Section 11.7: Strategy for Testing Series

WARNING: This may help in guessing the right test, BUT it is not a sufficient justification:

11.7.1 Step 1. Estimate the nth term: $\dfrac{n^2-1}{n^2+n} \simeq \dfrac{n^2}{n^2} = 1$

for very large n, $n^2-1 \simeq n^2$ and $n^2+n \simeq n^2$

Step 2. From step 1, we guess that the series diverges because $\dfrac{n^2-1}{n^2+n} \simeq 1 \neq 0$. Now, we can justify this by calcuting

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+n} = \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n}} = 1 \neq 0$$

Hence the series diverges by test for divergence

11.7.3 Step 1. Estimate the nth term: $\dfrac{1}{n^2+n} \simeq \dfrac{1}{n^2}$ for large n.

NOT a sufficient justification!

Step 2. From step 1, we see that the series is comparable to $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Now

$$0 \leq \frac{1}{n^2+n} \leq \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Hence $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ also converges by comparison test

11.7.5 $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{2^{3n}}$

is an alternating series with

(i) $0 \leq \underbrace{\dfrac{3^{(n+1)+1}}{2^{3(n+1)}} = \dfrac{3^{n+2}}{2^{3n+3}} \leq \dfrac{3^{n+1}}{2^{3n}}}$

Proof

$$\frac{\left(\frac{3^{n+2}}{2^{3n+3}}\right)}{\left(\frac{3^{n+1}}{2^{3n}}\right)} = \frac{3^{n+2}}{2^{3n+3}} \cdot \frac{2^{3n}}{3^{n+1}} = \frac{3}{2^6} < 1$$

(ii) $\displaystyle\lim_{n\to\infty} \frac{3^{n+1}}{2^{3n}} = \lim_{n\to\infty} \frac{3 \cdot 3^n}{8^n} = \lim_{n\to\infty} 3\left(\frac{3}{8}\right)^n = 0.$

By (i), (ii) and the alternating series test, the series converges.

Alternatively we can use the fact that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-3)^{n+1}}{2^{3n}}\right| = \sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{3n}}$ converges (as a consequence of ratio test) to conclude the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$.

Note: Absolute convergence of $\sum a_n$ implies convergence of $\sum a_n$.

But, we can show using integral test that this series is not absolutely convergent.

11.7.7 Setting $f(x) = \frac{1}{x\sqrt{\ln x}}$

for $x \geq 2$, we have

(i) f is continuous

(ii) $f(x) > 0$

(iii) $f'(x) = \frac{-1 - 2\ln x}{2x^2(\ln x)^{\frac{3}{2}}} < 0$

so f is decreasing

(More simply since $x\sqrt{\ln x}$ increases, so f decreases.)

Also,

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx \underset{du = \frac{1}{x}dx}{\overset{u = \ln x}{=}} \int_{\ln 2}^\infty \frac{1}{\sqrt{u}}\,du$$

$$= \left[2\sqrt{u}\right]_{\ln 2}^\infty \text{ diverges.}$$

So $\sum_{n=2}^\infty \frac{1}{n\sqrt{\ln n}}$ diverges by integral test.

11.7.9 Since

$$\lim_{k\to\infty} \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 \frac{1}{e}$$

$= \frac{1}{e} < 1$, the series $\sum_{k=1}^\infty \frac{k^2}{e^k}$ converges by ratio test.

11.7.11 $\sum_{n=2}^\infty \frac{(-1)^{n+1}}{n\ln n}$ is an alternating series

with (i) $0 \leq \frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n\ln n}$

(ii) $\lim_{n\to\infty} \frac{1}{n\ln n} = 0$

So the series converges by alternating series test.

11.7.13 Since

$$\lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}$$

$$= \lim_{n\to\infty} \left(\frac{3^{n+1}}{3^n}\right)\left(\frac{n+1}{n}\right)^2 \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} 3\left(1+\frac{1}{n}\right)^2 \frac{1}{n+1} = 0 < 1,$$

the series $\sum_{n=1}^\infty \frac{3^n n^2}{n!}$ converges by ratio test.

11.7.15 We have

$$\lim_{n\to\infty} \frac{(n+1)!}{(2)(5)(8)\cdots(3n+2)\underbrace{[3(n+1)+2]}_{3n+5}} \cdot \frac{(2)(5)\cdots(3n+2)}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{3n+5}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3+\frac{5}{n}} = \frac{1}{3} < 1.$$

So $\sum_{n=0}^\infty \frac{n!}{(2)(5)\cdots(3n+2)}$ Converges by ratio test.

11.7.17 Though $\sum_{n=1}^{\infty}(-1)^n 2^{1/n}$ is an alternating series, the alternating series test does NOT apply because $\lim_{n\to\infty} 2^{1/n} = 1 \neq 0$.

However, the fact that $\lim_{n\to\infty} 2^{1/n} \neq 0$ suffices to show that the series diverges by the test for divergence, since then $\lim_{n\to\infty} (-1)^n 2^{1/n}$ does not exist.

11.7.19 We consider $\sum_{n=1}^{\infty}\left|(-1)^n \frac{\ln n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ instead.

You can ignore this, unless you want to see that the series does not converge absolutely.

Set $f(x) = \frac{\ln x}{\sqrt{x}}$ for $x \geq 1$. Then

(i) f is continuous

(ii) $f(x) \geq 0$

(iii) $f'(x) = \frac{2 - \ln x}{2x^{3/2}} \leq 0$ for $x \geq e^2$, i.e., f is eventually decreasing

Also, $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\ln x}{\sqrt{x}}\,dx$

$x = e^u \leftrightarrow u = \ln x$

$du = \frac{1}{x}dx$

$\therefore du = \frac{1}{e^u}dx$

$e^u du = dx$

$$= \int_0^{\infty} \frac{u}{e^{\frac{1}{2}u}} e^u du$$

$$= \int_0^{\infty} u e^u du$$

$$= \left[ue^u - e^u\right]_0^{\infty}$$

$$= \left[e^u(u-1)\right]_0^{\infty} \text{ diverges.}$$

So $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$ does NOT converges ABSOLUTELY.

However, note that

(a) $0 \leq \frac{\ln(n+1)}{\sqrt{n+1}} \leq \frac{\ln n}{\sqrt{n}}$ for all sufficiently large n

(Recall that $f(x) = \frac{\ln x}{\sqrt{x}}$ is eventually decreasing)

(b) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\text{l'Hopital}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$

$$= \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0,$$

so that $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$.

Also, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ is an alternating series. Thus it is (conditionally) convergent.

11.7.21 We consider $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} 2^{2n}}{n^n}$

$$= \sum_{n=1}^{\infty} \frac{2^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n.$$

Note that for all $n \geq 5$,

$$0 \leq \left(\frac{4}{n}\right)^n \leq \left(\frac{4}{5}\right)^n.$$

Furthermore,
$\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ is a convergent geometric series with $a=\frac{4}{5}$ and $r=\frac{4}{5}$. Thus $\sum_{n=1}^{\infty}\frac{(-2)^{2n}}{n^n}$ converges by comparison test.

$\boxed{11.7.23}$ Note that

$\tan\left(\frac{1}{n}\right)>0$ for all $n\geq 1$.

Furthermore,

$$\lim_{n\to\infty}\frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\cdot\frac{1}{\cos\left(\frac{1}{n}\right)}$$

$$=1>0$$

[Remark: $\lim_{x\to 0}\frac{\sin x}{x}=1$]

So by the limit comparison test, $\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$ diverges because so does $\sum_{n=1}^{\infty}\frac{1}{n}$

$\boxed{11.7.25}$ Since

$$\lim_{n\to\infty}\frac{(n+1)!}{e^{(n+1)^2}}\frac{e^{n^2}}{n!}$$

$$=\lim_{n\to\infty}\frac{(n+1)!}{n!}\frac{e^{n^2}}{e^{n^2+2n+1}}$$

$$=\lim_{n\to\infty}\frac{n+1}{e^{2n+1}}=0<1,$$

$\sum_{n=1}^{\infty}\frac{n!}{e^{n^2}}$ converges by ratio test.

$\boxed{11.7.27}$ Note that

$$0\leq\frac{k\ln k}{(k+1)^3}\leq\frac{k\ln k}{k^3}=\frac{\ln k}{k^2}$$

So if $\sum_{k=1}^{\infty}\frac{\ln k}{k^2}$ converges, then $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$ also converges by comparison test. Now, set $f(x)=\frac{\ln x}{x^2}$. for $x\geq 1$. Then

(i) f is continuous

(ii) $f(x)\geq 0$

(iii) $f'(x)=\frac{1-2\ln x}{x^3}\leq 0$

for $\ln x\geq\frac{1}{2}$ or $x\geq e^{\frac{1}{2}}$.

So $f(x)$ is eventually decreasing.

Finally, $\int_1^{\infty}f(x)\,dx$

$$=\int_1^{\infty}\frac{\ln x}{x^2}dx\ \underset{\substack{du=\frac{1}{x}dx\\ x=e^u}}{\overset{u=\ln x}{=\!=}}\ \int_0^{\infty}ue^{-u}\,du$$

$$=\left[-ue^{-u}-e^{-u}\right]_1^{\infty}$$

$$=-\left[\frac{(u+1)}{e^u}\right]_0^{\infty}=1\quad\text{converges}$$

So $\sum_{k=1}^{\infty}\frac{\ln k}{k^2}$ converges by integral test, and this also implies the convergence of $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$

* $\boxed{11.7.29}$ Note that

$0 \leq \tan^{-1}(n) \leq \lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$.

[To see this more rigorously,
Let $g(x) = \tan^{-1}(x)$ for $x \geq 0$.
Then $g'(x) = \frac{1}{1+x^2} > 0$,
showing that g is increasing.
Therefore

$$g(x) = \tan^{-1}(x) \leq \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}.$$]

Hence $0 \leq \frac{\tan^{-1}(n)}{n\sqrt{n}} \leq \frac{\pi/2}{n^{3/2}}$.

But $\sum_{n=1}^{\infty} \frac{\pi/2}{n^{3/2}} = \pi/2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. So

$\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$ converges by Comparison test.

$\boxed{11.7.31}$ Since

$$\lim_{k\to\infty} \frac{5^k}{3^k+4^k} = \lim_{k\to\infty} \frac{(5/4)^k}{(3/4)^k+1} = +\infty \neq 0$$

because $\lim_{k\to\infty}\left(\frac{3}{4}\right)^k = 0$ and $\lim_{k\to\infty}\left(\frac{5}{4}\right)^k = \infty$,

$\sum_{k=1}^{\infty} \frac{5^k}{3^k+5^k}$ diverges by test for divergence.

$\boxed{11.7.33}$ Recall that

$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$, so we have the estimate $\frac{\sin(\frac{1}{n})}{\frac{1}{n}} \simeq 1$

or $\sin\left(\frac{1}{n}\right) \simeq \frac{1}{n}$ for all sufficiently large n.

Also,

$$\frac{\sin(\frac{1}{n})}{\sqrt{n}} \simeq \frac{\frac{1}{n}}{\sqrt{n}} \text{ for large } n.$$

This leads us to compare $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}}$ to $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$. Now,

$$\lim_{n\to\infty} \left| \frac{\sin(\frac{1}{n})}{\sqrt{n}} \Big/ \frac{1}{n\sqrt{n}} \right| = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}} \cdot \frac{n\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}}$$

$= 1 > 0$. So

$\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}}$ converges since so does $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$. (We use limit comparison test.)

11.7.35 Remark:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Now

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{\frac{n^2}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{n}$$

$$= \lim_{n\to\infty}\frac{1^n}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1.$$

So $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges by root test.

11.7.37 Remark:

$$\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty}\sqrt[n]{a} = 1$$

for any $a>0$.

Now $\lim_{n\to\infty}\sqrt[n]{\left(\sqrt[n]{2}-1\right)^n}$

$$= \lim_{n\to\infty}\left[\left(\sqrt[n]{2}-1\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\sqrt[n]{2}-1\right)$$

$$= 1-1 = 0 < 1.$$

So $\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)^n$ converges by root test.