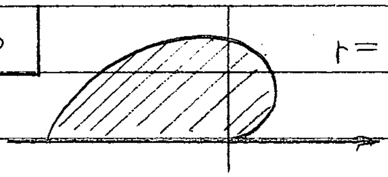
Section 10.4: Areas and Lengths in Polar Coordinates

p. 680

If $\mathcal{R}$ is the region bounded by the polar curve $r=f(\theta)\geq 0$ and by the rays $\theta=a$ and $\theta=b$, $0\leq b-a\leq 2\pi$, then the area of $\mathcal{R}$ is

$$A=\int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta$$

10.4.5

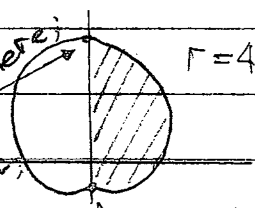


$r=\theta$

$$A=\int_0^{\pi}\frac{1}{2}\theta^2\,d\theta=\left[\frac{\theta^3}{6}\right]_0^{\pi}=\frac{\pi^3}{6}$$

10.4.7

$r=4+3\sin\theta$



r has a maximum here; so $\sin\theta$ must also be maximum; $\theta=\frac{\pi}{2}$

r has minimum here; so $\sin\theta$ must also be minimum; $\theta=-\frac{\pi}{2}$

$$A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}(4+3\sin\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(16+24\sin\theta+9\sin^2\theta)\,d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[16+24\sin\theta+9\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$

$$=\frac{1}{2}\left[\frac{41}{2}\theta-24\cos\theta-\frac{9\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{41}{4}\pi$$

10.4.11

$r^2=4\cos 2\theta$

See problem 10.3.41 for the sketch of the curve.

$$A=4\int_0^{\frac{\pi}{4}}\frac{1}{2}(4\cos 2\theta)\,d\theta$$

$$=8\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$=4$$

10.4.15

$r=1+2\sin 6\theta$

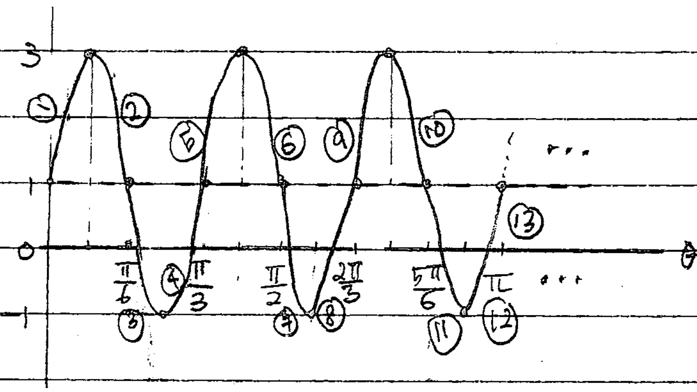
Observations:

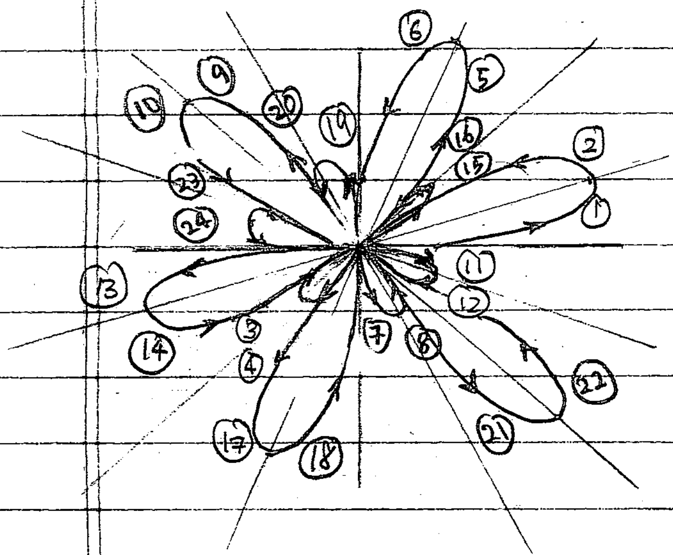
(1) r has period 2π in θ; so the curve is closed.

(2) Replacing θ by $\theta+\pi$:

$$r=1+2\sin(6\theta+6\pi)$$
$$=1+2\sin 6\theta,$$ so there is a symmetry about the pole

(3) Consider the graph of $1+2\sin 6\theta$:





Note that the curve $r = |1+2\sin 6\theta|$ has the same graph. Thus the enclosed area is

$$A = \int_0^{2\pi} \frac{1}{2}\left(1+2\sin 6\theta\right)^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left|1+2\sin 6\theta\right|^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1+4\sin 6\theta + 4\sin^2 6\theta\right) d\theta$$

$$= \cdots$$

$$= 3\pi$$

$\boxed{10.4.17}$ $r = \sin 2\theta$

See problem 10.3.37 for the sketch.

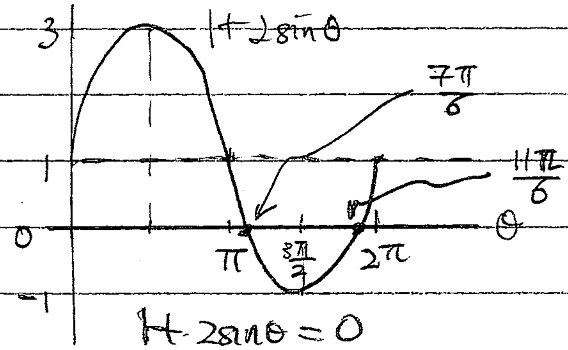
The area of one loop is

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2 2\theta \, d\theta$$

$$= \cdots$$

$$= \frac{\pi}{8}$$

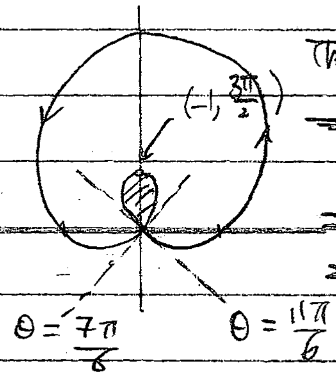
$\boxed{10.4.21}$ $r = 1 + 2\sin\theta$



$$1+2\sin\theta = 0$$

$$\iff \sin\theta = -\frac{1}{2}$$

$$\iff \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$$



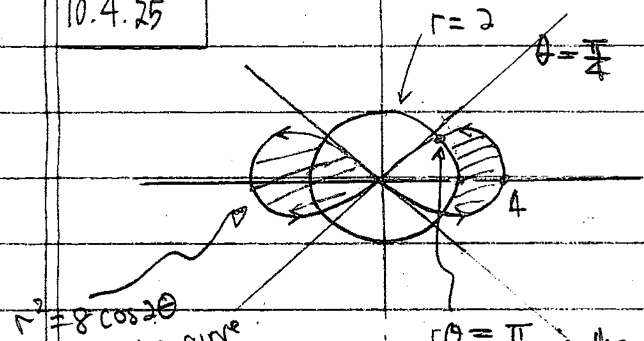
The area of the inner loop

$$= \int_{\frac{7\pi}{6}}^{\frac{11}{6}\pi} \frac{1}{2}\left(1+2\sin\theta\right)^2 d\theta$$

$$= \cdots$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

10.4.25



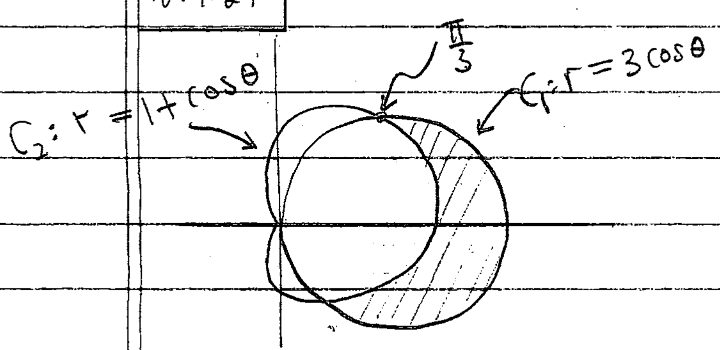
$\theta=\frac{\pi}{6}$ is the smallest positive angle where the curves intersect.

$$8\cos 2\theta = 2^2$$
$$\Leftrightarrow \cos 2\theta = \frac{1}{2}$$
$$\Leftrightarrow 2\theta = \pm\frac{\pi}{3}, \dots$$
$$\Leftrightarrow \theta = \pm\frac{\pi}{6}, \dots$$

The area of the shaded region is

$$A = 4\int_0^{\frac{\pi}{6}}\left[\frac{1}{2}(8\cos 2\theta)-\frac{1}{2}(2)^2\right]d\theta$$
$$= 2\left[4\sin 2\theta - 4\theta\right]_0^{\frac{\pi}{6}}$$
$$= 4\sqrt{3} - 4\pi/3$$

10.4.27

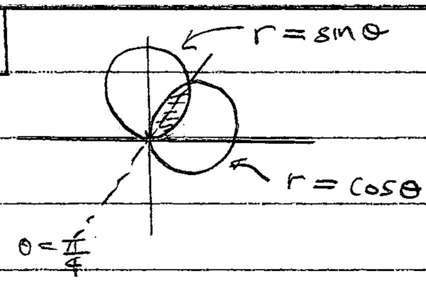


$$3\cos\theta = 1+\cos\theta$$
$$\Leftrightarrow \cos\theta = \frac{1}{2}$$
$$\Leftrightarrow \theta = \pm\frac{\pi}{3}, \dots$$

The area of the shaded region is

$$A = 2\int_0^{\frac{\pi}{3}}\left[\frac{1}{2}(3\cos\theta)^2-\frac{1}{2}(1+\cos\theta)^2\right]d\theta$$
$$= \int_0^{\frac{\pi}{3}}\left(9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right)d\theta$$
$$= \int_0^{\frac{\pi}{3}}\left(8\cos^2\theta - 2\cos\theta - 1\right)d\theta$$
$$= \dots$$
$$= \pi.$$

10.4.29

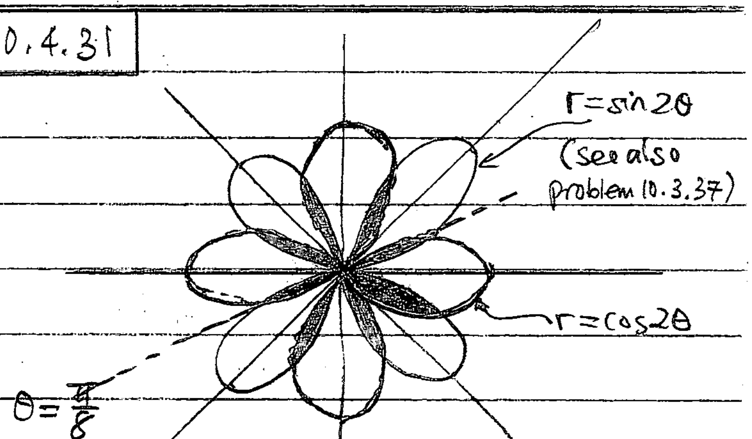


$$\sin\theta = \cos\theta \Leftrightarrow \tan\theta = 1 \Leftrightarrow \theta = \frac{\pi}{4}, \dots$$

Area of the shaded region

$$A = 2\int_0^{\frac{\pi}{4}}\frac{1}{2}(\sin\theta)^2 d\theta$$
$$= \dots$$
$$= \frac{\pi}{8}-\frac{1}{4}$$

10.4.31



The smallest positive angle where the two curves intersect

$\sin 2\theta = \cos 2\theta$

$\tan 2\theta = 1$

$2\theta = \frac{\pi}{4}, \ldots$

$\theta = \frac{\pi}{8}, \ldots$

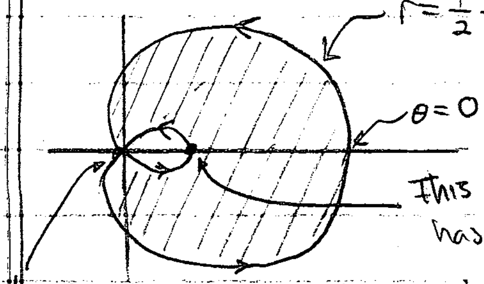
Area of the shaded region

$$A = 16\int_0^{\frac{\pi}{8}} \frac{1}{2}(\sin 2\theta)^2\, d\theta$$

$$= \ldots$$

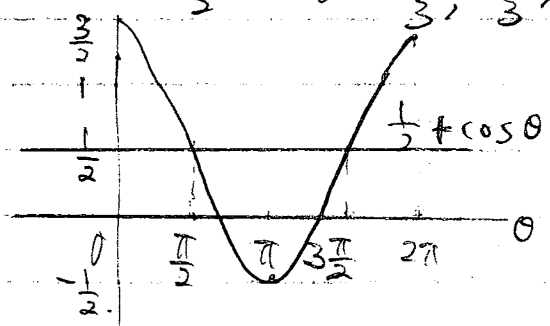
$$= \frac{\pi}{2} - 1$$

$\boxed{10.4.35}$ Compare with problem 10.3.53



this is where $r=0$ or, equivalently,

$\cos\theta = -\frac{1}{2}$ or $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$



Area of the shaded region

$$A = 2\left\{\int_0^{\frac{2\pi}{3}} \frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta - \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta\right\}$$

$$= \ldots$$

$$= \frac{\pi}{4} + \frac{3\sqrt{3}}{4}$$

$\boxed{10.4.41}$ $r = \sin\theta, \quad r = \sin 2\theta$

Find all intersection points.

$$\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta$$

$$\Leftrightarrow \sin\theta(1 - 2\cos\theta) = 0$$

$$\Leftrightarrow \sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$

$\theta = 0, \pm\pi, \pm 2\pi, \ldots$

$\theta = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \ldots$

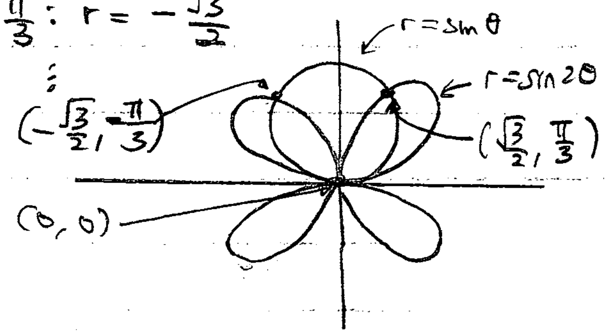
$\theta = 0$: $r = \sin\theta = 0$ ← we don't need to check the case $\theta = \pm 2\pi$ since $\sin\theta$ has period 2π

$\theta = \pm\pi$: $r = 0$

$\theta = +\frac{\pi}{3}$: $r = \frac{\sqrt{3}}{2}$

$\theta = -\frac{\pi}{3}$: $r = -\frac{\sqrt{3}}{2}$

$\vdots$



Page 5

p. 682

The length of a polar curve $r = f(\theta)$, $a \le \theta \le b$, with f' being continuous, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$\boxed{10.4.47}$ $r = \theta^2$, $0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \theta \sqrt{\theta^2 + 4}\, d\theta$$

$$\overset{u = \theta^2 + 4}{=} \int_4^{4+4\pi^2} \frac{1}{2}\sqrt{u}\, du$$

$$= \cdots$$

$$= \frac{8}{3}\left[(\pi^2+1)^{3/2} - 1\right].$$