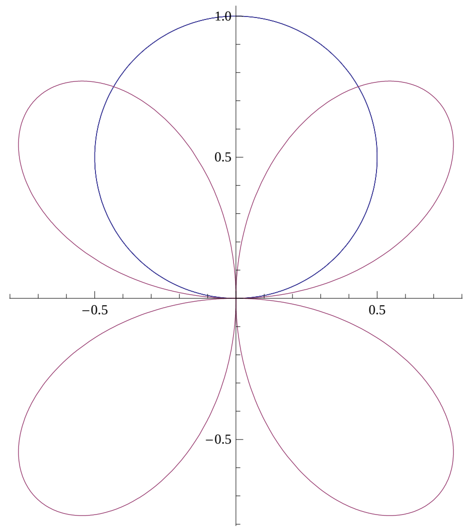
Problem 10.4.41. Find the intersection points of the curves $r = \sin\theta$ and $r = \sin 2\theta$.



<u>Solution.</u> To find the intersection points, we set

$$\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta$$

and get that

$$\sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$

as we did in class. From the equation $\sin\theta = 0$ we have

$$\theta = 0, \pm\pi, \pm 2\pi, \cdots.$$

We need only to consider the point(s) of intersection that correspond to[1] $\theta = 0,\ \pi$ since the given curves $r = \sin\theta$ and $r = \sin 2\theta$ have period 2π in θ, so that *any two values of θ that differ by an integer multiple of 2π yield the same point!* Using the formula for the curve $r = \sin\theta$, these values of θ give the polar coordinates

$$(\sin 0, 0) = (0, 0) \text{ and } (\sin\pi, \pi) = (0, \pi),$$

both of which clearly correspond to the same point, i.e., the pole. Of course, we could also have used the formula for the curve $r = \sin 2\theta$ to obtain

$$(\sin 2\cdot 0, 0) = (0, 0) \text{ and } (\sin 2\cdot\pi, \pi) = (0, \pi).$$

On the other hand, the equation $\cos\theta = 1/2$ yields

$$\theta = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \cdots.$$

As before, we need only to consider the point(s) of intersection that correspond to[2] $\theta = \pi/3,\ 5\pi/3$. Using the formula for the curve $r = \sin\theta$, these values of θ give the points of intersection

$$\left(\sin\frac{\pi}{3}, \frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right) \text{ and } \left(\sin\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right).$$

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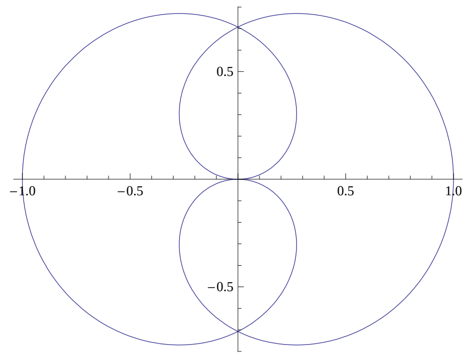
[1] We can also choose $\theta = 0,\ -\pi$.

[2] We can also choose the combinations (1) $\theta = \pi/3,\ -\pi/3$, (2) $\theta = -\pi/3,\ -5\pi/3$, or $\theta = 5\pi/3,\ -5\pi/3$.

While if we use the formula for the curve $r = \sin 2\theta$, then

$$\left(\sin 2 \cdot \frac{\pi}{3}, \frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right) \text{ and } \left(\sin 2 \cdot \frac{5\pi}{3}, \frac{5\pi}{3}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right).$$

Problem 10.4.51. Find the length of the curve $r = \sin(\theta/2)$.



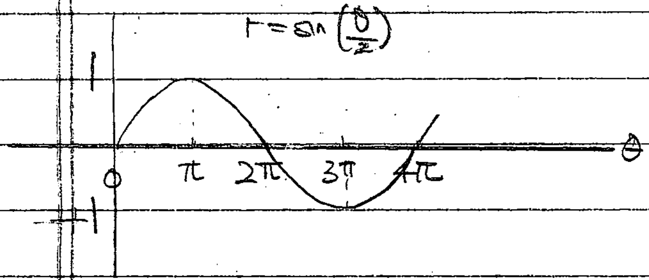
Solution. Essentially, we will have to integrate

$$\int_0^{\pi} \sqrt{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{4}}\, d\theta,^{3}$$

which is one quarter of the length of the curve, and which can only be approximated numerically.

[3] Refer to the next page to see how this can be derived.

$\boxed{10.4.51}$ Find the length of the curve $r = \sin\left(\frac{\theta}{2}\right)$.

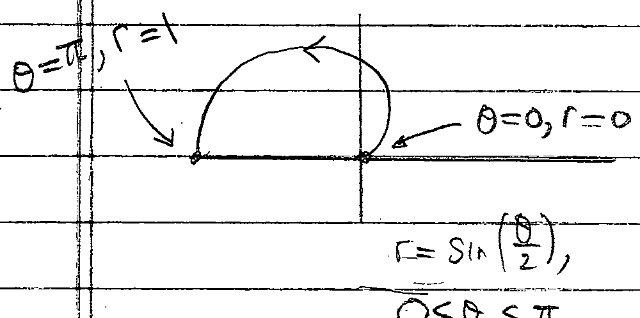


Observations:

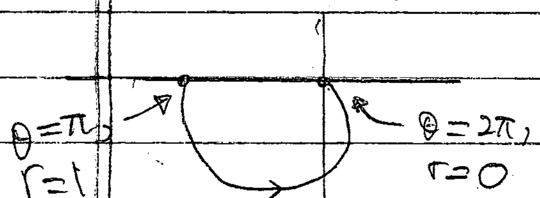
1. $r(\theta) = \sin\left(\frac{\theta}{2}\right)$ has a period 4π in θ, i.e.,

$$\begin{aligned} r(\theta+4\pi) &= \sin\left(\frac{\theta+4\pi}{2}\right) \\ &= \sin\left(\frac{\theta}{2}+2\pi\right) \\ &= \sin\left(\frac{\theta}{2}\right) \\ &= r(\theta). \end{aligned}$$

2. As θ increases from 0 to π, r increases from 0 to 1:



3. As θ increases from π to 2π, r decreases from 1 to 0:

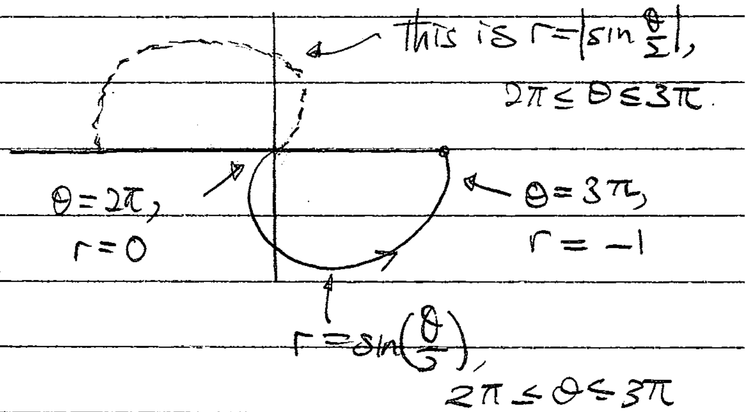


$r=\sin\left(\frac{\theta}{2}\right)$

$\pi\le\theta\le 2\pi$

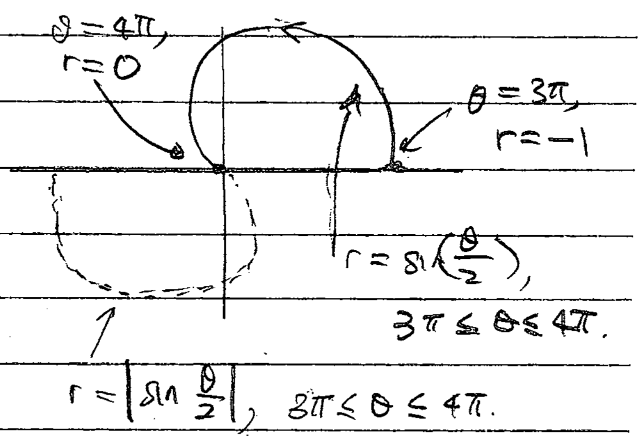
The tricky part begins ...

4. As θ increases from 2π to 3π, r decreases from 0 to -1:

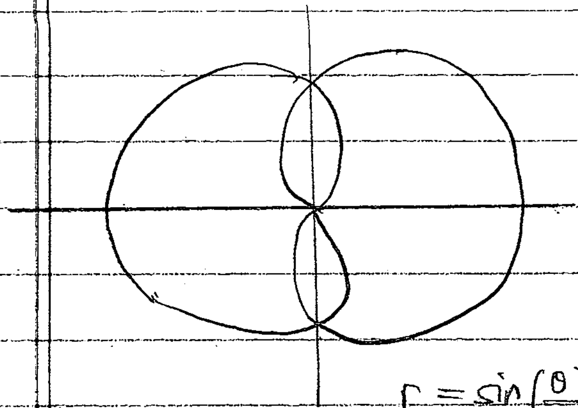


Note that this part of the curve (for $2\pi\le\theta\le3\pi$) is similar to the part of the curve for $0\le\theta\le2\pi$ being reflected across the pole.

5. Likewise, as θ increases from 3π to 4π, r increases from -1 to 0:



Combining all parts of the curve, we have



$r = \sin\left(\frac{\theta}{2}\right)$

So the length of the curve is

$$L = \int_0^{4\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

or $$L = 2\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

or $$L = 4\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

where $r = \sin\left(\frac{\theta}{2}\right)$

$$\frac{dr}{d\theta} = \frac{1}{2}\cos\left(\frac{\theta}{2}\right)$$