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Section 11.4: The Comparison Tests

p.731 **The Comparison Test.** Given $\sum a_n, \sum b_n$ with positive terms.

(i) $\sum b_n$ converges, $a_n \le b_n$ for all $n \ge n_0$ for some $n_0 \Longrightarrow$ $\sum a_n$ also converges

(ii) $\sum b_n$ diverges, $a_n \ge b_n$ for all $n \ge n_0$ for some n_0 $\Rightarrow \sum a_n$ also diverges

p.732 **Limit Comparison Test** Given $\sum a_n, \sum b_n$ with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where c is a positive number, then ~~either~~ both $\sum a_n, \sum b_n$ converge or both $\sum a_n, \sum b_n$ diverge.

$\boxed{11.4.1}$ Given $\sum a_n, \sum b_n$ with positive terms and $\sum b_n$ converges.

(a) If $a_n > b_n$ for all n, then there is no conclusion can be made on the convergence of $\sum a_n$.

Counter-example 1 Let $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n^3}$. Then $a_n > b_n$ for all $n > 1$ and $\sum a_n$ converges.

Counter example 2. Let $a_n = n$, $b_n = \frac{1}{n^2}$. Then $a_n > b_n$ for all $n > 1$. But $\sum a_n$ diverges.

(b) If $a_n < b_n$ for all n, then $\sum a_n$ converges (the comparison test.) Further, $0 \le \sum a_n \le \sum b_n$.

$\boxed{11.4.7}$ $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges

Comparison Test: $0 \le \frac{1}{n} = \frac{n}{n^2} < \frac{n+1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Longrightarrow \sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges

Limit Comparison Test: $\frac{1}{n} > 0$, $\frac{n+1}{n^2} > 0$;

$$\lim_{n\to\infty} \frac{\left(\frac{n+1}{n^2}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+1}{n^2} \cdot n$$

$$= \lim_{n\to\infty} \frac{n+1}{n} = 1 > 0;$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges

$\boxed{11.4.17}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges

Comparison Test: $0 \le \frac{1}{\sqrt{2}\,n} = \frac{1}{\sqrt{n^2+n^2}} \le \frac{1}{\sqrt{n^2+1}}$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2}\,n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges

Limit Comparison Test: $\frac{1}{\sqrt{n^2+1}} > 0$, $\frac{1}{n} = \frac{1}{\sqrt{n^2}} > 0$;

$$\lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n^2}}\right)}{\left(\frac{1}{\sqrt{n^2+1}}\right)} = \lim_{n\to\infty} \sqrt{\frac{n^2+1}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n^2}} = \sqrt{1+0}$$

$= 1 > 0$. and

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.

$\boxed{11.4.23}$ $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ converges.

Comparison Test: $0 \le \frac{5+2n}{(1+n^2)^2} \le \frac{5n+2n}{(n^2)^2} = \frac{7}{n^3}$

and $\sum_{n=1}^{\infty} \frac{7}{n^3}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ converges

Limit Comparison Test: $\frac{5+2n}{(1+n^2)^2} > 0$, $\frac{1}{n^3} > 0$ and

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^3}\right)}{\left[\frac{5+2n}{(1+n^2)^2}\right]} = \lim_{n\to\infty} \frac{(1+n^2)^2}{5+2n} \cdot \frac{1}{n^3}$$

$$= \lim_{n\to\infty} \frac{1+2n^2+n^4}{\frac{5}{n}+2} \cdot \frac{1}{n^4}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^4}+\frac{2}{n^2}+1}{\frac{5}{n}+2} = \frac{0+0+1}{0+2}$$

$= \frac{1}{2} > 0$.

So $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ converges

$\boxed{11.4.27}$ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

Observe that $\left(1+\frac{1}{n}\right)^2 e^{-n} = \left(1+\frac{2}{n}+\frac{1}{n^2}\right)e^{-n}$
$\simeq e^{-n}$ for large n.

Limit Comparison Test $\frac{1}{e^n} > 0$, $\left(1+\frac{1}{n}\right)^2 e^{-n} > 0$.

and $$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 e^{-n}}{e^{-n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2$$

$= 1 > 0$.

So $\sum_{n=1}^{\infty} e^{-n}$ converges $\Rightarrow \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$ converges

a geometric series with $a = e^{-1}$ and $r = \frac{1}{e}$

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11.4.33 Approximate $\sum_{n=1}^{\infty} \frac{1}{n^4+n^2}$

$$\simeq s_{10} = \sum_{k=1}^{10} \frac{1}{k^4+k^2}$$

$$= \frac{122\,131\,997\,547\,449}{215\,030\,621\,332\,800}$$

$$\simeq 0.5679749088$$

Estimating the error

For $n \geq 1$, define $a_n = \frac{1}{n^4+n^2}$

$b_n = \frac{1}{n^4}$

$s = \sum_{n=1}^{\infty} \frac{1}{n^4+n^2}$ $\quad s_n = \sum_{k=1}^{n} a_k$

$t = \sum_{n=1}^{\infty} \frac{1}{n^4}$ $\quad t_n = \sum_{k=1}^{n} b_k$

remainders $\begin{cases} R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots \\ T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots \end{cases}$

Note that $0 < a_n < b_n$ for all $n \geq 1$

So $0 < R_n < T_n$ for all $n \geq 1$ also

In particular,

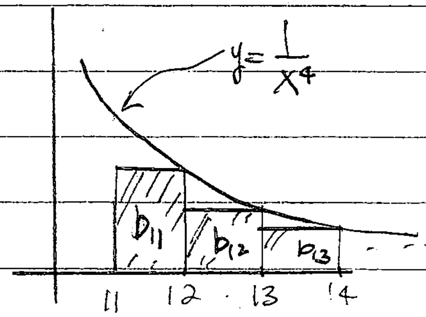
$0 < R_{10} < T_{10}$, so that

the error in estimating S

using S_{10} does NOT

exceed

$$T_{10} = \sum_{n=11}^{\infty} b_n = \sum_{n=11}^{\infty} \frac{1}{n^4}$$



On the other hand, we know (see $\boxed{2}$, p. 727)

$$T_{10} \leq \int_{10}^{\infty} \frac{1}{x^4}\,dx$$

$$= \left[-\frac{1}{3x^3}\right]_{10}^{\infty}$$

$$= \frac{1}{3000}$$

So the error in estimating S using S_{10} does NOT exceed $\frac{1}{3000} \simeq 0.000\overline{3}$

Note*: We can also use $b_n = \frac{1}{n^2}$. However, in this case, we can only show that the error does not exceed $\int_{10}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{10} = 0.1$, not as good as the preceding estimate.

$\boxed{11.4.37}$ Given the decimal representation of a number

$$0.d_1d_2d_3\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots$$
$$= \sum_{n=1}^{\infty} \frac{d_n}{10^n},$$

where each d_n is one of the numbers $0, 1, 2, \ldots, 9$.

It is always convergent by the Comparison test:

$$0 \le \frac{d_n}{10^n} \le \frac{9}{10^n}$$

and $\sum_{n=1}^{\infty} \frac{9}{10^n}$ is a geometric series with $a = \frac{9}{10}$ and $r = \frac{1}{10}$

So $\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$

is convergent. Therefore

$\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ is also convergent.