Section 11.12: Applications of Taylor Polynomials

If a function $f(x)$ is equal to its Taylor series at a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The nth partial sum of this series is the so-called nth-degree Taylor polynomial of f at a:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

11.12.5 Given $f(x) = \sin x$, $a = \pi/6$, $n = 3$.

$f(x) = \sin x \quad f(\pi/6) = \frac{1}{2}$

$f'(x) = \cos x \quad f'(\pi/6) = \sqrt{3}/2$

$f''(x) = -\sin x \quad f''(\pi/6) = -\frac{1}{2}$

$f'''(x) = -\cos x \quad f'''(\pi/6) = -\sqrt{3}/2$

So the 3rd-degree Taylor polynomial of f at $a = \pi/6$ is

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(\pi/6)}{k!}(x-\pi/6)^k$$

$$= \frac{1}{2} + \frac{\sqrt{3}/2}{1!}(x-\pi/6) - \frac{1/2}{2!}(x-\pi/6)^2 - \frac{\sqrt{3}/2}{3!}(x-\pi/6)^3$$

11.12.9 Given $f(x) = xe^{-2x}$, $a = 0$, $n = 3$

$f(x) = xe^{-2x} \quad f(0) = 0$

$f'(x) = (1-2x)e^{-2x} \quad f'(0) = 1$

$f''(x) = (-2-2+4x)e^{-2x} \quad f''(0) = -4$

$= (-4+4x)e^{-2x} \quad f'''(0) = 12$

$f'''(x) = (4+8-8x)e^{-2x}$

$= (12-8x)e^{-2x}$

So the 3rd-degree Taylor polynomial of f at $a = 0$ is

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

$$= \frac{1}{1!}x - \frac{4}{2!}x^2 + \frac{12}{3!}x^3$$

11.12.19 Given $f(x) = e^{x^2}$, $a = 0$, $n = 3$, $0 \le x \le 0.1$

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

So $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

$$= 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots$$

and thus the Taylor polynomial of degree 3 of f at $a = 0$ is

$$T_3(x) = 1 + x^2$$

More explicitly, we compute

$f(x) = e^{x^2}$

$f'(x) = 2xe^{x^2}$

$f''(x) = (2+4x^2)e^{x^2}$

$f'''(x) = (8x+4x+8x^3)e^{x^2}$
$= (12x+8x^3)e^{x^2}$

This is needed for error estimate → $f^{(4)}(x) = (12+24x^2+24x^2+16x^4)e^{x^2}$
$= (12+48x^2+16x^4)e^{x^2}$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = 2$

$f'''(0) = 0$

So $T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!}(x-0)^k$
$= 1+x^2$

Note that for $0 \le x \le 0.1$
or, more generally, $|x-0| \le 0.1$,

$|f^{(4)}(x)| \le f^{(4)}(0.1) = 12.4816e^{0.1^2}$
$\simeq 12.6070421654 7775$

(This can be verified by showing that
(i) $f^{(4)}(x)$ is even
(ii) $f^{(4)}(x)$ is increasing on $[0, 0.1]$)

So the Taylor's inequality (p. 763) gives

$$|R_3(x)| = |f(x) - T_3(x)| \le \frac{f^{(4)}(0.1)}{(3+1)!}|x-0|^4 = \frac{12.4816\,e^{0.1^2}}{4!}x^4 \simeq 0.525293423561 57x^4$$

for $|x-0| \le 0.1$

$\boxed{11.12.23}$ Use the information from problem 5 to estimate $\sin 35^\circ$ to five decimal places.

Note that $\sin 35^\circ = \sin \frac{35}{180}\pi$
$= \sin \frac{7}{36}\pi$

On the other hand, by Taylor's inequality,

$$|R_3(x)| = |f(x) - T_3(x)| \le \frac{M}{4!}\left(x-\frac{\pi}{6}\right)^4$$

for any M such that $|f^{(4)}(x)| \le M$
wherever
$|x - \frac{\pi}{6}| \le d$
for some d (such that $x = \frac{7}{36}\pi$ satisfies this inequality

Clearly, for our purpose, we can choose $d = \frac{7}{36}\pi - \frac{\pi}{6}$

$$= \frac{1}{36}\pi.$$

So $|R_3(x)| \le \frac{M}{4!}(x - \frac{\pi}{6})^4$

for $|x - \frac{\pi}{6}| \le \frac{1}{36}\pi$.

Furthermore, we can choose $M=1$ since $|f^{(4)}(x)| = |-\sin(x)| \le 1$.

So $|R_3(x)| \le \frac{(x-\frac{\pi}{6})^4}{4!}$

for $|x - \frac{\pi}{6}| \le \frac{1}{36}\pi$. In particular,

$$|R_3(\tfrac{7\pi}{36})| \le \frac{(\frac{7\pi}{36} - \frac{\pi}{6})^4}{4!}$$

$$= \frac{\pi^4}{36^4\, 4!}$$

$$= 2.41645 \times 10^{-6}$$

$$< 3 \times 10^{-6}$$

So up to 5 decimal places,

$$\sin 35^\circ = \sin \frac{7\pi}{36}$$

$$\approx T_3(\tfrac{7\pi}{36})$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 = 0.57358$$

$\boxed{11.12.25}$ Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So the nth-degree polynomial is

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

with

$$R_n(x) = e^x - T_n(x)$$

and

$$|R_n(x)| \le \frac{M}{(n+1)!}|x|^n$$

for any M such that $|f^{(n+1)}(x)| \le M$ whenever $|x| \le 0.1$.

However, since $f^{(n+1)}(x) = e^x$ and thus $|f^{(n+1)}(x)| \le e^1 < 3$ for $|x| \le 0.1$.

So we can choose $M = 3$. Thus

$$|R_n(x)| \le \frac{3|x|^{n+1}}{(n+1)!}$$

In particular, for $x = 0.1$, we have

$$|R_n(0.1)| \le \frac{0.1^{n+1}(3)}{(n+1)!}$$

So to make sure that $|R_n(0.1)| < 0.00001$, it suffices to require $\frac{(3)\,0.1^{n+1}}{(n+1)!} < 0.00001$.

→ next page

n	$(3)\,0.1^{n+1}/(n+1)!$
1	0.015
2	0.0005
3	0.0000125
4	$2.5\times10^{-7} < 0.00001$

Hence $T_4(0.1)$

$$= \sum_{k=0}^{4} \frac{0.1^k}{k!}$$

$$= 1 + \frac{0.1}{1!} + \frac{0.1^2}{2!} + \frac{0.1^3}{3!} + \frac{0.1^4}{4!}$$

$$= 1.10517$$

approximates $e^{0.1}$ to within 0.00001

$\boxed{11.12.27}$ $\sin x \approx x - \frac{x^3}{6}$

$|\text{error}| < 0.01$

Recall that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

In order to apply the alternating series estimation theorem (p. 738), we need to make sure

$$0 \le \frac{x^{2n+3}}{(2n+3)!} \le \frac{x^{2n+1}}{(2n+1)!}$$

or equivalently

$$(1)\quad 0 \le x \text{ and } \underbrace{\frac{x^{2n+3}}{x^{2n+1}} \le \frac{(2n+3)!}{(2n+1)!}}_{x^2 \le (2n+2)(2n+3)}$$

for all $n \ge 0$

Also,

$$(2)\quad \lim_{n\to\infty} \frac{x^{2n+1}}{(2n+1)!} = 0$$

this actually follows from the fact that the Maclaurin series converges for any x.

If (1) and (2) holds, we have

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \le \frac{x^5}{5!}$$

So to make sure

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| < 0.01$$

(under the conditions (1) & (2)), we can demand that

$$0 \le \frac{x^5}{5!} < 0.01$$

$$\text{or } 0 \le x^5 < 5!\; 0.01 = 1.2$$

$$\text{or } \underbrace{0 \le x < (1.2)^{\frac{1}{5}}}\approx 1.03714$$

We need to check that this satisfies (1) & (2).

Page 5

Clearly, if $0 \le x < (1.2)^{\frac{1}{5}}$, then

$$x^2 = (1.2)^{\frac{2}{5}} \simeq 1.07565$$
$$< (2n+2)(2n+3)$$
for all n.

Also, $\lim_{n\to\infty} \frac{x^{2n+1}}{(2n+1)!} = 0$.

Hence we have shown that if $0 \le x < (1.2)^{\frac{1}{5}}$, then

$$\left| \sin x - \left(x - \frac{x^3}{6}\right) \right| < 0.01.$$

We claim that this is true also if $-(1.2)^{\frac{1}{5}} < x < 0$.
In fact, if $u = -x$, then $0 < u < (1.2)^{\frac{1}{5}}$ if $-(1.2)^{\frac{1}{5}} < x < 0$. So

$$\left| \sin u - \left(u - \frac{u^3}{6}\right) \right| < 0.01$$

according to what we have shown. But

$$\sin u = \sin(-x) = -\sin x$$
$$u - \frac{u^3}{6} = (-x) - \frac{(-x)^3}{6}$$
$$= -\left(x - \frac{x^3}{6}\right).$$

Hence we also have

$$\left| \sin x - \left(x - \frac{x^3}{6}\right) \right|$$
$$= \left| -\left[\sin x - \left(x - \frac{x^3}{6}\right)\right] \right|$$
$$= \left| \sin u - \left(u - \frac{u^3}{6}\right) \right| < 0.01$$

Therefore for $|x| < 1.2^{\frac{1}{5}}$,

$$\left| \sin x - \left(x - \frac{x^3}{6}\right) \right| < 0.01$$