The integration of $\int \frac{\sqrt{u^2+1}}{u^2}\,du$ pertaining to 10.2.43

We need the following facts:

Definitions: $\cosh w \overset{\text{def}}{=} \frac{e^w+e^{-w}}{2}$, $\sinh w \overset{\text{def}}{=} \frac{e^w-e^{-w}}{2}$

Identity: $\cosh^2 w - \sinh^2 w = 1$

$\coth^2 w - 1 = \operatorname{csch}^2 w$ $\left(\coth w = \frac{\cosh w}{\sinh w}, \operatorname{csch} w = \frac{1}{\sinh w}\right)$

Derivatives: $\frac{d}{dw}\cosh w = \sinh w$

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Now set $u = \sinh w$.

Then $u^2+1 = \sinh^2 w + 1$

$= \cosh^2 w$;

$\sqrt{u^2+1} = \sqrt{\cosh^2 w}$

$= |\cosh w|$

$= \cosh w$

because $\cosh w > 0$;

$du = \cosh w\, dw$

So $\int \frac{\sqrt{u^2+1}}{u^2}\,du$

$= \int \frac{\cosh w}{\sinh^2 w} \underbrace{\cosh w\, dw}_{du}$

$= \int \frac{\cosh^2 w}{\sinh^2 w}\,dw$

$= \int \frac{\sinh^2 w + 1}{\sinh^2 w}\,dw$

$= \int (1 + \operatorname{csch}^2 w)\,dw$

$= w - \coth w + C$

$= w - \frac{\cosh w}{\sinh w} + C$

$= w - \frac{\sqrt{u^2+1}}{u} + C$

$\int \operatorname{csch}^2 w\,dw \overset{(110)}{=} \underbrace{-\coth w + C}$

This can be verified by differentiating $\coth w$ with respect to w.

Observe the similarity to $\int \csc^2 w\,dw = -\cot w + C$

To write w in terms of u, note that

$u = \sinh w = \frac{e^w - e^{-w}}{2}$

$\Rightarrow e^w - 2u - e^{-w} = 0$

$\Rightarrow (e^w)^2 - 2u e^w - 1 = 0$

$\overset{x=e^w}{\Longrightarrow} x^2 - 2ux - 1 = 0$ ← quadratic in x

$\Rightarrow x = \frac{2u \pm \sqrt{4u^2+4}}{2}$

$e^w = u + \sqrt{u^2+1}$ ← $e^w > 0$

So $w = \ln\left|u + \sqrt{u^2+1}\right|$