Section 10.2: Calculus with Parametric Curves

p. 660 If $x = f(t)$, $y = g(t) = F(f(t))$, then

$$\boxed{F'(x) = \frac{g'(t)}{f'(t)}}$$

provided $f'(t) \neq 0$. Equivalently,

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

provided $dx/dt \neq 0$.

10.2.1 $x = t - t^3$

$y = 2 - 5t$

$$\frac{dx}{dt} = 1 - 3t^2$$

$$\frac{dy}{dt} = -5$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{5}{3t^2 - 1}$$

10.2.5 $x = e^{\sqrt{t}}$,

$y = t - \ln t^2$; $t = 1$

$$dx/dt = e^{\sqrt{t}}/(2\sqrt{t}) \overset{t=1}{=\!=} e/2$$

[Also, since we're only interested in the part of the curve near $t = 1 > 0$, we can take $t > 0$ and write

$$y = t - 2\ln t.$$

So, $dy/dt = 1 - \frac{2}{t} \overset{t=1}{=\!=} -1$

The slope of the tangent to the curve at the point $(x, y) \overset{t=1}{=\!=} (e, 1)$ is thus

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \overset{t=1}{=\!=} \frac{-2}{e}$$

Hence an equation of the tangent is

$$y - 1 = -\frac{2}{e}(x - e)$$

10.2.7 $x = e^t$, $y = (t-1)^2$; $(1,1)$

Approach #1: At $(x, y) = (1,1)$,

$$\left.\begin{array}{r} e^t = 1 \\ (t-1)^2 = 1 \end{array}\right\} \; t = 0$$

$$dy/dt = 2(t-1) \overset{t=0}{=\!=} -2$$

$$dx/dt = e^t \overset{t=0}{=\!=} 1$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -2$$

So the tangent is given by

$$y - 1 = -2(x-1)$$

Approach #2

From $x = e^t$, we have

$$t = \ln x.$$

So

$$y = (\ln x - 1)^2$$

$$\frac{dy}{dx} = 2(\ln x - 1)\frac{1}{x} \overset{x=1}{=} -2$$

Hence we obtain the same equation for the tangent

$\boxed{10.2.11}$ $x = 4 + t^2$, $y = t^2 + t^3$;

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2t + 3t^2;$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+3t^2}{2t} = 1 + \frac{3}{2}t;$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3}{2};$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$$

The curve is concave upward when

$$\frac{d^2y}{dx^2} = \frac{3}{4t} > 0$$

or $t > 0$

$\boxed{10.2.15}$ $x = 2\sin t$, $y = 3\cos t$, $0 < t < 2\pi$;

$$\frac{dx}{dt} = 2\cos t, \quad \frac{dy}{dt} = -3\sin t;$$

$$\frac{dy}{dx} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t;$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\frac{3}{2}\sec^2 t$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = -\frac{3}{4}\sec^3 t$$

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The curve is concave upward when

$$\frac{d^2y}{dx^2} = \frac{-3}{4}\sec^3 t > 0$$

$\Leftrightarrow \sec^3 t < 0$

$\Leftrightarrow \cos^3 t < 0$

$\Leftrightarrow \frac{\pi}{2} < t < \frac{3\pi}{2}$

$\boxed{10.2.17}$ $x = 10 - t^2,$

$y = t^3 - 12t;$

$$\frac{dx}{dt} = -2t,$$

$$\frac{dy}{dt} = 3t^2 - 12;$$

The tangents are horizontal when

$$\frac{dy}{dt} = 0$$

$\Leftrightarrow 3t^2 - 12 = 0$

$\Leftrightarrow t = \pm 2$

$\Leftrightarrow x \overset{t=+2}{=\!=} 6,\ y \overset{t=+2}{=} -4$

or $x \overset{t=-2}{=\!=} 6,\ y \overset{t=-2}{=} +4$

$\Leftrightarrow (x,y) = (6,-4)$ or $(6,+4)$

Actually, need to check that $\left.\frac{dx}{dt}\right|_{t=\pm 2} = \mp 4 \neq 0$

The tangent is vertical when

$$\frac{dx}{dt} = 0$$

$\Leftrightarrow -2t = 0 \Leftrightarrow t = 0$

$\Leftrightarrow (x,y) \overset{t=0}{=\!=} (10, 0)$

Need to verify that $\left.\frac{dy}{dt}\right|_{t=0} = -12 \neq 0$

$\boxed{10.2.25}$ $x = \cos t,\ y = \sin t \cos t$

$= \frac{1}{2}\sin 2t;$

$$\frac{dx}{dt} = -\sin t,$$

$$\frac{dy}{dt} = \cos 2t;$$

At the point $(0,0)$, $\cos t = 0$ and thus

$$t = t_n = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

<u>Case n is even:</u>

$$\frac{dx}{dt} \overset{t=t_n}{=\!=} -\sin\frac{\pi}{2} = -1$$

$$\frac{dy}{dt} \overset{t=t_n}{=\!=} \cos 2\left(\frac{\pi}{2}\right) = -1$$

$$\frac{dy}{dx} \overset{t=t_n}{=\!=} \frac{-1}{-1} = 1$$

The corresponding tangent is given by

$$y - 0 = 1\,(x - 0)$$

or $y = x$.

<u>Case n is odd:</u>

$$\frac{dx}{dt} \overset{t=t_n}{=\!=} -\sin\left(\frac{\pi}{2} + \pi\right) = 1$$

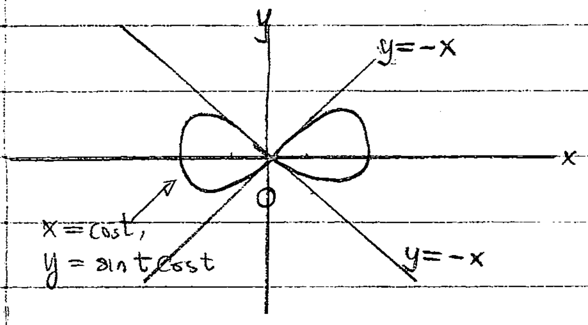
$$\frac{dy}{dt} \overset{t=t_n}{=\!=} \cos 2\left(\frac{\pi}{2} + \pi\right) = -1$$

$$\frac{dy}{dx} \overset{t=t_n}{=\!=} \frac{-1}{1} = -1$$

The corresponding tangent is given by

$$y - 0 = -1\,(x - 0)$$

or $y = -x$



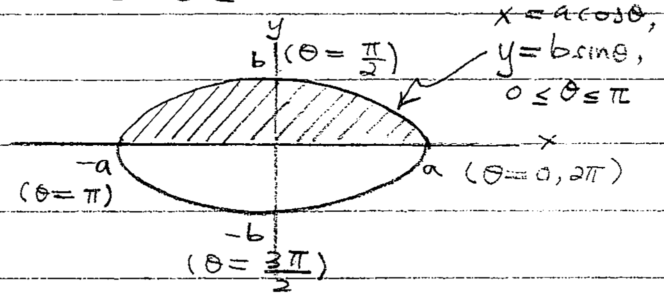
Exercise: Show that the parametric curve is symmetric about the x-axis as well as about the y-axis.

p. 662

If $x = f(t)$, $y = g(t) = F(f(t)) \geq 0$, and if the corresponding curve is traversed once as t increases from α to β, then the area under the curve $y = F(x)$ from $a = f(\alpha)$ to $b = f(\beta)$ is given by

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$$

$\boxed{10.2.31}$ $x = a\cos\theta$, $y = b\sin\theta$, $0 \leq \theta \leq 2\pi$



Area of the shaded region

$$= \int_{-a}^{a} \underset{y \geq 0}{y}\,dx$$

$$= \int_\pi^0 b\sin\theta\underbrace{(-a\sin\theta)d\theta}_{\frac{dx}{d\theta}d\theta}$$

$$= \int_0^\pi ab\sin^2\theta\,d\theta$$

$$= ab\underbrace{\int_0^\pi \sin^2\theta\,d\theta}$$

area under the graph of $\sin^2\theta$ (as a function of θ) from $\theta = 0$ to $\theta = \pi$; note that this is the same to the area under the graph of $\cos^2\theta$ from $\theta = 0$ to $\theta = \pi$.

$$= \frac{\pi}{2}ab$$

(See the following argument)

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$$\int_0^\pi \sin^2\theta \, d\theta = \int_0^\pi \cos^2\theta \, d\theta \quad \text{— (1)}$$

$$\int_0^\pi \sin^2\theta + \cos^2\theta \, d\theta = \int_0^\pi 1 \, d\theta = \pi \quad \text{— (2)}$$

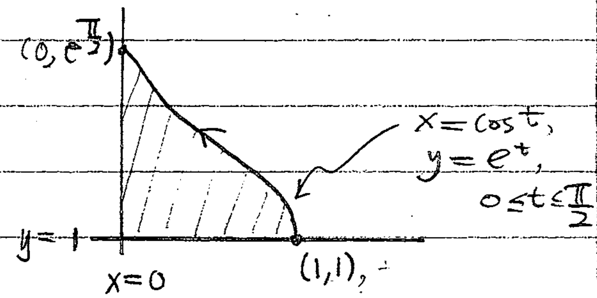
Using (1) & (2) we get

$$\int_0^\pi \sin^2\theta \, d\theta = \frac{\pi}{2}$$

So the area of the ellipse is

$$2\left(\frac{\pi}{2}ab\right) = \pi ab$$

* $\boxed{10.2.33}$



The area of the shaded region is given by

$$A = \int_0^1 (y-1)\, dx$$

$$= \int_{\frac{\pi}{2}}^0 (e^t - 1)\underbrace{(-\sin t)\, dt}_{\frac{dx}{dt}dt}$$

$$= \int_0^{\frac{\pi}{2}} (e^t \sin t - \sin t)\, dt$$

$$= \left[\frac{e^t(\sin t - \cos t)}{2} + \cos t\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}(e^{\pi/2} - 1)$$

p. 664

If $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, and f' and g' are continuous, and if the corresponding curve C is traversed precisely once as t increases from α to β, then the length of C is

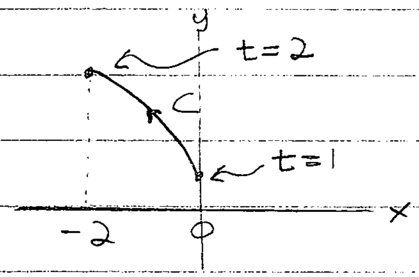
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\boxed{10.2.37}$ $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$

$$\cancel{\frac{dx}{dt} = 1 - 2t < 0 \text{ for } 1 \le t \le 2}$$

$$\frac{dy}{dt} = 2t^{1/2} > 0 \text{ for } 1 \le t \le 2$$

So the curve C is traversed exactly once as t varies from 1 to 2



The length of C is

$$L = \int_1^2 \sqrt{(1-2t)^2 + (2t^{1/2})^2}\, dt$$

$\boxed{10.2.41}$ $x = 1+3t^2,\ y = 4+2t^3,$
$0 \le t \le 1;$

$$\left.\begin{aligned}\frac{dx}{dt} &= 6t > 0\\ \frac{dy}{dt} &= 6t^2 > 0\end{aligned}\right\}\text{ for } t > 0$$

So the corresponding curve is traversed precisely once as t varies from $t=0$ to $t=1$. The length of the curve is

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\,dt$$
$$= 6\int_0^1 \sqrt{t^2+t^4}\,dt$$
$$= 6\int_0^1 t\sqrt{1+t^2}\,dt$$
$$\overset{u=1+t^2}{=\!=} 6\int_1^2 \sqrt{u}\,\frac{du}{2}$$
$$= 2\,(2\sqrt{2}-1)$$

$\boxed{10.2.43}$ $x = \dfrac{t}{1+t},\ y = \ln(1+t),$
$0 \le t \le 2$

$$\left.\begin{aligned}\frac{dx}{dt} &= \frac{1}{(1+t)^2} > 0\\ \frac{dy}{dt} &= \frac{1}{1+t} > 0\end{aligned}\right\}\text{ for } 0 \le t \le 2$$

So the curve is traversely exactly once as t varies from $t=0$ to $t=2$, and the length of the curve is

$$L = \int_0^2 \sqrt{\frac{1}{(1+t)^4} + \frac{1}{(1+t)^2}}\,dt$$
$$= \int_0^2 \sqrt{\frac{t^2+2t+2}{(1+t)^4}}\,dt$$
$$= \int_0^2 \frac{\sqrt{t^2+2t+2}}{(1+t)^2}\,dt$$
$$\overset{u=1+t}{=\!=} \int_1^3 \frac{\sqrt{u^2+1}}{u^2}\,du$$
$$= \left[-\frac{\sqrt{u^2+1}}{u} + \ln\left(u+\sqrt{u^2+1}\right)\right]_{u=1}^3$$

formula (24) on the table of integrals

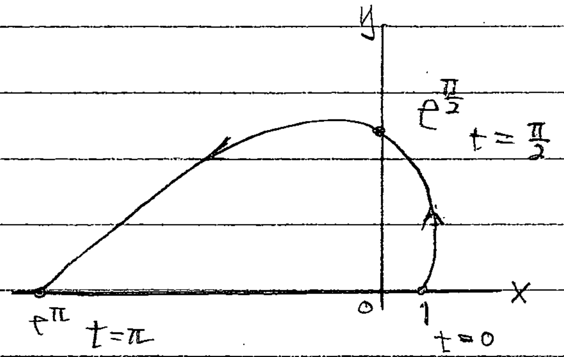
$$= -\frac{\sqrt{10}}{3} + \ln(3+\sqrt{10}) + \sqrt{2} - \ln(1+\sqrt{2})$$

10.2.45 $x = e^t \cos t,\ y = e^t \sin t,$

$0 \le t \le \pi$

$x^2 + y^2 = e^{2t}(\cos^2 t + \sin^2 t)$

$= e^t$; this shows that the distance of the point (x, y) from the origin increases with t exponentially. In particular, the curve is traversely precisely once as t varies from 0 to π.



$$\frac{dx}{dt} = e^t(\cos t - \sin t)$$

$$\frac{dy}{dt} = e^t(\sin t + \cos t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}$$

The length of the curve is

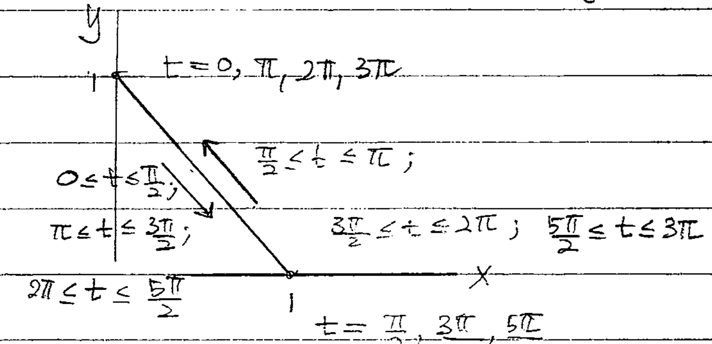
$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{2}\int_0^{\pi} e^t\, dt$$

$$= \sqrt{2}\,(e^{\pi} - 1)$$

10.2.51 * $x = \sin^2 t,\ y = \cos^2 t$

$0 \le t \le 3\pi$ Note: $x + y = 1$



The distance traveled

$= 6 \times$ the length of the curve

$= 6\sqrt{1^2 + 1^2}$

$= 6\sqrt{2}$

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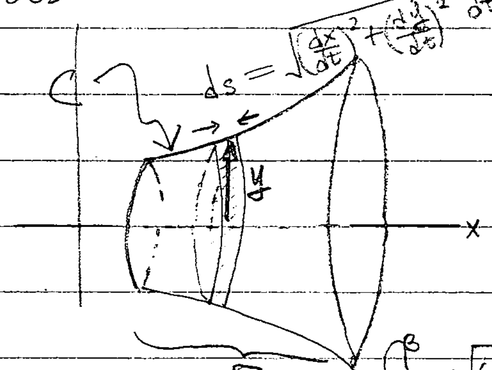
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

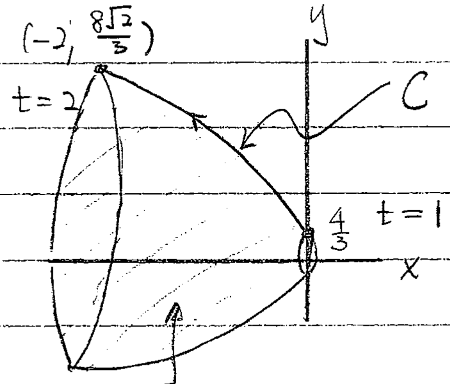


$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If the curve C : $x = f(t),\ y = g(t),\ \alpha \le t \le \beta$, is rotated about the x-axis, f' and g' are continuous, and $g(t) \ge 0$, then the resulting surface has an area S.

$\boxed{10.2.57}$ $x = t - t^2$, $y = \frac{4}{3} t^{3/2}$,

$1 \le t \le 2$



$$S = \int_1^2 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

But $\frac{dx}{dt} = 1 - 2t$

$\frac{dy}{dt} = 4t^{\frac{1}{2}}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 - 4t + 4t^2 + 4t$$

$$= 1 + 4t^2$$

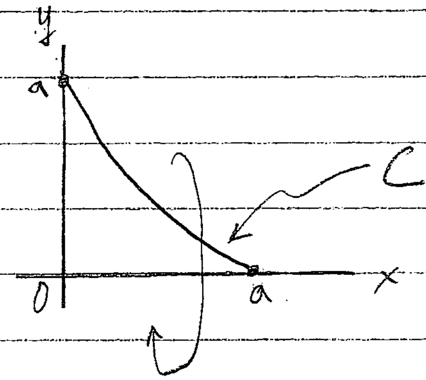
So $S = 2\pi \int_1^2 \frac{4}{3} t^{\frac{3}{2}} \sqrt{1 + 4t^2}\, dt$

$\boxed{10.2.61}$ $x = a\cos^3\theta$,

$y = a\sin^3\theta$,

$0 \le \theta \le \frac{\pi}{2}$

We assume $a > 0$.



$\frac{dx}{d\theta} = -3a\cos^2\theta \sin\theta$

$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{9a^2(\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)}$$

$$= \sqrt{9a^2\cos^2\theta\sin^2\theta}$$

$$= 3a\cos\theta\sin\theta$$

For $0 \le \theta \le \frac{\pi}{2}$

So the surface generated by rotating C about the x-axis has an area

$$S = 2\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta \cdot 3a\cos\theta\sin\theta\, d\theta$$

$$= 6\pi a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\, d\theta$$

$u = \sin\theta$

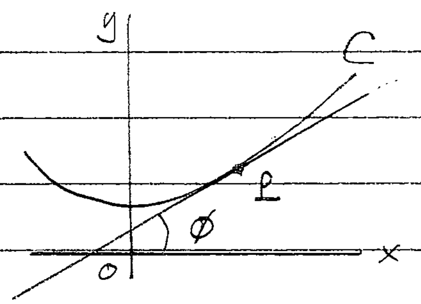
$$= 6\pi a^2 \int_0^1 u^4\, du$$

$$= \frac{6\pi a^2}{5}$$

10.2.69 The curvature at a point P of a curve is defined as

$$\kappa = \left| d\phi/ds \right|,$$

where ϕ is the angle of inclination of the tangent line at P:



(a) For a parametric curve C: $x = x(t)$, $y = y(t)$,

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

where $(\ )^{\cdot} = \frac{d}{dt}(\ )$.

Proof. Since $\phi = \tan^{-1}\left(\frac{dy}{dx}\right)$,

$$\frac{d\phi}{dt} = \frac{1}{1+\left(\frac{dy}{dx}\right)^2} \frac{d}{dt}\left(\frac{dy}{dx}\right) \quad \text{—— (1)}$$

Also,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)$$

$$= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} \quad \text{—— (2)}$$

So,

$$\frac{d\phi}{dt} = \frac{1}{1+(\dot{y}/\dot{x})^2} \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}$$

$$= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} \quad \text{—— (3)}$$

On the other hand,

$$s = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2}\, dt;$$

so that

$$\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2} \quad \text{—— (4)}$$

In consequence,

$$\left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{dt}\frac{dt}{ds}\right|$$

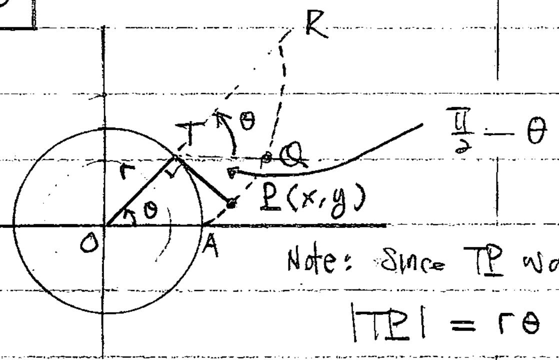
$$= \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

(b) If the curve has the parametrization $x = x$, $y = f(x)$, then

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}.$$

Proof. Set $t = x$ in part (a)

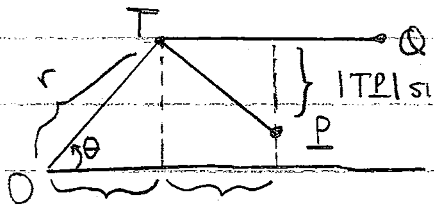
10.2.73



Note: Since $\underline{TP}$ was unwound from arc TA,

$$|T\underline{P}| = r\theta$$

$$T = (r\cos\theta,\ r\sin\theta)$$



$$|T\underline{P}| \sin \measuredangle \underline{P}TQ = r\theta \sin\left(\frac{\pi}{2}-\theta\right) = r\theta\cos\theta$$

$$\text{so } y = r(\sin\theta - \theta\cos\theta)$$

$r\cos\theta$ $\quad |T\underline{P}| \cos \measuredangle \underline{P}TQ$

$$= r\theta \cos \measuredangle \underline{P}TQ$$

$$= r\theta\cos\left(\frac{\pi}{2}-\theta\right)$$

$$= r\theta\sin\theta$$

$$\text{So } x = r(\cos\theta + \theta\sin\theta)$$