Section 11.9: Representations of Functions as Power Series

p. 756 **Theorem.** If

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has a radius of convergence $R>0$, then it is differentiable on the interval $(a-R, a+R)$ and

$$\text{(i) } f'(x) = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$\text{(ii) } \int f(x)\,dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots$$

have the same radii of convergence.

Note: The intervals of convergence of f, f', and $\int f\,dx$ may differ, though the radii of convergence are the same. See problem 36.

$\boxed{11.9.1}$ Both $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=1}^{\infty} n c_n x^{n-1}$ have the same radius of convergence, since

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} c_n \frac{d}{dx} x^n = \frac{d}{dx} \sum_{n=0}^{\infty} c_n x^n .$$

See theorem on p. 756.

Another useful fact (p. 754)

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

$\boxed{11.9.5}$

$$f(x) = \frac{1}{1-x^3} = 1 + x^3 + (x^3)^2 + \cdots = \sum_{n=0}^{\infty} (x^3)^n \text{ for } |x^3|<1 = \sum_{n=0}^{\infty} x^{3n} \text{ for } \underbrace{|x|<1}_{\text{interval of convergence}}$$

11.9.6

$$f(x)=\frac{1}{1+9x^2}$$
$$=\frac{1}{1-(-9x^2)}$$
$$=1+(-9x^2)+(-9x^2)^2+\cdots$$
$$=\sum_{n=0}^{\infty}(-9x^2)^n \text{ for } |-9x^2|<1$$
$$=\sum_{n=0}^{\infty}(-1)^n 9^n x^{2n} \text{ for } 9x^2<1$$
$$=\sum_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n} \text{ for } x^2<\frac{1}{9}$$
$$=\sum_{n=0}^{\infty}(-1)^n (3x)^{2n} \text{ for } \underbrace{-\frac{1}{3}<x<\frac{1}{3}}_{\text{interval of convergence}}$$

11.9.7

$$f(x)=\frac{1}{x-5}$$
$$=\frac{1}{5}\,\frac{1}{\frac{x}{5}-1}$$
$$=-\frac{1}{5}\cdot\frac{1}{1-\frac{x}{5}}$$
$$=-\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \text{ for } \left|\frac{x}{5}\right|<1$$
$$=-\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \text{ for } \underbrace{|x|<5}_{\text{interval of convergence}}$$

11.9.9

$$f(x)=\frac{x}{9+x^2}$$
$$=\frac{x}{9}\,\frac{1}{1+\left(\frac{x}{3}\right)^2}$$
$$f(x)=\frac{x}{9}\,\frac{1}{1-\left[-\left(\frac{x}{3}\right)^2\right]}$$
$$=\frac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n \text{ for } \left|-\left(\frac{x}{3}\right)^2\right|<1$$
$$=\frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{3^{2n}} \text{ for } x^2<9$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{3^{2n+2}} \text{ for } \underbrace{-3<x<3}_{\text{interval of convergence}}$$

11.9.11

$$f(x)=\frac{3}{x^2+x-2}$$
$$=\frac{3}{(x+2)(x-1)}$$
$$=\frac{A}{x+2}+\frac{B}{x-1}$$
$$=\frac{(A+B)x+2B-A}{(x+2)(x-1)}$$

So $\left.\begin{array}{l}2B-A=3\\ A+B=0\end{array}\right\}\ \begin{array}{l}B=1\\ A=-1\end{array}$

and $f(x)=\dfrac{1}{x-1}-\dfrac{1}{x+2}$
$$=\frac{-1}{1-x}-\frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right)$$
$$=-\left(\frac{1}{1-x}\right)-\frac{1}{2}\left[\frac{1}{1-(-\frac{x}{2})}\right]$$
$$=-\sum_{n=0}^{\infty}x^n-\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$
for $|x|<1$ and $\left|-\frac{x}{2}\right|<1$

$$f(x) = -\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n}$$

for $\underbrace{|x|<1 \text{ and } |x|<2}_{\text{so } |x|<1}$

$$= -\left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}\right]$$

for $|x|<1$

$$= -\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} + 1\right] x^n$$

for $\underbrace{|x|<1}_{\text{interval of convergence}}$

$\boxed{11.9.13\,(a)}$ To use differentiation to find a power series representation for $\frac{1}{(1+x)^2}$,

We first find a function $g(x)$ such that

$$g'(x) = \frac{1}{(1+x)^2}.$$

Thus
$$g(x) = \int \frac{1}{(1+x)^2}\,dx$$
$$\overset{u=1+x}{=} \int \frac{1}{u^2}\,du$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{1+x} + C$$
$$= -\sum_{n=0}^{\infty} (-x)^n + C$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} x^n + C \quad \text{for } |x|<1$$

Therefore
$$\frac{1}{(1+x)^2} = g'(x)$$
$$= \frac{d}{dx}\sum_{n=0}^{\infty} (-1)^{n+1} x^n$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{d}{dx} x^n$$
$$= \sum_{n=1}^{\infty} n(-1)^{n+1} x^{n-1}$$

with radius of convergence being identical to the radius of convergence of the series representation of $g(x)$, i.e., 1.

$\boxed{11.9.13\,(b)}$ From

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} n(-1)^{n+1} x^{n-1},$$

we have

$$\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right] = \sum_{n=1}^{\infty} n(-1)^{n+1} \frac{d}{dx} x^{n-1}$$

or
$$\frac{-2}{(1+x)^3} = \sum_{n=2}^{\infty} (n-1)n(-1)^{n+1} x^{n-2}.$$

Thus
$$\frac{1}{(1+x)^3} = -\frac{1}{2}\sum_{n=2}^{\infty} (n-1)n(-1)^{n+1} x^{n-2}$$

within the interval of convergence.

11.9.13(c)

$$\frac{x^2}{(1+x)^3}$$

$$= x^2 \cdot -\frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(-1)^{n+1} x^{n-2}$$

$$= -\frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(-1)^{n+1} x^n$$

within the interval of convergence

11.9.17

$$f(x) = \frac{x^3}{(x-2)^2}$$

$$= \frac{x^3}{(2-x)^2}$$

$$= \frac{x^3}{[2(1-\frac{x}{2})]^2}$$

$$= \frac{x^3}{2^2} \cdot \frac{1}{(1-\frac{x}{2})^2}$$

Use problem 11.9.13(a)

$$= \frac{x^3}{2}\sum_{n=1}^{\infty}(-1)^{n+1} n \left(\frac{-x}{2}\right)^{n-1}$$

for $\left|-\frac{x}{2}\right| < 1$

$$= \frac{x^3}{2}\sum_{n=1}^{\infty}(-1)^{n+1} n \frac{(-1)^{n-1} x^{n-1}}{2^{n-1}} \text{ for } |x| < 2$$

$$= \sum_{n=1}^{\infty}(-1)^{2n} n \frac{x^{n-1+3}}{2^{n-1+1}} \text{ for } |x| < 2$$

$$= \sum_{n=1}^{\infty} n \frac{x^{n+2}}{2^n} \text{ for } |x| < 2$$

radius of convergence $= 2$

It can be shown that the interval of convergence is $(-2, 2)$.

11.9.19

$$f(x) + C$$

$$= \ln(3+x) + C$$

$$= \int \frac{1}{3+x} dx$$

$$= \frac{1}{3}\int \frac{1}{1+\frac{x}{3}} dx$$

$$= \frac{1}{3}\int \sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n dx, \quad \left|\frac{-x}{3}\right| < 1$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}\int x^n dx, \quad |x| < 3$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{3^n(n+1)}, \quad |x| < 3$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}, \quad |x| < 3$$

$$\overset{k=n+1}{=} \sum_{k=1}^{\infty}\frac{(-1)^{k-1} x^k}{3^k k}, \quad |x| < 3$$

$$\overset{n=k}{=} \sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^n}{n 3^n}, \quad |x| < 3$$

So

$$f(x) = \ln(3+x)$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^n}{n 3^n} - C$$

for $|x| < 3$

But $\underbrace{f(0)}_{\ln 3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 0^n}{n 3^n} - C$

So $f(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n 3^n}$

$$= \ln 3 + \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{x}{3}\right)^n}$$

for $|x| < 3$

~~Note that this is an~~ alternating series for $x > 0$.

$\boxed{11.9.21}$ Note that

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

$$= \ln(1+x) - \ln(1-x)$$

for $|x| < 1$

Now, $\ln(1+x) + C_1$

$$= \int \frac{1}{1+x}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \,dx, \quad |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^n \,dx, \quad |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, \quad |x| < 1$$

$$\overset{k=n+1}{=} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}, \quad |x| < 1$$

$$\overset{n=k}{=} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1$$

and likewise,

$$\ln(1-x) + C_2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-x)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-1)^n x^n}{n}$$

$$= -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1$$

So $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n} + C$ where $C = C_1 + C_2$

$$= \sum_{n=1}^{\infty} \frac{[1 + (-1)^{n-1}]}{n} x^n + C$$

$$= \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} x^n + C, \quad |x| < 1$$

Note that $1 - (-1)^n = \begin{cases} 0 & (n \text{ even}) \\ 2 & (n \text{ odd}) \end{cases}$

For n odd, we can set

$$n = 2m+1, \quad m = 0, 1, 2, \ldots$$

So $f(x) = \sum_{m=0}^{\infty} \frac{2}{2m+1} x^{2m+1} + C$ for $|x| < 1$

* show that $C = 0$ by considering $f(0) = \ln\left(\frac{1+0}{1-0}\right) = 0$.

nth partial sum is

$$s_n(x) = \sum_{m=0}^{n} \frac{2}{2m+1} x^{2m+1}$$

$$s_0(x) = 2x$$

$$s_1(x) = 2x + \frac{2}{3} x^3$$

$$\vdots$$

11.9.27 Set s to

$$\int_0^{0.2} \frac{1}{1+x^5}\,dx$$

This is valid because the integral is evaluated over $[0, 0.2]$ that is contained in $(-1,1)$

$$= \int_0^{0.2} \sum_{n=0}^{\infty} (-1)^n x^{5n}\,dx$$

$$= \sum_{n=0}^{\infty} \int_0^{0.2} (-1)^n x^{5n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{5n+1}}{5n+1}\right]_0^{0.2}$$

$$s = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n\, 0.2^{5n+1}}{5n+1}}$$

This is an alternating series with nth partial sum given by

$$s_n = \sum_{k=0}^{\infty} \frac{(-1)^k\, 0.2^{5k+1}}{5k+1}$$

To approximate s using s_n to six decimal places, we need at least

$$|s - s_n| < 10^{-6}$$

On the other hand, recall that

$$|s - s_n| < b_{n+1},$$

where $b_n = \dfrac{0.2^{5n+1}}{5n+1}$.

n	b_n
0	0.2
1	0.0000106667
2	$1.86182 \times 10^{-9} < 10^{-6}$
3	4.096×10^{-13}

So a possible approximation to s up to six decimal places is

$$s_1 = 0.199989$$

Since $b_2 < 10^{-7}$, $s = s_1$ up to six decimal places.

11.9.29 We have

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for} \quad -1 \le x \le 1$$

[See example 7 on pp. 757–758, where it is shown that

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \cdots \text{]}.$$

So
$$s = \int_0^{1/3} x^2 \tan^{-1} x^4\,dx$$

$$= \int_0^{1/3} x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^{1/3} x^2 \cdot x^{8n+4}\,dx$$

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^{1/3} x^{8n+6}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[\frac{x^{8n+7}}{8n+7}\right]_0^{1/3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(8n+7)\,3^{8n+7}}$$

Set $S_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)(8k+7)3^{8k+7}}$

and $b_n = \frac{1}{(2n+1)(8n+7)3^{8n+7}}$

To approximate S using S_n to six decimal places, we need at least

$$|S - S_n| < 10^{-6}.$$

To this end, it suffices to require

$$b_{n+1} < 10^{-6}$$

because S is an alternating series.

n	b_n
0	0.0000653211
1	$1.5487 \times 10^{-9} < 10^{-6}$

Since $b_1 < 10^{-7}$,

$S_0 = 0.000065$ (6 decimal places)

approximates S up to 6 decimal places.

$\boxed{11.9.37}$ Given $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

$$= x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots$$

Note that

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = |x|.$$

By ratio test, $f(x)$ converges provided $|x| < 1$.

When $x = 1$, $f(x) = f(1) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

When $x = -1$, $f(x) = f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ also converges.

So the interval of convergence of $f(x)$ is $[-1, 1]$.

Now, $f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$

and it converges at least on $(-1, 1)$ since it has the same radius of convergence as $f(x)$.

When $x = 1$, $f'(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

When $x = -1$, $f'(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges

So $f'(x)$ has interval of convergence $[-1, 1)$.

With $f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$
$= 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots$

we have

$$f''(x) = \sum_{n=2}^{\infty} \frac{n-1}{n} x^{n-2}$$

Since it has the same radius of convergence as $f'(x)$, it converges at least on $(-1, 1)$.

When $x=1$, $f''(x) = \sum_{n=2}^{\infty} \frac{n-1}{n} = \sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)$ diverges because

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right) = 1 \neq 0$$

When $x = -1$, $f''(x) = \sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)(-1)^{n-2}$

diverges also. So $f''(x)$ has interval of convergence $(-1, 1)$.

| 11.9.39 | We have

Example 7, pp. 757 - 758

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{ on } [-1, 1]$$

Recall that $\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

So

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\sqrt{3}^{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3}^2)^n \sqrt{3}}$$

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

$$\boxed{\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}}$$

For additional information, recall that $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. But it is <u>NOT</u> valid to write

$$\frac{\pi}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}^{2n+1}}{2n+1}$$

because $\sqrt{3} \cong 1.73205 > 1$.