Page 1

Section 10.3 : Polar Coordinates

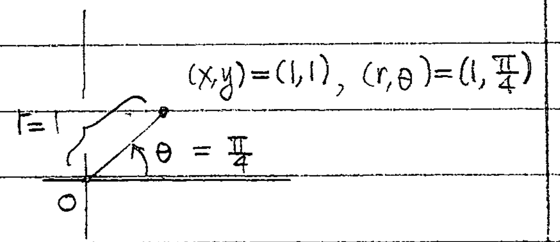
10.3.5 (a) $(x,y) = (1,1)$

$x = r\cos\theta$

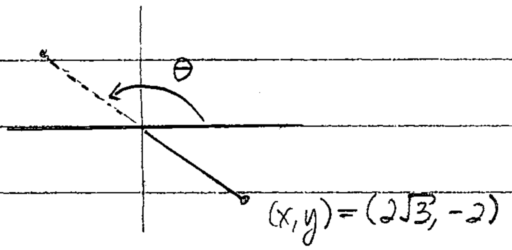
$y = r\sin\theta$

$r = \sqrt{x^2+y^2} = 1$

$\tan\theta = \dfrac{y}{x} = 1$



10.3.5(b) $(x, y) = (2\sqrt{3}, -2)$



$r = -\sqrt{x^2+y^2} = -4$

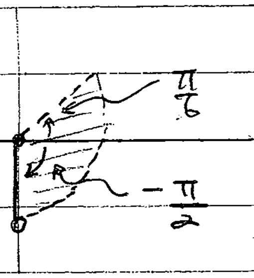
$\theta = \pi - \tan^{-1}(2\sqrt{3}/2)$

$= \pi - \tan^{-1}\sqrt{3}$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

10.3.9 $0 \le r < 4, \; -\frac{\pi}{2} \le \theta < \frac{\pi}{6}$



10.3.17 $r = 3\sin\theta$

$r^2 = 3r\sin\theta$

$x^2 + y^2 = 3y$

$x^2 + y^2 - 3y = 0$

$x^2 + (y-\frac{3}{2})^2 - (\frac{3}{2})^2 = 0$

$x^2 + (y-\frac{3}{2})^2 = (\frac{3}{2})^2$

The circle centered at $(x,y) = (0, \frac{3}{2})$ with radius $= \frac{3}{2}$

10.3.19 $r = \csc\theta$ ← Note that this is not well-defined when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

$r\sin\theta = 1$

$y = 1$

10.3.21 $x = 3$

$r\cos\theta = 3$

$r = 3\sec\theta$

* $\boxed{10.3.25:}$ $x^2+y^2 = 2cx$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2cr\cos\theta$$

$$r^2 = 2cr\cos\theta$$

$$r(r - 2c\cos\theta) = 0$$

$$r = 0 \quad \text{or} \quad r = 2c\cos\theta$$

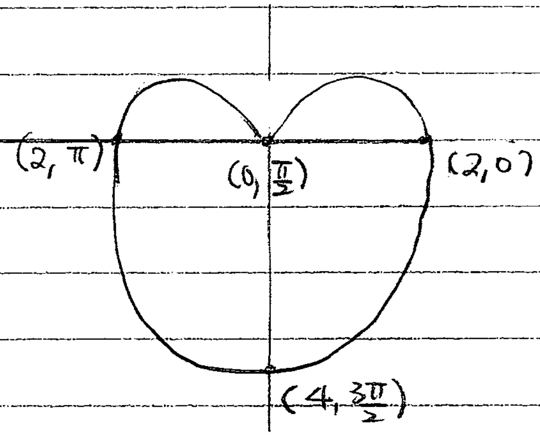
Note that the point $r=0$ is also included in the curve $r = 2c\cos\theta$, say when $\theta = \frac{\pi}{2}$.

So the curve $x^2+y^2 = 2cx$ can be described by $r = 2c\cos\theta$ completely.

Note that $r = 2(1-\sin\theta)$

(i) is increasing on $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$;

(ii) is decreasing on $0 \le \theta \le \frac{\pi}{2}$ and on $\frac{3\pi}{2} \le \theta \le 2\pi$.

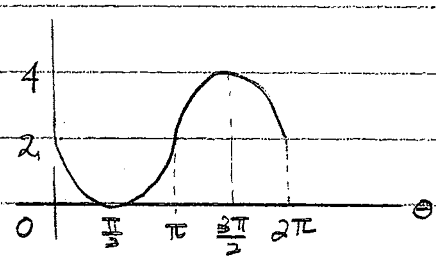


* $\boxed{10.3.33}$ $r = 2(1-\sin\theta)$

Observations:

(1) r has a period of 2π in θ; so, the curve is closed

(2) Consider the graph of the function $2(1-\sin\theta)$:



(3) Replacing θ by $\pi - \theta$:

$$r = 2[1 - \sin(\pi - \theta)]$$

$$= 2(1 - \sin\theta)$$

because $\sin(\pi-\theta) = \underbrace{\sin\pi}_{0}\cos\theta - \underbrace{\cos\pi}_{-1}\sin\theta$

$$= \sin\theta$$

So the curve is symmetric about the line $\theta = \frac{\pi}{2}$.

10.3.37 $r = \sin 2\theta$

observations:

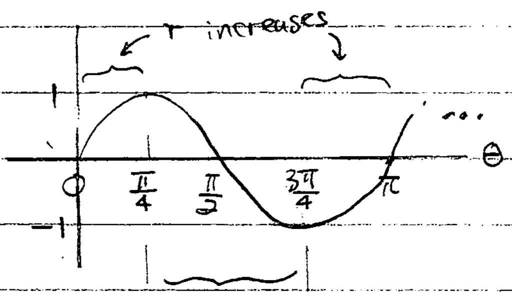
(1) r has a period of 2π in θ; so the curve is closed

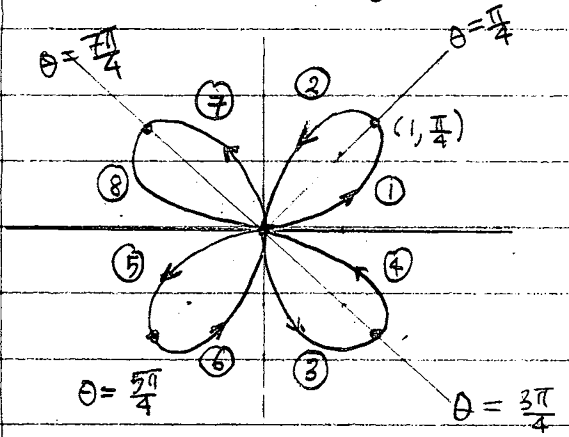
(2) Replacing θ by $\theta + \pi$:

$$r = \sin 2(\theta + \pi) = \sin 2\theta.$$

So the curve is symmetric about the pole

(3) Consider the graph of $\sin 2\theta$:





① : $0 \le \theta \le \frac{\pi}{4}$

② : $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

③ : $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$

④ : $\frac{3\pi}{4} \le \theta \le \pi$

⑤ : $\pi \le \theta \le \frac{5\pi}{4}$

⑥ : $\frac{5\pi}{4} \le \theta \le \frac{3\pi}{2}$

⑦ : $\frac{3\pi}{2} \le \theta \le \frac{7\pi}{4}$

⑧ : $\frac{7\pi}{4} \le \theta \le 2\pi$

10.3.39 $r = 2\cos 4\theta$

Observations:

(1) r has a period of 2π in θ; so the curve is closed

(2) Replacing θ by $\theta + \pi$:

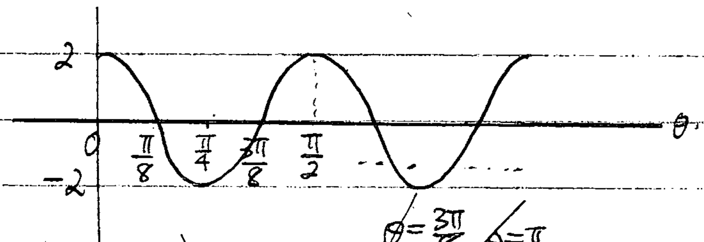
$$r = 2\cos(4\theta + 4\pi) = 2\cos 4\theta,$$ so there is a symmetry about the pole

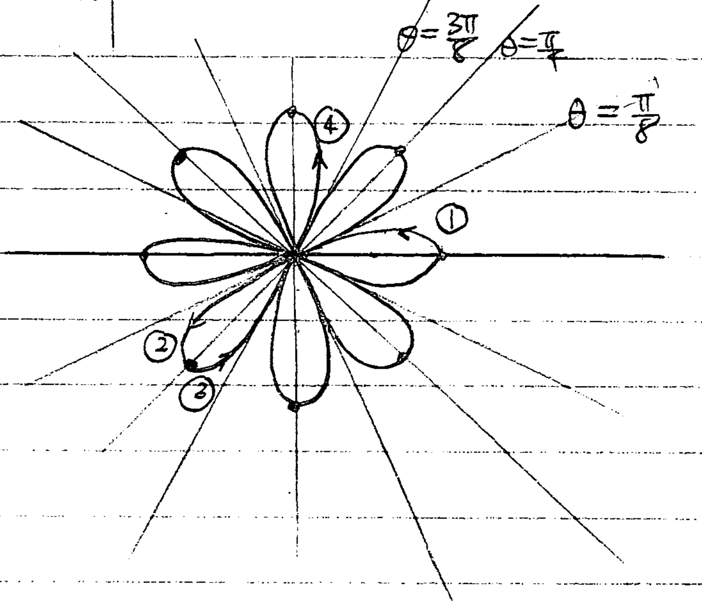
(3) Replacing θ by $-\theta$:

$$r = 2\cos(-4\theta) = 2\cos 4\theta,$$ so there is a symmetry about the polar axis.

Note: (2)+(3) $\Rightarrow$ a symmetry about the line $\theta = \frac{\pi}{2}$ as well.

(4) Consider the graph of $2\cos 4\theta$:





* $\boxed{10.3.41}$ $r^2 = 4\cos 2\theta$

$r = 2\sqrt{\cos 2\theta}$ or $r = -2\sqrt{\cos 2\theta}$

Observations:

(1) Replacing θ by $-\theta$:

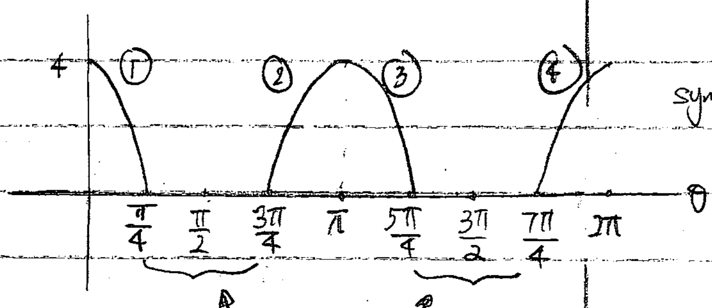
$$r^2 = 4\cos(-2\theta)$$
$$= 4\cos 2\theta,$$ so there is a symmetry about the polar axis

(2) Replacing r by $-r$:

$$(-r)^2 = 4\cos 2\theta$$
$$r^2 = 4\cos 2\theta,$$ so there is a symmetry about the pole

Note: $(1)+(2) \Rightarrow$ there is a symmetry about $\theta = \frac{\pi}{2}$

(3) Consider the graph of r^2 as a function of θ:



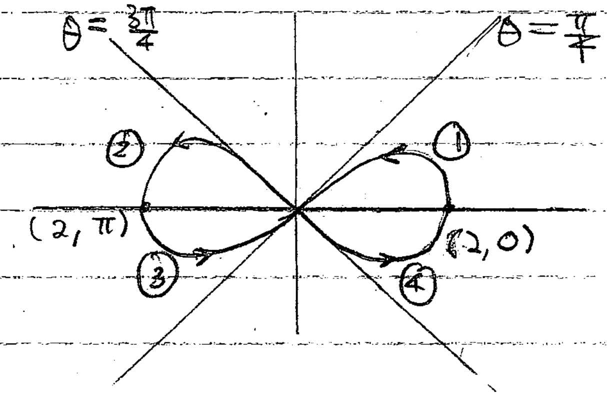
$\cos 2\theta \le 0$

Note that $r^2 = 0$

$\Leftrightarrow \cos 2\theta \Leftrightarrow 0$

$\Leftrightarrow 2\theta = \frac{\pi}{2} + n\pi, \quad n = 0, \pm1, \pm2, \ldots$

$\Leftrightarrow \theta = \frac{\pi}{4} + n\frac{\pi}{2}, \quad n = 0, \pm1, \pm2, \ldots$



$\boxed{10.3.45}$ $r = 1 + 2\cos 2\theta$

(1) r has a period 2π in θ; So the curve is closed

(2) Replacing θ by $-\theta$:

$$r = 1 + 2\cos(-2\theta)$$
$$= 1 + 2\cos 2\theta,$$ so the is a symmetry about the polar axis

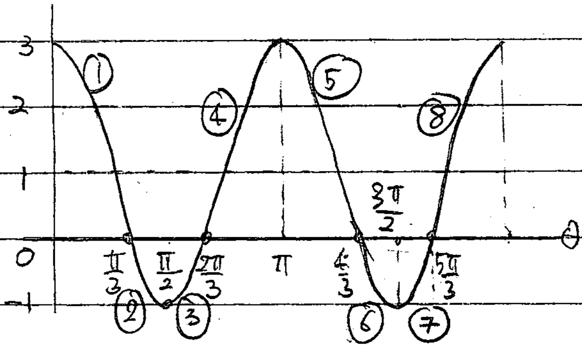
(3) Replacing θ by $\pi - \theta$:

$$r = 1 + 2\cos(2\pi - 2\theta)$$
$$= 1 + 2\cos(-2\theta)$$
$$= 1 + 2\cos 2\theta,$$ So there is

A symmetry about the line $\theta = \frac{\pi}{2}$

(4) Consider the graph of

$1+2\cos 2\theta$:



Note that $r = 0$

$\Longleftrightarrow \quad 1+2\cos 2\theta = 0$

$\Longleftrightarrow \quad \cos 2\theta = -\frac{1}{2}$

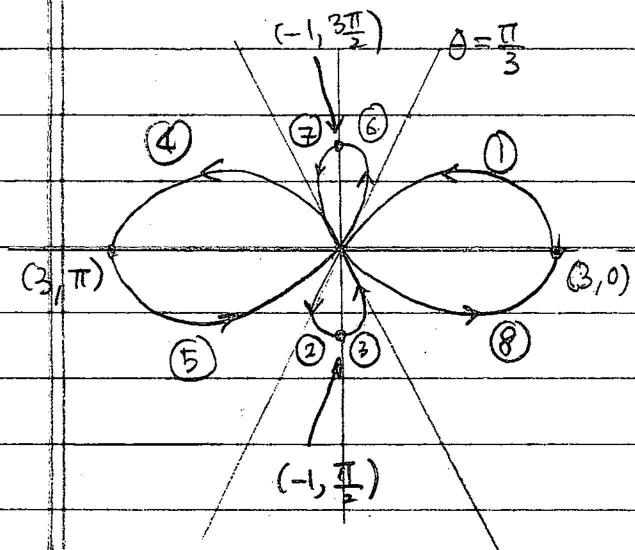
$\Longleftrightarrow \quad 2\theta = \frac{2\pi}{3} + n\theta,\ \frac{4\pi}{3} + n\theta,$

$n = 0, \pm 1, \pm 2, \ldots$

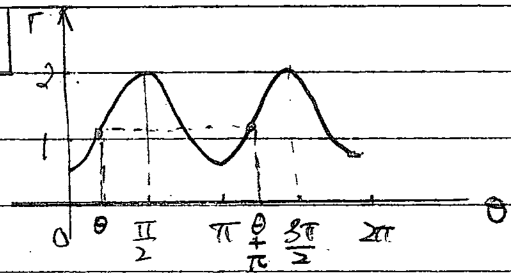
$\Longleftrightarrow \quad \theta = \frac{\pi}{3} + n\theta,\ \frac{2\pi}{3} + n\theta,$

$n = 0, \pm 1, \pm 2, \ldots$

$\Longleftrightarrow \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$



10.3.47

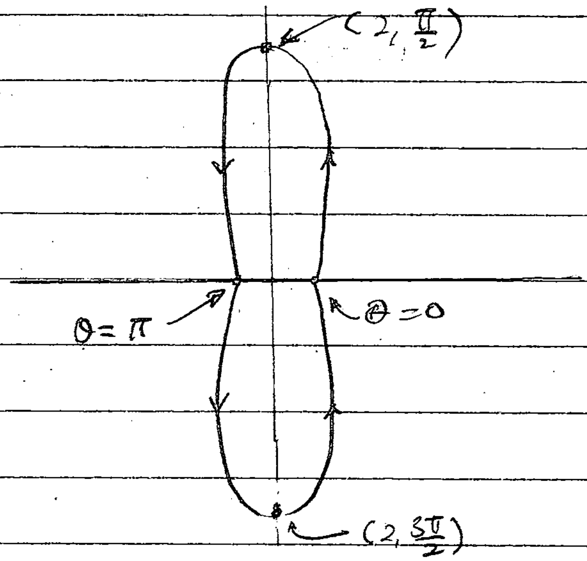


Observations:

(1) $r(\theta+\pi) = r(\theta)$ for $0 \le \theta \le \pi$,
so there is a symmetry about the pole

(2) $r(\pi - \theta) = r(\theta)$ for $0 \le \theta \le \pi$,
so there is a symmetry about the line $\theta = \frac{\pi}{2}$ (at least for the part of curve when $0 \le \theta \le \pi$.)

(3) $r(2\pi) = r(0)$, so the curve is closed.



Page 6

This has a period 2π in θ. Hence it suffices to consider the curve for $0 \le \theta \le 2\pi$.

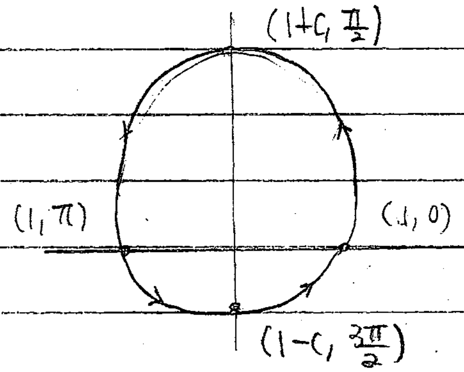
$\boxed{10.3.53(a)}$ $r = 1 + c\sin\theta$

Clearly, if $|c| \le 1$, then

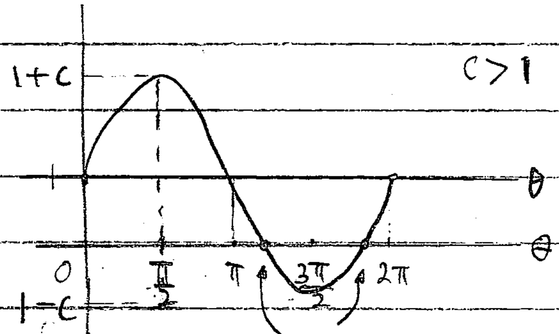
$$|r-1| = |c\sin\theta| \le 1$$

$\Rightarrow \quad -1 \le r - 1 \le 1$

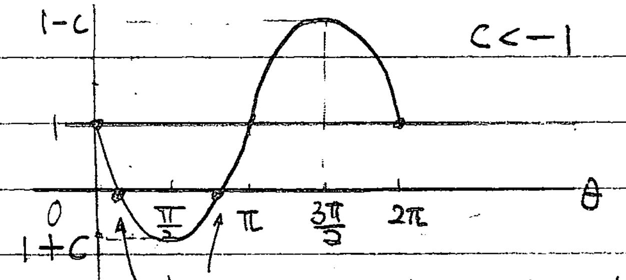
$\Rightarrow \quad 0 \le r \le 2$



Suppose $|c| > 1$. Consider r as a function of θ:



These are where the inner loop starts and ends



These are where the inner loop starts and ends.

The inner loop begins and ends when $r = 0$:

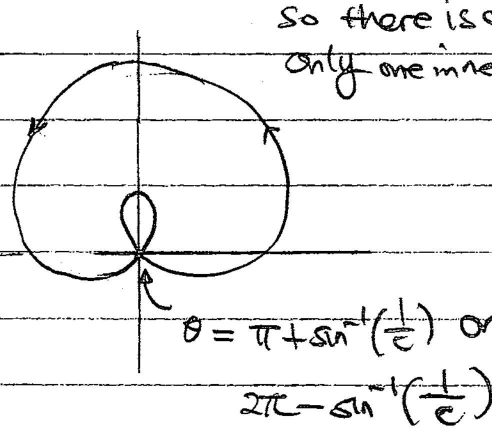
$$1 + c\sin\theta = 0$$

$$\sin\theta = -\frac{1}{c}$$

$$\theta = \sin^{-1}\left(-\frac{1}{c}\right)$$

$$= \underbrace{\pi + \sin^{-1}\left(\frac{1}{c}\right),\ 2\pi - \sin^{-1}\left(\frac{1}{c}\right)}_{\text{precisely two solutions in } 0 \le \theta \le 2\pi}$$

So there is one and only one inner loop.



10.3.53(b) Note that the limacon has a dimple only if $|c| \le 1$.

The dimple is characterized by y having a local maximum value — so $\frac{dy}{d\theta} = 0$ and $\frac{d^2y}{d\theta^2} < 0$, where

In fact, this occurs only if $\theta = \frac{3\pi}{2}$

$$y = r \sin\theta = (1 + c\sin\theta)\sin\theta = \sin\theta + c\sin^2\theta$$

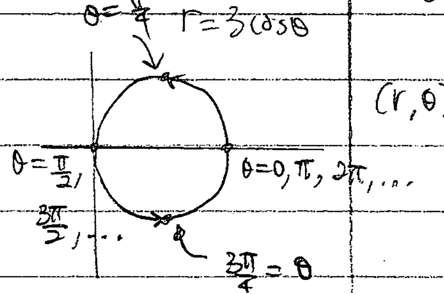
In fact,

$$\frac{dy}{d\theta} = \cos\theta + 2c\sin\theta\cos\theta.$$

$$\frac{d^2y}{d\theta^2} = -\sin\theta + 2c\sin^2\theta - 2c\cos^2\theta \overset{\theta = \frac{3\pi}{2}}{=} -1 + 2c$$

$$\begin{cases} < 0 & (c < \frac{1}{2}) \\ = 0 & (c = \frac{1}{2}) \\ > & (c > \frac{1}{2}) \end{cases}$$

So the dimple exists only if $c < \frac{1}{2}$ and disappears when $c = \frac{1}{2}$.



10.3.57 $r = \frac{1}{\theta}$, $\theta = \pi$

$$x = r\cos\theta = \frac{1}{\theta}\cos\theta \overset{\theta=\pi}{=} -\frac{1}{\pi}$$

$$y = r\sin\theta = \frac{1}{\theta}\sin\theta \overset{\theta=\pi}{=} 0$$

p. 674:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\frac{1}{\theta^2}\sin\theta + \frac{1}{\theta}\cos\theta}{-\frac{1}{\theta^2}\cos\theta - \frac{1}{\theta}\sin\theta} \overset{\theta=\pi}{=} \frac{0 - \frac{1}{\pi}}{\frac{1}{\pi^2} - 0} = -\pi$$

So the tangent line is given by

$$y - 0 = -\pi\left(x + \frac{1}{\pi}\right)$$

or $y = -\pi x - 1$

10.3.61 $r = 3\cos\theta$

$$x = r\cos\theta = 3\cos^2\theta$$

$$\frac{dx}{d\theta} = -6\cos\theta\sin\theta.$$

The tangents are vertical where $\frac{dx}{d\theta} = 0$:

$\cos\theta = 0$ or $\sin\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\theta = 0, \pi$

$\downarrow$ $\downarrow$

$(r, \theta) = (0, \frac{\pi}{2})$, $(0, \frac{3\pi}{2})$ or $(r, \theta) = (3, 0)$, $(3, \pi)$

$$y = r\sin\theta = 3\cos\theta\sin\theta$$
$$= \frac{3}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = 6\cos 2\theta$$

The tangents are horizontal when $\frac{dy}{d\theta} = 0$:

$$\cos 2\theta = 0$$
$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \dots$$

$$(r, \theta) = \left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right) \text{ or } \left(\frac{-3}{\sqrt{2}}, \frac{3\pi}{4}\right)$$

10.3.67 $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, is a circle. The center is $(x,y) = \left(\frac{b}{2}, \frac{a}{2}\right)$ and the radius is $\frac{\sqrt{a^2+b^2}}{2}$.

Proof $r^2 = ar\sin\theta + br\cos\theta$

$$x^2 + y^2 = ay + bx$$
$$x^2 - bx + y^2 - ay = 0$$
$$\left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 = 0$$
$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right) = \frac{a^2+b^2}{4}$$