Sec 11.6: Absolute Convergence, the Ratio Test, and the Root Test

Definition

1. $\sum |a_n|$ converges $\Longrightarrow$ $\sum a_n$ converges absolutely.
2. $\left.\begin{array}{l}\sum |a_n| \text{ diverges,} \\ \sum a_n \text{ converges}\end{array}\right\} \Longrightarrow \sum a_n$ converges conditionally

p.741 * Remark It is a fact that if $\sum a_n$ converges absolutely, then it converges.

p.~~742~~ Ratio Test Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

1. $L < 1 \Longrightarrow \sum |a_n|$ converges
2. $L > 1$ or $L = +\infty \Longrightarrow \sum a_n$ diverges
3. $L = 1 \Longrightarrow$ Ratio Test fails.

The convergence of $\sum a_n$ has to be tested using other tests.

p.744 Root Test Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

1. $L < 1 \Longrightarrow \sum |a_n|$ converges
2. $L > 1$ or $L = +\infty$ $\Longrightarrow \sum a_n$ diverges
3. $L = 1 \Longrightarrow$ Root Test fails.

Other tests have to be used.

General Strategy

$\sum |a_n|$ converges?

YES → $\sum a_n$ converges absolutely

NO → $\sum a_n$ converges?

YES → $\sum a_n$ converges conditionally

NO → $\sum a_n$ diverges

11.6.5

Step 1 $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt[4]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ is a p-series with $p = \frac{1}{4} < 1$, so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ cannot converge absolutely.

Step 2 (i) $0 \le \frac{1}{\sqrt[4]{n+1}} \le \frac{1}{\sqrt[4]{n}}$

(ii) $\lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges conditionally.

11.6.7

Step 1 Since $\lim_{n\to\infty} \frac{n}{5+n} = 1 \neq 0$,

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{5+n} \right| = \sum_{n=1}^{\infty} \frac{n}{5+n}$ diverges by the test for divergence. So $\sum_{n=1}^{\infty} (-1)^n \frac{n}{5+n}$ cannot converge absolutely.

Step 2. In fact, the same test shows that $\sum_{n=1}^{\infty} (-1)^n \frac{n}{5+n}$ diverges.

11.6.9 Note that the terms of $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ are positive. So it either converges absolutely or diverges.

Now,

$$\lim_{n\to\infty} \frac{\frac{1}{[2(n+1)]!}}{\frac{1}{(2n)!}} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{(1)(2)\cdots(2n)}{(1)(2)\cdots(2n)(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)}$$

$$= 0 < 1.$$

So $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges absolutely by ratio test.

11.6.11 Note that

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n e^{\frac{1}{n}}}{n^3}\right|}$$

$$= \lim_{n\to\infty} \left(\frac{e^{\frac{1}{n}}}{n^3}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{e^{\frac{1}{n^2}}}{n^{\frac{3}{n}}}$$

$$= \frac{\lim_{n\to\infty} e^{\frac{1}{n^2}}}{\lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3}$$

$$= \frac{1}{1} = 1,$$ so

the root test FAILS!!

Facts $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$\lim_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 0$.

Consider $\sum_{n=1}^{\infty} \left|\frac{(-1)^n e^{\frac{1}{n}}}{n^3}\right| = \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^3}$.

Now $0 < \frac{e^{\frac{1}{n}}}{n^3} \leq \frac{e}{n^3}$ and

$\sum_{n=1}^{\infty} \frac{e}{n^3} = e\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

So $\sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{1}{n}}}{n^3}$ converges absolutely.

(Since $\sum_{n=1}^{\infty} \left|\frac{(-1)^n e^{\frac{1}{n}}}{n^3}\right|$ converges by comparison test.)

11.6.13 $\lim_{n\to\infty} \left|\frac{(n+1)(-3)^{n+1}}{4^{n+1-1}} \cdot \frac{4^{n-1}}{n(-3)^n}\right|$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{3^{n+1}}{3^n}\right)\left(\frac{4^{n-1}}{4^n}\right)$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)(3)\left(\frac{1}{4}\right)$$

$$= \frac{3}{4} < 1.$$

So $\sum_{n=1}^{\infty} \left|\frac{n(-3)^n}{4^{n-1}}\right|$ converges by ratio test.

Hence $\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$ converges absolutely.

11.6.15 $\lim_{n\to\infty} \frac{10^{n+1}}{(n+1+1)4^{2(n+1)+1}} \cdot \frac{(n+1)4^{2n+1}}{10^n}$

$= \lim_{n\to\infty} \left(\frac{10^{n+1}}{10^n}\right)\left(\frac{n+1}{n+2}\right)\frac{4^{2n+1}}{4^{2n+3}}$

$= \lim_{n\to\infty} (10)\left(\frac{n+1}{n+2}\right)\frac{1}{4^2}$

$= \frac{10}{16} < 1$, so

$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ converges absolutely by ratio test.

11.6.17 Step 1 Since

$0 < \frac{1}{n} \le \frac{1}{\ln n}$ for $n \ge 2$

and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ also diverges by comparison test. So $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ does NOT converge absolutely.

Remark. That $n > \ln n$ for $n \ge 2$ can be shown as follows: Let $f(x) = x - \ln x$. Then $f'(x) = 1 - \frac{1}{x} > 0$ for all $x \ge 2$.

So $f(x) = x - \ln x \ge f(2)$
$= 2 - \ln 2 > 0$ for all $x \ge 2$ or, equivalently, $x > \ln x$ for all $x \ge 2$.

Step 2 (i) $0 < \frac{1}{\ln(n+1)} < \frac{1}{\ln n}$

(ii) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$.

So $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges (conditionally) by Alternating Series Test.

11.6.19 Since

$\left|\frac{\cos\frac{n\pi}{3}}{n!}\right| \le \frac{1}{n!}$ and $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges, $\sum_{n=1}^{\infty} \left|\frac{\cos\frac{n\pi}{3}}{n!}\right|$ converges by Comparison test, and $\sum_{n=1}^{\infty} \frac{\cos\frac{n\pi}{3}}{n!}$ converges absolutely.

Remark The converges of $\sum_{n=1}^{\infty} \frac{1}{n!}$ can be validated using the ratio test. See also problem 11.4.29. In fact, we will see in later section that $\sum_{n=1}^{\infty} \frac{1}{n!} = e$.

11.6.21 Since

$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^{1+3n}}} = \lim_{n\to\infty} \left(\frac{n^n}{3^{1+3n}}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{n^{\frac{n}{n}}}{3^{\frac{1+3n}{n}}}$

$= \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+3}}$

$= \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}} \cdot 3^3} = +\infty$

So $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ diverges by root test.

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}$$

11.6.23 Since $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n}$

$$= \lim_{n\to\infty} \frac{n^2+1}{2n^2+1}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2} < 1,$$

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ converges absolutely by root test.

11.6.25 Consider

$$1 - \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} - \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots + (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!} + \cdots.$$

Since

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1-1}1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(2n+1)!}\cdot\frac{(2n-1)!}{(-1)^{n-1}1\cdot 3\cdot 5\cdots(2n-1)}\right|$$

$$= \lim_{n\to\infty} \frac{(2n+1)\,\cancel{(1)(2)\cdots}(2n-1)}{(1)(2)\cdots(2n-1)(2n)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{(2n+1)}{(2n)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{2n} = 0 < 1, \text{ so}$$

the series converges absolutely by ratio test.

11.6.27 $\displaystyle\lim_{n\to\infty} \frac{(2)(4)\cdots(2n)[2(n+1)]}{(n+1)!}\,\frac{n!}{(2)(4)\cdots(2n)}$

$$= \lim_{n\to\infty} \frac{2n+2}{n+1} = \lim_{n\to\infty} 2 = 2$$

> 1, so $\displaystyle\sum_{n=1}^{\infty} \frac{(2)(4)\cdots(2n)}{n!}$ diverges by the ratio test.

Alternatively Note that

$$\frac{(2)(4)(6)\cdots(2n)}{n!} = \frac{(2\times 1)(2\times 2)(2\times 3)\cdots(2\times n)}{n!}$$

$$= \frac{2^n\, n!}{n!} = 2^n$$

So $\displaystyle\lim_{n\to\infty} \frac{(2)(4)\cdots(2n)}{n!} = +\infty.$

Hence the series diverges by test for divergence.

11.6.29 Consider $\sum a_n$, where $a_1 = 2$ and $a_{n+1} = \frac{5n+1}{4n+3}a_n$. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5n+1}{4n+3}$$

$= \frac{5}{4} > 1$, so the series diverges by ratio test