Section 11.3: The Integral Test and the Estimates of Sums.

p. 724 **Integral Test**

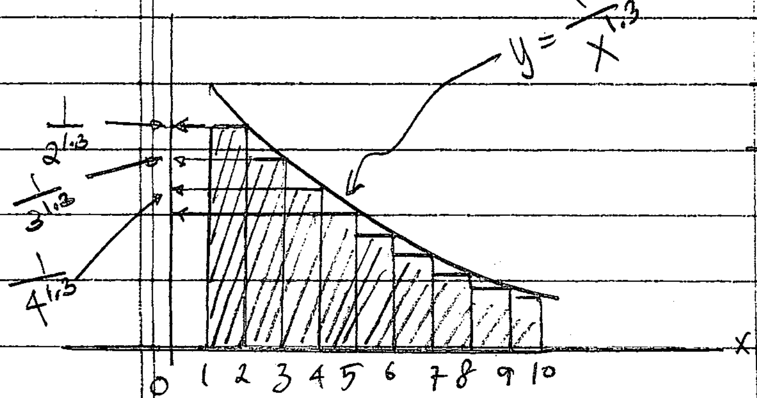
$a_n = f(n)$.

Suppose f is continuous, positive, and decreasing on $[n_*, \infty)$ for an integer $n_* \geq 1$.

(i) If $\int_{n_*}^{\infty} f(x)\,dx$ is convergent, then so is $\sum_{n=n_*}^{\infty} a_n$

(ii) If $\int_{n_*}^{\infty} f(x)\,dx$ is divergent, then so is $\sum_{n=n_*}^{\infty} a_n$

$\boxed{11.3.1}$



$\int_1^{\infty} \frac{1}{x^{1.3}}\,dx$ = ~~the~~ area under the graph of $y = \frac{1}{x^{1.3}}$ from $x = 1$ to $x = +\infty$

$>$ the ~~total~~ area of the inscribed rectangles $= \sum_{n=2}^{\infty} \frac{1}{n^{1.3}}$

$\boxed{11.3.7}$ Consider $\sum_{n=1}^{\infty} n e^{-n}$. Note that

$f(n) = n e^{-n}$ if $f(x) = x e^{-x}$ for $x \geq 1$.

Further,
1. $f(x) > 0$
2. f is continuous
3. f is decreasing

$\left[f(x) = \frac{1}{xe^x}, \quad f'(x) = -\frac{1+x}{x^2 e^x} < 0 \right]$.

Now $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x e^{-x}\,dx$

integration by parts $= \left[\frac{x e^{-x}}{-1} \right]_1^{\infty} - \int_1^{\infty} -e^{-x}\,dx$

$= \left[-x e^{-x} - e^{-x} \right]_1^{\infty}$

$= 0$ is convergent

So $\sum_{n=1}^{\infty} n e^{-n}$ is also convergent by the integral test

$\boxed{11.3.11}$ Consider $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

$= \sum_{n=1}^{\infty} \frac{1}{n^3}$

This is a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 3 > 1$.

So it is convergent.

p-series test (p. 725):

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{is convergent if } p > 1 \\ \text{is divergent if } p \leq 1. \end{cases}$$

$\boxed{11.2.15}$ Consider $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$.

Note that $f(n) = \frac{1}{n^2+4}$ if $f(x) = \frac{1}{x^2+4}$ for $x \geq 1$. Further,

1. f is continuous
2. $f(x) > 0$ for all $x \geq 1$
3. f is decreasing

$$\left[f'(x) = -\frac{2x}{(x^2+4)^2} < 0\right].$$

Now $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2+4}\,dx$

$$\overset{\text{integral (17)}}{=} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_1^{\infty}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right)$$

is convergent. So

$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ is also convergent by the integral test.

$\boxed{11.2.21}$ Consider $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. Note that $f(n) = \frac{1}{n \ln n}$ if $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$.

Further, 1. f is continuous
2. $f(x) > 0$ for $x \geq 2$
3. f is decreasing

$$\left[f'(x) = -\frac{1 + \ln x}{x^2(\ln x)^2} < 0\right].$$

Now $\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x \ln x}\,dx$

$$\underset{du = \frac{1}{x}dx}{\overset{u = \ln x}{=}} \int_{\ln 2}^{\infty} \frac{1}{u}\,du$$

$x = 2 \rightarrow u = \ln 2$

$x \to \infty \rightarrow u \to \infty$

$$= \left[\ln u\right]_{\ln 2}^{\infty}$$

diverges.

So $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges also by the integral test.

P. 727: Remainder Estimate for the Integral Test: If f is continuous, positive, and decreasing for $x \geq n$ and $\sum a_n$ is convergent, where $a_n = f(n)$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \overset{\text{def}}{=} S - S_n \leq \int_n^{\infty} f(x)\,dx$$

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Note that

$$\underbrace{S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_n^{\infty} f(x)\,dx}_{(3) \text{ on p. 727}}$$

Note: the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\frac{\pi^2}{6} \simeq 1.644934067$

$\boxed{11.3.31}$ We want to approximate $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(a) Set $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$.

Then $S_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{10^2}$

$$= \frac{1\,968\,329}{1\,270\,080}$$

$$\simeq 1.549767731$$

In fact, we need to check that f satisfies the hypothesis of the integral test

To calculate / approximate the error / remainder R_n, we calculate

$$\int_n^{\infty} f(x)\,dx = \int_n^{\infty} \frac{1}{x^2}\,dx$$

where $f(x) = \frac{1}{x^2}$

$$= \left[-\frac{1}{x}\right]_n^{\infty}$$

$$= \frac{1}{n}$$

and recall that

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n = \underbrace{S}_{S=\sum_{n=1}^{\infty}\frac{1}{n^2}} - S_n \le \int_n^{\infty} f(x)\,dx$$

So the remainder R_{10} associated with S_{10} is bounded as follows:

$$0.09090909091 \simeq \frac{1}{11} \le R_{10} = S - S_{10} \le \frac{1}{10} \simeq 0.1$$

This shows that the error is at most 0.1

(The exact error is $\frac{\pi^2}{6} - \left(\frac{1}{1^2} + \cdots + \frac{1}{10^2}\right)$ $\simeq 0.095166335568$; but this is usually unknown unless the exact value of the series can be calculated)

(b) Note that we have

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_n^{\infty} f(x)\,dx$$

$$S_{10} + \int_{11}^{\infty} f(x)\,dx \le S \le S_{10} + \int_{10}^{\infty} f(x)\,dx$$

$$\underbrace{S_{10} + \frac{1}{11}}_{\cancel{1.640676822}} \le S \le \underbrace{S_{10} + \frac{1}{10}}_{\cancel{1.649767731}}$$

So a better approximation to S is

$$\frac{S_{10} + \frac{1}{11} + S_{10} + \frac{1}{10}}{2} \simeq 1.645222277$$

(c) To make sure that the error R_n is less than 0.001, it suffices to require that

$$\int_n^\infty f(x)\,dx = \frac{1}{n} < 0.001$$

Since

$$0 < \frac{1}{n+1} = \int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx = \frac{1}{n}$$

Now, $\frac{1}{n} < 0.001$

$\Leftrightarrow\ n > \frac{1}{0.001} = 1000$

$\Leftrightarrow\ n \ge 1001.$

(In fact, $S_{1001} \cong 1.643935565$ and $S - S_{1001} \simeq 0.0009985021637$)

$\boxed{11.3.33}$ We want to estimate $\sum_{n=1}^\infty n^{-\frac{3}{2}}$ to within 0.01.

See also example 6 on p. 728

Let $f(x) = x^{-\frac{3}{2}} = \frac{1}{x^{\frac{3}{2}}}$ for $x \ge 1$.

Then 1. f is continuous

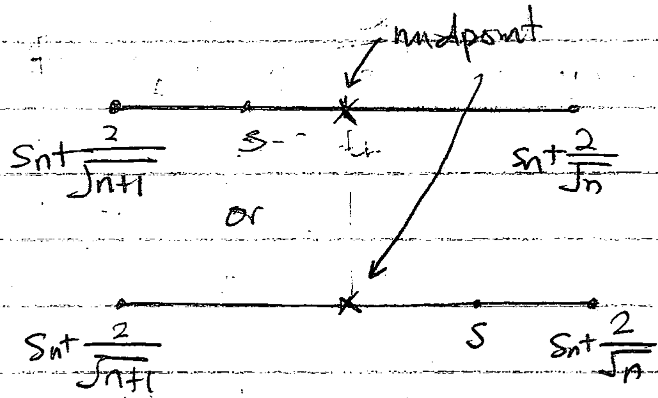
2. $f(x) > 0$ for $x \ge 1$

3. f is decreasing

So

$$\int_{n+1}^\infty f(x)\,dx \le R_n \overset{\text{def}}{=} \underbrace{S - S_n}_{S_n = \sum_{k=1}^n k^{-\frac{3}{2}},\ S = \sum_{n=1}^\infty n^{-\frac{3}{2}}} \le \int_{n+1}^\infty f(x)\,dx$$

$$\int_{n+1}^\infty f(x)\,dx = \left[-\frac{2}{\sqrt{x}}\right]_{n+1}^\infty = \frac{2}{\sqrt{n+1}}, \qquad \left[-\frac{2}{\sqrt{x}}\right]_n^\infty = \frac{2}{\sqrt{n}}.$$



Note that if we estimate S by the midpoint of the interval $\left(S_n + \frac{2}{\sqrt{n+1}}, S_n + \frac{2}{\sqrt{n}}\right)$, then the error is at most half of the length of the interval. Hence to estimate S to within 0.01, it suffices to make sure that

$$\frac{\left(\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}\right)}{2} < 0.01$$

or $\underbrace{\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}}$ < 0.01

[This can be shown to be decreasing by considering the function $g(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$ (for $x \geq 1$). whose derivative is

$$g'(x) = -\frac{1}{2x^{3/2}} + \frac{1}{2(x+1)^{3/2}}$$

< 0 for $x \geq 1$.]

Now

n	$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$ (approximate)
1	0.292893
2	0.129757
3	0.0773503
4	0.0527864
5	0.0389653
6	0.0302838
7	0.0244111
8	0.0202201
9	0.0171056
10	0.0147164
11	0.0128362
12	0.011325
13	0.0100889
14	0.00906235 < 0.01

So

$S_{14} \simeq 2.087227588$

approximates S to within 0.01.

Of course, so do

$S_{15} \simeq 2.104440848$

$S_{16} \simeq 2.120065848$

$\vdots$