Section 11.5: Alternating Series

Page 736

Alternating Series Test (AST). If

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

an alternating series

satisfies

(i) $0 \le b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$,

then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

11.5.5 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges.

(i) $0 \le \frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$ for all n

(ii) $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

So $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by AST.

11.5.7 $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$ diverges.

Note that

$$\lim_{n\to\infty}\frac{3n-1}{2n+1} = 3 \ne 0$$

and thus

$$\lim_{n\to\infty}(-1)^n\frac{3n-1}{2n+1} \text{ does not exist.}$$

Hence $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$ diverges by

the Test for Divergence (p. 718):

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \ne 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

11.5.9 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2+1}$ converges.

(i) $0 \le \frac{1}{4(n+1)^2+1} \le \frac{1}{4n^2+1}$ for all n

(ii) $\lim_{n\to\infty}\frac{1}{4n^2+1} = 0$.

So the series converges by AST.

* **11.5.11** $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$

(i) Consider $f(x) = \frac{x^2}{x^3+4}$, $x \ge 1$.

Then $f'(x) = \frac{8x - x^4}{(4+x^3)^2} < 0$

for all sufficiently large x.

(i.e., x such that $x^4 > 8x$ or $x^3 > 8$ or $x > 2$).

So $0 \le \frac{(n+1)^2}{(n+1)^3+4} \le \frac{n^2}{n^3+4}$

for all $n \ge 3$.

(ii) $\lim_{n\to\infty}\frac{n^2}{n^3+4} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{4}{n^3}}$

$= 0$.

So the series converges by AST

Page 2

$\boxed{11.5.13}$ $\sum_{n=2}^{\infty}(-1)^n \frac{n}{\ln n}$ diverges

This step is not needed → 1. Consider $f(x) = \frac{x}{\ln x}$, $x \geq 2$

$$f'(x) = \frac{\ln x - 1}{[\ln(x)]^2} > 0$$

for $\ln x > 1$ or $x > e$.

2. Furthermore,

$$\lim_{x\to\infty} \frac{x}{\ln x} \overset{\text{l'Hopital}}{=\!=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}}$$

$$= +\infty$$

Hence $\lim_{n\to\infty} \frac{n}{\ln n}$ does not exist (as a finite number) So $\sum_{n=2}^{\infty}(-1)^n \frac{n}{\ln n}$ diverges by the test for divergence.

$\boxed{11.5.15}$ $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}$ converges.

Hint: 1. $\cos(n\pi) = (-1)^n$ for all integer n.

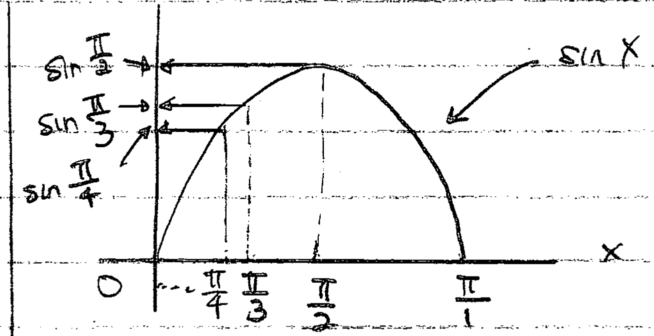
2. Use AST.

Interesting

$\boxed{11.5.17}$ $\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)$ converges

(i) Note that

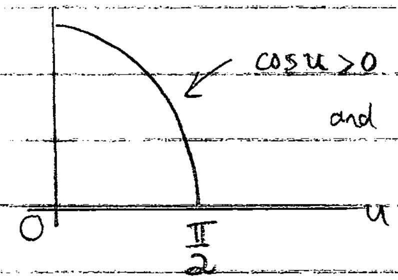
$$0 \leq \sin \frac{\pi}{n+1} \leq \sin \frac{\pi}{n} \text{ for } n \geq 2$$



More rigorously, Consider $f(x) = \sin\left(\frac{\pi}{x}\right)$, $x \geq 1$.

Then $f'(x) = -\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right) < 0$

for $x > 2$



$\cos u > 0$ for $0 < u < \frac{\pi}{2}$

and note that

$0 < \frac{\pi}{x} < \frac{\pi}{2}$

for $x > 2$.

(ii) $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$

$= \sin 0$

$= 0$

this can be done because the sine function is continuous

So $\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)$ converges by AST

$\boxed{11.5.19}$ $\sum_{n=1}^{\infty}(-1)^n \frac{n^n}{n!}$ diverges

Page 3

Consider $\dfrac{n^n}{n!} = \dfrac{\overbrace{n \cdot n \cdot n \cdots n}^{n \text{ Factors}}}{1 \cdot 2 \cdot 3 \cdots n}$

$$= \left(\frac{n}{1}\right)\left(\frac{n}{2}\right)\left(\frac{n}{3}\right)\cdots\left(\frac{n}{n}\right)$$

For $n = 2m$ for some m,

$$\frac{n^n}{n!} = \underbrace{\left(\frac{2m}{1}\right)}_{\geq 2}\underbrace{\left(\frac{2m}{2}\right)}_{\geq 2}\underbrace{\left(\frac{2m}{3}\right)}_{\geq 2}\cdots\underbrace{\left(\frac{2m}{m}\right)}_{\geq 2}\underbrace{\left(\frac{2m}{m+1}\right)}_{\geq 1}\cdots\underbrace{\left(\frac{2m}{2m}\right)}_{\geq 1}$$

$$\geq 2^m \xrightarrow[(m\to\infty)]{n\to\infty} +\infty$$

n	S_n
1	1
2	$\simeq 0.646\,446\,6094$
3	$\simeq 0.838\,896\,6991$
4	$\simeq 0.713\,896\,6991$
5	$\simeq 0.803\,339\,4182$
6	$\simeq 0.735\,298\,0365$
7	$\simeq 0.789\,292\,9612$
8	$\simeq 0.745\,098\,7874$
9	$\simeq 0.782\,135\,8244$
10	$\simeq 0.750\,513\,0478$

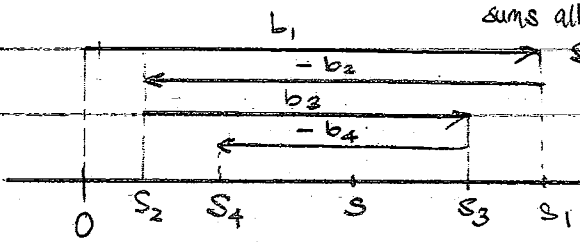
For $n = 2m-1$ for some m,

$$\frac{n^n}{n!} = \underbrace{\left(\frac{2m-1}{1}\right)}_{\geq 2}\underbrace{\left(\frac{2m-1}{2}\right)}_{\geq 2}\cdots\underbrace{\left(\frac{2m-1}{m-1}\right)}_{\geq 2}\underbrace{\left(\frac{2m-1}{m}\right)}_{\geq 1}\cdots\underbrace{\left(\frac{2m-1}{2m-1}\right)}_{\geq 1}$$

$$\geq 2^{m-1} \xrightarrow[(m\to\infty)]{n\to\infty} +\infty$$

$b_n \overset{\text{def}}{=} \dfrac{1}{n^{3/2}}$

Remark: Given any S_n and S_{n+1}, the subsequent partial sums all lie between S_n and S_{n+1}.



Hence $\lim\limits_{n\to\infty} \dfrac{n^n}{n!} = +\infty$. Thus the series diverges by the test for divergence

Note that S lies between ANY two consecutive partial sums S_n and S_{n+1}.

So ~~either~~

$\boxed{11.5.21}$ Set

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}}$$

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{3/2}}$$

$S_n \leq S \leq S_{n+1} \longrightarrow S - S_n \leq S_{n+1} - S_n$

or $S_{n+1} \leq S \leq S_n \longrightarrow S - S_n \leq S_n - S_{n+1}$

In either case

$$|S - S_n| \leq |S_{n+1} - S_n| = b_{n+1}$$

$$n! \overset{\text{def}}{=} \begin{cases} 1 & \text{if } n=0 \\ (1)(2)(3)\cdots(n-1)(n) & \text{if } n=1,2,3,\ldots \end{cases}$$

So $|S - S_{10}| \le b_{11} = \dfrac{1}{11^{3/2}}$

$\simeq 0.027410122$

* In fact, we need to make sure the series is a convergent alternating series

$\boxed{11.5.23}$ Set $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

and $S_n = \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n^2}$. Then

$$|S - S_n| \le b_{n+1} \overset{\text{def}}{=} \frac{1}{(n+1)^2}.$$

So, to make sure the error $|S - S_n| < 0.01$, it suffices to demand that

$$b_{n+1} = \frac{1}{(n+1)^2} < 0.01$$

or $(n+1)^2 > \frac{1}{0.01} = 100$

or $n+1 > \sqrt{100} = 10$

or $n > 9$

or $n \ge 10$, i.e.,

we need at least 10 terms to find the sum to the indicated accuracy.

$\boxed{11.5.25}$ Set $S = \sum_{n=1}^{\infty} \frac{(-2)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$

and $S_n = \sum_{k=1}^{n} \frac{(-2)^k}{k!} = \sum_{k=1}^{n} \frac{(-1)^k 2^k}{k!}$ and $b_n = \frac{2^n}{n!}$

(i) $$\frac{b_{n+1}}{b_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \frac{2^{n+1}}{2^n} \cdot \frac{(1)(2)(3)\cdots(n)}{(1)(2)(3)\cdots(n)(n+1)}$$

$$= \frac{2}{n+1} \le 1 \text{ for } n \ge 1$$

So $0 \le b_{n+1} \le b_n$

(ii) $$b_n = \frac{\overbrace{(2)(2)(2)\cdots(2)}^{n \text{ factors}}}{(1)(2)(3)\cdots(n)}$$

For $n \ge 4$

$$= \underbrace{\left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)}_{\frac{4}{3}}\underbrace{\left(\frac{2}{4}\right)}_{\le\frac{1}{2}}\cdots\underbrace{\left(\frac{2}{n-1}\right)}_{\le\frac{1}{2}}\underbrace{\left(\frac{2}{n}\right)}_{\le\frac{1}{2}}$$

$$\le \frac{4}{3}\left(\frac{1}{2}\right)^{n-3}$$

$$= \frac{4}{3}\left(\frac{1}{2^n}\right)$$

So for $n \ge 4$, $0 \le b_n \le \frac{4}{3}\left(\frac{1}{2^n}\right)$.

Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{4}{3}\left(\frac{1}{2^n}\right)$,

$\lim_{n\to\infty} b_n = 0$ by Squeeze Theorem

Therefore S converges by AST.

To make sure $|S - S_n| < 0.01$, we simply demand that

$$b_{n+1} = \frac{2^{n+1}}{(n+1)!} < 0.01$$

(See previous problem for the reasoning)

***Remark:** For 11.5.27, we actually need to check that the series are CONVERGENT alternating series

n	b_n
1	2
2	2
3	$1.\overline{3}$
4	$0.\overline{6}$
5	$0.2\overline{6}$
6	$0.0\overline{8}$
7	0.0253968
8	$0.00634921 < 0.01$

Since $b_8 < 0.01$, we need only to compute S_7 to approximate s with an error < 0.01.

Observation. Let $s = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$, $t = \sum_{n=1}^{\infty} \frac{b_n}{10^n}$ be the decimal representations of two numbers s and t. If $a_1 = b_1$, $a_2 = b_2, \ldots, a_d = b_d$, then $|s-t| \leq 10^{-d}$.

Proof. We may suppose $s \geq t$. Then

$$0 \leq s - t = \sum_{n=d+1}^{\infty} \frac{a_n - b_n}{10^n}$$

$$\leq \sum_{n=d+1}^{\infty} \frac{9}{10^n}$$

$$= \frac{9}{10^{d+1}} \cdot \frac{1}{1-\frac{1}{10}}$$

$$= \frac{1}{10^d}$$

$\boxed{11.5.27}$ Set $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$, $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^5}$, and $b_n = \frac{1}{n^5}$. To make sure S_n approximate s correct to four decimal places, we need at least

$$|s - S_n| < 0.0001 = 10^{-4}$$

To make sure this is the case, we can demand that

$$b_{n+1} = \frac{1}{(n+1)^5} < 0.0001$$

or $\quad n > \sqrt[5]{10\,000} - 1$

$\quad\quad \simeq 5.3096\ldots$

or $\quad n \geq 6$

Now, $S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^5} \simeq 0.972\,080\,0630$

$$b_7 = \frac{1}{7^5} \simeq 0.000\,059\,499 < 0.0001$$

and hence adding b_7 to S_6 does NOT change the fourth decimal place of S_6. So

$s = 0.9721$ (correct to four decimal places)

Note that $0 < b_7 - b_8 < b_7$

$$0 < \overbrace{b_7 - b_8}^{>0} \underbrace{+ b_9}_{} < b_7$$

$$0 < \overbrace{b_7 - b_8}^{>0} + \underbrace{b_9 - b_{10}}_{<0}{}^{>0} < b_7$$

$$0 < \overbrace{b_7 - b_8}^{>0} \overbrace{+ b_9 - b_{10}}^{>0} + b_{11} < b_7$$

$\vdots$

Hence the fact that adding b_7 to S_6 does

Page 6

NOT change the fourth decimal place of S_6 also shows that

$$S_n = 0.9721,$$

when corrected to four decimal places, for all $n \geq 6$.

11.5.29 Set $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$,

$S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1} k^2}{10^k}$, and $b_n = \frac{n^2}{10^n}$

To make sure S_n approximate S correct to four decimal places, we need at least

$$|S - S_n| < 10^{-4}.$$

To ensure this is the case, we demand that

$$b_{n+1} = \frac{(n+1)^2}{10^{n+1}} < 10^{-4}$$

n	b_n
1	0.1
2	0.04
3	0.009
4	0.001 6
5	0.000 25
6	0.000 036 $< 10^{-4}$
7	4.9×10^{-6}

n	S_n
5	0.067 650 000 0
6	0.067 614 000 0
7	0.067 618 900 0

Nevertheless, note that

$S_5 = 0.0677$ (corrected to four decimal places)

$S_6 = S_5 - b_6 = 0.0676$ (corrected to four decimal places).

So, we cannot take S_5 as an approximation to S corrected to four decimal places.

However, since

$$S_7 = S_6 + b_7 = 0.0676\ldots$$

agrees with S_6 up to four decimal places,

$$S = 0.0676$$

corrected to four decimal places.

Note: $0 < b_7 - b_8 < b_7$

$0 < \underbrace{b_7 - b_8}_{>0} + b_9 < b_7$

$0 < \overbrace{b_7 - b_8}^{>0} + \underbrace{b_9 - b_{10}}_{>0} < b_7$

$0 < \overbrace{b_7 - b_8}^{>0} + \overbrace{b_9 - b_{10}}^{>0} + b_{11} < b_7$

The fact that adding b_7 to S_6 does NOT change the fourth decimal place of S_6 shows that when $n \geq 6$, $S_n = 0.0676$ (corrected to four decimal places)