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Section 11.11 : The Binomial Series

p. 773 The Binomial series. For any real number k and $|x|<1$,

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

radius of convergence is 1

where $\binom{k}{n} = \begin{cases} 1 & \text{if } n=0 \\ \dfrac{k(k-1)\cdots(k-n+1)}{n!} & \text{if } n\geq 1. \end{cases}$

11.11.3

$$\frac{1}{(2+x)^3}$$

$$= \frac{1}{2^3(1+\frac{x}{2})^3}$$

$$= \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$$

$$= \frac{1}{8}\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\binom{-3}{n}\frac{x^n}{2^{n+3}} \quad \text{for } \left|\frac{x}{2}\right|<1$$

or $|x|<2$

with radius of convergence $= 2$.

11.11.7

$$\frac{x}{\sqrt{4+x^2}}$$

$$= \frac{x}{\sqrt{4(1+\frac{x^2}{4})}}$$

$$= \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{x}{2}\left[1+\left(\frac{x}{2}\right)^2\right]^{-\frac{1}{2}}$$

$$= \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^{2n}$$

$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^{2n+1}$$

for $\left|\left(\frac{x}{2}\right)^2\right|<1$ or $\frac{x^2}{4}<1$

or $x^2<4$

or $-2<x<2$

with radius of convergence $= 2$

11.11.17 (a)

$$f(x) = \sqrt{1+x^2}$$

$$= (1+x^2)^{\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^{2n} \quad (*)$$

for $|x^2|<1$ or $|x|<1$

with radius of convergence $=1$

(b) Recall that the Maclaurin series of f has the form $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$. In particular, observe that the coefficient of x^{10} is $\frac{f^{(10)}(0)}{10!}$. On the other hand, the coefficient of x^{10} in $(*)$ is $\binom{\frac{1}{2}}{5}$. So

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$$\frac{f^{(10)}(0)}{10!} = \binom{\frac{1}{2}}{5}$$

or $f^{(10)}(0) = 10! \binom{\frac{1}{2}}{5}$

$$= 10! \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{5!}$$

$$= 99225$$