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Mixed hyperbolic-elliptic systems in self-similar flows

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— Dedicated to Constantine Dafermos on his 60^{th} birthday

Abstract. From the observation that self-similar solutions of conservation laws in two space dimensions change type, it follows that for systems of more than two equations, such as the equations of gas dynamics, the reduced systems will be of mixed hyperbolic-elliptic type, in some regions of space. In this paper, we derive mixed systems for the isentropic and adiabatic equations of compressible gas dynamics. We show that the mixed systems which arise exhibit complicated nonlinear dependence. In a prototype system, the nonlinear wave system, this behavior is much simplified, and we outline the solution to some typical Riemann problems.

Keywords: Two dimensional Riemann problems, equations of mixed type, nonlinear wave equation.

Mathematical subject classification: Primary: 35L65, 35L37; Secondary: 35M10.

1 Background: The Occurrence of Mixed Systems

For a large class of equations governing unsteady compressible flow in two space dimensions, study of self-similar problems (such as Riemann problems

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and shock reflection problems) results in boundary value problems for reduced systems which are of hyperbolic type far from the origin but change type at a sonic line or shock, determination of which is part of the solution. This class, consisting of systems with acoustic and linear waves, was characterized in [1]. In particular, for the compressible Euler equations of gas dynamics (either isentropic or adiabatic), and for a simpler nonlinear wave system, there are one or more characteristic families which are linearly degenerate and which remain real inside the subsonic region determined by the acoustic wave speeds. These families govern the evolution of shear and entropy waves; although they are also degenerate from the point of view of wave propagation, in that the characteristic normals form a plane rather than a cone supported in a half-plane, they are not trivial. For example, they are responsible for the intricate patterns of swirls seen inside the subsonic regions in shock reflection experiments and numerical simulations.

As part of our program to study multidimensional conservation laws by solving self-similar problems in two space dimensions, in this paper we formulate boundary value problems for some reduced equations of this mixed type. In [1], Čanić and Keyfitz demonstrated that if one linearizes such a system at a constant state, then a second-order, degenerate elliptic equation can be obtained by taking some combination of the variables, while a complementary set forms a hyperbolic system. However, this formal reduction did not show how the nonlinear equations could be reduced, nor how to formulate boundary value problems for these systems.

In this paper, we lay out the problem and outline the solution for a simple case. We do not tackle the important question of how to solve the free boundary problems that couple shock evolution with the flow in the subsonic regime, a problem for which we have found solutions in some cases involving the unsteady transonic small disturbance (UTSD) equation, [2, 3]. The advantage of the UTSD equation is that linear waves have been eliminated, so that the subsonic flow is governed by an elliptic equation. The disadvantage of that equation is that the subsonic region is typically unbounded; that is one reason we were able to obtain only local solutions (near the shock interaction point) where the UTSD equation correctly models the flow. To extend the methods we are developing to a full flow field, one needs to solve more realistic equations, such as the compressible gas dynamics equations for ideal flow. Many obstacles remain before we can complete this program.

In this paper, we apply recent work of Čanić and Kim, [6], which solves a class of degenerate elliptic problems with fixed boundaries. This work provides existence of a weak solution for domains satisfying a uniform exterior cone condition but does not deal with continuity up to the boundary for non-convex domains. As we show in the present paper, even for the simplest Riemann problems the subsonic regions are not convex and extension of the results in [6] to non-convex domains is needed. This is the subject of a companion paper, [4], where continuity up to the sonic boundary of a non-convex domain is proved for a set of Riemann problems for the nonlinear wave system. The present paper focusses on the complications which are added by coupling linearly degenerate hyperbolic with quasilinear elliptic equations.

To our knowledge, there is no general theory for quasilinear mixed systems. Specific problems from gas dynamics can guide our ideas about how to reduce the system. In particular, the hyperbolic part of the system, we find, linearly convects variables from the sonic boundary toward the origin, and is singular, in self-similar coordinates, either at the origin or at points determined by the elliptic part of the flow. In the nonlinear wave system, no new singularities are introduced into the solution in the interior of the subsonic region. However, the quantities convected by the hyperbolic equation are themselves quite singular because of the nature of the flow variables in self-similar coordinates. For the equations of compressible gas dynamics, which we do not solve in this paper, these singularities are the centers of the spirals seen in visualizations of shock reflection problems. Serre, [10], has pointed out that, for example, the vorticity cannot be in L^p for any p.

The structure of this paper is as follows. Sections 2, 3 and 4 carry out the reduction for several conservation law systems: the nonlinear wave system and the isentropic and the adiabatic Eulerian gas dynamics equations. In the final section, we present a solution for one problem for the nonlinear wave system.

The work of Serre [11] on reformulating the self-similar Euler equations in pressure-angle variables has been very useful to us. Serre derived the quasilinear equation which governs the elliptic part of the problem, using either the pressure or (in the steady case) the flow angle as the state variable. Serre also proved some useful maximum principles for this equation.

The paper of Zhang and Zheng [13], see also earlier work [7], by Chang (Zhang) and Chen, performs the service of displaying and reducing the gas dynamics equations, calculating the characteristics and identifying the sonic and subsonic states. Zhang and Zheng classify the different types of Riemann problems satisfying two conditions: (1) the initial data are constant in quadrants, and (2) the discontinuities propagate as single waves (shock, rarefaction or contact) at infinity.

Following Zhang and Zheng's work, Schultz-Rinne, Collins and Glaz [9], performed a series of numerical calculations which offer instructive comparison with Zhang and Zheng's conjectures. Later work further refined these conjectures [8]. Recent progress in theory and computation of multidimensional problems is detailed in the papers of Serre, Zhang and Zheng, and Schultz-Rinne, Collins and Glaz cited here.

2 The Nonlinear Wave System

The nonlinear wave system (NLWS) is obtained either by starting with the isentropic gas dynamics equations and neglecting terms which are quadratic in the velocity, or by writing the nonlinear wave equation as a first-order system. In terms of the conserved quantities (density and momenta) the system is

$$\begin{aligned} \rho_t + m_x + n_y &= 0 \\ m_t + p_x &= 0 \\ n_t + p_y &= 0; \end{aligned} \tag{2.1}$$

here $p = p(\rho)$ with $p' = c^2(\rho)$, $m = u\rho$, and $n = v\rho$. Thus (m, n) is the momentum vector and (u, v) the velocity vector.

To change to self-similar coordinates, define $(\xi, \eta) = (x/t,\, y/t)$, so

$$t\partial_t = -\xi\partial_\xi - \eta\partial_\eta, \quad t\partial_x = \partial_\xi, \quad t\partial_y = \partial_\eta.$$

The system in self-similar coordinates reads

$$\begin{aligned} -\xi\rho_\xi - \eta\rho_\eta + m_\xi + n_\eta &= 0 \\ -\xi m_\xi - \eta m_\eta + p_\xi &= 0 \\ -\xi n_\xi - \eta n_\eta + p_\eta &= 0. \end{aligned} \tag{2.2}$$

We obtain a second-order equation which changes type at the sonic line by eliminating m and n from (2.2). This equation, the nonlinear wave equation, in physical coordinates reads:

$$\rho_{tt} = -(m_x + n_y)_t = -(m_t)_x - (n_t)_y = p_{xx} + p_{yy} = \nabla \cdot (c^2(\rho)\nabla\rho).$$

Then we obtain

$$\big((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi + \big((c^2 - \eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0, \tag{2.3}$$

the self-similar nonlinear wave equation, here written with the principal part in divergence form.

To find a third equation, which, together with (2.3), will give a system equivalent to the reduced nonlinear wave system, (2.2), observe that a quantity which evolves independently of ρ is the specific vorticity, $\overline{w} = n_x - m_y = (\rho v)_x - (\rho u)_y$, which is time-independent: $\overline{w}_t = 0$. Working in self-similar coordinates, we replace this with a self-similar expression,

$$w = t\overline{w} = n_\xi - m_\eta,$$

which satisfies

$$\xi w_\xi + \eta w_\eta + w = (\xi, \eta) \cdot \nabla w + w = 0. \tag{2.4}$$

This could be written in divergence form as $(\xi w)_\xi + (\eta w)_\eta - w = 0$. Equations (2.3) and (2.4) form our prototype system in which (2.3) changes type at $\xi^2 + \eta^2 = c^2$ and the system is of mixed type for $\xi^2 + \eta^2 < c^2$. We discuss appropriate boundary conditions and the solution of this problem in Section 5.

This problem is somewhat artificial, because it came from dropping some terms in the Euler equations. Before studying it further, we look at the Euler equations to see if similar systems arise. It appears that they do.

3 The Isentropic Gas Dynamics Equations

The equations for isentropic flow again form a system of three equations, conserving the same quantities as in the nonlinear wave system, and respecting the same thermodynamic relation $p = p(\rho)$. The system is

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y &= 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y &= 0; \end{aligned} \tag{3.1}$$

Reduction to a self-similar system is carried out in [11] and in [13].

The self-similar equations, ignoring conservation form, are

$$\begin{aligned} (u-\xi)\rho_\xi + \rho u_\xi + (v-\eta)\rho_\eta + \rho v_\eta = U\rho_\xi + \rho u_\xi + V\rho_\eta + \rho v_\eta = 0 \\ (u-\xi)u_\xi + p_\xi/\rho + (v-\eta)u_\eta = U u_\xi + p_\xi/\rho + V u_\eta = 0 \\ (u-\xi)v_\xi + (v-\eta)v_\eta + p_\eta/\rho = U v_\xi + V v_\eta + p_\eta/\rho = 0; \end{aligned}$$

$U = u - \xi$, $V = v - \eta$ are the components of the 'pseudovelocity'. This term is somewhat deceptive, as it buries the fact that, unlike steady transonic flow, and like the nonlinear wave system, the distinction between supersonic and subsonic regions depends on position in space as well as on the states. A version that refers only to the pseudovelocity is

$$U\rho_\xi + \rho U_\xi + V\rho_\eta + \rho V_\eta + 2\rho = 0 \tag{3.2}$$
$$UU_\xi + p_\xi/\rho + VU_\eta + U = 0 \tag{3.3}$$
$$UV_\xi + VV_\eta + p_\eta/\rho + V = 0. \tag{3.4}$$

A second-order equation for ρ, whose coefficients depend on U and V, is

$$\begin{aligned}\partial_\xi\big((U^2-c^2)\rho_\xi + UV\rho_\eta + \rho U\big) + \partial_\eta\big((V^2-c^2)\rho_\eta + UV\rho_\xi + \rho V\big)\\ + (UV_\eta - VU_\eta)\rho_\xi + (VU_\xi - UV_\xi)\rho_\eta + 2(U_\xi V_\eta - V_\xi U_\eta)\rho = 0.\end{aligned} \tag{3.5}$$

This equation, which is similar to the equation for steady transonic potential flow, was obtained by Serre, [11]. Serre makes the pressure the dependent variable. This is the better choice for the adiabatic case. In the isentropic case it makes no difference.

Next, we obtain the analog of the vorticity equation, (2.4); a simpler equation results when we use the vorticity itself instead of the specific vorticity (that is, modified by the density) as we did for the nonlinear wave system. The procedure is to eliminate ρ from (3.3) and (3.4) by differentiating the first with respect to η, the second with respect to ξ and subtracting. We define the self-similar (pseudo) vorticity

$$W = U_\eta - V_\xi = u_\eta - v_\xi,$$

and the equation obtained is

$$UW_\xi + VW_\eta + (U_\xi + V_\eta + 1)W = (UW)_\xi + (VW)_\eta + W = 0. \tag{3.6}$$

The first form represents the operator as the directional derivative of W in the direction (U, V) of the pseudovelocity. This is a clear analogy to (2.4) for w in the nonlinear wave system. The operator in equation (3.6) is also singular, now when the pseudovelocity is zero. (But recall, as in our comment on the term ‘pseudovelocity’ that (U, V) is alternatively the ‘pseudoposition’.) For a constant solution, this occurs at the center of the sonic circle, $(u-\xi)^2 + (v-\eta)^2 = c^2$. For a nonconstant subsonic flow, there may be one or several points where the pseudovelocity vanishes.

Now, (3.5) and (3.6) couple an elliptic and a hyperbolic equation in the subsonic region. The coupling is through coefficients which depend on U and V and their derivatives. This is to be expected: since the type of (3.5) depends on (U, V), these functions must appear as the coefficients of (3.5). In this problem, we must solve equations for U and V simultaneously with (3.5) and (3.6), instead of afterwards, as in the nonlinear wave system. To recover U and V from the conserved quantities ρ and W, we may use

$$U_\eta - V_\xi = W \tag{3.7}$$

and (3.2) in the form

$$U_\xi + V_\eta = -\frac{1}{\rho}(U\rho_\xi + V\rho_\eta) - 2 \tag{3.8}$$

and this pair of equations forms a linear elliptic system for U and V with a source term determined by W and coefficients depending on ρ. However, we shall see in Section 5 that this may not be the best way to view the problem. As an alternative, we may regard (3.3) and (3.4) as transport equations for the components U and V. The coupling of (3.5) and (3.6) via U and V does not occur in the nonlinear wave system, and we have not yet considered this problem. We shall see in Section 5 that finding (u, v) by solving transport equations is an effective procedure for the nonlinear wave system.

The analogy with the nonlinear wave system suggests that the possibly singular behavior in (3.6) may be similar to that of w in (2.4). We shall see that (2.4), while singular, does not pose any difficulties, and we conjecture that the solutions of (3.6) will also be well-behaved.

We note that the system (3.5), (3.6) does not conserve the same quantities as the original system. This does not matter for the solution ρ of (3.5) in the subsonic region, where we expect ρ to be continuous. It is possible to replace W by a specific vorticity, for example $(\rho v)_\xi - (\rho u)_\eta$, at the cost of obtaining a more complicated equation. However, since the only discontinuities in the subsonic region are linear, we conjecture that maintaining the correct conservation form is unnecessary.

4 The Adiabatic Gas Dynamics Equations

We also consider the system of four equations governing the evolution of an ideal gas with a polytropic equation of state, $p = e(\gamma - 1)\rho$, in which an equation for conservation of energy is added to those for mass and momentum. In the subsonic region, there are two coincident real characteristics. In this situation, we supplement the elliptic equation for the density or pressure with a pair of equations for the evolution of shear (vorticity) and entropy variables.

A standard form of the system is, [13],

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y &= 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y &= 0 \\ (\rho E)_t + (\rho u H)_x + (\rho v H)_y &= 0; \end{aligned} \tag{4.1}$$

with the notation $E = e + q^2/2,\ q^2 = u^2 + v^2$, $H = e + p/\rho + q^2/2$, and the relations

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}, \quad E = \frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}q^2, \quad H = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}q^2.$$

We work with the variables ρ, u, v, and p.

In self-similar coordinates, one can reduce (4.1) to a simplified form, [13]:

$$(\rho U)_\xi + (\rho V)_\eta + 2\rho = 0 \tag{4.2}$$

$$U U_\xi + V U_\eta + U + \frac{p_\xi}{\rho} = 0 \tag{4.3}$$

$$U V_\xi + V V_\eta + V + \frac{p_\eta}{\rho} = 0 \tag{4.4}$$

$$(p^{1/\gamma} U)_\xi + (p^{1/\gamma} V)_\eta + 2p^{1/\gamma} = 0. \tag{4.5}$$

(This reduction does not conserve the correct quantities; however, it is a convenient way to examine the characteristic structure of the system.) We have again used the pseudovelocities, $U = u - \xi$, $V = v - \eta$.

The equation for p, derived by Serre in [11], with $\vec{u} = (U, V)$, $Q^2 = U^2 + V^2$, and $c^2 = \gamma p/\rho$, is

$$\nabla \cdot \left(\frac{1}{\rho Q^2} \left[\nabla p - \frac{1}{c^2}\vec{u}(\vec{u} \cdot \nabla p) \right] - \frac{1}{Q^2}\vec{u} \right) = 0.$$

Here the principal part, exhibiting change of type at the sonic circle $U^2 + V^2 = c^2$, is the familiar form

$$\left(1 - \frac{U^2}{c^2}\right) p_{\xi\xi} - \frac{UV}{c^2} p_{\xi\eta} + \left(1 - \frac{V^2}{c^2}\right) p_{\eta\eta}.$$

Rewriting (4.2) and (4.5), we have

$$\begin{aligned} (U, V) \cdot \nabla \rho + \rho(\nabla \cdot (U, V) + 2) &= 0 \\ (U, V) \cdot \nabla p + \gamma p(\nabla \cdot (U, V) + 2) &= 0, \end{aligned}$$

from which we obtain

$$(U, V) \cdot \nabla p = \frac{\gamma p}{\rho}(U, V) \cdot \nabla \rho = c^2 (U, V) \cdot \nabla \rho \,. \tag{4.6}$$

Now, (4.6) leads immediately to the first hyperbolic equation:

$$(U, V) \cdot \left(\gamma \frac{\nabla \rho}{\rho} - \frac{\nabla p}{p} \right) = 0.$$

This can also be written as an equation for ρ (as the second-order equation is an equation for p):

$$(U, V) \cdot \nabla \log \left(\frac{\rho}{p^{1/\gamma}} \right) = 0. \tag{4.7}$$

This equation expresses the transport of entropy, $S = \rho^\gamma / p$, along streamlines (pseudostreamlines in this case).

Finally, the other hyperbolic equation describes the evolution of the self-similar vorticity, as in the previous two examples. Define

$$W = U_\eta - V_\xi, \tag{4.8}$$

differentiate (4.3) and (4.4) with respect to η and ξ respectively and subtract. The result is

$$\nabla \cdot (UW, VW) + W = \frac{p_\xi \rho_\eta - p_\eta \rho_\xi}{\rho^2} \tag{4.9}$$

which is now more complicated than before, because it involves p and ρ, but is, as before, a transport equation for W along the vector field (U, V) and is, like the equation for the entropy, (4.7), singular at points where $(U, V) = 0$.

Using a slightly different form of the second-order equation, also derived by Serre, we collect the three equations governing the flow in the subsonic region:

$$\left.\begin{aligned} \nabla \cdot \left(\frac{1}{\rho Q^2} \left[\nabla p - \frac{1}{c^2} \vec{u} (\vec{u} \cdot \nabla p) \right] \right) - \frac{2c^2 - Q^2}{\rho c^2 Q^4} \vec{u} \cdot \nabla p &= 0 \\ (U, V) \cdot \nabla \log \left(\frac{\rho}{p^{1/\gamma}} \right) &= 0 \\ \nabla \cdot (UW, VW) + W - \frac{p_\xi \rho_\eta - p_\eta \rho_\xi}{\rho^2} &= 0. \end{aligned}\right\} \tag{4.10}$$

This form has some flaws: the first equation, as written, appears singular at $(U, V) = 0$, as observed by Serre, [11]. On the other hand, dividing by Q^2 was simply a convenient way to obtain the equation. One could remedy this difficulty by multiplying the equation by Q^2 or by ρQ^2. The same difficulty observed at the end of the previous section, that this system does not conserve the corrrect quantities, occurs in this reduction also.

As in the previous sections, it is necessary to solve (4.10) along with a pair of equations for U and V. We can recover U_ξ and V_η from (4.2), and the other partial derivatives from the definition of W, (4.8):

$$\begin{aligned} U_\xi + V_\eta &= -\frac{\rho_\xi}{\rho}U - \frac{\rho_\eta}{\rho}V - 2 \\ U_\eta - V_\xi &= W \end{aligned}$$

exactly as in the isentropic case. However, as mentioned there and as will be discussed in the next section, it may be preferable to recover U and V from the transport equations, (4.3) and (4.4). That is, rather than solve (4.10), we couple the second-order equation for p with (4.7), (4.3) and (4.4).

5 The Nature of the Solution of the Nonlinear Wave System

We examine the mixed type problem formulated in Section 2. In this problem, the part that changes type (the second-order equation for ρ, (2.3)) and the hyperbolic part, (equation (2.4) for w) are uncoupled. Although this problem is oversimplified and somewhat artificial, it may serve as a prototype for the mixed problems arising from the gas dynamics equations.

Consider sectorially constant Riemann data for $u = (\rho, m, n)$. We illustrate three-sector data in Figure 5.1(a). Label the sector boundaries $x = \kappa_i y$, $i = a, b, c$.

Recall that $\overline{w} = w/t$ is constant in time and so the vorticity equation, (2.4), has the explicit solution

$$w(\xi, \eta) = t\bar{w}(x, y, t) = \bar{w}(\xi, \eta, 1) = \bar{w}(\xi, \eta, 0) = \bar{w}^{(0)}(\xi, \eta).$$

For sectorially constant initial data, w is a measure supported on the discontinuity set of the data. However, no additional singularity is introduced by the fact that the operator in (2.4) is degenerate at the origin. It is possible that the degenerate equations (3.6), (4.7) and (4.9) will have solutions of this nature.

The solution far from the origin is locally one-dimensional. Construction of one-dimensional Riemann solutions is standard, see Smoller [12]; we give the formulas in Appendix A to fix notation.

A one-dimensional Riemann problem at $x = \kappa_i y$ gives rise to three waves, at $\xi = \kappa_i \eta + \chi_i^-$, $\xi = \kappa_i \eta$, and $\xi = \kappa_i \eta + \chi_i^+$. Denote the state adjacent to u_j, for

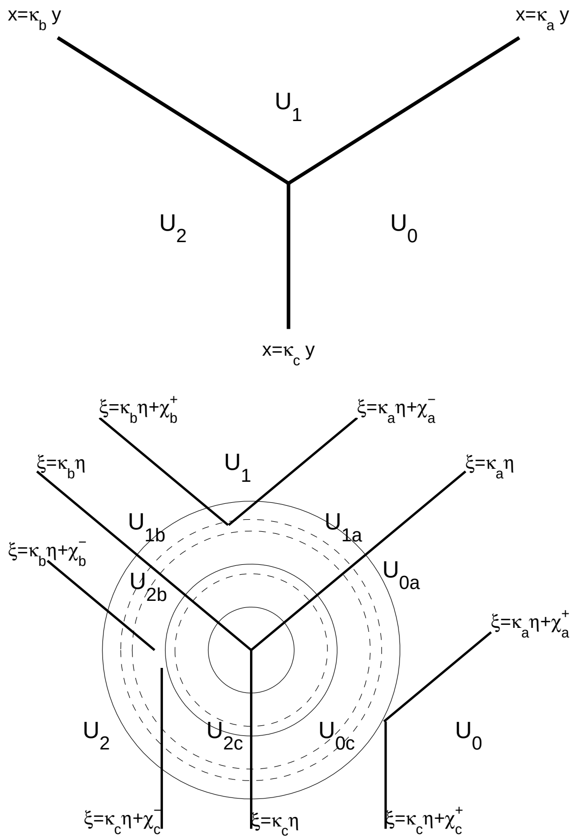

Figure 5.1: (a) Sectorially Constant Data and (b) The Far-Field Solution.

the Riemann problem at κ_i, by u_{ji}. (Rarefaction waves have finite thickness; for simplicity, we use the same notation.) Up to six sonic circles are generated by the far-field data, corresponding to the three original states and to the three new values of ρ which appear in the solution. These are indicated in Figure 5.1(b).

The one-dimensional waves interact near the origin to produce interesting subsonic behavior of the solution. Some analysis of these interactions is possible by elementary means, [1].

No matter how many sectors we consider, the first set of interactions takes place within a single sector, between the incoming waves generated by the two sector boundaries. Using the one-dimensional solutions, the interaction points in similarity space can be calculated in a straightforward way, as the shocks are of the form $\xi = \kappa\eta + \chi^{\pm}$, and $\chi^{\pm}$ is determined from the densities (only) on either side by equation (6.1). Similarly, the leading edge of a rarefaction entering a region with density ρ is $\xi = \kappa\eta \pm \sqrt{1+\kappa^2}c(\rho)$, where $c(\rho)$ is the local sound speed. We denote by Ξ_i the intersection point of the waves in sector i. Each wave is either a shock or a rarefaction, so there are three kinds of interactions; we discuss each in turn.

Shock-Shock Interactions

The interaction of two one-dimensional shocks produces a quasi-one-dimensional Riemann problem, as described in [1]. For the NLWS, the fact that w, the vorticity, is constant in time means that no linear waves are generated, and a solution of the quasi-one-dimensional Riemann problem, when it exists, consists of two waves, each of which may be a shock or a rarefaction centered at the interaction point, Ξ_i. However, hyperbolic theory does not predict a solution if one or both states are subsonic at Ξ_i. In addition, not all hyperbolic quasi-one-dimensional Riemann problems have solutions. Explicit conditions can be given for both of these obstructions. In either case, the solution may not be a pair of waves from Ξ_i but may be qualitatively different. Nonexistence of solutions to this type of quasi-one dimensional Riemann problem is the genesis of the 'von Neumann paradox' in weak shock reflection.

Rarefaction-Rarefaction Interactions

The interaction begins where the leading edges of the two waves intersect; if it were localized at a point, it would give a quasi-one-dimensional Riemann problem consisting of hyperbolic data (since rarefactions exist only in the supersonic region), and the solution would typically consist of two centered quasi-one-dimensional rarefaction waves, with a new state between them. We conjecture that this describes the qualitative behavior. The outgoing waves are approximately centered. They emerge from the interaction zone as simple waves, since they are adjacent to the same constant states that lie behind the trailing edges of the incoming rarefactions. (Theorem 3.2 of [1] does not apply here, as we are dealing with more than two equations; however, it can be extended to this case, since there are no linear waves in this open sector.) Furthermore, since both outgoing waves are expanding (rarefaction) waves, the solution in the hyperbolic region is continuous up to the sonic line. Thus, one can estimate the value of the new approximately constant intermediate state, the shape of the sonic boundary and the data on that boundary.

Shock-Rarefaction Interactions

The third type of interaction is also nonlocal, since it begins at Ξ_i where the shock intersects the leading edge of the rarefaction, but is not centered there. At this point the state behind the shock may be subsonic, so that even if the problem were approximated by a quasi-one-dimensional Riemann problem, a solution might not exist. Finally, unlike the previous case the solution will not be continuous beyond the interaction point. We conjecture that in the case of supersonic states the solution will be similar to a quasi-one-dimensional Riemann problem at Ξ_i, and will typically give rise to a shock, a rarefaction, and a new intermediate state. However, in the case that one state is subsonic, then, as in the case of shock-shock interactions with a subsonic state, qualitatively different behavior may occur.

Examples

We expect each wave interaction in the hyperbolic region to produce a pair of waves, simple or shock. The flow in the hyperbolic region could then be tracked, by an adaptation of front-tracking (on a discrete scale), until the flow becomes sonic. Beyond that point, transonic shocks, which correspond to free boundary problems, will occur. To prove existence of a solution in the subsonic region, then, would require extensions of the method we used in solving free boundary problems for the transonic small disturbance equation, [2, 3] (extensions are needed because uniform obliqueness typically fails at points on the shock); we have yet to tackle this problem.

We illustrate one case with a finite-difference simulation; contours of ρ and n are shown in Figure 5.2. For this set of data, there are two rarefactions (in the southeast corner) and four shocks. Two shocks entering the picture from the top have intersected in the hyperbolic region, producing a reflected shock (traveling southwest) and a rarefaction. (We know there is a rarefaction by explicitly solving the quasi-one-dimensional Riemann problem here.) The two shocks which enter from the south and west sides and intersect also produce a reflected shock, traveling northeast. These two reflected shocks merge to form a transonic shock; proving the existence of this free boundary poses an interesting challenge.

We note that the primary intersection of the two shocks in the southwest corner is already a situation which cannot be resolved using only hyperbolic techniques. This is because the shock entering from the south is transonic before the intersection takes place, and thus a quasi-one-dimensional Riemann problem at this

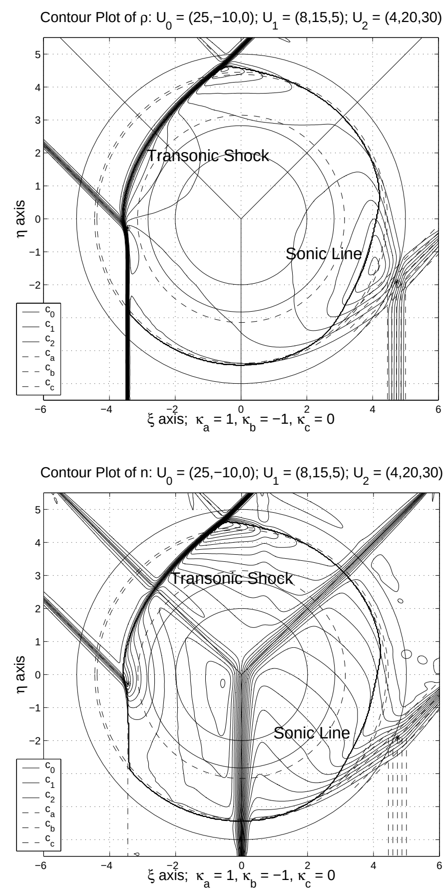

Figure 5.2: Contour Plots of ρ and n.

point does not have a solution unless the state behind the transonic shock is exactly sonic. In that case there may be a solution, consisting of a rarefaction and a shock, which does not violate the triple point paradox. This situation is similar to the scenario suggested for the UTSD equation in our earlier paper, [5]. To complete the solution, one needs to prove that an elliptic free boundary problem has a global solution.

Finally, the contour plot for n shows clearly the linear waves along the original sector boundaries, and shows that they appear to interact with the nonlinear waves via linear superposition.

In one case we have been able to establish existence of a weak solution to the nonlinear wave system in the entire plane, [4]. This case is given by a three-parameter family of Riemann data, $U_0 = U_2 = (\rho_0, m_0, n_0)$; $U_1 = (\rho_1, m_0, n_1)$, $\kappa_a = -\kappa_b$, $\kappa_c = 0$, with $\rho_1 < \rho_0$; ρ_0, m_0 and n_0 are free parameters and n_1 is chosen so that the far-field solution consists of two outgoing rarefaction waves (the χ_a^+ and χ_b^- waves) and two linear waves along κ_a and κ_b. There are no supersonic wave interactions in this case: Each one-dimensional rarefaction fills a half-strip in similarity space, terminating where it becomes sonic. Thus the sonic boundary consists of sectors of the sonic circles corresponding to ρ_1 and ρ_0, and the ends of the strips, as pictured in Figure 5.3.

According to the calculation in Appendix C we expect the solution near the smaller circle to exhibit a square-root singularity in ρ, while near the maximum ρ_1 the density should decrease linearly with slope given by equation (6.4). The theory in [6], which provides existence of a weak solution but does not yield continuity of the solution up to the boundary, is extended in [4] to prove that a solution ρ to this subsonic problem exists, is continuous up to the boundary of the subsonic region, except possibly at the inner corners, and lies in a Sobolev space consistent with these estimates.

The Linear Waves

Once the solution component ρ has been determined, the solution can be completed by finding the momentum vector (m, n). The most straightforward way to do this is to note that the original self-similar system, (2.2), gives transport equations for m and n:

$$\frac{\partial m}{\partial s} = p_\xi; \quad \frac{\partial n}{\partial s} = p_\eta; \tag{5.1}$$

where $s = (\xi^2 + \eta^2)/2$ is a radial variable and $\partial/\partial s$ is differentiation in the radial direction. In problems like the two examples above, the subsonic region is (weakly) starshaped with respect to the origin. In the supersonic region, m and n are found by solving quasi-one-dimensional problems (in the second example) or by other hyperbolic techniques such as front-tracking (we conjecture). Then (m, n) is known at the sonic line: by continuity where the flow is continuous and from the Rankine-Hugoniot relation where the sonic boundary is a shock. We show in Appendix C that if ρ is continuous at the sonic line, then so are m and n. Thus, since ρ is known in the subsonic region and is continuous up to the boundary, [4], m and n can be recovered by integrating (5.1) from the sonic line inward to the origin. Since ρ is smooth in the interior of the subsonic region, the values of m and n obtained this way are consistent with the vorticity $w(\xi, \eta) = w^{(0)}(\xi, \eta)$.

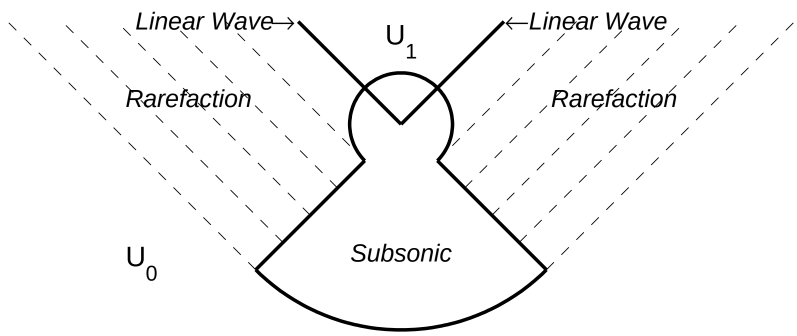

Figure 5.3: Subsonic Region for Outgoing Rarefaction Data

The form of the problem suggests the alternative of recovering (m, n) from ρ and w by the following procedure. From the first equation of (2.2) and the definition of w we have

$$\begin{aligned} m_\xi + n_\eta &= \xi\rho_\xi + \eta\rho_\eta \\ m_\eta - n_\xi &= w. \end{aligned}$$

Decoupling m and n results in Poisson problems:

$$\begin{aligned} \Delta m &= (\xi\rho_\xi + \eta\rho_\eta)_\xi + w_\eta \\ \Delta n &= (\xi\rho_\xi + \eta\rho_\eta)_\eta - w_\xi. \end{aligned}$$

Each problem comes with Dirichlet data: the values of m and n at the sonic line. Regularity of solutions found this way is low, since the terms on the right hand side are derivatives of measures. In particular, this approach suggests possible singularities in (m, n) which, on the basis of the transport equation approach, are not realized. In principle, this approach is effective if the subsonic region is not starshaped with respect to the origin; however, if the functions found this way are not compatible with the original transport equations, then it is not at clear how to interpret them as solutions.

As a variant of this approach, one may note that $w \equiv 0$ in the interior of each sector, and so $(m, n) = \nabla\psi_i$ in the i-th sector; then the first equation of (2.2) implies

$$\Delta\psi_i = \xi\rho_\xi + \eta\rho_\eta$$

in sector i. Now, the value of $\nabla\psi_i$ is given on the sonic line, and the jump in w across the sector boundary j is equal to its initial value:

$$[n+\kappa_j m]=[n^{(0)}+\kappa_j^{(0)}]$$

on L_j. That is,

$$(\kappa_j,1)\cdot(\nabla\psi_i-\nabla\psi_{i-1})=[n^{(0)}+\kappa_j^{(0)}].$$

Since this is a tangential derivative, we can integrate to obtain

$$\psi_i(\kappa_j\eta,\eta)-\psi_{i-1}(\kappa_j\eta,\eta)=\eta\big(n_i^{(0)}-n_{i-1}^{(0)}+\kappa_j(m_i^{(0)}-m_{i-1}^{(0)})\big)$$

where we adopt the convention $\psi_i(0,0)=0$ for each potential. This formulation has the disadvantage that the boundary conditions do not immediately lead to a solution, as the problem is overspecified on the sonic boundary, and underspecified on the radial lines.

We conclude that coupling the elliptic equation for ρ in the subsonic region with transport equations for the conserved quantities m and n is an effective method for this problem, when the subsonic region is starshaped with respect to the origin. However, in examples (which are easy to construct) where the subsonic region is not starshaped, some compatibility must be demonstrated. The pair of equations (5.1) is overdetermined, but can be seen to be consistent because the evolution equation for w, (2.4), is linear.

We conclude with a comment on the relevance of this simplified problem to the gas dynamics equations. In the case of the gas dynamics equations, the transport equations are also linear in the characteristic variables, and the discontinuities in the subsonic region are also linear. We conjecture that our conclusions on transport in the nonlinear wave system will be relevant there. That is, we expect to see low regularity, essentially linear behavior, and no additional complications due to the singularities in the coefficients of the hyperbolic equations.

6 Appendices

A: One-Dimensional Riemann Problems: Nonlinear Wave System

One-dimensional waves solve Riemann problems along lines transverse to $x=\kappa y$; self-similar solutions $U(x-\kappa y,t)=U((x-\kappa y)/t)=U(\chi)$ to $U_t+F_x+G_y=0$ satisfy the Rankine-Hugoniot relation (the notation $[\cdot]$ refers to the jump in a quantity) $\chi[U]=[F-\kappa G]$ at discontinuities. For our problem, this leads to the equations

$$\chi[\rho]=[m-\kappa n],\quad \chi[m]=[p],\quad \chi[n]=-\kappa[p].$$

In the standard way, [12], we have

$$\chi^- = -\sqrt{1+\kappa^2}\sqrt{\frac{[p]}{[\rho]}}, \quad \text{or} \quad \chi^+ = \sqrt{1+\kappa^2}\sqrt{\frac{[p]}{[\rho]}} \tag{6.1}$$

along shock curves S^- and S^+, and

$$[m] = \frac{\chi}{1+\kappa^2}[\rho] \quad \text{and} \quad [n] = -\kappa[m].$$

These are curves in U-space; we can parameterize them by ρ for fixed U_0. The shock admissibility conditions apply: $\lambda_L > \chi > \lambda_R$ where λ is the characteristic speed, $\lambda^\pm = \pm\sqrt{1+\kappa^2}c(\rho)$, $c^2 = dp/d\rho$. We have $\rho_L > \rho_R$ for a $+$-shock and $\rho_L < \rho_R$ for a $-$-shock. The backward portions of the curves, $S_*^\pm$, correspond to choosing U_0 to be the right state, U_R.

The continuous solutions are rarefaction waves; for fixed U_0 these can also be parameterized by ρ; the curves are

$$m = m_0 \pm \frac{1}{\sqrt{1+\kappa^2}}\int_{\rho_0}^{\rho} c(r)\,dr, \quad n = n_0 \mp \frac{\kappa}{\sqrt{1+\kappa^2}}\int_{\rho_0}^{\rho} c(r)\,dr.$$

We have $\rho_R > \rho_L$ for R^+ and $\rho_R < \rho_L$ for R^-. Figure 6.1(a) shows projections of these curves in the (m, ρ) plane.

The linear waves are simply contact discontinuities at $\chi = 0$; they also form a one-parameter family with $[\rho] = 0$ and $[m] = \kappa[n]$.

For some data, even the one-dimensional problem will not have a solution, as a vacuum state will arise. The condition for this can be given explicitly. Define

$$G(0) = \frac{1}{\sqrt{1+\kappa^2}}\left(\int_{\rho_R}^{0} c(r)dr + \int_{\rho_L}^{0} c(r)dr\right) + \\ + \frac{1}{1+\kappa^2}\big(m_R - m_L - \kappa(n_R - n_L)\big).$$

Then we have

Proposition 6.1. *For a pair of states $\{U_L, U_R\}$ and a given slope κ, there is a unique self-similar solution $U(\chi)$ to the Riemann problem if $G(0) < 0$ and no solution otherwise.*

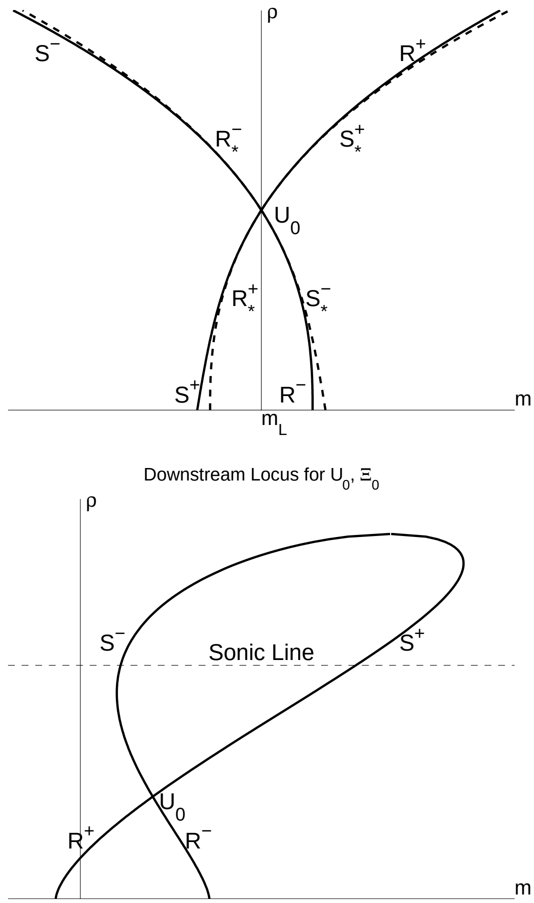

Figure 6.1: (a) Shock and Rarefaction Curves and (b) The Downstream Locus

B: The Quasi-One-Dimensional Problem

We solve the Riemann problems generated by two states U_L and U_R and a point Ξ_0 in the forward timelike direction: a sector pointing toward the origin in the (ξ, η) plane. The orientations 'left' and 'right' refer to an observer at Ξ_0 facing the origin. The new middle state, U_M, is separated from U_L and U_R by '$-$' or '$+$' waves (shocks or rarefactions) respectively.

A useful approach, developed in [1], is to construct the downstream locus of a state U_0: this locus consists of all the states that could serve as U_M whether U_0 is on the left or the right at Ξ_0. See Figure 6.1(b). The Riemann problem has a solution if the downstream loci of U_L and U_R intersect at a point which is a '$-$' wave from U_L and a '$+$' wave from U_R. Parameterizing the points on the downstream locus of U_0 by ρ, we have $\rho > \rho_0$ for shocks and

$$[m] = \frac{[p]}{\xi_0^2 + \eta_0^2}\left(\xi_0 \pm |\eta_0|\sqrt{\frac{(\xi_0^2+\eta_0^2)[\rho]}{[p]} - 1}\right),$$

$$\chi = \frac{[p]}{[m]}, \kappa = \frac{\xi_0 - \chi}{\eta_0}, [n] = -\kappa[m]$$

where $[f] = f - f_0$ and the shock equation is $\xi = \kappa\eta + \chi$; the $-$ sign goes with the $-$-shock. The points on the downstream locus with $\rho < \rho_0$ are centered rarefactions. Along a line $\xi = \kappa\eta + \chi$ in the rarefaction, the states are

$$m = m_0 + \int_{\rho_0}^{\rho} c(r)\left(\frac{c(r)}{\xi_0^2+\eta_0^2}\xi_0 - |\eta_0|\sqrt{\xi_0^2+\eta_0^2 - c^2(r)}\right) dr,$$
$$n = n_0 + \frac{[p] - \xi_0[m]}{\eta_0};$$

here χ and κ are

$$\chi = \frac{\xi_0 c^2 + \eta_0\sqrt{c^2(\xi_0^2+\eta_0^2-c^2)}}{c^2 - \eta_0^2}, \quad \kappa = \frac{\xi_0 - \chi}{\eta_0}.$$

C: Weak Solutions at the Sonic Line

In our earlier paper about self-similar solutions, [1], we noted that the nonlinear wave equation appears to have both regular (Lipschitz continuous) and singular (Hölder continuous) solutions at the sonic circle. We amplify on that here. The wave system (linear or nonlinear) in matrix form is

$$\begin{pmatrix} -\xi & 1 & 0 \\ c^2 & -\xi & 0 \\ 0 & 0 & -\xi \end{pmatrix}\begin{pmatrix} \rho_\xi \\ m_\xi \\ n_\xi \end{pmatrix} + \begin{pmatrix} -\eta & 0 & 1 \\ 0 & -\eta & 0 \\ c^2 & 0 & -\eta \end{pmatrix}\begin{pmatrix} \rho_\eta \\ m_\eta \\ n_\eta \end{pmatrix} = 0\,;$$

the sonic line is given by $\xi^2 + \eta^2 = c^2$. We examine the solution on the subsonic side of a sonic line at which U is constant on the supersonic side. Introducing polar coordinates $\xi = r\cos\theta$, $\eta = r\sin\theta$ to straighten the sonic line, we obtain

$$\begin{pmatrix} -r & \cos\theta & \sin\theta \\ c^2\cos\theta & -r & 0 \\ c^2\sin\theta & 0 & -r \end{pmatrix} U_r + \frac{1}{r}\begin{pmatrix} 0 & -\sin\theta & \cos\theta \\ -c^2\sin\theta & 0 & 0 \\ c^2\cos\theta & 0 & 0 \end{pmatrix} U_\theta = 0,$$

for $U = (\rho, m, n)$. The second-order equation for $\rho(r,\theta)$ is

$$\big((c^2 - r^2)\rho_r)_r + \frac{c^2}{r}\rho_r + \left(\frac{c^2}{r^2}\rho_\theta\right)_\theta = 0. \tag{6.2}$$

In [1] we showed that if $c'(\rho) \neq 0$ then this equation has a solution with finite slope at the sonic line. We also expect to see a solution with a square root singularity; in the case of a linear equation, this behavior characterizes the fundamental solution. For the linear equation, c^2 is constant and if we seek a solution $\rho(r)$, then (6.2) becomes a first-order linear equation for ρ_r whose general solution is

$$\rho = a_1 \log\left(\frac{c + \sqrt{c^2 - r^2}}{r}\right) + a_2.$$

Since we are interested in solutions near the sonic line, $r = c$, these solutions all have square-root (not logarithmic) singularities. If we seek a solution of the form

$$\rho = \rho_m + \alpha(r_m - r)^\beta, \tag{6.3}$$

inside the sonic line (circle) $r = r_m = c_m$ for a general nonlinearity with $c^2 = dp/d\rho$ (ignoring the θ dependence of the solution, which will not influence the leading term), then the equation becomes, to leading order,

$$p''(\rho)\rho_r + \big[p''(\rho_m)(\rho - \rho_m) + r_m^2 - r^2\big]\frac{\rho_{rr}}{\rho_r} + \frac{p' - 2r^2}{r} = 0.$$

(We have written $c^2 = p'(\rho)$ for convenience and used Taylor's theorem to expand $c^2 - r^2 = c^2 - c_m^2 + r_m^2 - r^2$ near r_m.) Now using (6.3) we have

$$-(2\beta - 1)[(r_m - r)^{\beta-1}\alpha p''(\rho_m) + r_m] = 0.$$

If $p''(\rho_m) = 0$ (the linear case) then we must have $\beta = 1/2$ and we obtain a square root singularity, exactly as in the linear solution given above; the value of α is undetermined. On the other hand, if $p''(\rho_m) \neq 0$ there is still a solution of this form, but now there is a second solution with $\beta = 1$. This solution, whose first derivative will be Lipschitz continuous at the sonic line, has slope α there, with

$$\alpha = -\frac{r_m}{p''(\rho_m)}, \tag{6.4}$$

the value found in [1].

To determine the complete solution U from the solution for ρ, we write the system in conservation form as

$$\widetilde{F}_r + \widetilde{G}_\theta = S\,.$$

Then the condition for a weak solution is $dr/d\theta[\widetilde{G}] = [\widetilde{F}]$. At the sonic line $r =$ const, then, $\widetilde{F}$ is continuous; now

$$\widetilde{F} = (-r\rho + \cos\theta m + \sin\theta n,\ p(\rho)\cos\theta - rm,\ p(\rho)\sin\theta - rn)$$

and so this means that all three components of U are continuous across the sonic line.

References

[1] S. Čanić and B. L. Keyfitz. *Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problems*. Archive for Rational Mechanics and Analysis, **144:** (1998), 233–258.

[2] S. Čanić, B. L. Keyfitz and E. H. Kim. *Free boundary problems for the unsteady transonic small disturbance equation: Transonic regular reflection*. Methods and Applications of Analysis, **7:** (2000), 313–336.

[3] S. Čanić, B. L. Keyfitz and E. H. Kim. *A free boundary problem for a quasilinear degenerate elliptic equation: Regular reflection of weak shocks*. Communications on Pure and Applied Mathematics, **LV:** (2002), 71–92.

[4] S. Čanić, B. L. Keyfitz and E. H. Kim. *The two-dimensional nonlinear wave equation: Global Riemann solutions with noninteracting rarefaction waves*. In preparation, 2001.

[5] S. Čanić, B. L. Keyfitz and E. H. Kim. *Weak shock reflection modeled by the unsteady transonic small disturbance equation*. In H. Freistühler and G. Warnecke, (eds), Proceedings of the Eighth International Conference on Hyperbolic Problems: Theory, Numerics, and Applications. To appear, 2001.

[6] S. Čanić and E. H. Kim. *A class of quasilinear degenerate elliptic equations*. Submitted, 2001.

[7] T. Chang and G.-Q. Chen. *Diffraction of planar shock along compressive corner*. Acta Mathematica Scientia, **3:** (1986), 241–257.

[8] T. Chang, G.-Q. Chen and S. Yang. *On the 2-D Riemann problem for the compressible Euler equations I*. Interaction of shocks and rarefaction waves. Discrete and Continuous Dynamical Systems, **1:** (1995), 555–584.

[9] C. W. Schultz-Rinne, J. P. Collins and H. M. Glaz. *Numerical solution of the Riemann problem for two-dimensional gas dynamics*. SIAM Journal on Scientific Computing, **14:** (1993), 1394–1414.

[10] D. Serre. Remark at 2001 Coloquio.

[11] D. Serre. *Écoulements de fluides parfaits en deux variables indépendentes de type espace.* Réflexion d'un choc plan par un dièdre compressif. Archive for Rational Mechanics and Analysis, **132:** (1995), 15–36.

[12] J. A. Smoller. *Shock Waves and Reaction-Diffusion Equations*. Springer-Verlag, New York, 1983.

[13] T. Zhang and Y.-X. Zheng. *Conjecture on the structure of solutions of the Riemann problem for two-dimensional gas dynamics systems*. SIAM Journal on Mathematical Analysis, **21:** (1990), 593–630.

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