Quiz # 1 : Nov 16 – 11.1 – 11.5

Section 11.1 : Sequences

11.1.13 $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\}$

$= \{(\frac{2}{3})^0, -(\frac{2}{3}), (\frac{2}{3})^2, -(\frac{2}{3})^3, \ldots\}$

$= \{(-\frac{2}{3})^0, (-\frac{2}{3})^1, (-\frac{2}{3})^2, (-\frac{2}{3})^3, \ldots\}$

$= \{a_n\}$, where $a_n = \left(-\frac{2}{3}\right)^n$ for $n \geq 0$.

11.1.23 $\{a_n\}$, where $a_n = \cos\left(\frac{n}{2}\right)$ is divergent.

Proof by contradiction. To the contrary, we suppose that $L = \lim_{n\to\infty} a_n$ exists. Note that this implies that $\lim_{n\to\infty} a_{2n} = L$ also.

On the other hand, recall that

$$\cos n = 2\cos^2\frac{n}{2} - 1.$$

But $\cos n = a_{2n}$. Hence

$$a_{2n} = 2a_n^2 - 1.$$

Passing to the limit $n \to \infty$ yields

$$L = 2L^2 - 1$$

or, equivalently,

$$2L^2 - L - 1 = 0.$$

Therefore $L = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)} = 1$ or $-\frac{1}{2}$.

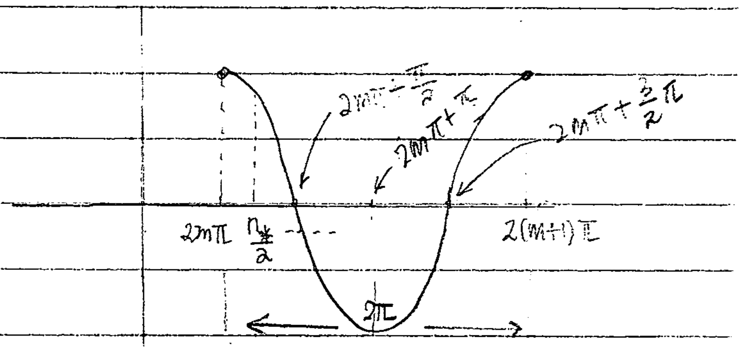
This means that if $L = \lim_{n\to\infty} a_n$ exists, then it is necessary that

$$L = 1 \quad \text{or} \quad L = -\frac{1}{2}.$$

We would show that NEITHER of these is possible. We do this by using the following claims:

Claim # 1. Within any interval I_m of the form $2m\pi \leq x \leq 2(m+1)\pi$, where m is an integer, there is always a number of the form $\frac{n_m}{2}$, where n_m is an integer, such that $\cos\frac{n_m}{2} < 0$.

This is clear intuitively by considering the graph of the cosine function:



To validate the claim, let n_* be the smallest integer such that

$$2m\pi < \frac{n_*}{2}.$$

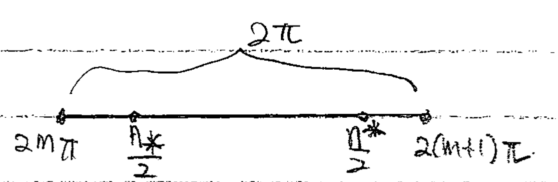
Note that $\frac{n_*}{2} - 2m\pi < \frac{1}{2}$, otherwise $n_* - 1$ would be an integer smaller that n_* such that $2m\pi < \frac{n_*-1}{2}$. Likewise, let n^* be the largest integer such that

$$\frac{n^*}{2} < 2(m+1)\pi.$$

Then $2(m+1)\pi - \frac{n^*}{2} < \frac{1}{2}$ also. Now, it is clear that

$$\left[\frac{n_*}{2} - 2m\pi\right] + \left[\frac{n^*}{2} - \frac{n_*}{2}\right] + \left[2(m+1)\pi - \frac{n^*}{2}\right] = 2\pi,$$

the length of the interval I_m.



Rearranging the equation we get

$$2\pi = \left[\frac{n^*}{2} - \frac{n_*}{2}\right] + \left[\frac{n_*}{2} - 2m\pi\right] + \left[2(m+1)\pi - \frac{n^*}{2}\right]$$

$$< \left[\frac{n^*}{2} - \frac{n_*}{2}\right] + 1$$

or $$\frac{n^*}{2} - \frac{n_*}{2} > 2\pi - 1 \simeq 5.2832$$

Using the facts that n^* and n_* are integers, we have

$$\frac{n^*}{2} - \frac{n_*}{2} \geq 5.5$$

or $$n^* - n_* \geq 11.$$

This shows that the interval I_m contains at least 12 numbers of the form $\frac{n}{2}$, where n is an integer, i.e.,

$$\frac{n_*}{2}, \frac{n_*+1}{2}, \ldots, \frac{n_*+11}{2}.$$

On the other hand, it is clear that the subinterval $J_m (2m\pi + \frac{\pi}{2}, 2m\pi + \frac{3\pi}{2})$ contains no more that 7 numbers of the form $\frac{n}{2}$, where n is an integer. To see this, let $\underline{n}$ and $\bar{n}$ be the smallest and largest integers respectively such that

$$\frac{\underline{n}}{2} > 2m\pi + \frac{\pi}{2} \text{ and } \frac{\bar{n}}{2} < 2m\pi + \frac{3\pi}{2}.$$

Then $\frac{\bar{n}}{2} - \frac{\underline{n}}{2} < \pi$ or

$$\bar{n} - \underline{n} \leq 6$$

To summarize:

1. $I_m = (2m\pi, 2m\pi + 2\pi)$ contains at least 12 numbers of the form $\frac{n}{2}$, where n is an integer.
2. The subinterval $J_m = (2m\pi + \frac{\pi}{2}, 2m\pi + \frac{3\pi}{2})$ contains no more than 7 numbers of the same form.
3. Therefore, there is at least one number of the form $\frac{n_m}{2}$, where n_m is an integer that is contained in ~~either~~ $(2m\pi, 2m\pi + \frac{\pi}{2})$ or $(2m\pi + \frac{3\pi}{2}, 2m\pi + 2\pi)$. In particular, $\cos \frac{n_m}{2} < 0$. This completes the proof for claim # 1.

Claim #2 There are infinitely many terms of the sequence $\{a_n\}$, where $a_n = \cos \frac{n}{2}$, that are negative.

For each integer $m \geq 0$, pick a number $\frac{n_m}{2}$, where n_m is an integer, from the interval $I_m = (2m\pi, 2m\pi + 2\pi)$ such that $\cos \frac{n_m}{2} < 0$:

$$0 < \frac{n_0}{2} < 2\pi, \quad a_{n_0} = \cos \frac{n_0}{2} < 0;$$
$$2\pi < \frac{n_1}{2} < 4\pi, \quad a_{n_1} = \cos \frac{n_1}{2} < 0;$$
$$4\pi < \frac{n_2}{2} < 6\pi, \quad a_{n_2} = \cos \frac{n_2}{2} < 0;$$
$$\vdots \qquad \vdots$$

This is possible by claim #1. Clearly, $n_i = n_j$ only if $i = j$. So there are infinitely may terms of $\{a_n\}$ that are negative.

Claim #3 $L = \lim_{n\to\infty} a_n \neq 1$.

This follows immediately from claim #2 because if $n_0, n_1, \ldots$ is the list of integers from claim #2, then

Page 4

$a_{n_0} < 0,\ a_{n_1} < 0, \ldots$, from which it follows that

$$\lim_{m\to\infty} a_{n_m} \le 0 < 1$$

even if the limit exists

Likewise, it can be shown that

$$L = \lim_{n\to\infty} a_n \neq -\frac{1}{2}$$

Hence we may conclude that $\{a_n\}$, where $a_n = \cos\frac{n}{2}$, is indeed divergent.

$\boxed{11.1.27}$ $\displaystyle\lim_{n\to\infty} \frac{e^n + e^{-n}}{e^{2n} - 1} = 0$

Approach #1. Use the fact that

p. 704 $\lim_{x\to\infty} f(x) = L,\ f(n) = a_n \Rightarrow \lim_{n\to\infty} a_n = L$

$$\text{Now } \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^{2x} - 1} \overset{\text{l'Hopital}}{=} \lim_{x\to\infty} \frac{e^x - e^{-x}}{2e^{2x}}$$

$$= \lim_{x\to\infty} \left(\frac{1}{2e^x} - \frac{1}{2e^{3x}} \right)$$

$$= 0.$$

Approach #2 Use Squeeze Theorem

p. 705: If $a_n \le b_n \le c_n$ for all $n \ge n_0$ for some n_0, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Now

$$0 \le \frac{e^n + e^{-n}}{e^{2n} - 1} \le \frac{e^n + e^{-n}}{e^{2n} - \frac{1}{2}e^{2n}} = 2\left(\frac{1}{e^n} + \frac{1}{e^{3n}}\right)$$

$$0 \xrightarrow{n\to\infty} 0, \quad \tfrac{1}{2}e^{2n} > 1, \quad 2\left(\frac{1}{e^n} + \frac{1}{e^{3n}}\right) \to 0$$

So $\displaystyle\lim_{n\to\infty} \frac{e^n + e^{-n}}{e^{2n} - 1} = 0$

$\boxed{11.1.37}$ $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$ diverges.

Since the terms of the sequence are either 0's or 1's, the limit must be ~~either~~ 0 or 1 if it exists. Nevertheless, since there are infinitely many 0's and 1's, neither 0 or 1 can be the limit. Hence the limit does not exist.

11.1.41 Given $a_n = (-1)^n \frac{n+1}{n}$.

Note that for n even, $(-1)^n = 1$. So for even n, a_n lies on the curve $y = \frac{x+1}{x} = 1 + \frac{1}{x}$, $x \geq 1$

Likewise, for odd n, a_n lies on the curve $y = -\frac{x+1}{x} = -1 - \frac{1}{x}$, $x \geq 1$.

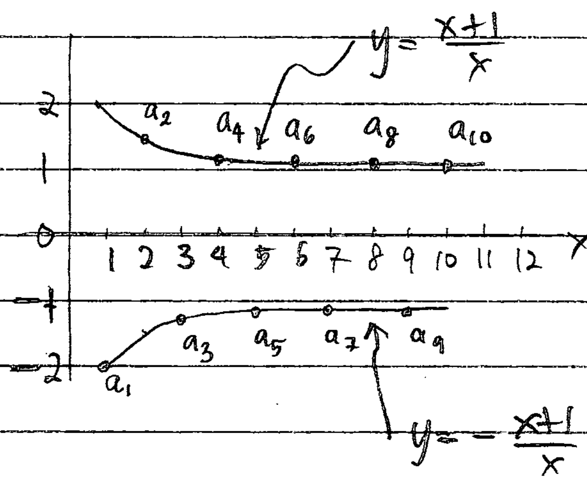
Also, for $x \geq 1$,

$\frac{x+1}{x} = 1 + \frac{1}{x} \geq 1$ and

$-\frac{x+1}{x} = -1 - \frac{1}{x} \leq -1$

with $\lim_{x\to\infty} \frac{x+1}{x} = 1$ and

$\lim_{x\to\infty} -\frac{x+1}{x} = -1$.



* In Mathematica, this can be plotted using the following lines of codes:

```
Clear[f];
f[n_] = (-1)^n (n+1) / n;
ListPlot[Table[{n, f[n]}, {n, 1, 100}],
    PlotJoined -> False, PlotRange -> {-2, 2}]
```

This specifies the range of n

The sequence is divergent.

On one hand, $\lim_{m\to\infty} a_{2m} = \lim_{m\to\infty} (-1)^{2m} \frac{2m+1}{2m}$

$= \lim_{m\to\infty} 1 + \frac{1}{2m}$

$= 1 + 0 = 1$

On the other hand, $\lim_{m\to\infty} a_{2m+1} = \lim_{m\to\infty} (-1)^{2m+1} \frac{(2m+1)+1}{2m+1}$

$= -\lim_{m\to\infty} \left(1 + \frac{1}{2m+1}\right)$

$= -(1+0)$

$= -1$.

Using the fact that <u>a sequence has AT MOST ONE limit</u>, we conclude that $\left\{(-1)^n \frac{n+1}{n}\right\}$ is divergent.

11.1.53 Suppose $\{a_n\}$

- is decreasing,
- and has its terms lie between 5 and 8.

We use Monotonic Sequence Theorem.

p.709 If a sequence is bounded and monotonic (increasing or decreasing), then it is convergent.

1. Since $\{a_n\}$ is decreasing, it is monotonic

2. All its terms lie between 5 and 8, so it is bounded.

Hence it is convergent. Furthermore, since $5 \le a_n \le 8$, we have

$$5 \le \lim_{n\to\infty} a_n \le 8.$$

$\boxed{11.1.57}$ $\{a_n\} = \{\cos(\frac{n\pi}{2})\}$

$$= \{0, -1, 0, 1, 0, -1, \ldots\}$$

is not monotonic

In fact, $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even but not divisible by } 4 \\ 1 & \text{if } n \text{ is divisible by } 4 \end{cases}$

Clearly, it is bounded because

$$-1 \le \cos\left(\frac{n\pi}{2}\right) \le 1$$

for all n.

$\boxed{11.1.63}$ Given $a_1 = 1$

$$a_{n+1} = 3 - \frac{1}{a_n}$$

1. $\{a_n\}$ is increasing

Proof. Note that $a_2 = 3 - \frac{1}{a_1} = 2 > a_1$.
Suppose $a_{n+1} > a_n$. Then

$$a_{n+2} = 3 - \underbrace{\frac{1}{a_{n+1}} > 3 - \frac{1}{a_n}}_{(\text{because } a_{n+1} > a_n)} = a_{n+1}.$$

So $a_{n+1} > a_n$ for all n by induction.

2. $1 \le a_n < 3$ for all n.

Proof. Note that $1 \le a_1 = 1 < 3$. Suppose $1 \le a_n < 3$. Then

$$\underbrace{1 \le a_{n+1}}_{(\text{because } a_n \ge 1 \text{ implies } \frac{1}{a_n} \le 1)} = 3 - \overbrace{\frac{1}{a_n} < 3}^{\text{because } a_n > 0.}$$

Hence $1 \le a_n < 3$ for all n by induction

So by the Monotonic Sequence Theorem,

$$L = \lim_{n\to\infty} a_n \text{ exists.}$$

Now it is clear that

$$\lim_{n\to\infty} a_{n+1} = L$$

also. So the relation

$$a_{n+1} = 3 - \frac{1}{a_n}$$

implies that

$$L = 3 - \frac{1}{L}$$

or $L^2 - 3L + 1 = 0$.

Hence

$$L = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$

But $\frac{3-\sqrt{5}}{2} \simeq 0.381966\ldots$

< 1.

So $L = \frac{3+\sqrt{5}}{2}$.