Section 11.2: Series

11.2.3 $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = -2$

p. 715 — For a geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

See also examples 2 and 3 on pp. 715–716

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

$$a = \frac{12}{-5}, \quad |r| = \left|-\frac{1}{5}\right| < 1$$

So
$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{\frac{12}{-5}}{1-\left(-\frac{1}{5}\right)} = \frac{-12}{6} = -2$$

The partial sums are $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$

$S_1 = -\frac{12}{5}$	$S_6 = -\frac{31248}{15625}$
$S_2 = -\frac{48}{25}$	$S_7 = -\frac{156252}{78125}$
$S_3 = -\frac{252}{125}$	$S_8 = -\frac{781248}{390625}$
$S_4 = -\frac{1248}{625}$	$S_9 = -\frac{3906252}{1953125}$
$S_5 = -\frac{6252}{3125}$	$S_{10} = -\frac{19531248}{9765625}$

11.2.11 $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$

$$= 3 + \frac{2}{3}(3) + \left(\frac{2}{3}\right)^2(3) + \left(\frac{2}{3}\right)^3(3) + \cdots$$

a geometric series with $a = 3$, $r = \frac{2}{3}$

$$= \frac{3}{1-\frac{2}{3}} = \frac{3}{\left(\frac{1}{3}\right)} = 9$$

(Note that $|r| = \frac{2}{3} < 1$.)

11.2.17 $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{1}{-3}\left(\frac{-3}{4}\right)^n$

$$= \left(-\frac{1}{3}\right)\left(\frac{-3}{4}\right) + \left(-\frac{1}{3}\right)\left(\frac{-3}{4}\right)^2 + \cdots$$

$$= -\frac{1}{3}\Big[\underbrace{\left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \cdots}_{\text{a geometric series with } a=-\frac{3}{4},\ |r|=\left|-\frac{3}{4}\right|<1}\Big]$$

$$= -\frac{1}{3} \cdot \frac{-\frac{3}{4}}{1-\left(-\frac{3}{4}\right)} = \frac{1}{7}$$

11.2.23 $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

We use the method of telescoping. See also example 6 on p. 717.

Consider the partial sum

$$S_n = \sum_{k=2}^{\infty} \frac{2}{k^2-1} \quad \text{for } n \geq 2$$

Note that

$$\frac{2}{k^2-1} = \frac{2}{(k+1)(k-1)}$$
$$= \frac{A}{k+1} + \frac{B}{k-1}$$
$$= \frac{A(k-1) + B(k+1)}{(k+1)(k-1)}$$
$$= \frac{(A+B)k + B - A}{k^2-1}$$

So $(A+B)k + (B-A) = 2$

for all $k \geq 2$. So

$$\left.\begin{aligned} A+B &= 0 \\ B-A &= 2 \end{aligned}\right\} \begin{aligned} A &= -1 \\ B &= 1 \end{aligned}$$

and $\dfrac{2}{k^2-1} = \dfrac{1}{k-1} - \dfrac{1}{k+1}$

Hence

$$S_n = \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k+1} \right)$$

NOTE: The signs of these terms must be opposite for this method to work

or
$$S_n = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1}$$
$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

So
$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} S_n$$
$$= 1 + \frac{1}{2} + 0 + 0$$
$$= \frac{3}{2}.$$

11.2.33 We first consider $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$, the limit of whose can be calculated using the telescoping method.

Set $S_n = \sum_{k=1}^{n} \frac{3}{k(k+3)}$. Note that

$$\frac{3}{k(k+3)} = \frac{A}{k} + \frac{B}{k+3} = \frac{A(k+3)+Bk}{k(k+3)} = \frac{(A+B)k+3A}{k(k+3)}$$

So $\left.\begin{array}{l} 3A = 3 \\ A+B=0 \end{array}\right\} \begin{array}{l} A=1 \\ B=-1 \end{array}$

Hence

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+3}\right)$$
$$= 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \frac{1}{5} - \frac{1}{8} + \cdots + \frac{1}{n-2} - \frac{1}{n+1} + \frac{1}{n-1} - \frac{1}{n+2} + \frac{1}{n} - \frac{1}{n+3}$$

or

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

So $\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{n\to\infty} S_n$
$$= 1 + \frac{1}{2} + \frac{1}{3} + 0 + 0 + 0 = \frac{11}{6}.$$

Next, note that $\sum_{n=1}^{\infty} \frac{5}{4^n}$ is a geometric series with $a = \frac{5}{4}$ and $|r| = \frac{1}{4} < 1$.

So
$$\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{\frac{5}{4}}{1-\frac{1}{4}} = \frac{5}{3}.$$

Now
$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+3)} + \frac{5}{4^n}\right] = \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \frac{5}{4^n}$$

since these are convergent

$$= \frac{11}{6} + \frac{5}{3} = \frac{7}{2}$$

11.2.39 $0.123\overline{456} = ?$

$$0.123\overline{456} = 0.123 + 0.000\overline{456}$$
$$= \frac{123}{1000} + (0.000456 + 0.000000456 + \cdots)$$

Note that

$0.000456 + 0.000000456 + \cdots$

is a geometric series with

$a = 0.000456$ and $|r| = \frac{1}{1000}$

So

$$0.000456 + 0.000000456 + \cdots$$

$$= \frac{0.000456}{1 - \frac{1}{1000}}$$

$$= \frac{456/1\,000\,000}{999/1000}$$

$$= \frac{19}{41\,625}$$

Hence $0.123\overline{456}$

$$= \frac{123}{1000} + \frac{19}{41625}$$

$$= \frac{41\,111}{333\,000}$$

See also example 4 on p. 716

$\boxed{11.2.43}$ Note that $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$

is a geometric series with $a = 1$ and $r = 4x$. So it is convergent if and only if

$$|r| = |4x| < 1 \quad \text{or} \quad |x| < \frac{1}{4}.$$

In this case, $\sum_{n=0}^{\infty} 4^n x^n$

$$= \frac{1}{1-4x}$$

See also example 5 on p. 716.

$\boxed{11.2.49.}$ Suppose $S_n = \frac{n-1}{n+1}$ is the nth partial sum of $\sum_{n=1}^{\infty} a_n$, i.e.,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

Then $S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$, and thus

$$S_n = S_{n-1} + a_n$$

or $a_n = S_n - S_{n-1}$

$$= \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1}$$

$$= \frac{2}{n+n^2}.$$

By definition, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

$$= \lim_{n\to\infty} \frac{n-1}{n+1}$$

$$= \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}}$$

$$= \frac{1-0}{1+0} = 1$$

$\boxed{11.2.57}$

$$\begin{aligned} 0 &= 0+0+0+\cdots \\ &= (1-1)+(1-1)+(1-1)+\cdots \\ &= 1-1+1-1+1-1+\cdots \\ &= 1+(-1+1)+(-1+1)+(-1+1)+\cdots \\ &= 1+0+0+0+\cdots \\ &= 1 \end{aligned}$$

~~this series is divergent~~;

so the equality

$0 = 1-1+1-1+1-1+\cdots$

is NOT well defined.