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Section 11.10: Taylor and Maclaurin Series

p. 761 If f has a power series representation (expansion) at a, that is, if

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$$

for $|x-a|<R$ for some $R>0$ and a sequence $\{c_n\}$, then

$$c_n=\frac{f^{(n)}(a)}{n!}$$

The series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the Taylor series of f at/about a. The series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

is called the Maclaurin series of f.

The polynomial

$$T_n(x)=\sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$

is called the nth degree Taylor polynomial of f at a.

The function f is the sum of its Taylor series if

$$f(x)=\lim_{n\to\infty}T_n(x)$$

In this case,

$$\lim_{n\to\infty}R_n(x)=0,$$

where $R_n(x)=f(x)-T_n(x)$ is called the remainder of the Taylor series.

Conversely, if $\lim_{n\to\infty}R_n(x)=0$ for $|x-a|<R$ for some $R>0$, then $f(x)=\lim_{n\to\infty}T_n(x)$ for $|x-a|<R$.

p. 763. If $|f^{(n+1)}(x)|\le M$ for $|x-a|\le d$ for some $M>0$ and $d>0$, then

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}$$

for $|x-a|\le d$.

11.10.5 Given $f(x)=(1+x)^{-3}$. Then

$f(0)=1=\frac{(-1)^0 2!}{2}$

$f'(x)=-3(1+x)^{-4}$ $\quad f'(0)=-3=\frac{(-1)^1 3!}{2}$

$f''(x)=(3)(4)(1+x)^{-5}$ $\quad f''(0)=(3)(4)=\frac{(-1)^2 4!}{2}$

$f'''(x)=-(3)(4)(5)(1+x)^{-6}$ $\quad f'''(0)=-(3)(4)(5)$

$\vdots$ $\quad \vdots \quad =\frac{(-1)^3 5!}{2}$

In general, it can be shown that

$$f^{(n)}(0) = \frac{(-1)^n (n+2)!}{2}$$

for $n \geq 0$, where $f^{(0)}(0) \stackrel{\text{def}}{=} f(0)$

So the Maclaurin series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!}{n!\, 2} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^n$$

Now,

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+2)(n+3)x^{n+1}}{2} \cdot \frac{2}{(-1)^n (n+1)(n+2)x^n} \right|$$

$$= \lim_{n\to\infty} \frac{n+3}{n+1} |x|$$

$$= |x|.$$

So, by ratio test, the radius of convergence is 1.

$\boxed{11.10.13}$ Given $f(x) = e^x$, $a = 3$

Then $\qquad f(3) = e^3$

$f'(x) = e^x \qquad f'(3) = e^3$

$f''(x) = e^x \qquad f''(3) = e^3$

$f'''(x) = e^x \qquad f'''(3) = e^3$

$\vdots \qquad \vdots$

So the Taylor series of f about $a=3$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^3}{n!} (x-3)^n$$

Using ratio test, it can be shown that this series converges for $-\infty < x < \infty$

$\boxed{11.10.17}$ Given $f(x) = \frac{1}{\sqrt{x}}$, $a = 9$. Then

$f(9) = \frac{1}{3}$

$f'(x) = -\frac{1}{2} x^{-\frac{3}{2}} \qquad f'(9) = -\frac{1}{2}(3^{-3})$

$f''(x) = \frac{1}{2} \cdot \frac{3}{2} x^{-\frac{5}{2}} \qquad f''(9) = \frac{1}{2} \cdot \frac{3}{2} (3^{-5})$

$f'''(x) = -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} x^{-\frac{7}{2}} \qquad f'''(9) = -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} (3^{-7})$

In general, for $n \geq 1$,

$$f^{(n)}(9) = \frac{(-1)^n (1)(3)(5)\cdots(2n-1)}{2^n} 3^{-(2n+1)}$$

$$= \frac{(-1)^n (1)(2)(3)(4)\cdots(2n-1)(2n)}{(2)(4)\cdots(2n)\, 2^n\, 3^{2n+1}}$$

$$= \frac{(-1)^n (2n)!}{[2(1)][2(2)]\cdots[2(n)]\, 2^n 3^{2n+1}}$$

$$= \frac{(-1)^n (2n)!}{2^n n!\, 2^n 3^{2n+1}}$$

$$= \frac{(-1)^n (2n)!}{3\, n!\, 6^{2n}}$$

This is true also when $n=0$

So the Taylor series of f at $a = 9$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{3(n!)^2 6^{2n}} (x-9)^n$$

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Now

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2n+2)!}{3[(n+1)!]^2 6^{2n+2}}(x-9)^{n+1}\cdot\frac{3(n!)^2\,6^{2n}}{(-1)^n(2n)!\,(x-9)^n}\right|$$

$$=\lim_{n\to\infty}\frac{(2n+2)!}{(2n)!}\frac{(n!)^2}{[(n+1)!]^2}\frac{6^{2n}}{6^{2n+2}}|x-9|$$

$$=\lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)(n+1)6^2}|x-9|$$

$$=\frac{4}{6^2}|x-9|$$

$$=\frac{1}{9}|x-9|$$

So by ratio test, the radius of convergence is 9.

$\boxed{11.10.27}$ Given $f(x)=x^2e^{-x}$

Since $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ on $(-\infty,\infty)$,

$$e^{-x}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$$

and $x^2e^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!}$

on $(-\infty,\infty)$

$\boxed{11.10.35}$ Recall that

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

on $(-\infty,\infty)$

(See p. 767) So

$$\cos x^2=\sum_{n=0}^{\infty}(-1)^{2n}\frac{(x^2)^{2n}}{(2n)!}$$
$$=\sum_{n=0}^{\infty}(-1)^{2n}\frac{x^{4n}}{(2n)!}$$
$$=1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\cdots$$

On $(-\infty,\infty)$

Specifically, the radius of convergence is $+\infty$

$\boxed{11.10.41}$ It is more advantageous to consider the Maclaurin series of $f(x)=(1+x)^k$, k any real number.

We have

$f(x)=(1+x)^k$

$f'(x)=k(1+x)^{k-1}$

$f''(x)=k(k-1)(1+x)^{k-2}$

$f'''(x)=k(k-1)(k-2)(1+x)^{k-3}$

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$\vdots$

$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$

Then

$f(0) = 1$

$f'(0) = k$

$f''(0) = k(k-1)$

$f'''(0) = k(k-1)(k-2)$

$\vdots$

$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

So the Maclaurin series of $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where we define, for $n \geq 0$,

$$\binom{k}{n} = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{k(k-1)\cdots(k-n+1)}{n!} & \text{if } n \geq 1 \end{cases}$$

In fact, it can be shown (see p. 773) that

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \text{ for } |x| < 1$$

for any real number k.

So for

$$\sqrt{1+x^3} = (1+x^3)^{\frac{1}{2}}$$
$$= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x^3)^n$$
$$= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{3n} \text{ for } |x| < 1$$

Hence

$$\int \sqrt{1+x^3}\,dx = \int \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{3n}\,dx$$
$$= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \int x^{3n}\,dx$$
$$= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{x^{3n+1}}{3n+1} + C$$

$\boxed{11.10.47}$ We have (p. 767)

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for $-1 \leq x \leq 1$

So

$$\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots}{x^3}$$
$$= \lim_{x \to 0} \left(\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \cdots \right)$$
$$= \frac{1}{3}$$

$\boxed{11.10.51}$ We have

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

on $(-\infty, \infty)$

and

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

on $(-\infty, \infty)$.

So

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$= 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots$$

$$= 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

and $e^{-x^2} \cos x$

$$= \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$- \frac{x^2}{1!} - \frac{x^4}{1!2!} + \cdots$$

$$+ \frac{x^4}{2!} + \cdots$$

$$= 1 - \frac{3}{2}x^2 + \frac{x^4}{24} + \cdots$$ (first three nonzero terms)

on $(-\infty, \infty)$

$\boxed{11.10.59}$ We have

$$3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$$

$$= \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots$$

$$= \left(\frac{3^0}{0!} + \frac{3}{1!} + \cdots \frac{3^4}{4!} + \cdots\right) - \frac{3^0}{0!}$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} - 1$$

$$= e^3 - 1$$

because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$