} = 0$$

ρu_{tt} force
proportional to
u_{xx} curvature



Equation $(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$ predicts

- waves travelling with characteristic speeds ($\pm c = \sqrt{T/\rho}$)

which is not obvious from the local description

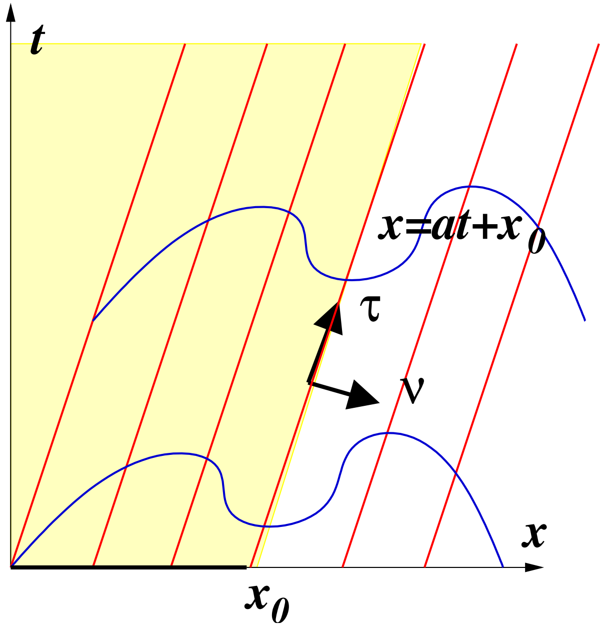
Note conservation form $\partial_t \begin{pmatrix} u_t \\ u_x \end{pmatrix} + \partial_x \begin{pmatrix} -c^2 u_x \\ -u_t \end{pmatrix} = 0$

Hyperbolicity

Model $u_t + au_x = 0, \quad u = f(x - at)$
Characteristics

1. Propagation of information
2. Barrier to information

Linear Theory:



Characteristic normals for linear eqns and systems

$$P(\partial)u = f, \qquad \partial = \partial_{x_1}\partial_{x_2}\ldots\partial_{x_n}$$

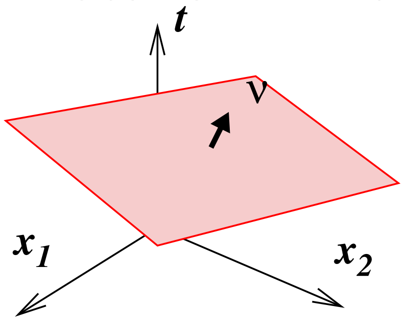
$P_0(\nu) = 0$: characteristic normal
First-order system:

$$\sum_i A_i \partial_{x_i} u + Bu = f, \quad P_0(\nu) = \det\left(\sum A_i \nu_i\right)$$

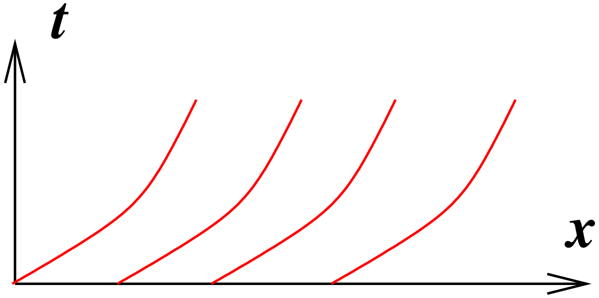
Quasilinear Hyperbolic Equations

Specialize to space & time $(x,t) = (x_1, \dots, x_n, t)$

Picture in multi-D



Variable coefficients



Characteristics are surfaces; still feature

1. Propagation of information (inside envelope)
2. Barrier to information (domain of dependence)

Burgers Equation $u_t + uu_x = 0$

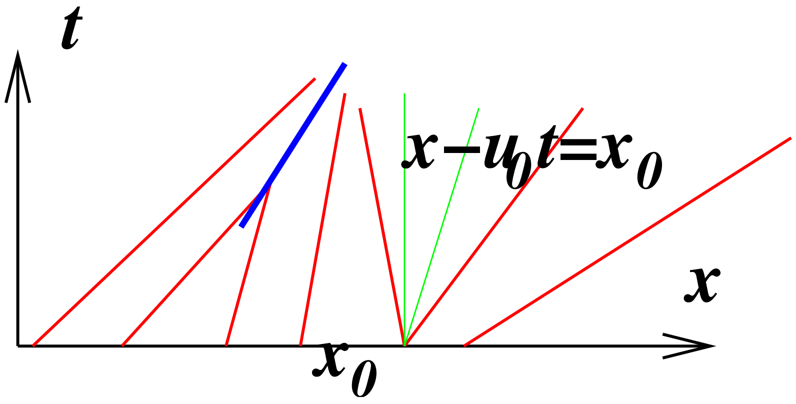
$$u(x_0 + u_0(x_0)t, t) = u_0(x_0)$$

Converging characteristics:

form shock, weak solution

Diverging characteristics:

form rarefaction



Loss of time reversibility: information is lost in forward time

Weak Solutions

Notion of weak solution central to modern PDE

- smooth categories (C^∞, C^ω) not correct for well-posedness
- “Derivative bad – Integral good”
- classical spaces not closed under taking of classical derivatives (unbounded operators)
- spaces of distributions allow definition of weak derivatives ($\int u'\varphi = -\int u\varphi'$) for linear operators

$$u_t + au_x = 0 \qquad \int (u\varphi_t + au\varphi_x)\,dx\,dt = 0, \quad \forall\varphi \in C_0^\infty$$

The Weakness of Weak Solutions

Three facts about linear theory:

1. useful spaces are Sobolev spaces (Banach or Hilbert, not merely topological) – $W^{m,p}$: m weak derivatives in L^p
2. “weak convergence” is useful, and is a different concept from “weak solution”
3. combine with regularity to get classical solutions (especially for elliptic equations)

Three difficulties with nonlinear equations:

1. $\mathcal{D}'$ is too broad (need to define $f(u)$)
2. weak convergence does not preserve nonlinear relations
3. hyperbolic and elliptic theory very different

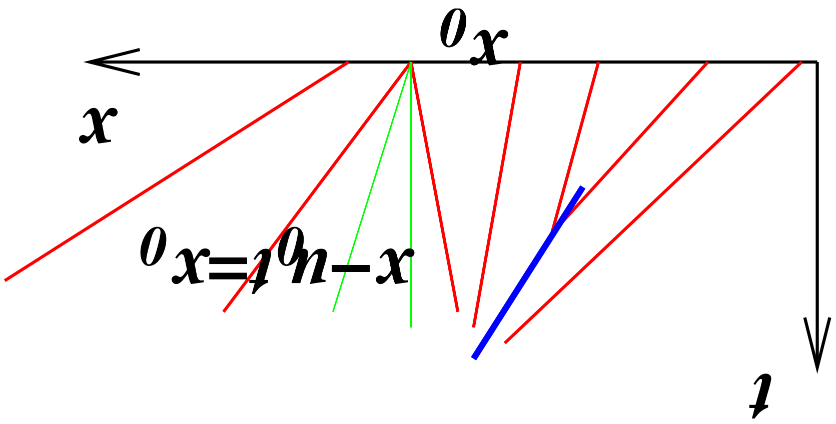
Parabolic equations and entropy

Quasilinear system $u_t + \sum A_i(u)\partial_i u + B(u) = 0$
Weak solutions defined if each $A_i = \mathrm{d}f_i$ (conservation laws)
Shocks are a type of weak solution:

$$\int \left\{ u\varphi_t + \sum f_i(u)\partial_i\varphi - B(u)\varphi \right\} dx\, dt = 0$$

Discontinuity at shock – RH relation $s[u] = [f(u) \cdot \nu]$

Time reversal: like backward
heat equation



Vanishing viscosity

$$u_t + \sum A_i(u)\partial_i u + B(u) = \varepsilon \Delta u$$

Entropy: convex function $\eta(u)$, $\eta(u)_t + \sum \partial_{x_i} q_i(u) \leq 0$

Analysis of Conservation Laws

Geometric approaches:

- existence of solutions to RH relation – bifurcation theory
- admissibility of shocks – phase plane analysis
- resolution of a discontinuity (Riemann problem) – IFT

Analytic tools:

- function spaces, L^1, L^∞, BV
- geometric measure theory
- nonlinear semigroup theory
- compactness: Helly's theorem, compensated compactness

Bifurcation Theory

Existence of solutions of RH equation $s[u] = [f(u)]$ (1 D)

$V(u, s; u_\ell) \equiv f(u) - f(u_\ell) - s(u - u_\ell) = 0$ R-H relation

$V : u \in \mathbf{R}^n \to \mathbf{R}^n$, parameterized by s

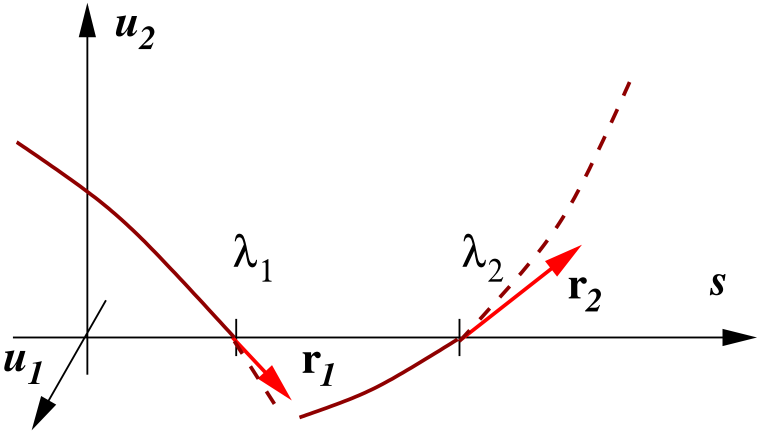
States joined to u_ℓ by a shock: soln set (u, s) of $V = 0$

$\exists$ Trivial solution $u = u_\ell$ for all s

"t-equivalence"; transcritical bifurcation

IFT $\Rightarrow$ u_ℓ unique soln if $\mathrm{d}V_u(u_\ell, s) = A(u_\ell) - sI$ nonsingular

canonical form $h(x, \lambda) = x^2 - \lambda x$: $h_x = 0$, $h_{xx} \neq 0$, $h_{x\lambda} \neq 0$



$s = \lambda_i(u_\ell)$

Liapunov-Schmidt reduction to single equation — follows from distinct eigenvalues

$h_x = 0$ implies $\dot{u} = \mathbf{r}_i$

$h_{xx} \neq 0$ follows from $\mathbf{r}_i \cdot \nabla\lambda_i \neq 0$ (genuine nonlinearity)

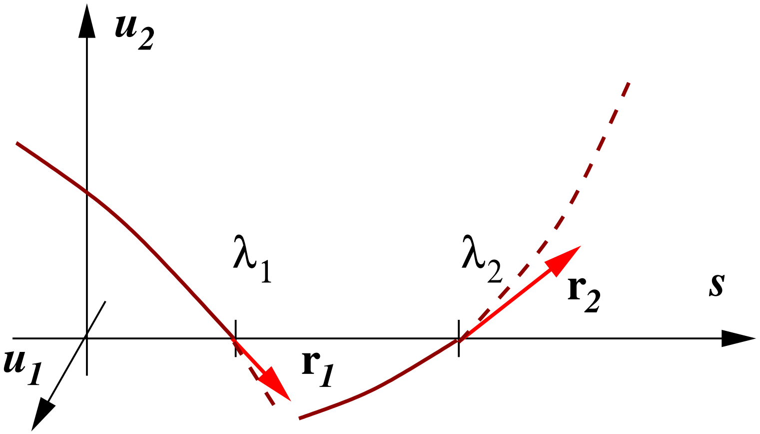
$h_{x\lambda} \neq 0$ follows from $\ell_i \mathbf{r}_i \neq 0$

Riemann Problems

One space dimension $u_t + f(u)_x = 0$, $u \in \mathbf{R}^n$
Travelling waves for shocks $u_t + f(u)_x = \varepsilon u_{xx}$; $u = u\left(\frac{x-st}{\varepsilon}\right)$
Note determination of time direction

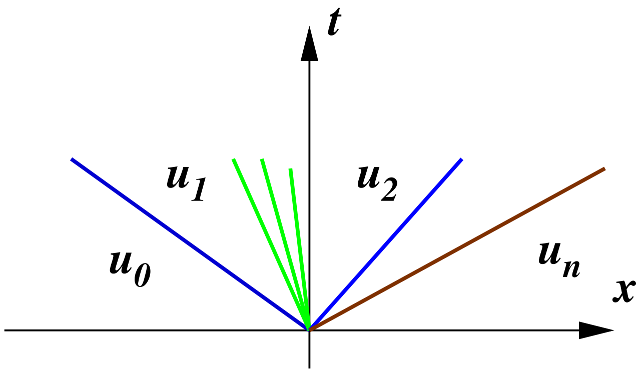
Riemann Data

$$u(x,0) = \begin{cases} u_\ell, & x < 0 \\ u_r, & x \geq 0 \end{cases}$$



$$u_0 = u_\ell, \quad u_1 = W_1(\epsilon_1; u_0), \quad \ldots, \quad u_n = W_n(\epsilon_n; u_{n-1})$$

Solve $u_n(\epsilon_1, \ldots \epsilon_n) = u_r$
by IFT for small ϵ



Random Choice and Wave Front Tracking

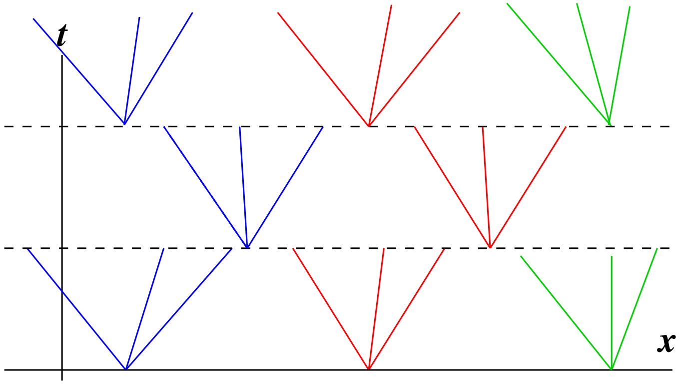
Weak solutions defined for $u \in L^\infty$ (bdd, mble)

1-D, more regularity: $u(x,0) \in BV \Rightarrow$ sol'n in BV

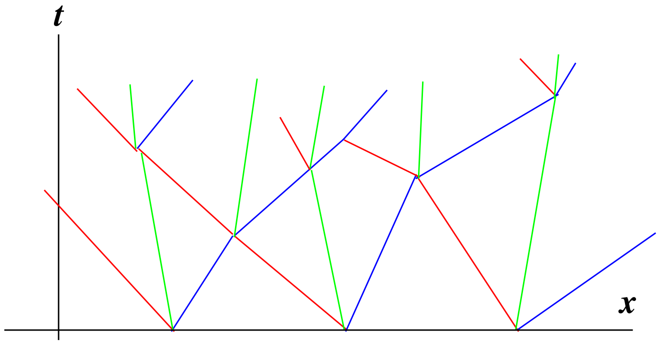
"Outside a set of 1-D Hausdorff measure 0, a BV fn is either approx continuous or has an approx jump discont."

Use Riemann solutions to prove existence:

Glimm's random choice



Risebro-Bressan's wave front tracking



$\operatorname{Var} u(\cdot,0) \le \varepsilon \Rightarrow \operatorname{Var} u(\cdot,t) \le M$, $\int |u(t,x) - u(s,x)| \le L|t-s|$

Helly's theorem $\Rightarrow$ subsequence cvges ptwise to BV soln.

Bressan: SRS (Standard Riemann Semigroup) – uniqueness, well-posedness, & regularity (cont's except for countable set of shock curves & interaction points)

Nonlinear Semigroups

Solutions to scalar eqn form L^1-contractive semigroup:

$$\int |u(x,t) - v(x,t)|\, dx \le \int |u(x,s) - v(x,s)|\, dx, \qquad t > s$$

Basis for existence theorem (Crandall): abstract Cauchy problem

$$\frac{du}{dt} + A(u) = 0, \qquad \text{in} \quad L^1$$

False for systems. But for systems, Bressan and Liu & Yang found a nonlinear functional equiv to L^1 dist and such that:

$$\Phi\big(u(t), v(t)\big) \le \Phi\big(u(s), v(s)\big) + L(t-s), \qquad \forall \quad s < t$$

and showed that any stable solution coincides with front-tracking solution

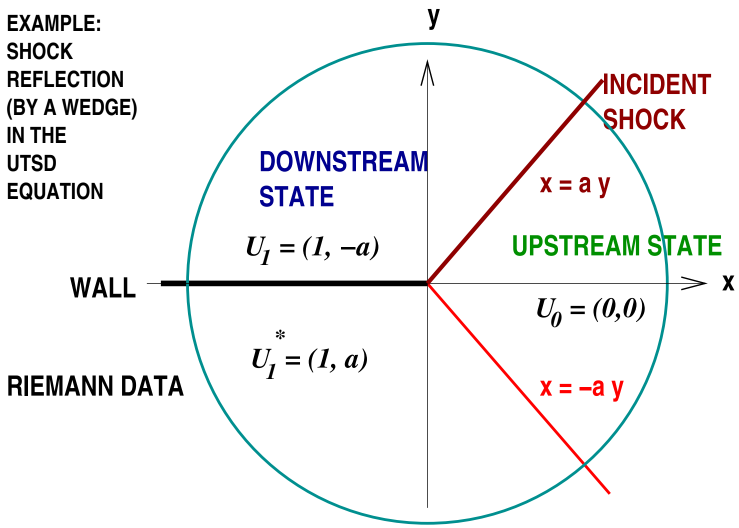
Multidimensional problems

Two-D Riemann problems

Self-similar problems

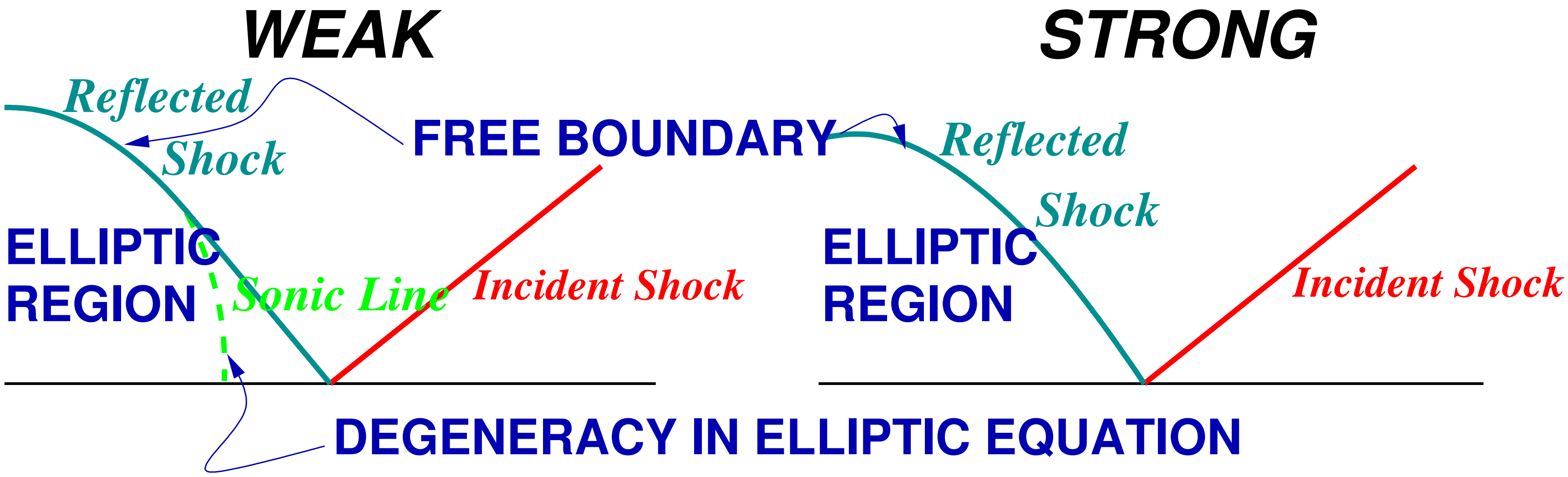
Model equations

- Canic, K & Kim
- T Chang (D Zhang)
- S-X Chen
- Y Zheng, K-W Song
- G-Q Chen & M Feldman



EXAMPLE:
SHOCK
REFLECTION
(BY A WEDGE)
IN THE
UTSD
EQUATION

RIEMANN DATA



DEGENERACY IN ELLIPTIC EQUATION

Future Directions

- Large Data: obstructions to existence of weak solutions
- “Resonances” among different wave families
- Relation to kinetic theory and other “more physical” continuum mechanics theories
- Multidimensional problems:
  - BV not the correct space: what are good candidates?
  - what are good model problems?
  - what information can numerical simulations give?

Slides for talk
http://www.math.uh.edu/$\sim$blk

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