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Existence and Uniqueness of Entropy Solutions to the Riemann Problem for Hyperbolic Systems of Two Nonlinear Conservation Laws*

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This paper contains a proof of the existence and uniqueness of solutions to the Riemann problem for systems of two hyperbolic conservation laws in one space variable. Our main assumptions are that the system is strictly hyperbolic and genuinely nonlinear. We also require that the system satisfy standard conditions on the second Fréchet derivatives, and one other hypothesis, which we have called the half-plane condition. This hypothesis replaces other, more restrictive hypotheses required by previous authors. The methods and results of this paper are designed to be applicable to systems of conservation laws which are not strictly hyperbolic.

1. INTRODUCTION

This paper contains a proof of the existence of solutions to the Riemann problem for systems of two hyperbolic conservation laws in one space variable,

$$U_t + F(U)_x = 0 \tag{1.1}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad F = \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix}.$$

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Our main assumptions are that the system is strictly hyperbolic and genuinely nonlinear for all $U \in \mathbf{R}^2$. We also require that the system satisfy standard conditions on the second Fréchet derivatives of F, and one other hypothesis, which we have called the half-plane condition. This hypothesis replaces other, apparently more restrictive hypotheses required by previous authors ([1], [2]. [4], [5], [7], [15]–[18]). For the Riemann problem with arbitrary initial data

$$U(x, 0) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0, \end{cases} \tag{1.2}$$

we show (Theorem 3.2) that these four assumptions are sufficient to determine existence and uniqueness of centered solutions satisfying the Lax entropy condition at discontinuities. An important subsidiary result (Theorem 5.6) is that any discontinuity which can occur in a weak solution of (1.1) is either itself an entropy shock or would become an entropy shock after a left-right reversal.

In proving this result we were helped greatly by studying the techniques first developed by Smoller and Johnson in a series of papers ([7], [16]–[18]) on the Cauchy and Riemann problems for hyperbolic conservation laws. Our main innovation was in recognizing that the description of shock and rarefaction curves in the u, v-plane is much simplified by expressing the curves parametrically. While Smoller and Johnson had to impose an explicit entropy hypothesis called "condition (L)" on the shock curves in order to establish the existence of solutions, we have been able to eliminate the need for any *a priori* assumption at all on the shock curves or speeds. Our solutions do nevertheless satisfy the appropriate entropy conditions because we prove that condition (L) is in fact always satisfied. In addition, we have replaced the requirement $f_v g_u > 0$ by an apparently weaker ***half-plane condition*** described in Section 2 which says that the eigenvectors of the matrix $A(U) = dF$ point into opposite fixed half-planes.

Our interest in extending the work of Smoller and Johnson arose from a consideration of non-strictly hyperbolic systems. Conservation laws in which a pair of eigenvalues become equal for certain values of U arise in physical contexts such as nonlinear elasticity, magneto-fluid dynamics and crystal optics. We began our study by looking at some model systems of two conservation laws in which the characteristic speeds (eigenvalues of A) become equal along a curve in the U-plane. On this curve the system exhibits a parabolic degeneracy in the sense that the matrix A is not diagonalizable; moreover, $f_v g_u \leqslant 0$ there. We were able to solve a few cases explicitly (see [9]), and conjectured that a condition of ***opposite variation*** of the eigenvalues (as defined in Section 2) was sufficient for existence of solutions to the Riemann problem. This requirement is closely related to the half-plane condition for hyperbolic systems. In future work we expect to apply the results in this paper to a proof of existence for solutions to the Riemann problem for opposite-variation non-strictly-hyperbolic conservation laws. Meanwhile, we feel that the results presented here have independent interest as an extension of earlier results.

In the next section we give precise definitions of the assumptions stated above, and establish the convexity of the rarefaction curves (in the U-plane) for the systems we are considering. We also relate opposite variation to the half-plane condition and both of these to the assumption $f_v g_u > 0$. Proofs of these relations, which are not needed in the main line of development, are deferred to Section 7.

Section 3 reviews the definitions of shock waves and entropy, and states the main existence and uniqueness result (Theorem 3.2).

In Section 4 we define the Hugoniot locus of solutions to the Rankine–Hugoniot jump condition in the U-plane, derive an ordinary differential equation (4.4) for the Hugoniot locus, and establish the existence of the shock curves: four branches of the Hugoniot locus which extend to infinity, which are star-shaped with respect to their common point of origin U_0, and whose points can each be joined to U_0 by an entropy shock (Theorems 4.5 and 4.6).

In Section 5 we show that the four shock curves constitute the entire Hugoniot locus (Theorem 5.1), and establish a reciprocity relation among them. Section 6 completes the proof of the main theorem of the paper.

We include an appendix to show that convexity of the shock curves need not hold even for the special systems considered by Johnson and Smoller in [7]. In the erratum [8] Johnson and Smoller note that convexity is not actually needed for the proofs in [7] and [16]–[18].

It is perhaps worthwhile to remind the reader that the extensions considered here are quite different from those of Liu ([12]–[14]) and Dafermos and DiPerna [3], who relaxed the condition of genuine nonlinearity for systems of the $f_v g_u > 0$ type.

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2. Properties of the Rarefaction Curves

We shall assume throughout that the system (1.1) is strictly hyperbolic, genuinely nonlinear, and satisfies the Smoller–Johnson condition on the second Fréchet derivative of F. We begin with a precise statement of these three hypotheses and a description of their consequences for the field of rarefaction curves connected with the system. Later in this section we shall introduce and discuss a fourth hypothesis, the half-plane condition.

Expand equation (1.1) in the form

$$U_t + AU_x = 0, \tag{2.1}$$

with

$$A = F_U = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}.$$

The system is called *strictly hyperbolic* if for every $U \in \mathbf{R}^2$ the matrix A possesses two distinct real eigenvalues $\lambda_1 = \lambda_1(U)$ and $\lambda_2 = \lambda_2(U)$. For definiteness we shall take

$$\lambda_1 < \lambda_2 . \tag{2.2}$$

These eigenvalues are called the (local) *characteristic speeds*. The corresponding right eigenvectors r_1, r_2 must then be linearly independent. A strictly hyperbolic system thus gives rise to two direction fields $r_1(U)$ and $r_2(U)$ in the (u, v)-plane. The integral curves R_1 and R_2 of these respective direction fields are called *rarefaction curves*. There are two families of rarefaction curves, each family filling the U-plane smoothly, so that two distinct curves of the same family never intersect. Curves of opposite families are never tangent to one another; therefore no two curves of opposite families can intersect more than once, and no rarefaction curve can close on itself. We shall denote the R_i-curve passing through any particular point U_0 in the U-plane by $R_i(U_0)$.

The system (2.1) is said to be *genuinely nonlinear* if each characteristic speed λ_i varies in strictly monotone fashion along every rarefaction curve R_i of its own family: that is, $r_i \cdot \nabla\lambda_i \neq 0$, where ∇ denotes the gradient operator in the U-plane. We then normalize the r_i by choosing

$$| r_i | = 1, \qquad r_i \cdot \nabla\lambda_i > 0, \qquad i = 1, 2. \tag{2.3}$$

We also introduce the left eigenvectors l_1, l_2 of A, and normalize them by

$$| l_i | = 1, \qquad l_i r_i > 0, \qquad i = 1, 2; \tag{2.4}$$

observe that

$$l_i r_j = 0, \qquad i \neq j. \tag{2.5}$$

Smoller and Johnson [18] have shown how these left eigenvectors can be used to establish an alternative formulation of genuine nonlinearity. Beginning with the definition of right eigenvector:

$$(A - \lambda_i) r_i = 0, \tag{2.6}$$

differentiate in the direction of r_i:

$$[r_i \cdot \nabla(A - \lambda_i)] r_i + (A - \lambda_i)(r_i \cdot \nabla r_i) = 0$$

or

$$d^2F(r_i, r_i) = (r_i \cdot \nabla\lambda_i) r_i - (A - \lambda_i)(r_i \cdot \nabla r_i), \tag{2.7}$$

where $d^2F(r_i, r_i) = (r_i \cdot \nabla A) r_i$ denotes the second Fréchet derivative of F in the direction of r_i. Then multiply by l_i, obtaining

$$l_i \, d^2F(r_i, r_i) = (r_i \cdot \nabla\lambda_i)(l_i r_i). \tag{2.8}$$

From (2.3) and (2.4) we see that genuine nonlinearity is equivalent to the inequality

$$l_i\, d^2F(r_i, r_i) > 0, \qquad i = 1, 2. \tag{2.9}$$

The *Smoller–Johnson condition* can also be expressed in terms of the eigenvectors; it is

$$l_j\, d^2F(r_i, r_i) > 0, \qquad j \neq i. \tag{2.10}$$

Condition (2.10) was introduced by Smoller and Johnson in [18] and shown there to be equivalent to the Glimm–Lax shock interaction condition [6]. Its geometric significance appears upon multiplying (2.7) by l_j:

$$l_j\, d^2F(r_i, r_i) = (\lambda_i - \lambda_j)\, l_j(r_i \cdot \nabla r_i) = (\lambda_i - \lambda_j)\, l_j r_i', \tag{2.11}$$

where the prime denotes differentiation in the direction of r_i (i.e. along the rarefaction curve R_i). Since r_i is a unit vector, r_i' must be perpendicular to r_i, therefore parallel to l_j; indeed $r_i' = \kappa_i l_j$, where $|\kappa_i|$ is the curvature of R_i. Thus

$$\kappa_i = \frac{l_j\, d^2F(r_i, r_i)}{\lambda_i - \lambda_j} \tag{2.12}$$

and $\kappa_1 < 0$, $\kappa_2 > 0$ by (2.10) and (2.2). Hence all rarefaction curves of both families are convex, and more particularly the R_1 curves bend toward $-l_2$ (i.e.

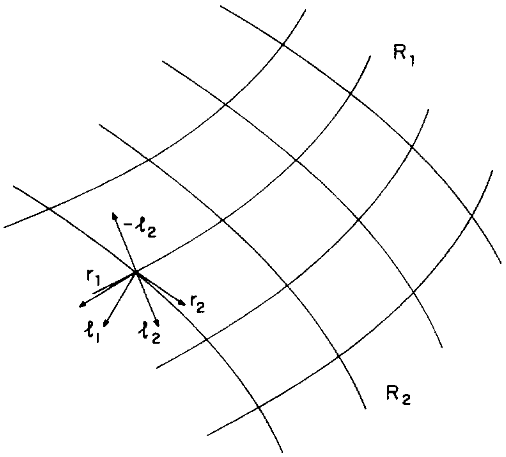

FIGURE 2-1

away from r_2), while the R_2 curves bend toward l_1 (and r_1), as in Figure 2-1. As a consequence, each R_i divides the plane into two unbounded regions, the "inside" (concave side) and the "outside" (convex side) of R_i. Rarefaction curves of the same family are *nested* in the sense that one of every pair of them will lie entirely inside the other.

A 2×2 system of conservation laws which is strictly hyperbolic, genuinely nonlinear, and satisfies the Smoller–Johnson condition (2.10) will be called an *admissible* system. In this paper we treat only admissible systems.

Further restrictions on the class of conservation laws are imposed by the need to deal effectively with shocks. For example, Smoller and Johnson in [18] require $f_v g_u > 0$, which implies that r_i is never parallel to either of the coordinate axes. Liu [14] observes that the condition $f_v g_u > 0$ can be met through an affine change of coordinates if r_1 and r_2 always point into two fixed (supplementary) sectors of the u,v-plane. In this paper, we require only that r_1 and r_2 lie in distinct open half-planes. Systems satisfying $f_v g_u > 0$ or Liu's sector condition clearly meet this requirement.

DEFINITION 2.1. An admissible system (2.1) is said to satisfy the *half-plane condition* if there exists a fixed vector w, independent of U, such that $r_1 \cdot w < 0$ and $r_2 \cdot w > 0$ for all U.

THEOREM 2.2. *The half-plane condition is satisfied if and only if*

$$r_1(U_1) \neq r_2(U_2) \tag{2.13}$$

for every pair of points U_1 and U_2.

Proof. The necessity of (2.13) is immediate. To show sufficiency, let K_i be the locus (on the unit circle K) of the endpoints of $r_i(U)$ for all U. The loci K_1 and K_2 are connected arcs which by (2.13) do not intersect. Moreover, each K_i must lie within some open semicircle, for if K_i contained a closed or even half-closed semicircle it would also contain all of $-K_j$, $j \neq i$, and so (by the Brouwer fixed-point theorem) we would have $r_i(U) = -r_j(U)$ for some U, which would contradict strict hyperbolicity. Hence K_1 and K_2 can be separated by a diameter of K, one of whose perpendiculars yields w. ▌

The half-plane condition appears somewhat unnatural and requires knowledge of the global behavior of the eigenvectors. In applications it may be more convenient to use a local property called opposite variation which is sufficient (though not necessary) to imply the half-plane condition. Roughly speaking, opposite variation says that the two characteristic speeds λ_1 and λ_2 vary in opposite senses as U changes. Many commonly occurring conservation laws (e.g. all nonlinear wave equations) have this property.

Definition 2.3. An admissible system (2.1) is said to display *opposite variation* if

$$r_i \cdot \nabla\lambda_j < 0, \qquad i \neq j. \tag{2.14}$$

Opposite variation may also be described in terms of left and right eigenvectors. Beginning with $(A - \lambda_j)r_j = 0$, differentiate along r_i:

$$d^2F(r_i, r_j) = (r_i \cdot \nabla\lambda_j)r_j - (A - \lambda_j)(r_i \cdot \nabla r_j), \tag{2.15}$$

where $d^2F(r_i, r_j) = (r_i \cdot \nabla A)r_j$ denotes the mixed second Fréchet derivative, which is symmetric in its two arguments r_i and r_j. Next, multiply (2.15) by l_j to obtain

$$l_j\, d^2F(r_i, r_j) = (r_i \cdot \nabla\lambda_j)(l_j r_j). \tag{2.16}$$

Thus we have proved

Lemma 2.4. *Opposite variation is equivalent to the inequality*

$$l_j\, d^2F(r_i, r_j) < 0, \qquad i \neq j. \tag{2.17}$$

The point of introducing opposite variation lies in the following theorem, whose proof may be found in Section 7.

Theorem 2.5. Any admissible system which displays opposite variation must satisfy the half-plane condition.

How is opposite variation related to Smoller and Johnson's assumption $f_v g_u > 0$? For systems in which rarefaction curves of opposite families always intersect, we shall prove (Theorem 7.5) that opposite variation implies not only the half-plane condition but also a stronger sector condition (Definition 7.3). Liu's affine transformation of coordinates ([14], p. 80) will then yield $f_v g_u > 0$ for the transformed system.

3. STATEMENT OF THE MAIN THEOREM

The *Riemann problem* for the system (1.1) is to find a weak solution of the system for $t > 0$ given the initial conditions (1.2), where U_l and U_r are arbitrary constant 2-vectors. Solutions consist of constant states separated by either rarefaction waves or lines of discontinuity. A *rarefaction wave* is a smooth region of the x, t-plane in which all values of U lie on a single rarefaction curve R_i; the corresponding characteristic speed λ_i must increase in the direction of increasing x. Across a discontinuity U must satisfy the *Rankine–Hugoniot condition*

$$s[U] = [F], \tag{3.1}$$

where $s = dx/dt$ is the speed of propagation of the discontinuity and $[\cdot]$ denotes the jump of the enclosed quantity across the discontinuity. Because of the restricted nature of the initial conditions, only *centered* solutions, containing centered rarefaction waves (U depending on x/t) and centered uniform discontinuities (s constant), need be considered in a Riemann problem. Either kind of wave can be completely specified by giving the states $U_\pm$ on both sides of the wave: U_- denotes the value of U immediately to the left of the wave in the x, t-plane, while U_+ is the value just to the right.

The choice of left and right sides of a wave is not arbitrary. For a rarefaction wave associated with an R_k-curve (called a *k wave*), we have already remarked that we must have

$$\lambda_k(U_-) < \lambda_k(U_+) \tag{3.2}$$

to insure smoothness. For discontinuities, the conservation law and Rankine–Hugoniot condition alone are not sufficient to distinguish between U_- and U_+, but for mathematical well-posedness and physical relevance it is customary (cf. Lax [11]) to impose an additional entropy condition. A discontinuity is called a *shock* wave if its propagation speed s satisfies the Lax entropy conditions for some value of k:

$$\lambda_k(U_-) > s > \lambda_k(U_+), \tag{3.3a}$$

$$s < \lambda_2(U_+), \qquad k = 1, \qquad \text{or} \qquad s > \lambda_1(U_-), \qquad k = 2. \tag{3.3b}$$

Such a shock is also referred to as a *k-shock*.

DEFINITION 3.1. A weak solution of a system of conservation laws (2.1) will be called an *entropy solution* if all discontinuities occurring in the solution are shocks.

We are now prepared to state the principal result of this paper.

THEOREM 3.2. *For an admissible system of conservation laws* (1.1) *which satisfies the half-plane condition, the Riemann problem* (1.2) *has one and only one centered entropy solution when the rarefaction curves* $R_1(U_l)$ *and* $R_2(U_r)$ *intersect, and none when they do not.*

This theorem thus asserts existence and uniqueness for all initial data when the following intersection property holds.

DEFINITION 3.3. An admissible system is said to possess the *intersection property* if every R_1-curve intersects every R_2-curve.

Remark. Most conservation laws do have the intersection property, though Smoller in [17] has produced some interesting examples of systems which do not.

In the absence of the intersection property, we will at least always have uniqueness, and we can guarantee existence in three of the four possible cases (Figures 3-2 through 3-4) into which the initial data may fall. Why ? Because Lemma 3.4, stated and proved below, assures in these cases the existence of the needed point of intersection. Existence can fail only if U_l lies outside $R_2(U_r)$ and U_r simultaneously outside $R_1(U_l)$, as in Figure 3-1.

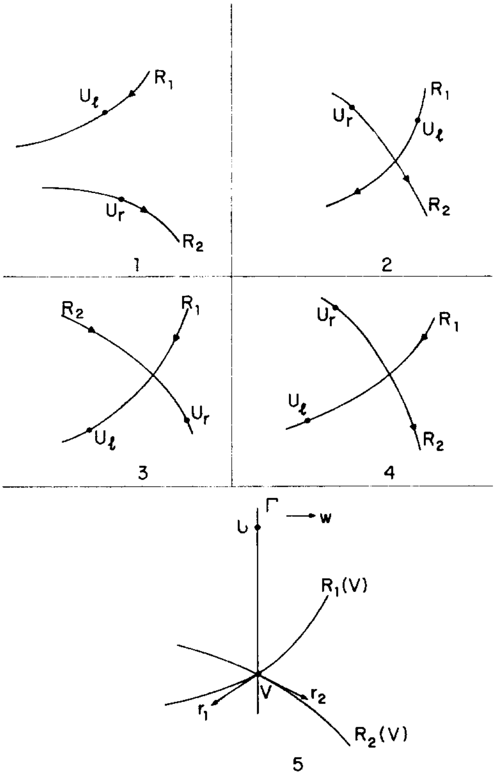

FIGURE 3

LEMMA 3.4. *Suppose a point U lies inside a rarefaction curve R of the family R_i. Then $R_j(U)$ intersects R for $j \neq i$.*

Proof. We shall produce a point V on R which lies inside $R_j(U)$. This would be sufficient to prove the lemma, since two convex curves, each of which contains a point inside the other, clearly must intersect.

Let Γ be the line through U in the direction orthogonal to w. Since U is inside the convex curve R, any line through U intersects R at least once, while Definition 2.1 insures that a line perpendicular to w cannot intersect any rarefaction curve more than once. Thus Γ intersects R in a unique point which we call V.

Next consider the eigenvectors $r_1(V)$ and $-r_2(V)$. By the half-plane condition, these point toward the same side of Γ, while the Smoller–Johnson condition says that r_1 points toward the inside of $R_2(V)$ and $-r_2$ toward the inside of $R_1(V)$. Hence the arc of $R_1(V)$ which is inside $R_2(V)$ lies on the same side of Γ as the arc of $R_2(V)$ which is inside $R_1(V)$. From this it follows geometrically (Figure 3-5) that each of the two rays into which V divides Γ lies entirely inside one of the curves $R_k(V)$ and outside the other. Since U is on Γ and inside $R = R_i(V)$, it follows that U is outside $R_j(V)$, and the fact that the R_j-curves are nested then implies that V lies inside $R_j(U)$ as required. ▌

Sections 4 through 6 will be devoted to the proof of Theorem 3.2.

4. The Shock Curves and Their Properties

Central to the treatment of shock waves is the *Hugoniot locus* $H(U_0)$, defined as the set of all states U which can appear in a weak solution on one side of a discontinuity when the state on the other side is U_0. The Rankine–Hugoniot condition (3.1) allows us to express the Hugoniot locus as

$$H(U_0) = \{U \mid \exists s\colon s(U - U_0) = F(U) - F(U_0)\}. \tag{4.1}$$

The point $U = U_0$ is itself on the Hugoniot locus, with s arbitrary, but any other point on the locus must have a well-determined value of s which we may call $s(U, U_0)$. Accordingly we might consider $H(U_0)$ to be the projection on the U-plane of the set

$$K(U_0) = \{(U, s) \mid s(U - U_0) = F(U) - F(U_0)\} \tag{4.2}$$

in the three-dimensional U, s-space. Since $K(U_0)$ is defined in 3-space by two scalar equations, it will behave locally like a one-dimensional manifold, i.e. a curve, except for possible singular points such as U_0 itself. Thus its projection $H(U_0)$ will consist of smooth arcs and singular points. These singular points will be projections of either singular points of K or points at which K has a tangent parallel to the s-axis. We begin by deriving a differential equation for the smooth arcs and characterizing the possible locations of the singularities.

On a smooth arc of H, we may differentiate equation (4.1) with respect to arc length μ in the U-plane:

$$s\frac{dU}{d\mu} + (U - U_0)\frac{ds}{d\mu} = \frac{d}{d\mu}F(U). \tag{4.3}$$

Setting $T = U - U_0$ and denoting $d/d\mu$ by a dot, we observe that $\dot{U} = \dot{T} = t$

is the unit tangent vector to H, so that (4.3) becomes $st + \dot{s}T = F_U\dot{U} = At$ or

$$\dot{s}T = (A - s)t. \tag{4.4}$$

Equation (4.4) is the basic differential equation which is valid along any smooth branch of the Hugoniot locus, and will play a majot role in all that follows.

To locate the singularities of H, we may at first regard (4.4) as a differential relation defining a vector field for $(t, \dot{s})$ in three-space as follows:

$$t = \pm \frac{(A - s)^{-1} T}{|(A - s)^{-1} T|}, \qquad \dot{s} = \pm \frac{1}{|(A - s)^{-1} T|}, \tag{4.5}$$

where the sign is determined by choosing an orientation along the curve. This vector field is defined and nonsingular except at points where $T = 0$ $(U = U_0)$ or s is equal to an eigenvalue λ_i of A. When s does equal λ_i but $l_i \cdot T \neq 0$, (4.4) still determines a unique $t = \pm r_i$ and $\dot{s} = 0$, so that singularities of the three-dimensional vector field can occur only at $U = U_0$ and at points U where T is parallel to one of the right eigenvectors r_j of $A(U)$. If U is any point of the Hugoniot locus other than one of these just named, and s the corresponding propagation speed determined by (4.1), then (U, s) will be a regular point of the vector field defined by (4.4). Therefore (U, s) will lie on an integral curve of this three-dimensional field, i.e. a regular curve Γ along which $s[U] - [F]$ is constant. But this constant is zero at (U, s), and Γ is therefore a regular branch of the locus $K(U_0)$ defined by (4.2). Moreover, Γ does not have a vertical tangent at (U, s), since $\dot{s}$ is finite. Thus we may conclude that the point U is itself a regular point of $H(U_0)$. We state this conclusion formally as a theorem.

THEOREM 4.1. *A point $U \neq U_0$ of the Hugoniot locus $H(U_0)$ at which $T = U - U_0$ is parallel to neither of the right eigenvectors r_1, r_2 of $A(U)$ must be an interior point of a regular arc of $H(U_0)$.*

Theorem 4.1 required no assumptions on the underlying conservation laws beyond mere differentiability of the coefficients. When we add our basic hypotheses, a much stronger result emerges.

THEOREM 4.2. *For an admissible system of conservation laws which satisfies the half-plane condition, T is never parallel to r_1 or r_2 at any point $U \neq U_0$ on the Hugoniot locus $H(U_0)$.*

COROLLARY. *Under these hypotheses, all points of $H(U_0)$ are regular except for U_0 itself.*

Proof of Theorem 4.2. Suppose $T \neq 0$ is parallel to $r_i(U)$ for $U \in H(U_0)$, so that

$$T = kr_i(U), \qquad k \neq 0. \tag{4.6}$$

Substitution into (4.1) yields

$$skr_i(U) = F(U_0 + T) - F(U_0).$$

Multiply by $l_j(U), j \neq i$, obtaining

$$0 = l_j(U)F(U_0 + T) - l_j(U)F(U_0).$$

Applying the mean value theorem to the scalar function

$$l_j(U)F(U_0 + \theta T), \qquad 0 \leqslant \theta \leqslant 1,$$

we deduce that

$$0 = l_j(U)\frac{d}{d\theta}F(U_0 + \theta T) = l_j(U)\,A(U_0 + \theta T)\,kr_i(U)$$

for some value $\theta = \theta_1 \in (0, 1)$. Setting $U_1 = U_0 + \theta_1 T$ and dividing by $k \neq 0$, we see that $A(U_1)\,r_i(U)$ is perpendicular to $l_j(U)$, hence parallel to $r_i(U)$, and therefore $r_i(U)$ is an eigenvector of $A(U_1)$ as well as of $A(U)$. Moreover, this common eigenvector is parallel to $U - U_1$. The following lemma will show that this situation is impossible.

Lemma 4.3. *Under the hypotheses of Theorem* 4.2, *for two distinct points* U_1 *and* U_2 *the matrices* $A(U_1)$ *and* $A(U_2)$ *cannot each have an eigenvector parallel to* $U_2 - U_1$.

Proof. Suppose $A(U_1)$ has an eigenvector $r_i(U_1)$ parallel to $U_2 - U_1$. Then the line joining U_1 and U_2 is tangent to the convex curve $R_i(U_1)$ at U_1, and therefore U_2 lies outside $R_i(U_1)$. Hence U_1 is inside $R_i(U_2)$, so that this line cannot also be tangent to $R_i(U_2)$. Thus $r_i(U_2)$ is not parallel to $U_2 - U_1$.

We must also show that $r_j(U_2)$ cannot be parallel to $U_2 - U_1$ for $j \neq i$. If it were, the argument of the preceding paragraph would imply that U_2 lies inside $R_j(U_1)$. Then both $R_j(U_1)$ and $R_i(U_2)$ would be concave toward the segment $\overline{U_1U_2}$, which is parallel to $r_i(U_1)$ and $r_j(U_2)$—see Figure 4-1. But the geometrical formulation of the Smoller–Johnson condition (2.10) says that, if R_j is concave toward $\pm r_i$, then R_i is concave toward $\mp r_j$. Letting $\nu = |\,U_2 - U_1\,|^{-1}$, this yields $\nu(U_2 - U_1) = \pm r_i(U_1)$ and $\nu(U_1 - U_2) = \mp r_j(U_2)$, or $r_i(U_1) = r_j(U_2)$, contradicting the half-plane condition (2.13). ∎

Theorem 4.2 and its corollary are thus established, and we may proceed to examine the individual branches of the Hugoniot locus, each of which satisfies (4.4). At the singular point $U = U_0$, (4.4) reduces to $(A - s)t = 0$, so that there are four solutions $s = \lambda_i$, $t = \pm r_i$, $i = 1, 2$. Thus (cf. Lax [10]) there are four branches of $H(U_0)$ which originate at U_0, and we call these branches *shock curves*. In Section 5 we shall justify this nomenclature by proving that the discontinuities joining all points U on the shock curves to U_0 are actually shocks, and that the four shock curves and the point U_0 constitute the entire Hugoniot locus. We denote the shock curves by $S_i(U_0)$, $S_i^*(U_0)$, $i = 1, 2$, where the *unstarred* curves S_i are those which leave U_0 in the $-r_i$ direction (so that they correspond to decreasing λ_i and therefore to i-shocks with the state U_0 on the *left*). The shock curves are sketched in Figure 4-2.

We first investigate the behavior of S_1 and S_2 near U_0, and then extend the results to the entire length of these curves. The corresponding properties of S_1^* and S_2^*, which follow in a symmetric fashion, will be stated afterwards. Our chief tool will be the differential equation (4.4) together with the representation

$$t = \alpha_1 r_1 + \alpha_2 r_2 \tag{4.7}$$

of the tangent vector in terms of the (local) right eigenvectors. Substituting (4.7) into (4.4) yields the fundamental equation

$$\dot{s}T = \alpha_1(\lambda_1 - s)r_1 + \alpha_2(\lambda_2 - s)r_2 , \tag{4.8}$$

valid along each S_i.

Theorem 4.4. *For an admissible system, the shock curve $S_i(U_0)$, $i = 1, 2$, makes third-order contact at U_0 with the corresponding rarefaction curve $R_i(U_0)$. (That is, S_i and R_i have the same tangent and curvature at U_0.) Moreover, near U_0, S_1 lies inside R_1, while S_2 lies outside R_2.*

Proof. At U_0 we have $T = 0$, $s = \lambda_i$, $t = -r_i$, so that $\alpha_i = -1$ and $\alpha_j = 0$, $j \neq i$. Differentiating (4.8) once along S_i yields

$$\begin{aligned} \ddot{s}T = {} & \dot{\alpha}_i(\lambda_i - s)r_i + \dot{\alpha}_j(\lambda_j - s)r_j \\ & + \alpha_i(\dot{\lambda}_i - 2\dot{s})r_i + \alpha_j(\dot{\lambda}_j - 2\dot{s})r_j \\ & + \alpha_i(\lambda_i - s)\dot{r}_i + \alpha_j(\lambda_j - s)\dot{r}_j , \end{aligned} \tag{4.9}$$

which at U_0 reduces to

$$0 = \dot{\alpha}_j(\lambda_j - \lambda_i)r_j - (\dot{\lambda}_i - 2\dot{s})r_i . \tag{4.10}$$

Thus $\dot{\alpha}_j = 0$ and

$$\dot{s} = \tfrac{1}{2}\dot{\lambda}_i = \tfrac{1}{2}(t \cdot \nabla)\lambda_i = -\tfrac{1}{2}(r_i \cdot \nabla)\lambda_i < 0. \tag{4.11}$$

The curvature of S_i at U_0 is found by differentiating (4.7) and then setting $U = U_0$:

$$\dot{t} = \dot{\alpha}_i r_i - \dot{r}_i \qquad \text{at} \quad U = U_0 . \tag{4.12}$$

Multiplying (4.12) with $t = -r_i$ and observing that $|\,t\,| = |\,r_i\,| = 1$, so that $t \cdot \dot{t} = r_i \cdot \dot{r}_i = 0$, we establish $\dot{\alpha}_i = 0$ at U_0, and therefore $\dot{t} = -\dot{r}_i$ or

$$(t \cdot \nabla)t = -(t \cdot \nabla)r_i = (r_i \cdot \nabla)r_i \tag{4.13}$$

there. This says that S_i and R_i have the same curvature vector at U_0 and thus make third-order contact.

To determine the direction in which S_i separates from R_i upon leaving U_0, we differentiate (4.9) yet again, set $U = U_0$ in the result, and multiply by l_j. The surviving terms yield

$$\begin{aligned} 0 &= \alpha_i(\dot{\lambda}_i - \dot{s})\, l_j\dot{r}_i + \ddot{\alpha}_j(\lambda_j - s)\, l_j r_j \\ &= -\dot{s} l_j \dot{r}_i + \ddot{\alpha}_j(\lambda_j - \lambda_i)\, l_j r_j\,. \end{aligned} \tag{4.14}$$

But at U_0

$$\dot{r}_i = (t \cdot \nabla)r_i = -(r_i \cdot \nabla)r_i = -r_i{}', \tag{4.15}$$

so that (2.11) and (4.14) yield

$$\ddot{\alpha}_j(U_0) = \frac{\dot{s} l_j\, d^2F(r_i\,, r_i)}{(\lambda_j - \lambda_i)^2\, l_j r_j} < 0 \tag{4.16}$$

by (2.10), (2.4) and (4.11). Since $\alpha_j(U_0) = \dot{\alpha}_j(U_0) = 0$, this means that $\alpha_j(U) < 0$ for U near U_0. Hence $S_i(U_0)$ crosses R_i-curves from the r_j side toward the $-r_j$ side, and therefore lies on the $-r_j$ side of $R_i(U_0)$. But now the Smoller–Johnson condition identifies this $-r_j$ side as the inside if $i = 1$ but the outside for $i = 2$ (cf. Figure 4-3). ▌

THEOREM 4.5. *Each curve $S_i(U_0)$ consists of a simple arc extending from U_0 to infinity. It is star-shaped with respect to U_0, and lies entirely inside $R_1(U_0)$ and outside $R_2(U_0)$. At each point $U \neq U_0$ on $S_i(U_0)$ the following inequalities are satisfied*:

$$\alpha_i < 0; \tag{4.17}$$

$$\alpha_j < 0, \qquad j \neq i; \tag{4.18}$$

$$\dot{s} < 0; \tag{4.19}$$

$$\lambda_i(U_0) > s(U) > \lambda_i(U). \tag{4.20}$$

The shock speed on $S_1(U_0)$ also satisfies

$$s(U) < \lambda_2(U), \qquad i = 1. \tag{4.21}$$

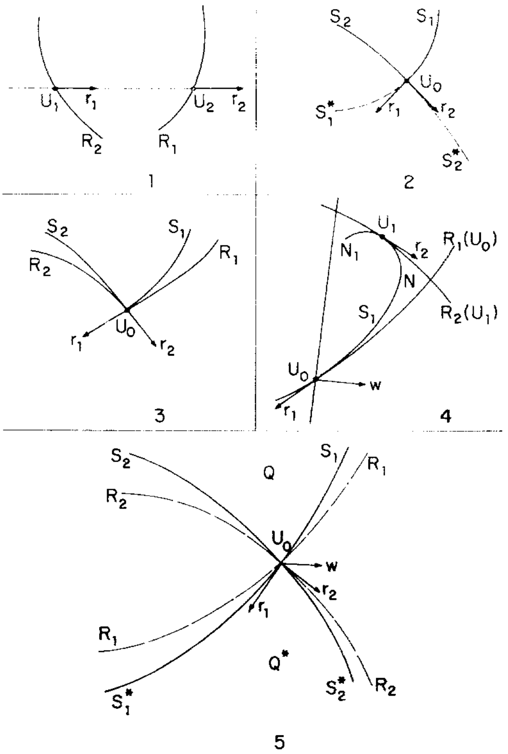

FIGURE 4

Remark. (4.20) and (4.21) together constitute the Lax entropy conditions for 1-shocks, and show that discontinuities having $U_- = U_0$ and U_+ on $S_1(U_0)$ are actual shocks. The condition corresponding to (4.21) for 2-shocks terminating on $S_2(U_0)$ would be

$$s(U) > \lambda_1(U_0), \qquad i = 2. \tag{4.22}$$

Condition (4.22) is in fact true, but its proof depends on ideas to be developed in Section 5. Hence we do not yet formally assert it.

Proof of Theorem 4.5. The Corollary to Theorem 4.2 shows that S_i, once established near U_0, must either extend as a simple arc without singularities to infinity or return eventually to U_0. But (4.17) and (4.18) imply that S_i does not cross any rarefaction curve twice. Thus S_i cannot return to U_0. Hence the entire theorem will be established if we prove that (4.17)–(4.21) hold and that S_i is starshaped.

Now (4.17), (4.19) and (4.21) all hold at U_0, and we have just seen that (4.18) holds near U_0. Also (4.11), which is valid at U_0, implies (4.20) near U_0. Thus

(4.17)–(4.21) hold on S_i in some (one-sided) neighborhood N of U_0 on S_i. We will show that N contains the whole of S_i.

If it does not, there will then be a first point $U_1 \neq U_0$ on S_i at which one of (4.17)–(4.21) is violated. Now the first half of (4.20) cannot be violated at U_1, since (4.19) holds on S_i between U_0 and U_1. Since U_1 is on the Hugoniot locus, Theorem 4.2 asserts that T at U_1 is not parallel to r_1 or r_2. But if any of (4.17)–(4.21) except (4.19) and the first half of (4.20) is first violated at U_1, one of the terms on the right-hand side of (4.8) will vanish there. Therefore (4.19) is invalid at U_1, and

$$\dot{s}(U_1) = 0. \tag{4.23}$$

From (4.8) it then follows that we are in one of two possible cases at $U = U_1$:

Case 1. $\dot{s} = 0,\ s = \lambda_i,\ \alpha_j = 0$ or

Case 2. $\dot{s} = 0, s = \lambda_j, \alpha_i = 0$.

In Case 1 we have $\alpha_i(U_1) \leqslant 0$ by continuity from (4.17), and therefore $\alpha_i = -1$ and $t = -r_i$. Hence

$$\dot{\lambda}_i(U_1) = -(r_i \cdot \nabla)\lambda_i < 0$$

and $(s - \lambda_i)^{\cdot} > 0$ at U_1. But in this case $s - \lambda_i = 0$ at U_1, so that $s - \lambda_i < 0$ for points of N near U_1. This contradicts (4.20) and shows that Case 1 cannot occur.

Case 2 is clearly impossible for S_2, since $s = \lambda_1$ at U_1 is incompatible with $s \geqslant \lambda_2 > \lambda_1$ obtained by continuity from (4.20). Thus inequalities (4.17)–(4.20) are established for all of S_2.

For S_1, we provisionally admit the possibility of a Case 2 point U_1 and look at the portion of $S_1(U_0)$ beyond U_1. We rewrite (4.8) for U near U_1 in the form

$$T = \frac{\alpha_1(\lambda_1 - s)}{\dot{s}} r_1 + \frac{\alpha_2(\lambda_2 - s)}{\dot{s}} r_2 . \tag{4.24}$$

The vector $T = U - U_0$ has a limiting value $U_1 - U_0$ at $U = U_1$, and Theorem 4.2 says that this limit is not parallel to r_1 or r_2. Therefore both coefficients on the right-hand side of (4.24) have nonzero limits at $U = U_1$:

$$T = \beta_1 r_1 + \beta_2 r_2 , \qquad \beta_1\beta_2 \neq 0, \tag{4.25}$$

and $\beta_1 < 0$, $\beta_2 > 0$ by continuity from N. But l'Hospital's rule at U_1, where $s = \lambda_2$, $\alpha_1 = 0$, $\alpha_2 = -1$, $t = -r_2$, says that

$$0 > \beta_1 = (\lambda_1 - \lambda_2)\,\dot{\alpha}_1/\ddot{s}, \qquad 0 < \beta_2 = (\dot{s} - \dot{\lambda}_2)/\ddot{s} = -\dot{\lambda}_2/\ddot{s}.$$

But at U_1 we have $\dot{\lambda}_2 = -(r_2 \cdot \nabla)\lambda_2 < 0$ and therefore $\ddot{s}(U_1) > 0$ and $\dot{\alpha}_1(U_1) > 0$. Hence beyond U_1 (4.24) again holds, but now

$$\alpha_1 > 0, \qquad \alpha_2 < 0, \qquad \dot{s} > 0, \qquad s(U) > \lambda_2(U) \tag{4.26}$$

are valid instead of (4.17)–(4.21). Thus there is a neighborhood N_1 on S_1 beyond U_1 in which (4.26) holds, and we suppose that $U_2 \neq U_0$, U_1 is the terminal point of N_1. ($U_2 \neq U_0$ because α_2 is consistently negative on $N \cup N_1$.) The arguments already used for U_1 show that $\dot{s}(U_2) = 0$, $s(U_2) = \lambda_2(U_2)$ and $\alpha_1(U_2) = 0$, so that $t = -r_2$ and $\dot{\lambda}_2 < 0$, hence $(s - \lambda_2)^{\cdot} > 0$ at U_2 and therefore $s < \lambda_2$ for U in N_1 near U_2, contradicting (4.26). Thus U_2 cannot exist, and if there is a U_1, then N_1 must extend to infinity.

Now from the half-plane condition (Definition 2.1), combined with (4.24) and (4.26), we may calculate that $w \cdot T > 0$ for U in N_1. However, the first inequality of (4.26) says that, in N_1, S_1 crosses the R_2-curves in the $+r_1$ direction, i.e. from outside to inside, and thus U remains inside the convex curve $R_2(U_1)$. Furthermore, $\alpha_2 < 0$ in $N \cup N_1$ implies in a similar manner that U remains inside the convex curve $R_1(U_0)$. Therefore N_1 is contained in the region bounded by the line $w \cdot (U - U_0) = 0$ and the two convex curves $R_1(U_0)$ and $R_2(U_1)$. Since this region is finite (Figure 4-4), N_1 cannot extend to infinity, and thus U_1 cannot exist. Therefore the original neighborhood N contains all of S_1, and inequalities (4.17)–(4.21) hold for all $U \neq U_0$ on S_1.

The star-shaped property of S_i follows now from (4.4). The property we seek to establish is that every ray through U_0 contains at most one point of $S_i(U_0)$. Were this not so, then T would be parallel to t at some point $U_3 \neq U_0$ on S_i, and (4.4) then asserts that $t(U_3)$ is an eigenvector of $A(U_3)$. Thus $T(U_3)$ would also be an eigenvector of $A(U_3)$, contradicting Theorem 4.2. ▌

The results analogous to Theorems 4.4 and 4.5 for the curves $S_i^*(U_0)$ are expressed in the following theorem. The proof proceeds in a manner entirely symmetric to that for the unstarred curves, except that λ_1 and λ_2 are interchanged and the signs of most inequalities reversed. (I.e. S_1^* behaves like S_2, and S_2^* like S_1.) The details are omitted.

THEOREM 4.6. *Each curve $S_i^*(U_0)$ consists of a simple arc extending from U_0 to infinity. It is star-shaped with respect to U_0, and makes third-order contact with $R_i(U_0)$ there. It lies entirely inside $R_2(U_0)$ and outside $R_1(U_0)$, and for $U \neq U_0$ the following inequalities hold on S_i^*:*

$$\alpha_i > 0; \tag{4.27}$$

$$\alpha_j > 0, \qquad j \neq i; \tag{4.28}$$

$$\dot{s} > 0; \tag{4.29}$$

$$\lambda_i(U_0) < s(U) < \lambda_i(U). \tag{4.30}$$

In addition, for $S_2{}^*$ we have

$$s(U) > \lambda_1(U), \qquad i = 2. \tag{4.31}$$

Figure 4-5 illustrates the four shock curves originating at U_0 and their relationship to the rarefaction curves $R_i(U_0)$. The shock curves, being part of the Hugoniot locus, cannot intersect one another except at U_0. Hence, $S_1(U_0)$, $S_1{}^*(U_0)$, $S_2(U_0)$, $S_2{}^*(U_0)$ divide the plane into four unbounded regions while the shock and rarefaction curves together form eight unbounded regions.

The following theorem will be useful in studying the global behavior of the shock curves. It tells us that each S_i and $S_i{}^*$ is confined to one of the two half-planes determined by a line through U_0 perpendicular to w.

THEOREM 4.7. *Let U lie on one of the shock curves originating at U_0, and set $T = U - U_0$. Then*

$$w \cdot T > 0, \qquad U \in S_1 \quad \textit{or} \quad S_2{}^* \tag{4.32}$$

and

$$w \cdot T < 0, \qquad U \in S_2 \quad \textit{or} \quad S_1{}^*. \tag{4.33}$$

Proof. Inequality (4.32) follows immediately from (4.24), the half-plane condition, and the various inequalities in Theorems 4.5 and 4.6. To prove (4.33), note first that the star shape of the shock curve S_i or $S_i{}^*$ and its third-order contact with $R_i(U_0)$ insure that the entire curve S_i or $S_i{}^*$ lies on the same side of the tangent $r_i(U_0)$ to R_i at U_0 as does R_i itself. The Smoller–Johnson condition tells us which side: for S_2, T lies on the r_1 side of $-r_2(U_0)$, while for $S_1{}^*$, T is on the $-r_2$ side of $r_1(U_0)$. Since $S_1{}^*$ and S_2 extend to infinity and do not cross, the T vectors for both curves must point into the quadrant between $r_1(U_0)$ and $-r_2(U_0)$. Then (4.33) follows from the half-plane condition. ▌

We may remark that the proof of (4.33) allows us to assert the somewhat stronger inequality

$$w \cdot T \leqslant -\epsilon\,|\,T\,|, \qquad \epsilon = \epsilon(U_0) > 0, \qquad U \in S_2\,,\, S_1{}^*, \tag{4.34}$$

where

$$\epsilon(U_0) = \min_{i=1,2} |\, w \cdot r_i(U_0)|.$$

Geometrically, (4.34) says that within their (open) half-plane $S_1{}^*$ and S_2 are further confined to a closed subsector of angular opening smaller than π.

5. Further Properties of the Shock Curves: Reciprocity and Entropy Conditions

Now that we have constructed the four shock curves S_i, $S_i{}^*$ through U_0, it is natural to ask whether the Hugoniot locus $H(U_0)$ always consists of just these four curves, or whether it could contain additional points or even whole branches. We begin this section with a demonstration that no additional points or branches can exist under our basic assumptions. This fact will be important in three ways. First, it is essential for uniqueness of solutions. This was already recognized by Smoller [16] and will be made clearer in Section 6. Second, it has as a corollary a new relationship of reciprocity between S_i and $S_i{}^*$ (Theorem 5.4) which will allow us to derive the entropy condition (4.22) on S_2. Finally, Theorem 5.4 will be used in the existence portion of our main result (Section 6), where it insures that certain shock curves originating at different points cannot intersect.

THEOREM 5.1. *For admissible systems satisfying the half-plane condition, $H(U_0)$ is precisely the union of the four shock loci $S_i(U_0)$, $S_i{}^*(U_0)$, $i = 1, 2$.*

Remark. Smoller in [16] proved under stronger hypotheses that $H(U_0)$ contained no points satisfying the entropy conditions (3.3) other than those on the shock curves. The present result, though largely based on Smoller's method of proof, extends his result by eliminating the entropy requirement.

We begin the proof of Theorem 5.1 by supposing that $\bar{U}$ is a point of $H(U_0)$ not on a shock locus. There are two possible cases (cf. Figure 4-5):

Case 1. $\bar{U}$ lies between $S_1(U_0)$ and $S_2{}^*(U_0)$ or between $S_2(U_0)$ and $S_1{}^*(U_0)$.

Case 2. $\bar{U}$ lies in one of the other two regions of the plane, i.e. between S_1 and S_2 or between $S_1{}^*$ and $S_2{}^*$.

In Case 1 we shall draw one of the shock curves originating at $\bar{U}$, show that it intersects an appropriate shock curve through U_0, and derive a contradiction. In Case 2, we will draw the branch of $H(U_0)$ which passes through $\bar{U}$ and show that it has no way to escape to infinity. In both cases we will use the Hugoniot locus $H(\bar{U})$ originating at $\bar{U}$. The symmetry of the Rankine–Hugoniot condition in (4.1) implies that $U \in H(\bar{U})$ if and only if $\bar{U} \in H(U)$, and that $s(U, \bar{U}) = s(\bar{U}, U)$.

The key to the first case is the following lemma.

LEMMA 5.2. *Let $\bar{U}$ be a point of $H(U_0)$ not on any of the shock curves through U_0. Then none of the following points of intersection can exist if different from U_0:*

$$\begin{aligned} U_1 &= S_1(U_0) \cap S_2(\bar{U}), \\ U_2 &= S_2(U_0) \cap S_1(\bar{U}), \\ U_3 &= S_1{}^*(U_0) \cap S_2{}^*(\bar{U}), \\ U_4 &= S_2{}^*(U_0) \cap S_1{}^*(\bar{U}). \end{aligned} \tag{5.1}$$

Proof. Let $U \neq U_0$ be any of the points $U_1, \ldots, U_4$. Then $U \in H(U_0)$, $U \in H(\bar{U})$, and $\bar{U} \in H(U_0)$ so there are three values s_1, s_2, s_3 such that

$$\begin{aligned} s_1(U - U_0) &= F(U) - F(U_0), \\ s_2(U - \bar{U}) &= F(U) - F(\bar{U}), \\ s_3(\bar{U} - U_0) &= F(\bar{U}) - F(U_0). \end{aligned} \tag{5.2}$$

Therefore $s_2(U - \bar{U}) + s_3(\bar{U} - U_0) = s_1(U - U_0)$ or

$$(s_2 - s_3)(U - \bar{U}) - (s_1 - s_3)(U - U_0) = 0. \tag{5.3}$$

Now $U - \bar{U} = T(U, \bar{U})$ and $U - U_0 = T(U, U_0)$ are linearly independent, because (4.24) implies that the secant vectors T to the shock curves S_1 and S_2 (or $S_1{}^*$ and $S_2{}^*$) point into adjacent quadrants with respect to the local coordinate system $r_i(U)$, $i = 1, 2$. Thus (5.3) implies

$$s_1 = s_2 = s_3\,. \tag{5.4}$$

But now s_1 is a shock speed on a shock curve originating at U_0, while s_2 is a shock speed at the same point U on a shock curve originating at $\bar{U}$. Thus the appropriate inequalities among (4.20), (4.21), (4.30), and (4.31) must hold for $s = s_1 = s_2$ and the respective $\lambda_i(U)$. But in all four cases these inequalities lead to a contradiction. ▌

One more lemma now suffices to eliminate Case 1.

LEMMA 5.3. *For a point $\bar{U}$ in the regions described by Case* 1, *one of the intersection points U_1, U_2, U_3, U_4 defined in* (5.1) *always exists and is different from U_0.*

Proof. (Cf. Figure 4-5 for orientation). If $\bar{U}$ lies between $S_1(U_0)$ and $S_2{}^*(U_0)$, inequalities (4.32) for U_0 and (4.34) for $\bar{U}$ insure that $S_2(\bar{U})$ and $S_1{}^*(\bar{U})$ each intersect either $S_1(U_0)$ or $S_2{}^*(U_0)$. Now if $\bar{U}$ is in addition on or outside of $R_2(U_0)$, then $S_2(\bar{U})$ lies outside $R_2(\bar{U})$ and therefore outside $R_2(U_0)$ and so cannot intersect $S_2{}^*(U_0)$ which lies inside $R_2(U_0)$. Hence $S_2(\bar{U})$ intersects $S_1(U_0)$, and U_1 exists. Otherwise, $\bar{U}$ must be on or outside $R_1(U_0)$, and similar reasoning shows that U_4 exists. The same type of argument applies if $\bar{U}$ is between $S_2(U_0)$ and $S_1{}^*(U_0)$: if $\bar{U}$ is *inside* $R_1(U_0)$ then U_2 will exist, while otherwise $\bar{U}$ will be inside $R_2(U_0)$ and we get U_3. ▮

Now we examine Case 2. We treat explicitly the situation where $\bar{U}$ is in the region $\underset{\sim}{Q}$ between S_1 and S_2; the proof for the region $\underset{\sim}{Q}^*$ between $S_2{}^*$ and $S_1{}^*$ is entirely symmetric and will be omitted.

Since $\bar{U}$ is inside $R_1(U_0)$, we know from Lemma 3.4 that $R_2(\bar{U})$ intersects $R_1(U_0)$, so that $S_2{}^*(\bar{U})$ also intersects the smooth curve $S_1(U_0) \cup S_1{}^*(U_0)$ by the argument of Lemma 5.3. This intersection cannot lie interior to $S_1{}^*(U_0)$, for if it did it would be a point of the type U_3 forbidden by Lemma 5.2. Thus the intersection, U, occurs at U_0 itself or in the interior of $S_1(U_0)$. If $U = U_0 \in S_2{}^*(\bar{U})$, from Theorem 4.6 for $S_2{}^*(\bar{U})$ we have

$$s(\bar{U}, U_0) > \lambda_2(\bar{U}). \tag{5.5}$$

If U is interior to $S_1(U_0)$ as in Figure 5-1, then (5.1) holds for the triple U, $\bar{U}$, U_0. We see that $U - U_0$ and $U - \bar{U}$ are linearly independent, since they both point into the quadrant between $-r_1(U)$ and $r_2(U)$, but cannot point in precisely the same direction since U is inside both $R_1(U_0)$ and $R_2(\bar{U})$. Hence (5.4) holds, and since $s_2 > \lambda_2(\bar{U})$ by (4.30), the same is true for s_3. We conclude that if $\bar{U} \in H(U_0) \cap \underset{\sim}{Q}$ then $s(\bar{U}, U_0)$ satisfies (5.5).

Now, Theorem 4.2 and its Corollary tell us that $\bar{U}$ lies on a regular branch B of the Hugoniot locus which is contained entirely in $\underset{\sim}{Q}$ and must either extend to infinity in both directions or have at least one endpoint at U_0. The latter case is not possible; we have already investigated the Hugoniot locus near U_0 and have found that the only branches emanating from that point are the shock curves. Hence B has both endpoints at infinity.

Furthermore, the differential equation (4.4) and its consequence (4.8) holds along B, and $\dot{s} \neq 0$ since (5.5) prevents s from ever being an eigenvalue of the local matrix A. Therefore (4.24), which we rewrite as

$$T = \beta_1 r_1 + \beta_2 r_2 \tag{5.6}$$

with $\beta_i = \alpha_i(\lambda_i - s)/\dot{s}$, also holds along B. Any $U \in B \subset \underset{\sim}{Q}$ is outside $R_2(U_0)$, so U_0 is inside $R_2(U)$ and thus T points outward across $R_2(U)$ as in Figure 5-2; hence $\beta_1 < 0$. If we choose the direction of differentiation along B so that $\dot{s} > 0$, we find that $\alpha_1 > 0$ (since $s > \lambda_2 > \lambda_1$), and that β_2 and α_2 also have opposite signs. Thus T and t always lie in opposite quadrants with respect to the local eigenvectors. Moreover, T is never parallel to t on B, since (just as for the shock curves) $T \parallel t$ would contradict Theorem 4.2. Therefore B is star-shaped with respect to U_0. Since $\alpha_1 > 0$, t points inside $R_2(U)$; since $R_2(U)$ intersects $R_1(U_0)$, t cannot point into the bounded region between $S_1(U_0)$ and the segment $\overline{UU_0}$. Hence, with increasing μ (arc-length on B), T turns toward $R_2(U_0)$ (counter-clockwise in Figure 5-2).

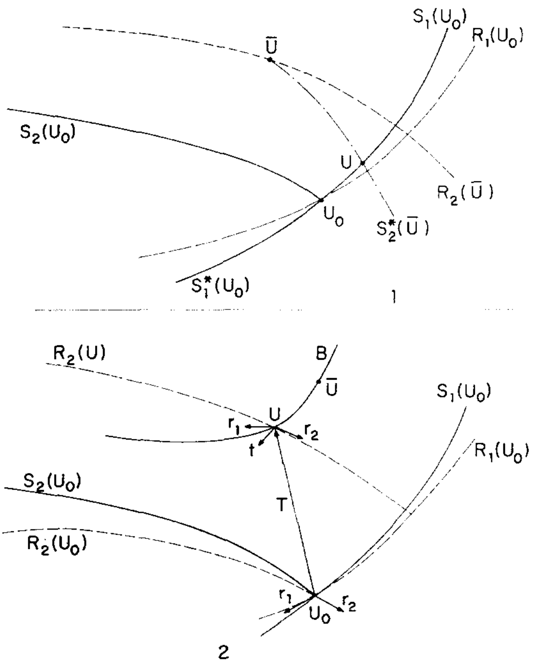

FIGURE 5

Since B is star-shaped, the directions of the vectors T form a monotonic bounded sequence which has a limit direction v, where v is chosen for definiteness to be a unit vector. Consequently, for some sequence of values of μ going to infinity, t approaches v. Since t and T lie in opposite quadrants with respect to the local eigenvectors r_1 and r_2, these eigenvectors must approach v or $-v$. Since B is in the half-strip bounded by $R_2(U_0)$, $R_1(U_0)$ and $R_2(U)$, the direction of T tends to that of $-r_2$, so r_2 approaches $-v$. Now in this half-strip $r_2 \cdot w > \epsilon > 0$ by the arguments used in the proof of Theorem 4.7. Therefore $v \cdot w \leqslant -\epsilon < 0$. Since $r_1 \cdot w < 0$ everywhere, r_1 cannot approach $-v$ and thus must approach $+v$. Equation (4.7) now implies that $\alpha_1 - \alpha_2 \to 1$, so that far enough out in our sequence we have

$$\alpha_1 > \alpha_2 . \tag{5.7}$$

The vector T also approaches r_1 and $-r_2$ in direction and sense (though not in magnitude) as $U \to \infty$, so that similarly $\beta_1 > \beta_2$ for all points U far enough out on B. Since $\dot{s} > 0$, this implies

$$0 < \alpha_1(s - \lambda_1) < \alpha_2(s - \lambda_2). \tag{5.8}$$

But, on B, $0 < s - \lambda_2 < s - \lambda_1$, so (5.8) yields $\alpha_1 < \alpha_2$, contradicting (5.7).

Thus the assumption $U \in Q$ leads eventually to a contradiction, and therefore $H(U_0) \cap Q$ is empty. This with the symmetric result for Q^*, shows that Case 2 is impossible and completes the proof of Theorem 5.1.

Now we may state the reciprocity relationship between shock curves which was mentioned at the beginning of this section.

THEOREM 5.4. *For an admissible system satisfying the half-plane condition, $U \in S_i(U_0)$ if and only if $U_0 \in S_i^*(U)$.*

Proof. Suppose $U \in S_i(U_0) \subset H(U_0)$. Then we must have $U_0 \in H(U)$ with the same value of s. Therefore Theorem 5.1 says that U_0 lies on one of the four shock curves originating at U. By Theorem 4.7, U_0 cannot lie on $S_i(U)$ or $S_j^*(U)$, $j \neq i$. If U_0 is on $S_j(U)$, then s would satisfy both $\lambda_i(U_0) < s < \lambda_i(U)$ and $\lambda_j(U) > s > \lambda_j(U_0)$, so that simultaneously $\lambda_i(U_0) > \lambda_j(U_0)$ and $\lambda_i(U) < \lambda_j(U)$, contradicting (2.2). Hence U_0 must be on $S_i^*(U)$. The converse is proved in the same way. ▌

As an immediate consequence we obtain the remaining entropy condition (4.22) on S_2 and its dual for S_1^*.

THEOREM 5.5. *Let $U_+ \in S_k(U_-)$ [or $U_- \in S_k^*(U_+)$]. Then there is a k-shock [satisfying both Lax entropy conditions* (3.3)] *which joins the state U_- on the left to the state U_+ on the right.*

Proof. Theorem 5.4 tells us that the two hypotheses of this theorem are equivalent, so both of them hold. The result for $k = 1$ follows from Theorem 4.5. When $k = 2$ we have $U_- \in S_2^*(U_+)$, and now (4.30) and (4.31) yield the Lax conditions (3.3) for a 2-shock. ▌

COROLLARY. All points U on $S_2(U_0)$ satisfy (4.22), while all points on $S_1^*(U_0)$ satisfy the dual inequality

$$s(U) < \lambda_2(U_0), \qquad i = 1. \tag{5.9}$$

Remark. Inequality (4.22) constitutes Smoller and Johnson's "Condition (L)" ([18], page 176). They use this condition as an additional hypothesis in their existence theorem, but speculate that it may be unnecessary. Now the basic hypothesis $f_v g_u > 0$ of [18] implies our half-plane condition, and our conditions for an admissible system are the same as those in [18]. Hence the above corollary contains the answer to the question raised by Smoller and Johnson on page 184 of [18]: is condition (L) always valid globally? The answer is yes.

Theorems 5.1 and 5.5 together give us the general result:

THEOREM 5.6. *For admissible systems satisfying the half-plane condition, every point U on the Hugoniot locus $H(U_0)$ can be joined to U_0 by an entropy shock.*

6. Existence and Uniqueness of Solutions

We are now ready to construct solutions to the Riemann problem (1.1), (1.2) and prove their uniqueness. By a solution we shall mean a *centered entropy solution*: a vector function $U(x, t)$, $t \geqslant 0$, consisting of constant states U_l, U_1, U_2,..., U_r separated by centered rarefaction waves or uniform shocks, which is a weak solution of (1.1). Now a (k-)rarefaction wave joining a state U_- on its left to a state U_+ on its right fills the region $\lambda_k(U_-) < x/t < \lambda_k(U_+)$, while a k-shock joining the states U_- and U_+ is situated on a ray $x/t = s$ with $\lambda_k(U_-) > s > \lambda_k(U_+)$. Hence there is room in the x, t-plane for at most one wave (shock or rarefaction) of each type k. The general solution therefore contains at most three constant states (U_l, U_r and an intermediate state U_m) separated by two waves, and since $\lambda_1 < \lambda_2$ these must be a type 1 wave connecting U_l to U_m and a wave of type 2 from U_m to U_r.

To construct this solution we employ the diagram shown in Figure 6-1. Through U_l we draw the two shock curves $S_i(U_l)$, $i = 1, 2$, and the semi-infinite arcs $R_i^+(U_l)$ of the two rarefaction curves $R_i(U_l)$ which start out from U_l in the directions of $+r_i$, $i = 1, 2$. According to the results of Sections 2 and 4, these four curves divide the U-plane into four regions (marked I, II, III, IV in Figure 6-1) meeting at U_l. If U_r is on one of these four curves, then the Riemann problem may be solved by a single wave (rarefaction or shock) connecting U_l to U_r. Otherwise two waves are required, with an intermediate state U_m on $R_1^+(U_l)$ or $S_1(U_l)$, and the final state U_r located on either $R_2^+(U_m)$ or $S_2(U_m)$. The nature of the solution will depend upon which of the regions I–IV contains U_r.

Region I is smoothly filled by R_2 curves. If U_r is in region I and $R_2(U_r)$ intersects $R_1(U_l)$—the first case in the hypothesis of Theorem 3.2—the Riemann problem has just one solution containing two rarefaction waves and an intermediate state $U_m = R_1^+(U_l) \cap R_2^-(U_r)$. If $R_2(U_r)$ fails to intersect $R_1(U_l)$, then (1.1), (1.2) has no solution of this type.

Region II is also filled smoothly with R_2 curves, and here all of them must intersect $S_1(U_l)$ because of their convexity, (4.32) and the half-plane condition. Moreover, for U_r in region II the point of intersection $U_m = S_1(U_l) \cap R_2(U_r)$ is unique because by (4.17) S_1 crosses each rarefaction curve R_2 at most once. The Riemann problem is then solved by a 1-shock from U_l to U_m and a 2-rarefaction from U_m to U_r; the shock is properly separated from the rarefaction wave in the x, t-plane because of (4.21).

In region III we consider the shock curves $S_2(\bar{U})$ originating at points $\bar{U} \in R_1^+(U_l)$. If two such curves $S_2(\bar{U}_1)$ and $S_2(\bar{U}_2)$ were to intersect, say at U_3, Theorem 5.4 would imply that $S_2^*(U_3)$ passes through both $\bar{U}_1$ and $\bar{U}_2$, and therefore intersects $R_1(U_l)$ twice, contrary to (4.27). The same argument shows that $S_2(\bar{U})$ cannot leave region III by crossing $S_2(U_l)$, and it also cannot get out by recrossing $R_1(U_l)$. Thus these curves $S_2(\bar{U})$ smoothly fill region III. They

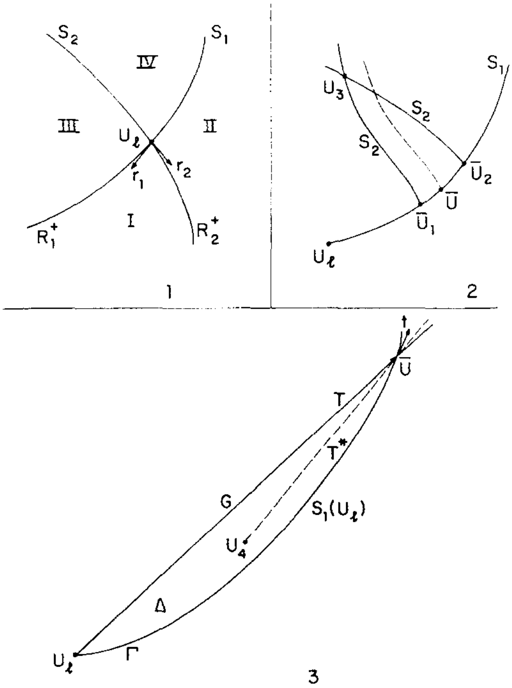

FIGURE 6

can leave no internal gaps, because they are solutions of the differential equation (4.4) with an initial condition depending smoothly on $\bar{U}$, and they can leave no external gap within region III as $\bar{U}$ approaches infinity because condition (4.33) and the convexity of $R_1^+(U_l)$ guarantee that infinitely many $S_2(\bar{U})$-curves originate and remain on the far side of an arbitrary line perpendicular to w. We may conclude that to each U_r in region III there corresponds a unique $\bar{U} = U_m \in R_1^+(U_l)$ such that $U_r \in S_2(\bar{U})$. We then solve (1.1), (1.2) with a 1-rarefaction from U_l to U_m, followed by a 2-shock from U_m to U_r, with $s > \lambda_1(U_m)$ by the Corollary to Theorem 5.5.

Finally we look at region IV. Again we consider the curves $S_2(\bar{U})$, this time for $\bar{U} \in S_1(U_l)$. We describe the behavior of the resulting family of curves in a series of lemmas, whose proofs will be given after the main argument is concluded.

LEMMA 6.1. *The curve* $S_2(\bar{U})$, $\bar{U} \in S_1(U_l)$, *lies entirely in the region* (IV) *of the U-plane bounded by* $S_1(U_l)$ *and* $S_2(U_l)$.

LEMMA 6.2. *The speed* s_2 *of the* 2-*shock joining* $\bar{U}$ *to an arbitrary point* $U \in S_2(\bar{U})$ *is always greater than the speed* s_1 *of the* 1-*shock joining* $\bar{U}$ *to* U_l.

LEMMA 6.3. *For two distinct points* $\bar{U}_1$ *and* $\bar{U}_2$ *on* $S_1(U_l)$, *the curves* $S_2(\bar{U}_1)$ *and* $S_2(\bar{U}_2)$ *do not intersect.*

From Lemma 6.3 we can argue, just as we did in region III, that the $S_2(\bar{U})$ curves smoothly fill region IV. (An external gap in region IV would contradict Lemma 3.4.) Therefore to each U_r in region IV there is a unique $\bar{U} = U_m \in S_1(U_l)$ such that $U_r \in S_2(\bar{U})$, and the corresponding solution to the Riemann problem consists of two shocks, whose speeds are correctly related because of Lemma 6.2. Thus the existence portion of Theorem 3.2 follows from Lemmas 6.1 through 6.3.

The uniqueness of the solution also follows from these lemmas, and particularly Lemma 6.1. For we have seen that with each of the regions I–IV is associated a particular type of solution to the Riemann problem, and that as long as U_r remains within the region there is only one solution of this type. We have also seen that no solution of types I through III can exist when U_r lies outside the associated region, and Lemma 6.1 implies the corresponding fact for solutions of type IV. Thus the proof of our main theorem (Theorem 3.2) will be finished as soon as proofs of Lemmas 6.1–6.3 are given.

Proof of Lemma 6.1. $S_2(\bar{U})$ enters region IV intially, since it starts out at $\bar{U}$ in the direction of $-r_2$ while the tangent t to S_1 points between $-r_2$ and $-r_1$ [relations (4.17) and (4.18)]. It cannot leave the region at U_l because $\bar{U} \notin S_2{}^*(U_l)$, so that if it leaves at all there would have to be a point $U \neq \bar{U}, U_l$ of $S_2(\bar{U})$ which lies on either $S_1(U_l)$ or $S_2(U_l)$. In either case, the non-collinear triple of points U, $\bar{U}$, U_l would satisfy an interlocking set of three Rankine–Hugoniot relations similar to (5.1). The three associated shock speeds would then all be equal. But if $U \in S_1(U_l)$ this would mean $s(U_l, U) = s(U_l, \bar{U})$ for $U \neq \bar{U}$ on $S_1(U_l)$, contradicting (4.19), while if $U \in S_2(U_l)$ we would have both a 1-shock and a 2-shock originating at U_l with the same speed, which contradicts (4.20) and (4.21). Therefore $S_2(\bar{U})$ can never leave region IV. ▌

Proof of Lemma 6.2. For U near $\bar{U}$ on $S_2(\bar{U})$, s_2 is close to $\lambda_2(\bar{U})$, while $s_1 < \lambda_2(\bar{U})$ by (4.21). Thus $s_2 > s_1$ near $\bar{U}$, and if ever $s_2 \leqslant s_1$ there must be a point $U_1 \in S_2(\bar{U})$ for which $s_2 = s_1$. But then $s_2(U_1 - \bar{U}) = F(U_1) - F(\bar{U})$ and $s_2(\bar{U} - U_l) = s_1(\bar{U} - U_l) = F(\bar{U}) - F(U_l)$. Add these two equations to obtain $s_2(U_1 - U_l) = F(U_1) - F(U_l)$, or $U_1 \in H(U_l)$. Therefore U_1 is on $S_i(U_l)$ or $S_i{}^*(U_l)$ and does not lie in region IV. This would contradict Lemma 6.1. ▌

Proof of Lemma 6.3. Suppose $S_2(\bar{U}_1)$ and $S_2(\bar{U}_2)$ did intersect, say at U_3. Then for points $\bar{U}$ between $\bar{U}_1$ and $\bar{U}_2$ on $S_1(U_l)$ the curve $S_2(\bar{U})$ could not escape to infinity without first crossing one of the curves $S_2(\bar{U}_n)$, $n = 1, 2$ (cf. Figure 6-2). Thus a point of intersection U_3 will continue to exist as $\bar{U}_2$ and $\bar{U}_1$ are allowed to approach each other along $S_1(U_l)$. Compactness then assures that for some such sequence of $\bar{U}_1$, $\bar{U}_2$ with $\bar{U}_1 - \bar{U}_2 \to 0$ the point U_3 will approach a (finite) limit, which we denote by U_4. Observing that U_3 is on both $S_2(\bar{U}_1)$ and

$S_2(\bar{U}_2)$ so that $S_2{}^*(U_3)$ contains both $\bar{U}_1$ and $\bar{U}_2$, we deduce upon passage to the limit that $S_2{}^*(U_4)$ has double contact with $S_1(U_l)$ at the common limit point $\bar{U} \neq U_l$ of $\bar{U}_1$ and $\bar{U}_2$. In other words, the tangent t^* to $S_2{}^*(U_4)$ at $\bar{U}$ is parallel to the tangent t to $S_1(U_l)$ at $\bar{U}$. Since t points strictly between $-r_1$ and $-r_2$ by (4.17) and (4.18), while t^* points between $+r_1$ and $+r_2$ by (4.27), (4.28), this means $t^* = -t$ and also $\bar{U} \neq U_4$. But now equation (4.4) may be applied to both curves at $\bar{U}$:

$$\dot{s}T = (A - s)t,$$

$$\dot{s}^*T^* = (A - s^*)t^*,$$

with the obvious notation $T^* = \bar{U} - U_4$ and s^* for the shock speed along $S_2{}^*(U_4)$. Adding these yields

$$\dot{s}T + \dot{s}^*T^* = (s^* - s)t. \tag{6.1}$$

But now $\dot{s} < 0$ and $\dot{s}^* > 0$ by (4.19) and (4.29), while $s^* > s$ by Lemma 6.2 since s^* is also the shock speed associated with U_4 on $S_2(\bar{U})$. Hence T^* is a convex linear combination of t and T. Geometrically (cf. Figure 6-3), this means that T^*, which terminates at $\bar{U}$, points out of the angle formed there by the arc $\Gamma = U_l\bar{U}$ of $S_1(U_l)$ and the straight line segment G joining U_l to $\bar{U}$. Now U_4, which is the initial point of T^*, must lie in region IV, since it is on $S_2(\bar{U})$. Therefore U_4 lies in the (compact) subregion Δ bounded by G and Γ. But this is impossible because $S_2(\bar{U})$, being star-shaped with respect to $\bar{U}$ and therefore unable to recross G, has no way to leave Δ and still remain within region IV, as Lemma 6.1 says it must. This contradiction establishes Lemma 6.3. ▌

The main result of the paper, Theorem 3.2, is now fully proved.

7. OPPOSITE VARIATION AND THE HALF-PLANE CONDITION

This section, which is independent of Sections 3 through 6, contains the arguments, examples and proofs supporting the connections among opposite variation, the half-plane condition, and $f_v g_u > 0$ which were asserted at the end of Section 2. We proceed first with the proof of

THEOREM 2.5. *Any admissible system which displays opposite variation must satisfy the half-plane condition.*

Assuming opposite variation, we can determine the direction of variation of r_j along R_i by multiplying (2.15) with l_i:

$$l_i\, d^2F(r_i, r_j) = (\lambda_j - \lambda_i)\, l_i(r_i \cdot \nabla r_j). \tag{7.1}$$

The left-hand side of (7.1) is negative because of (2.17), with i and j interchanged, and the symmetry of the Fréchet derivative. Hence, noting that r_j is a unit vector whose derivative $r_j' = r_i \cdot \nabla r_j$ must be parallel to l_i, we have proved

LEMMA 7.1. *In the case of opposite variation,*

$$r_i \cdot \nabla r_j = \gamma_i l_i \qquad \text{with} \quad \gamma_1 < 0, \quad \gamma_2 > 0. \tag{7.2}$$

This lemma may be combined with the inequalities following (2.12) to give a more complete picture of the rarefaction curves in the opposite-variation case. We already know that r_1 rotates away from r_2 as we follow an R_1 curve in the r_1-direction; but now (7.2) says that r_2 also rotates away from r_1. Similarly when following R_2 we know that r_2 rotates toward r_1, and now find that r_1 rotates toward r_2. Thus r_1 and r_2 rotate in opposite senses as one proceeds along any rarefaction curve.

The proof of Theorem 2.5 also depends on the following geometric property of the rarefaction curves.

LEMMA 7.2. *Given two points U_1 and U_2 in the U-plane, there exists a point U_0 such that U_1 and U_2 lie either on or inside both $R_1(U_0)$ and $R_2(U_0)$.*

Proof. Of the two curves $R_1(U_k)$, $k = 1, 2$, one lies nested inside the other; call the outer one Γ_1. Similarly let Γ_2 be the outer curve of the two $R_2(U_k)$. Should Γ_1 and Γ_2 correspond to the same point U_k, then this U_k will serve as U_0. If not, we will have $\Gamma_1 = R_1(U_i)$ and $\Gamma_2 = R_2(U_j)$ with $j \neq i$, so that U_j lies inside $R_1(U_i)$ and U_i inside $R_2(U_j)$. Hence Γ_1 and Γ_2 form a pair of convex curves, each of which contains a point inside the other as in Figure 7-1. Such a pair must intersect, and the point of intersection will be the required U_0. ▌

Proof of Theorem 2.5. Let U_1 and U_2 be arbitrary points of the plane, and let U_0 be the point determined by Lemma 7.2. Since each family of rarefaction curves is convex and fills the plane, we may construct two smooth directed arcs P_1 and P_2, leading from U_0 to U_1 and U_2 respectively, which never cross any rarefaction curve in the outward direction. More specifically, if p denotes the unit tangent vector to either P_k at any of its points, we require that

$$p = \delta_1 r_1 + \delta_2 r_2 \qquad \text{with} \quad \delta_1 \geqslant 0, \quad \delta_2 \leqslant 0. \tag{7.3}$$

(Of course if $U_0 = U_k$ the corresponding arc P_k reduces to a single point.) We can now compute the direction of rotation of r_1 and r_2 as one proceeds along P_k:

$$\begin{aligned} p \cdot \nabla r_1 &= \delta_1 r_1 \cdot \nabla r_1 + \delta_2 r_2 \cdot \nabla r_1 = \theta_1 l_2 \\ p \cdot \nabla r_2 &= \delta_1 r_1 \cdot \nabla r_2 + \delta_2 r_2 \cdot \nabla r_2 = \theta_2 l_1 \end{aligned} \tag{7.4}$$

with $\theta_1 = \delta_1\kappa_1 + \delta_2\gamma_2 \leqslant 0$ and $\theta_2 = \delta_1\gamma_1 + \delta_2\kappa_2 \leqslant 0$ by (2.12), (7.2) and (7.3). Therefore the eigenvectors r_1 and r_2 rotate away from one another as P_k is traversed. Since r_1 and r_2 never become parallel, the vector r_1 must remain within the (closed) angle formed by $r_1(U_0)$ and $-r_2(U_0)$ along the entire length of both arcs P_k, while r_2 must remain within the vertically opposite angle (cf. Figure 7-3). Therefore $r_1(U_1)$ and $r_2(U_2)$ cannot coincide, and we have established (2.13). ▌

While the half-plane condition will be sufficient to establish our main result on shock waves, it is interesting to note that in many significant cases we can go further and establish the existence of sectors for r_1 and r_2.

Definition 7.3. An admissible system satisfies the *sector condition* if there exist *two* fixed linearly independent vectors w_1 and w_2 such that for all U the following inequalities hold:

$$r_1 \cdot w_1 < 0, \qquad r_1 \cdot w_2 > 0$$
$$r_2 \cdot w_i > 0, \qquad i = 1, 2.$$

Remark: The sector condition always holds under the Smoller–Johnson assumption $f_v g_u > 0$, since we can then take w_1 and w_2 parallel to the coordinate

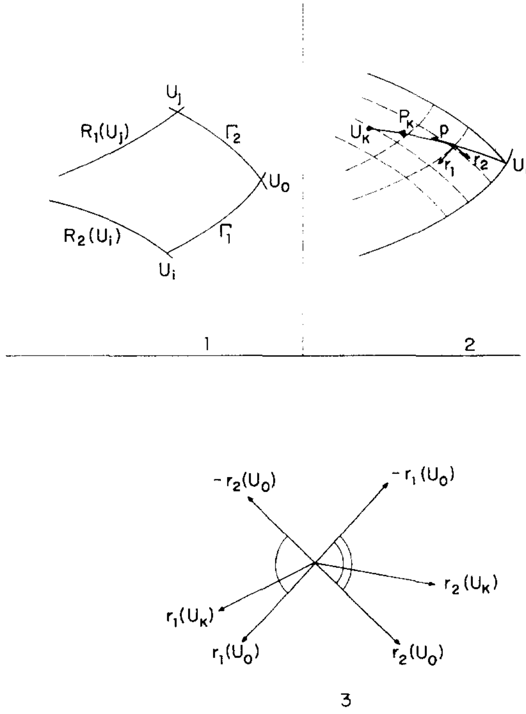

Figure 7

axes. A remark of Liu [14] shows this sector condition equivalent to the existence of an affine transformation of coordinates in the U-plane such that $f_v g_u > 0$ in the transformed coordinate system.

THEOREM 7.4. *The sector condition holds if and only if*

$$r_1(U_1) \neq \pm r_2(U_2) \tag{7.5}$$

for every pair of points U_1 *and* U_2.

The proof is similar to that of Theorem 2.2 and will be omitted.

THEOREM 7.5. *An admissible system which displays opposite variation and possesses the intersection property* (*Definition* 3.3) *must necessarily satisfy the sector condition.*

Proof. Condition (7.5) with the plus sign was established by Theorem 2.5. For the minus sign, we first assert that Lemma 7.2 now holds with the word "inside" replaced by "outside", since in the crucial case pictured in Figure 7-1 the intersection property insures that the curves $R_1(U_j)$ and $R_2(U_i)$ going to the left must also intersect. Then from the new U_0 thus obtained we construct paths P_k to U_k as in the proof of Theorem 2.5, but proceeding in the "outward" direction with respect to the rarefaction curves. Along these paths r_1 will rotate toward r_2 and vice versa, so that $r_1(U_1)$ and $r_2(U_2)$ both lie in the angle between $r_1(U_0)$ and $r_2(U_0)$ and cannot add up to zero. ▌

APPENDIX

AN EXAMPLE OF NON-CONVEX SHOCK CURVES

Many of the proofs in this paper, especially in Section 6, could have been simplified if the shock curves were known to be convex rather than merely star-shaped. Johnson and Smoller noted in [8] that the shock curves are not always convex under their assumptions, but stated that hyperbolic systems of the special form

$$\begin{aligned} u_t + f(v)_x &= 0, \\ v_t + g(u)_x &= 0, \end{aligned} \tag{A.1}$$

have convex shock curves. This appendix presents a counterexample.

Let us choose $f(v) = v$ and

$$g(u) = u + \log \cosh u. \tag{A.2}$$

Then

$$A = \begin{pmatrix} 0 & 1 \\ g' & 0 \end{pmatrix},$$

so that $-\lambda_1 = \lambda_2 = \sqrt{g'(u)} = (1 + \tanh u)^{1/2} > 0$ and we have a strictly hyperbolic system. The eigenvectors are

$$r_i = \frac{1}{\sqrt{1 + g'}} \begin{pmatrix} \mp 1 \\ \sqrt{g'} \end{pmatrix}, \qquad l_i = \frac{1}{\sqrt{1 + g'}} (\mp \sqrt{g'}, 1),$$

with the upper sign for $i = 1$, the lower for $i = 2$. Note that $(r_i \cdot \nabla)\lambda_i = (1 + g')^{-1/2} \cdot \frac{1}{2}(g')^{-1/2} g''$ with $g'' = \operatorname{sech}^2 u > 0$, so that the genuine nonlinearity condition (2.3) holds. Also

$$d^2F(r_i, r_i) = \frac{1}{1 + g'(u)} \begin{pmatrix} 0 \\ g''(u) \end{pmatrix},$$

so that

$$l_j \, d^2F(r_i, r_i) = (1 + g')^{-3/2} g'' > 0$$

and the condition (2.10) holds: our system (A.1) is admissible, both for us and for Smoller and Johnson [18].

Now the shock curves $S_i(0)$ and $S_i{}^*(0)$ are the solutions of $[f]/[u] = [g]/[v]$ or $fv \equiv v^2 = gu$, that is

$$v = \pm [ug(u)]^{1/2} \tag{A.3}$$

with the upper sign for the starred curves. Looking for example at $S_2{}^*$, we compute $v' = \frac{1}{2}(ug)^{-1/2}(g + ug')$ and

$$v'' = \tfrac{1}{4}(ug)^{-3/2} h(u) \tag{A.4}$$

with

$$h(u) = 2u^2 g g'' - (g - ug')^2. \tag{A.5}$$

Now near $u = 0$ we may expand $g(u)$ and $h(u)$ in power series to obtain $h(u) = 2u^3 + O(u^4)$ and

$$v''(u) = \tfrac{1}{2} + O(u) > 0. \tag{A.6}$$

However, for large positive u we have

$$\begin{aligned} g(u) &= 2u - \log 2 + O(e^{-2u}), \\ g'(u) &= 2 + O(e^{-2u}), \\ g''(u) &= 4e^{-2u}[1 + o(1)], \end{aligned}$$

and therefore

$$h(u) = -\log^2 2 + O(u^3 e^{-2u}),$$

so that

$$v''(u) = -\tfrac{1}{16}\sqrt{2}\,(\log 2)^2 u^{-3} + O(u^{-4}) < 0. \tag{A.7}$$

Thus $S_2^*(0)$ is concave upward near $u = 0$ and downward near $u = +\infty$, and so fails to be globally convex. Since by (A.3) the Hugoniot locus $H(0)$ is exactly symmetric with respect to both axes, its remaining branches $S_1(0)$, $S_2(0)$ and $S_1^*(0)$ are also non-convex.

Note added in proof. We would like to draw attention to the article "On the decomposition of a discontinuity for a system of two quasilinear equations" by V. A. Borovikov [*Trudy Moscov. Mat. Obsc. Tom* 27 (1972), translated in Trans. Moscow Math. Soc. Vol. 27], which constructs solutions to the Riemann problem under similar local and non-local assumptions on the system. Borovikov requires that the intersection property hold (Definition 3.3), and that no two rarefaction curves have a common tangent, a property which holds for the systems we consider (Lemma 4.3). He also considers systems in which the inequality is reversed in the Smoller-Johnson condition (2.10), and shows that further restrictions must then be imposed.

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