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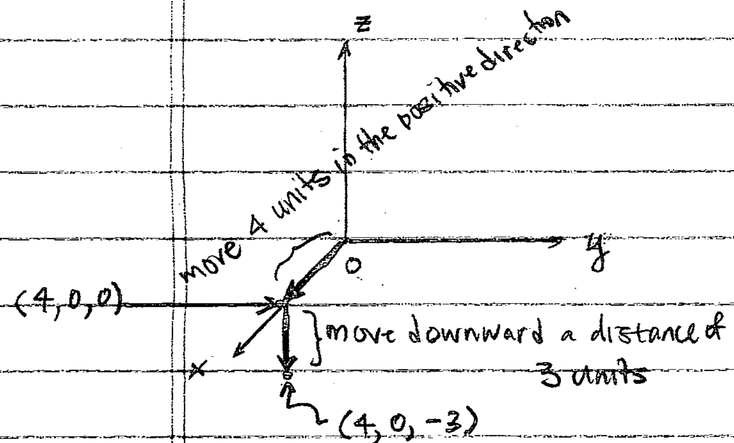
Section 12.1: Three-Dimensional Coordinate Systems

12.1.1

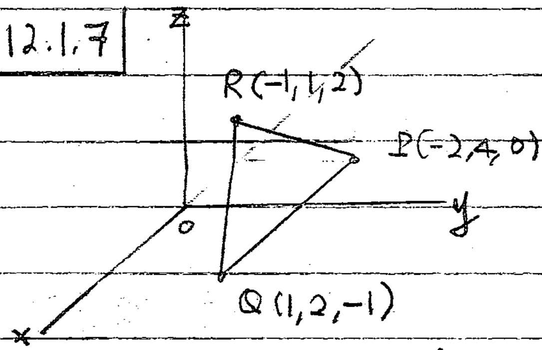


p. 795 **Distance Formula in Three Dimensions**

The distance between two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

12.1.7



$\triangle PQR$ is equilateral because

$$|PQ| = \sqrt{[1-(-2)]^2+(2-4)^2+(-1-0)^2}$$
$$= \sqrt{9+4+1} = \sqrt{14}$$

Similarly, $|QR| = |RP| = \sqrt{14}$.

So $|PQ| = |QR| = |RP|$.

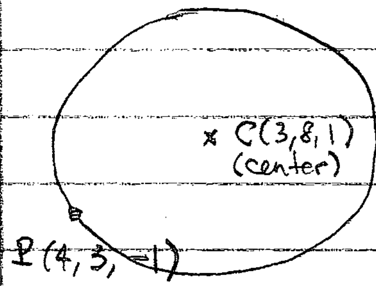
p. 796. **Equation of a Sphere** An equation of a sphere with center $C(h,k,l)$ and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2 = r^2.$$

Remark. The sphere is the set of points $P(x,y,z)$ whose distance from C is r

See example 4, p. 796.

12.1.13 The square of the radius r must be given by



$$r^2 = |PC|^2$$
$$= (4-3)^2+(3-8)^2+(-1-1)^2$$
$$= 30.$$

So an equation of the sphere is

$$(x-3)^2+(y-8)^2+(z-1)^2 = 30.$$

12.1.19 (a) Prove that the midpoint M of the line segment from $P_1(x_1,y_1,z_1)$ to $P_2(x_2,y_2,z_2)$ is given by

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Proof. To begin with, let $M=(x_M,y_M,z_M)$. We shall show that $x_M = (x_1+x_2)/2$. The values of y_M and z_M can be calculated similarly.

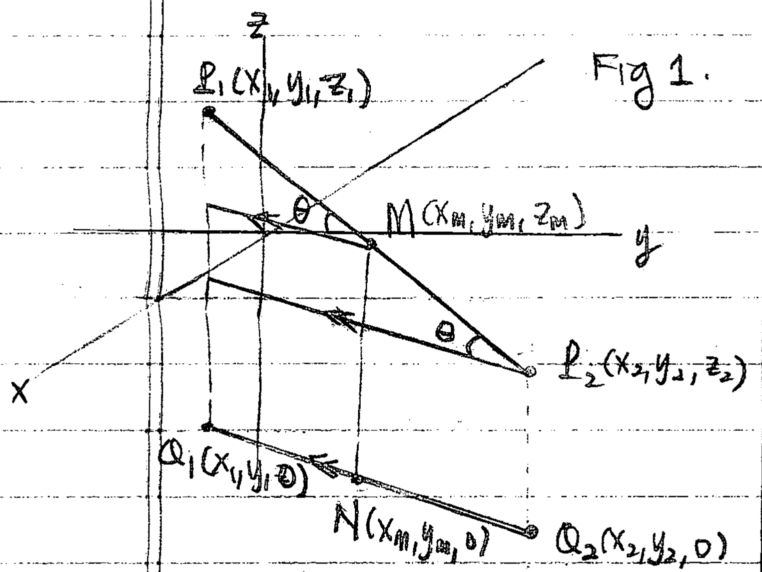


Fig 1.

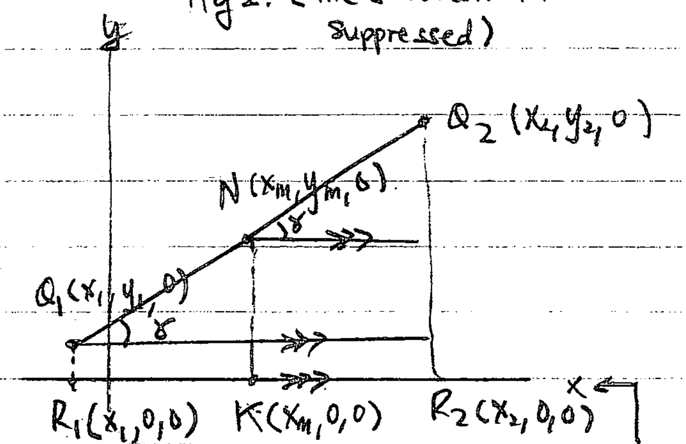
Project the line segment P_1P_2 onto the xy-plane. Then we obtain the points Q_1, Q_2, and N as the projections of the points P_1, P_2, and M respectively onto the xy-plane. See Fig 1. Note that

$$|Q_1N| = |P_1M| \cos\theta,$$
$$|NQ_2| = |MP_2| \cos\theta.$$

Since M is the midpoint of the line segment P_1P_2, $|P_1M| = |MP_2|$. Hence $|Q_1N| = |NQ_2|$ also, from which we see that N is the midpoint of the line segment Q_1Q_2.

Next, project the line segment Q_1Q_2 onto the xz-plane, obtaining the line segment R_1R_2. See Fig 2.

Fig 2. (the z-direction is suppressed)



Here is the location of the xz-plane, with the z-direction being suppressed.

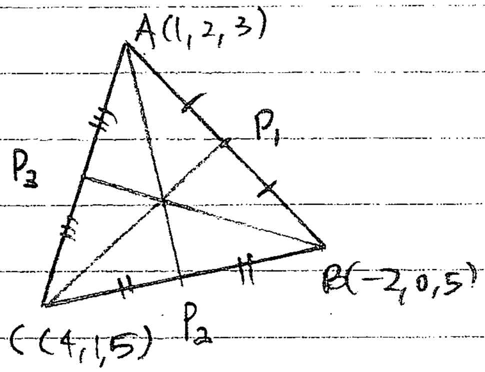
Note that

$$|R_1K| = |Q_1N| \cos\gamma,$$
$$|KR_2| = |NQ_2| \cos\gamma$$

Since N is the midpoint of the line segment Q_1Q_2, $|Q_1N| = |NQ_2|$. Hence $|R_1K| = |KR_2|$, from which we deduce that K is the midpoint of the line segment R_1R_2. In particular, it is clear that

$$x_m = \frac{x_1 + x_2}{2}.$$

12.1.19 (b) Recall that a <u>median</u> of a triangle is a line segment from a vertex to the midpoint of the opposite side.



If P_1, P_2, and P_3 are the midpoints of AB, BC, and CA respectively, then

$$P_1 = \left(\frac{1-2}{2}, \frac{2+0}{2}, \frac{3+5}{2}\right) = \left(-\frac{1}{2}, 1, 4\right)$$

$$P_2 = \left(\frac{-2+4}{2}, \frac{0+1}{2}, \frac{5+5}{2}\right) = \left(1, \frac{1}{2}, 5\right)$$

$$P_3 = \left(\frac{4+1}{2}, \frac{1+2}{2}, \frac{3+5}{2}\right) = \left(\frac{5}{2}, \frac{3}{2}, 4\right)$$

So the lengths of the medians are

$|AP_2| = \frac{5}{2}$,

$|BP_3| = \frac{1}{2}\sqrt{94}$

$|CP_1| = \frac{1}{2}\sqrt{85}$

12.1.23 The equation $y = -4$ represents a plane parallel to the xz-plane and 4 units to its left.

12.1.27 The inequality $0 \le z \le 6$ represents all points on or between the horizontal planes $z = 0$ (the xy-plane) and $z = 6$.

12.1.31

$$x^2 + y^2 + z^2 - 2z < 3$$

$$x^2 + y^2 + (z-1)^2 - 1^2 < 3$$

$$x^2 + y^2 + (z-1)^2 < 4$$

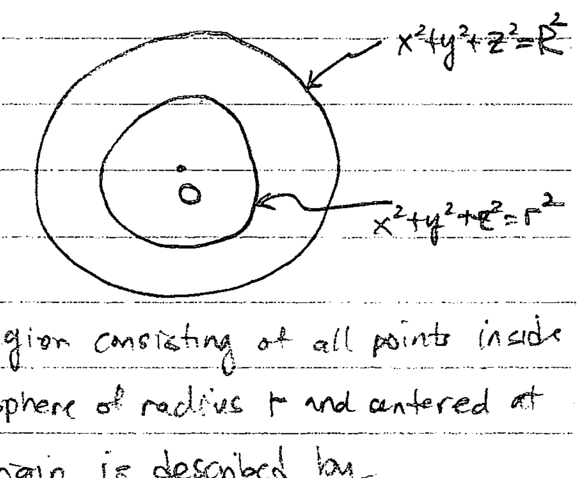
$$\sqrt{x^2 + y^2 + (z-1)^2} < 2$$

This shows that the inequality represents all points whose distance from the point $(0,0,1)$ is less than 2. It is actually the set of points in the interior of the sphere with radius 2 and center $(0,0,1)$.

12.1.37 Given $0 < r < R$.

The region consisting of all points in the interior of the sphere of radius R and centered at the origin is described by

$$x^2 + y^2 + z^2 < R^2$$



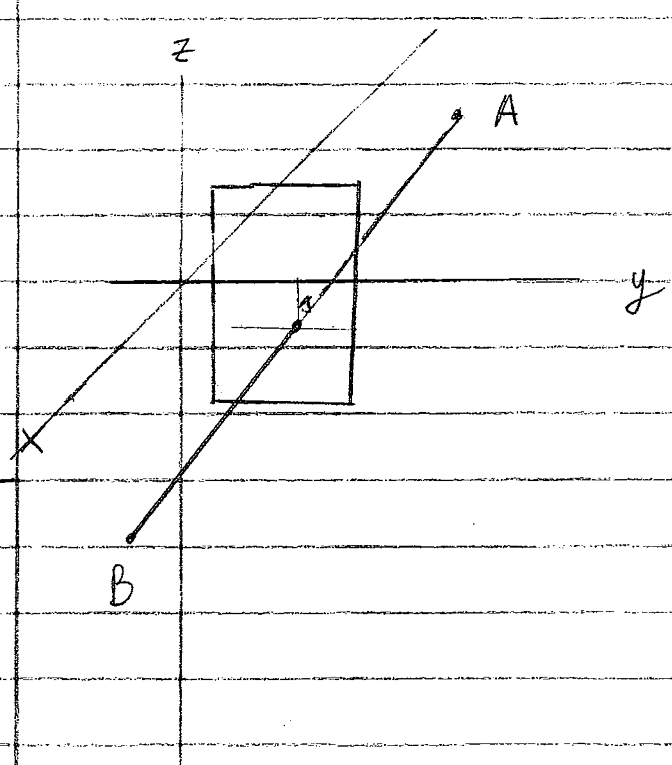
The region consisting of all points inside the sphere of radius r and centered at the origin is described by

$$x^2 + y^2 + z^2 \le r^2.$$

Hence the region consisting of all points between (but not on) the spheres of radii r and R centered at the origin is described by

$$r^2 < x^2 + y^2 + z^2 < R^2.$$

(or $r < \sqrt{x^2+y^2+z^2} < R$)



$\boxed{12.1.41}$ Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$.

Let $P(x, y, z)$ be a point in the set. Then $|AP| = |BP|$, so that

$$(x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2$$

$$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4$$

$$14x - 6y - 10z = 9.$$

Hence the set is actually a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).