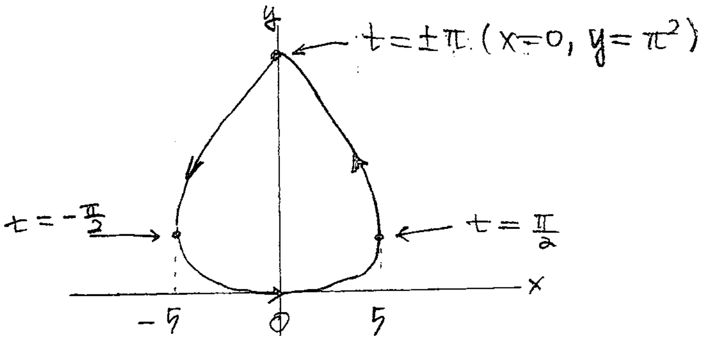
Section 10.1: Curves Defined by Parametric Equations

10.1.3

$$x = 5\sin t, \quad y = t^2 \qquad (-\pi \le t \le \pi)$$

Note: $-5 \le x \le 5$

$0 \le y \le \pi^2$



$$\frac{dx}{dt} = 5\cos t \begin{cases} \le 0 & (-\pi \le t \le -\frac{\pi}{2}) \\ \ge 0 & (-\frac{\pi}{2} \le t \le \frac{\pi}{2}) \\ \le 0 & (\frac{\pi}{2} \le t \le \pi) \end{cases}$$

$$\frac{dy}{dt} = 2t \begin{cases} \le 0 & (-\pi \le t \le 0) \\ \ge 0 & (0 \le t \le \pi) \end{cases}$$

Note: x increases when $\frac{dx}{dt} \ge 0$

x decreases when $\frac{dx}{dt} \le 0$

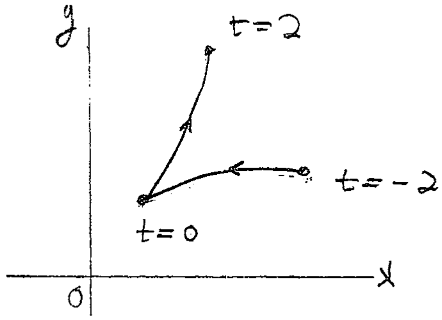
Likewise for y

10.1.4

$$x = e^{-t} + t, \quad y = e^t - t \qquad (-2 \le t \le 2)$$

$$\frac{dx}{dt} = -e^{-t} + 1 \begin{cases} \ge 0 & (t \ge 0) \\ \le 0 & (t \le 0) \end{cases}$$

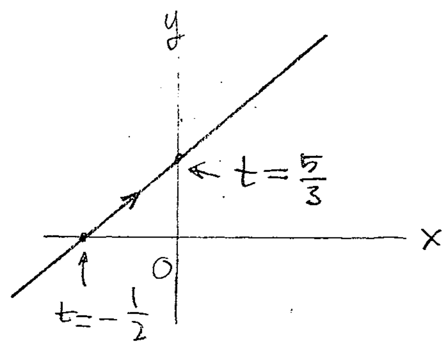
$$\frac{dy}{dt} = e^t - 1 \begin{cases} \ge 0 & (t \ge 0) \\ \le 0 & (t \le 0) \end{cases}$$



10.1.5

$$x = 3t - 5 \longrightarrow t = \tfrac{1}{3}(x+5)$$

$$y = 2t + 1 \longrightarrow y = \tfrac{2}{3}(x+5) + 1 = \tfrac{2}{3}x + \tfrac{13}{3}$$

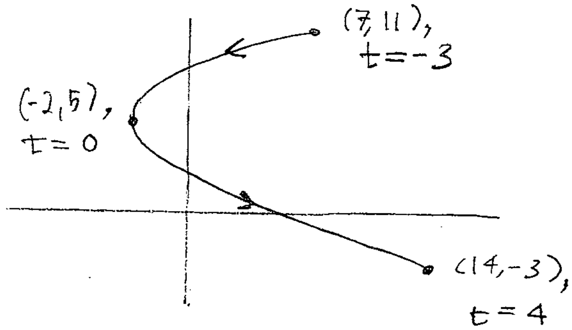


10.1.7

$$x = t^2 - 2, \quad y = 5 - 2t \qquad (-3 \le t \le 4)$$

$$\begin{cases} t = \dfrac{5-y}{2} \\ x = \left(\dfrac{5-y}{2}\right)^2 - 2 \\ \quad = \dfrac{1}{4}(y-5)^2 - 2 \end{cases}$$

Also
$$-3 \le t \le 4$$
$$-6 \le 2t \le 8$$
$$-8 \le -2t \le 6$$
$$-3 \le 5 - 2t \le 11$$
$$-3 \le y \le 11$$



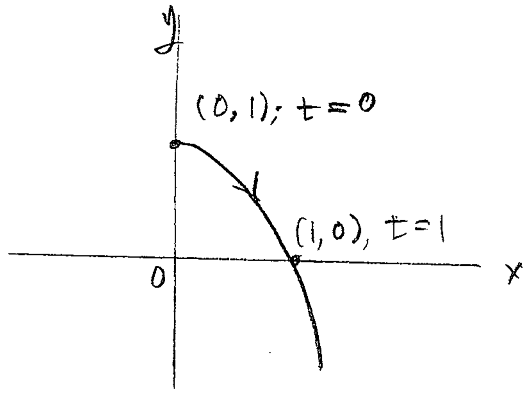
10.1.9

$x = \sqrt{t}$ ← $t \geq 0$ by assumption

$y = 1 - t$

$= 1 - (\sqrt{t})^2$

$= 1 - x^2$

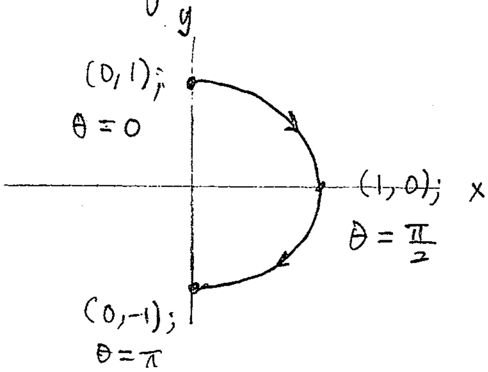


10.1.11

$x = \sin\theta$, $y = \cos\theta$ $\quad (0 \leq \theta \leq \pi)$

$x^2 + y^2 = \sin^2\theta + \cos^2\theta$

$= 1$, so the curve must be part of the unit circle centered at the origin



$$\frac{dx}{d\theta} = \cos\theta \begin{cases} \geq 0 & (0 \leq \theta \leq \frac{\pi}{2}) \\ \leq 0 & (\frac{\pi}{2} \leq \theta \leq \pi) \end{cases}$$

$$\frac{dy}{d\theta} = -\sin\theta \leq 0 \quad (0 \leq \theta \leq \pi)$$

10.1.13

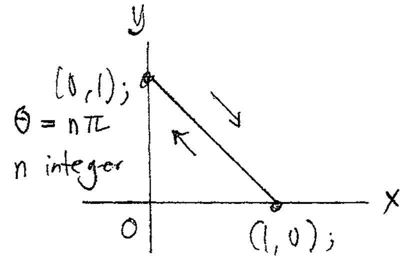
$x = \sin^2\theta$

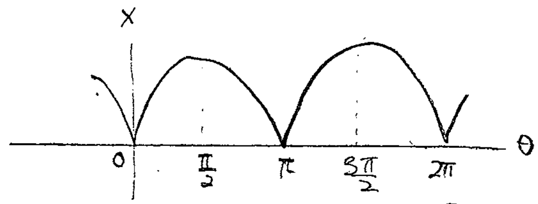
$y = \cos^2\theta$

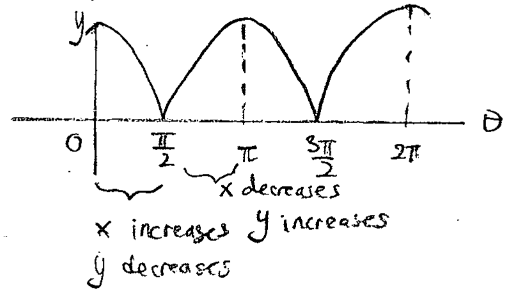
Note: $0 \leq x \leq 1$

$0 \leq y \leq 1$

$x + y = 1$







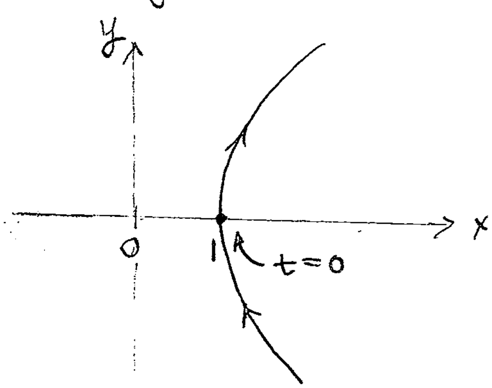
10.1.17

$x = \cosh t \overset{\text{def}}{=} \dfrac{e^t + e^{-t}}{2}$

$y = \sinh t = \dfrac{e^t - e^{-t}}{2}$

An identity: $\cosh^2 t - \sinh^2 t = 1$

So $x^2 - y^2 = 1$



Note: $x = \dfrac{e^t + e^{-t}}{2} \geq \dfrac{1+1}{2} = 1$

$\dfrac{dy}{dt} = \cosh t \geq 1$

(y is always increasing)

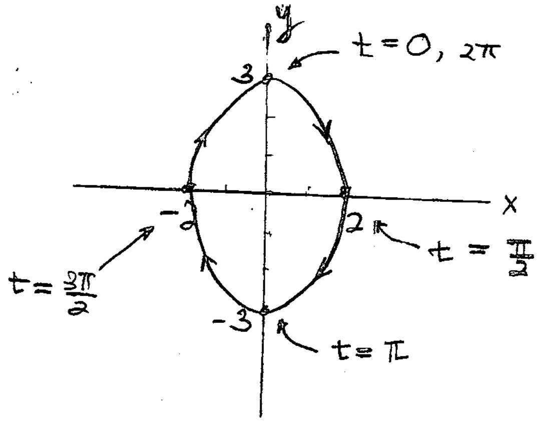
10.1.21 $x = 2\sin t$, $y = 3\cos t$, $0 \le t \le 2\pi$

Note: $-2 \le x \le 2$

$-3 \le y \le 3$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \sin^2 t + \cos^2 t$$

$$= 1$$



The particle moves once clockwise along the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ starting and ending at $(0, 3)$.

10.1.24

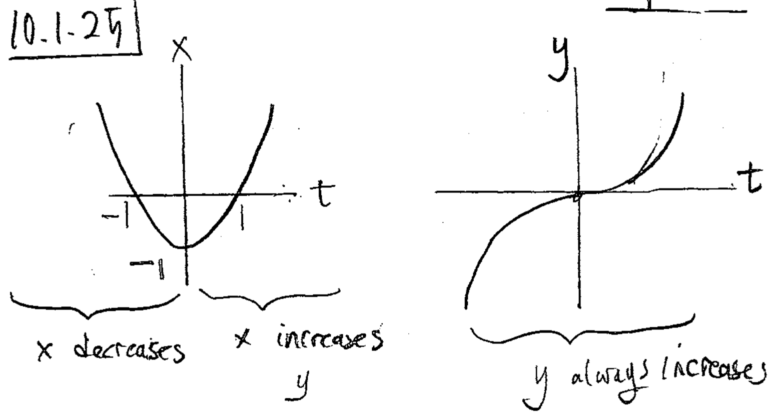
(a) $1 \le x \le 2 \longrightarrow$ the only option is III

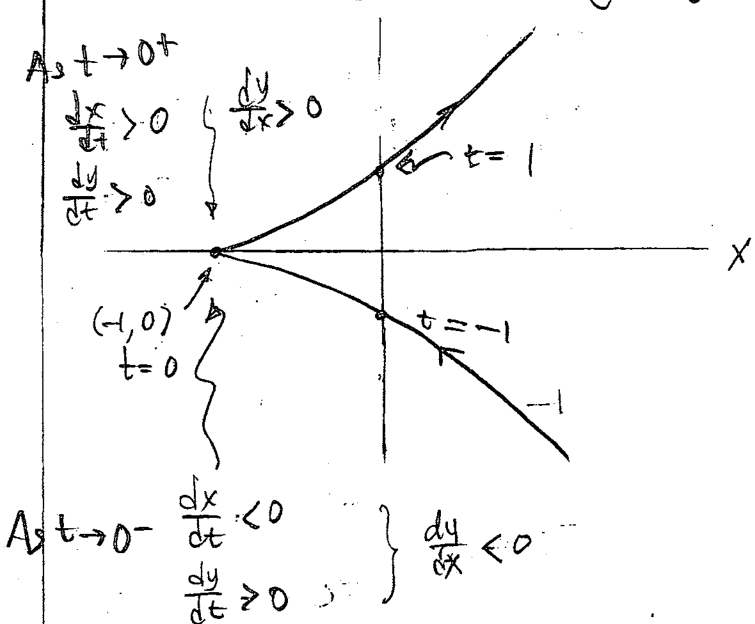
(c) $0 \le y \le 2 \longrightarrow$ the only option is IV

(b) x cycles through the values from -2 to 2 two times, while y cycles through the same range of values three times $\longrightarrow$ I

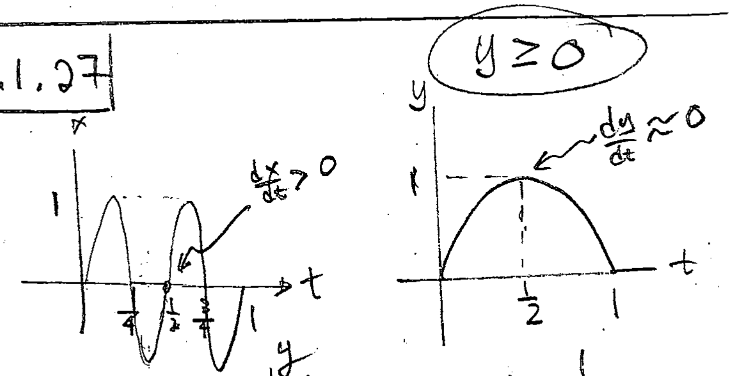
(d) While y decreases, x assumes the value 2 over an interval of the parameter values $\longrightarrow$ II

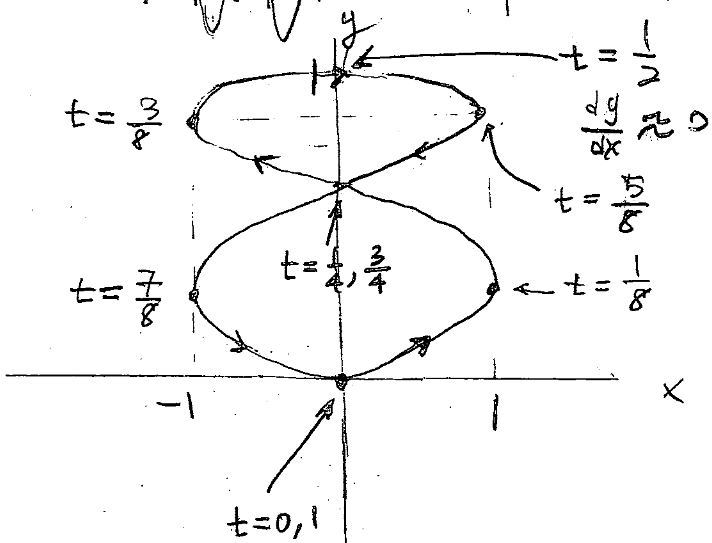
10.1.25





10.1.27





10.1.28

(c) $x = \sin 3t$, $y = \sin 4t$

① : $-1 \le x \le 1$, $-1 \le y \le 1$

② When $t = 0$, $x = y = 0$
(the point $(0,0)$ must be included)

② rules out I, II, VI

① rules out III, IV

So (c) $\longrightarrow$ V

(d) $x = t + \sin 2t$, $y = t + \sin 3t$

③ $x \xrightarrow{t\to\infty} +\infty$, $y \xrightarrow{t\to\infty} +\infty$;
this rules out I, II, VI

④ $x \xrightarrow{t\to-\infty} -\infty$, $y \xrightarrow{t\to-\infty} -\infty$;
this rules out IV.

So (d) $\longrightarrow$ III

(a) $x = t^3 - 2t$, $y = t^2 - t$

⑤ $x \xrightarrow{t\to\infty} +\infty$, $y \xrightarrow{t\to\infty} +\infty$;
this rules out I, II, VI.

So (a) $\longrightarrow$ IV

(b) $x = t^3 - 1$, $y = 2 - t^2$

⑥ : $x \xrightarrow{t\to\infty} +\infty$, $y \xrightarrow{t\to\infty} -\infty$;
this rules out I, II

So (b) $\longrightarrow$ VI

(f) $x = \cos t$
$y = \sin(t + 5\sin 5t)$

⑦ : $x(-t) = \cos(-t) = x(t)$

$$\begin{aligned} y(-t) &= \sin[-t + \sin 5(-t)] \\ &= \sin(-t - \sin 5t) \\ &= -\sin(t + \sin 5t) \\ &= -y(t) \end{aligned}$$

So the graph is symmetric about the x-axis; this rules out I.

(f) $\longrightarrow$ II

(e) $\longrightarrow$ I

We can also consider the intercepts, i.e. where either $x = 0$ or $y = 0$.

10.1.31 $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$ $\quad (0 \le t \le 1)$

Clearly, $x_1 \le x \le x_1 + (x_2 - x_1) = x_2$

$y_1 \le y \le y_1 + (y_2 - y_1) = y_2$

When $t = 0$, $x = x_1$, $y = y_1$

When $t = 1$, $x = x_2$, $y = y_2$

Furthermore, it can be shown that

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1,$$

a line with slope $\frac{y_2 - y_1}{x_2 - x_1}$.

So the parametric eqns is the line segment that joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

10.1.33 $x^2 + (y-1)^2 = 4$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$$

Using the fact that

$\cos^2 t + \sin^2 t = 1$,

we can set

$$\left.\begin{aligned} \frac{x}{2} &= \cos t \\ \frac{y-1}{2} &= \sin t \end{aligned}\right\} \quad \left.\begin{aligned} x &= 2\cos t \\ y &= 2\sin t + 1 \end{aligned}\right\} (*)$$

By restricting t to $0 \le t \le 2\pi$, this gives the representation of the circle with a counterclockwise orientation starting at $(2, 1)$.

Clearly, the circle is traced out only once as t varies from 0 to 2π.

(a) To trace the circle clockwise, use the change of parameter $\tau = -t$:

$$x = 2\cos(-\tau) = 2\cos\tau$$

$$y = 2\sin(-\tau) + 1 = 1 - 2\sin\tau$$

$$\underbrace{0 \le t \le 2\pi}$$

$$0 \le -\tau \le 2\pi$$

$$-2\pi \le \tau \le 0$$

Note that both sine and cosine have period 2π. So the parameter interval can also be taken as

$$0 \le \tau \le 2\pi$$

(b) To trace the circle counterclockwise three times, we need only to increase the size of the parameter interval in $(*)$ three times:

$$x = 2\cos t$$

$$y = 2\sin t + 1$$

$$(0 \le t \le 6\pi)$$

Alternatively,

Note that

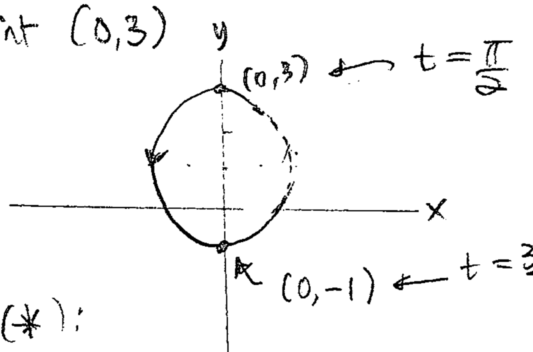
$$\underbrace{0 \le t \le 6\pi}$$

$$0 \le \frac{t}{3} \le 2\pi$$

So we can use the change of parameter $\tau = \frac{t}{3}$:

$$x = 2\cos 3\tau$$
$$y = 2\sin 3\tau + 1$$
$$0 \le \tau \le 2\pi$$

(c) Warning: Now we have to start at the point (0,3)



Consider(*):

$x=0 \to 2\cos t = 0 \to t = \frac{\pi}{2}, \frac{3\pi}{2}$

$y=3 \to 2\sin t = 2 \to t = \frac{\pi}{2}$

$y=-1 \to 2\sin t = -2 \to t = \frac{3\pi}{2}$

So to trace the circle halfway counter-clockwise, starting at (0,3), the parametrization is

$$x = 2\cos t$$
$$y = 2\sin t + 1$$
$$\left(\frac{\pi}{2} \le t \le \frac{3\pi}{2}\right)$$

$\boxed{10.1.35}$

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Use the identity $\cos^2\theta + \sin^2\theta = 1$.

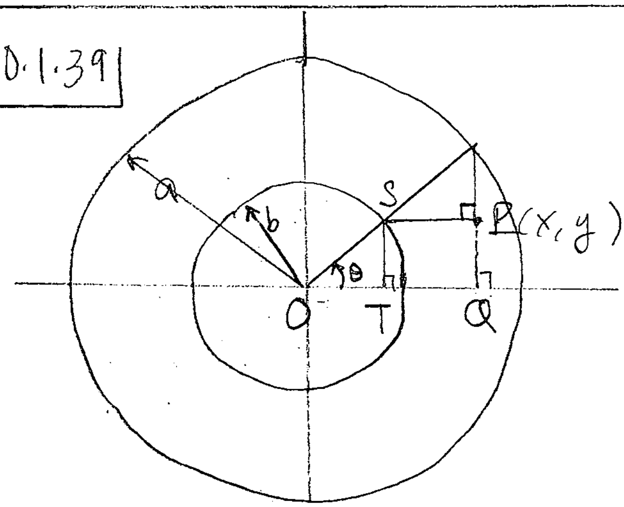
Set $\frac{x}{a} = \sin\theta \to x = a\sin\theta$

$\frac{y}{b} = \cos\theta \to y = b\cos\theta$

$$0 \le \theta \le 2\pi$$

(c) As b increases, the ellipse stretches vertically.

$\boxed{10.1.39}$



Note $x = |OQ|$

$y = |QP| = |ST|$

Then $x = a\cos\theta$

$y = b\sin\theta$

But $\cos^2\theta + \sin^2\theta = 1$

So $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$;

the possible positions of P form an ellipse.

$\boxed{\text{No: 1.45}}$ $x = t^2 \leftarrow$ this implies that $x \geq 0$

$y = t^3 - ct$

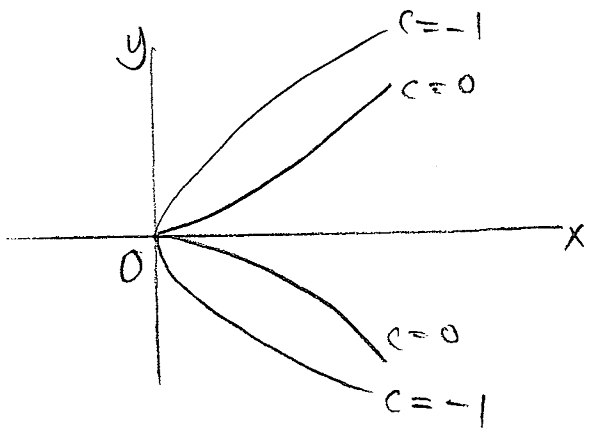
$= t(t^2 - c)$

With $t = \pm\sqrt{x}$, the graph can be split into two parts:

$$\left\{\begin{array}{l} y_1 = \sqrt{x}(x-c) \\ y_2 = -\sqrt{x}(x-c) \end{array}\right\} \text{ symmetric about the } x\text{-axis}$$

Note that since $x \geq 0$, if $c \leq 0$,

then $\left.\begin{array}{l} y_1 \geq 0 \\ y_2 \leq 0 \end{array}\right\}$ never intersect except at $(0,0)$.



If $c > 0$,

$y_1 = y_2$ when

$\sqrt{x}(x-c) = -\sqrt{x}(x-c)$

$\sqrt{x}(x-c) = 0$

$x = 0$ or $x = c$.

y_1 and y_2 intersect at $(0,0)$ and at $(c, 0)$